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On colouring integers avoiding t-AP distance-sets

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Abstract. A t-AP is a sequence of the form $a, a+d, \dots, a+(t-1)d$, where $a, d \in \mathbb{Z}$. Given a finite set X and positive integers d, t, $a_1, a_2, \dots, a_{t-1}$, define $\nu(X,d) = |\{(x,y) : x, y \in X, y > x, y - x = d\}|$, $(a_1, a_2, \dots, a_{t-1}; d) =$ a collection X s.t. $\nu(X, d\cdot i) \geqslant a_i$ for $1 \leqslant i \leqslant t-1$.

In this paper, we investigate the structure of sets with bounded number of pairs with certain gaps. Let $(t-1, t-2, \dots, 1; d)$ be called a *t-AP distance-set* of size at least t. A k-colouring of integers $1, 2, \dots, n$ is a mapping $\{1, 2, \dots, n\} \to \{0, 1, \dots, k-1\}$ where $0, 1, \dots, k-1$ are colours. Let $ww(k,t)$ denote the smallest positive integer n such that every k-colouring of $1, 2, \dots, n$ contains a monochromatic t-AP distance-set for some $d > 0$. We conjecture that $ww(2,t) \geqslant t^2$ and prove the lower bound for most cases. We also generalize the notion of $ww(k,t)$ and prove several lower bounds.

1. Introduction

A t-AP is a sequence of the form $a, a+d, \dots, a+(t-1)d$, where $a, d \in \mathbb{Z}$. For example, $3, 7, 11, 15$ is a 4-AP with $a = 3$ and $d = 4$.

Given a finite set X and positive integers d, t, $a_1, a_2, \dots, a_{t-1}$, define

$$\nu(X,d) = |\{(x,y) : x, y \in X, y > x, y - x = d\}|,$$
$$(a_1, a_2, \dots, a_{t-1}; d) = \text{a collection } X \text{ s.t. } \nu(X, d\cdot i) \geqslant a_i \text{ for } 1 \leqslant i \leqslant t-1.$$

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The t-AP $\{x, x+d, \ldots, x+(t-1)d\}$ (say T) has $\nu(T, d\cdot i) = t-i$ for $1 \leqslant i \leqslant t-1$. On the other hand, a set $(t-1, t-2, \ldots, 1; d)$ (say Y) has $\nu(Y, d\cdot i) \geqslant t-i$ for $1 \leqslant i \leqslant t-1$, but not necessarily contains a t-AP.

A k-colouring of integers $1, 2, \ldots, n$ is a mapping $\{1, 2, \ldots, n\} \to \{0, 1, \ldots, k-1\}$ where $0, 1, \ldots, k-1$ are colours. Let $ww(k,t)$ denote the smallest integer n such that every k-colouring of $1, 2, \ldots, n$ contains a monochromatic set $(t-1, t-2, \ldots, 1; d)$ for some $d > 0$. Here, $(t-1, t-2, \ldots, 1; d)$ is a *t-AP distance-set* of size at least t. The existence of $ww(k,t)$ is guaranteed by van der Waerden's theorem [1]. Given positive integers k, t, and n, a *good k-colouring* of $1, 2, \ldots, n$ contains no monochromatic t-AP distance set. We call such a good k-colouring, a *certificate* of the lower bound $ww(k,t) > n$. We write a certificate as a sequence of n colours each in $\{0, 1, \ldots, k-1\}$, where the i-th ($i \in \{1, 2, \ldots, n\}$) colour corresponds to the colour of the integer i.

A certificate of lower bound $ww(k,t) > n$ that avoids a monochromatic arithmetic progression, may still be invalid, since it may contain a monochromatic distance set. For example, while looking for a certificate of lower bound of $ww(2,4)$, if the set $X = \{1, 2, 3, 5, 9, 10\}$ (which does not contain a 4-AP) is monochromatic, then the colouring is "bad" as $\nu(X,1) = 3$, $\nu(X,2) = 2$, and $\nu(X,3) = 1$.

In this paper, we perform computer experiements to observe the patterns of certificates of $ww(k,t) > n$. We conjecture that $ww(2,t) \geqslant t^2$ and prove the lower bound for most cases. We also generalize the notion of $ww(k,t)$ and provide several lower bounds.

2. Some values and bounds

With a primitive computer search algorithm, we have computed the following values and bounds of $ww(k,t)$. Theorem 1 gives a lower bound for $ww(k,t)$. A computed lower bound is presented only if it improves over the bound given by Theorem 1.

Theorem 1. *Given $k \geqslant 2, t \geqslant 3$, if $t \leqslant 2k+1$, then*

$$ww(k,t) > k(t-1)(t-2).$$

Proof. Let $n = k(t-1)(t-2)$ and consider the colouring

$$f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k-1\}.$$

Let $X_i = \{x \in X : f(x) = i\}$. Take the certificate

$$(0^{t-1}1^{t-1}\cdots(k-1)^{t-1})^{t-2}.$$

TABLE 1. $ww(k,t)$

k/t	3	4	5	6	7	8
2	9	19	33	43	64	85
3	17	39	>56	>67	>97	>121
4	33	$\geqslant 70$	>85	>102	>134	
5	$\geqslant 44$	>86	>135	>141	>181	>242
6	>56	>106	>175	>221	>254	>287
7	>73	>142	>214	>278	>298	>380
8	>91	>168	>246	>338	>390	>484
9	>115	>198	>302	>398	>478	>567
10	>127	>233	>365	>464	>558	>691
11	>146	>275	>417	>581	>672	>806
12	>157	>315	>474	>649	>769	>927
13	>174	>337	>550	>760	>840	>1085
14	>198	>405	>594	>828	>949	>1220
15	>229	>434	>666	>904	>1087	>1334
16	>230	>493	>784	>1015	>1236	>1517
17	>270	>525	>849	>1082	>1375	>1676
18	>298	>589	>932	>1211	>1509	>1841
19	>337	>629	>988	>1338	>1635	>2027
20	>348	>689	>1098	>1445	>1850	>2249
21	>364	>756	>1179	>1561	>2014	>2487
22	>401	>824	>1288	>1701	>2153	>2632
23	>422	>890	>1354	>1868	>2249	>2820
24	>476	>948	>1459	>1952	>2563	>3107
25	>500	>1033	>1592	>2125	>2746	>3284

We show that for each d, there exists j with $1 \leqslant j \leqslant t-1$ such that $\nu(X_i, d\cdot j) < t-j$ for each $i \in \{0,1,\ldots,k-1\}$. The largest difference between two monochromatic numbers in the certificate is

$$n-(k-1)(t-1)-1 = (t-1)(k(t-2)-k-1)-1 < k(t-1)(t-3).$$

Since the existence of a monochromatic set $(t-1, t-2, \ldots, 1; d)$ in X requires $\nu(X_i, d\cdot(t-1)) \geqslant 1$, we have $d < k(t-3)$. We have the following cases:

(a) $1 \leqslant d \leqslant k-1$: Take $x, y \in \{1,2,\ldots,n\}$ such that $y = x + d(t-1)$. But by our choice of d, we have $f(y) = (f(x)+d) \mod k \neq f(x)$, that is, x and y cannot be monochromatic. So, $\nu(X_i, (t-1)\cdot d) = 0 < 1$ for each $i \in \{0,1,\ldots,k-1\}$.

(b) $k \leqslant d \leqslant t-3$: Take $x, y \in \{1,2,\ldots,n\}$ such that $y = x + d(t-a)$ where a is such that $(t-3)(t-a) \leqslant (t-1)(k-1)$ and $k(t-a) \geqslant k$, which gives us a bound

$$t - \frac{(t-1)(k-1)}{t-3} \leqslant a \leqslant t-k.$$

Such an a exists since

$$\frac{(t-1)(k-1)}{t-3} = k-1+\frac{2(k-1)}{t-3} \geqslant k-1+\frac{2(k-1)}{(2k+1)-3} \geqslant k.$$

(c) $d = t-2$: In each block of $t-1$ colours, there is one pair of integers at distance $t-2$, and there are $t-2$ such blocks for each colour. So, $\nu(X_i, 1\cdot d) = \nu(X, t-2) = t-2 < t-1$ for each $i \in \{0,1,\ldots,k-1\}$.

(d) $(t-1) \leqslant d \leqslant (k-1)(t-1)$: Take $x, y \in \{1,2,\ldots,n\}$ such that $y = x+d = x+q(t-1)+r$ where $1 \leqslant q \leqslant k-1$ and $0 \leqslant r \leqslant t-2$. Suppose $x = q_x(t-1)+r_x$ with $0 \leqslant r_x \leqslant t-2$. Then

$$\begin{aligned} f(x) &= q_x \mod k; \\ f(y) &= (f(x)+q+\lfloor (r+r_x)/(t-1)\rfloor) \mod k. \end{aligned}$$

If $r > 0$, then $q \leqslant k-2$, which implies $q+\lfloor (r+r_x)/(t-1)\rfloor \leqslant (k-2)+1 = k-1$. If $r=0$, then $q \leqslant k-1$, which implies $q+\lfloor (0+r_x)/(t-1)\rfloor \leqslant (k-1)+0 = k-1$. Therefore, $f(y) \neq f(x)$; and x and y cannot be monochromatic. So, $\nu(X_i, d\cdot 1) = 0 < t-1$ for each $i \in \{0,1,\ldots,k-1\}$.

Since $t \leqslant 2k+1$, we have

$$\begin{aligned} d < k(t-3) = kt-3k &= (kt-k-2k) \\ &\leqslant (kt-k-(t-1)) = (k-1)(t-1). \end{aligned}$$

Hence, we are done and there is no monochromatic t-AP distance set in X. □

Conjecture 1. For $t \geqslant 3$, $ww(2,t) \geqslant t^2$.

Lemma 1. *For $t \geqslant 3$ and $t \neq 2^u$ with $u \geqslant 2$, $ww(2,t) \geqslant t^2$.*

Proof. Let $t = 2^u + v$ with $1 \leqslant v \leqslant 2^u - 1$. Let $n = t^2-1$ and $X = \{1,2,\ldots,n\}$, and consider the colouring $f : X \to \{0,1\}$. Let $m = n-1-(t-1) = q\cdot 2^u + r$ with $0 \leqslant r \leqslant 2^u-1$.

Now, take the certificate

$$\begin{cases} 01^{t-1}(0^{2^u}1^{2^u})^{q/2}0^r, & \text{if } q \equiv 0 \pmod 2; \\ 01^{t-1}(0^{2^u}1^{2^u})^{\lfloor q/2\rfloor}0^{2^u}1^r, & \text{if } q \equiv 1 \pmod 2. \end{cases}$$

We need to show that for each d, there exists j with $1 \leqslant j \leqslant t-1$ such that $\nu(X, d \cdot j) < t-j$. Since the existence of a monochromatic set $(t-1, t-2, \ldots, 1; d)$ in X requires $\nu(X, d \cdot (t-1)) \geqslant 1$, we have $d(t-1) < t^2-1$, that is, $1 \leqslant d \leqslant t$. Let $X_i = \{x \in X : f(x) = i\}$.

Suppose $q \equiv 0 \pmod 2$. Then we have the following two cases:

(a) $d \equiv 1 \pmod 2$: Take $x, y \in X$ such that $y = x + d \cdot 2^u$.

(a1) $v+1 \leqslant x < y \leqslant n-r$: Since $f(x) \in \{0,1\}$ and $f(x+1 \cdot 2^u) = (f(x)+1) \mod 2 \neq f(x)$, we have $f(y) = f(x + d \cdot 2^u) \neq f(x)$. So, two monochromatic integers cannot both be in $\{v+1, v+2, \ldots, n-r\}$.

(a2) $2 \leqslant x \leqslant v$ and $y \leqslant n-r$: Since $f(x) = 1$ and $f(x+1 \cdot 2^u) = 1 = f(x)$, we have $f(y) = (x + d \cdot 2^u) = f(x)$. So, there are exactly $v-1$ pairs of integers with colour 1 at distance $d \cdot 2^u$.

(a3) $x = 1$ and $y \leqslant n-r$: Since $f(x) = 0$, $f(x+1 \cdot 2^u) = 1 \neq f(x)$, and $1 + 2^u > v$, using case (i) we have 0 pair of integers with colour 0 at distance $d \cdot 2^u$.

(a4) $x \geqslant 1$ and $n-r+1 \leqslant y \leqslant n$: Since $f(y) = 0$ and $r < 2^u$, we have $f(y - 1 \cdot 2^u) = 1$, which implies $f(y - d \cdot 2^u) = 1 \neq f(y)$. That is, adding r trailing zeros does not change the number of monochromatic pairs at distance $d \cdot 2^u$.

Therefore, for each $i \in \{0,1\}$, we have $\nu(X_i, d \cdot 2^u) \leqslant v-1 < v = t-2^u$.

(b) $d \equiv 0 \pmod 2$: Let $d = 2^w \cdot d_o$, with d_o being an odd number and $w \geqslant 1$. Then $\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) \leqslant v-1 < v = t-2^u$ (by case (i)) for each $i \in \{0,1\}$.

The case $q \equiv 1 \pmod 2$ is similar. □

Lemma 2. *Suppose $t = 2^u$ for some $u \geqslant 2$ and $t-1$ is prime. Then for each $r \in \{1, 2, \ldots, u\}$ and for each $d_o \in \{1, 3, \ldots, 2^{u-r}-1\}$, there exists $s \in \{1, 2, 3, \ldots, d_o - 1\}$ such that d_o divides $s(t-1) - 2^{u-r}$.*

Proof. Since $t-1$ is prime, we have $gcd(t-1, d_o) = 1$, and hence the linear congruence $(t-1)s \equiv 2^{u-r} \pmod{d_o}$ has a solution. Extended Euclid Algorithm yields $x, y \in \mathbb{Z}$ such that $(t-1) \cdot x + d_o \cdot y = 1$. Then $s = (x \cdot 2^{u-r}) \mod d_o$. Since $d_o \nmid x$ and $d_o \nmid 2^{u-r}$, we have $s \neq 0$. Hence $s \in \{1, 2, \ldots, d_o - 1\}$. □

Lemma 3. *If $t = 2^u$ for $u \geqslant 2$ with $t-1$ prime, then $ww(2,t) \geqslant t^2$.*

Proof. Take the certificate $0(1^{t-1}0^{t-1})^{t/2}1^{t-2}$. We have the following two cases:

(a) $d \equiv 1 \pmod 2$: Take $x, y \in X$ such that $y = x + d \cdot (t-1)$.

(a1) $2 \leqslant x \leqslant t^2 - t + 1 = n - (t-2)$: Since $f(x) \in \{0,1\}$ and $f(x + 1 \cdot (t-1)) = (f(x)+1) \mod 2 \neq f(x)$, we have $f(y) = f(x + d \cdot (t-1)) \neq f(x)$. So, two monochromatic integers cannot both be in $\{2, 3, \ldots, n-(t-2)\}$.

(a2) $x = 1$ and $y \leqslant n - (t-2)$: Since $f(x) = 0$, $f(x + 1 \cdot (t-1)) = 1 \neq f(x)$, using case (i) we have 0 pair of integers with colour 0 at distance $d \cdot (t-1)$.

(a3) $x \geqslant 1$ and $n - (t-2) + 1 \leqslant y \leqslant n$: Since $f(y) = 0$, we have $f(y - 1 \cdot (t-1)) = 1$, which implies $f(y - d \cdot (t-1)) = 1 \neq f(y)$. That is, adding $t-2$ trailing ones does not change the number of monochromatic pairs at distance $d \cdot (t-1)$.

Therefore, for each $i \in \{0,1\}$, we have $\nu(X_i, d \cdot (t-1)) = 0 < 1$.

(b) $d \equiv 0 \pmod 2$: For $d \in \{2, 4, \ldots, t\}$ and $j \in \{1, 2, \ldots, t-1\}$,

$$2 \leqslant d \cdot j \leqslant t(t-1) = t^2 - t.$$

For a given $d \in \{2, 4, \ldots, t\}$, we show that there exists (j, w) with $j \in \{1, 2, \ldots, t-1\}$ and $w \in \{1, 3, 5, \ldots, t-1\}$ such that $d \cdot j = w(t-1) - 1$. In that case, since $d \cdot j \leqslant t(t-1)$, we have $w \leqslant t$; and also since $d \cdot j$ is even, $w(t-1)$ is odd, which implies w is odd, that is, $w \in \{1, 3, 5, \ldots, t-1\}$.

Let $d = 2^r d_o$ with $1 \leqslant r \leqslant u$ and odd number $d_o \in \{1, 3, \ldots, 2^{u-r} - 1\}$ (since $d \leqslant t = 2^u$). For a w to exist and satisfy $d \cdot j = w(t-1) - 1$, we need

$$2^r d_o \cdot j = w(t-1) - 1 = wt - (w+1),$$

that is, 2^r divides $w + 1$ (since 2^r divides $wt = w2^u$). Let $w = s \cdot 2^r - 1$ with $s \in \{1, 2, \ldots, d_o - 1\}$. The chosen s requires to satisfy that d_o divides $(w(t-1) - 1)/2^r = s(t-1) - 2^{u-r}$. By Lemma 2, such an s exists.

It can be observed that for a given $d \in \{2, 4, \ldots, t\}$, if $d \cdot j_1 = w_1 \cdot (t-1) - 1$ for some $j_1 \in \{1, 2, 3, \ldots, t-1\}$ and $w_1 \in \{1, 3, 5, \ldots, t-1\}$, then

$$d \cdot j_2 = w_2 \cdot (t-1) + 1,$$

with $j_1 + j_2 = t - 1$ and $w_1 + w_2 = d$. We claim that $\nu(X_1, d \cdot t_i) < t - j_i$ for at least one $i \in \{1, 2\}$. If $\nu(X_1, d \cdot j_1) < t - j_1$, then we are done. Suppose

$$\nu(X_1, d \cdot j_1) = \nu(X_1, w_1(t-1) - 1) = \frac{t}{2} - \frac{w_1 - 1}{2} \geqslant t - j_1,$$

which implies $t/2 + (w_1 - 1)/2 \leqslant j_1$. Now,

$$\begin{aligned} \nu(X_1, d \cdot j_2) = \nu(X_1, w_2(t-1) + 1) &= \frac{t}{2} - \frac{w_2 - 1}{2} = \frac{t}{2} - \frac{d - w_1 - 1}{2} \\ &= \frac{t}{2} + \frac{w_1 - 1}{2} + 1 - \frac{d}{2} \leqslant j_1 + 1 - \frac{d}{2} = t - 1 - j_2 + 1 - \frac{d}{2} < t - j_2. \end{aligned}$$

Similarly, we can show that $\nu(X_0, d\cdot j) < t - j$ for some $j \in \{1, 2, \ldots, t-1\}$. $\square$

3. Generalized distance-sets

Here we consider variants of $ww(k,t)$ with different variations of parameters in a distance set.

Let $gww(k,t;a_1,a_2,\ldots,a_{t-1})$ (with $a_i \geqslant 1$) denote the smallest positive integer n such that any k-colouring of $1,2,\ldots,n$ contains monochromatic set $(a_1,a_2,\ldots,a_{t-1};d)$ for some $d>0$. In this definition,

$$gww(k,t;t-1,t-2,\ldots,1) = ww(k,t).$$

Observation 1. Let us write $gww(2,t;a_1,a_2,\ldots,a_{t-1})$ as $gww(2,t,r)$, where $a_i = r$ for $1 \leqslant i \leqslant t-1$. It is trivial that $gww(2,t,t-1) \geqslant ww(2,t)$. Table 2 contains a few computed values of $gww(2,t,r)$.

TABLE 2. $gww(2,t,r)$

t/r	1	2	3	4	5	6	7	8
3	9	13						
4	13	21	29					
5	33	37	41	49				
6	41	45	57	65	74			
7	49	53	69	85	92	$\geqslant 96$		
8	57	61	85	105	114	> 118	> 123	
9	129	133	137	> 140	> 144	> 148	> 152	> 156
10	145	149	153					
11	161	165	169					
12	177	181	185					
13	193	197	> 200					
14	209	213	> 216					
15	225	229	> 232					
16	241	245	> 248					
17	513	> 516						
33	> 2048	> 2052						
65	> 8192							

Lemma 4. *For $u \geqslant 1$ and $1 \leqslant v \leqslant 2^u$,*

$$gww(2, 2^u + v, 1) \geqslant (2^u + v - 1)2^{u+1} + 1.$$

Proof. Consider $t = 2^u + v$ $(t \geqslant 5)$ and let $n = (2^u + v - 1)2^{u+1} = (t-1)2^{u+1}$ and $X = \{1,2,\ldots,n\}$. Consider the colouring $f : X \to \{0,1\}$ and take the certificate $(0^{2^u}1^{2^u})^{t-1}$.

Let $X_i = \{x \in X : f(x) = i\}$. We claim that this 2-colouring of X does not contain a monochromatic set $(1, 1, \ldots, 1; d)$ for any $d > 0$, that is, for each d with $1 \leqslant d < 2^{u+1}$ and for each $i \in \{0, 1\}$, there exists $j \in \{1, 2, \ldots, t-1\}$ such that $\nu(X_i, d \cdot j) = 0$.

(a) $d \equiv 1 \pmod 2$: Take $x, y \in X$ such that $y = x + d \cdot 2^u$. Since d is odd, if $f(x) = 0$, then $f(x + d \cdot 2^u) = 1$ and vice-versa. Hence, $\nu(X_i, d \cdot 2^u) = 0$ for each $i \in \{0, 1\}$

(b) $d \equiv 0 \pmod 2$: Let $d = 2^w \cdot d_o$, with d_o being an odd number and $w \geqslant 1$. Then for each $i \in \{0, 1\}$,

$$\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) = 0 \text{ (by case } (a)).$$

So, X does not contain a monochromatic set $(1, 1, \ldots, 1; d)$ for any $d > 0$. □

Conjecture 2. For $u \geqslant 1$ and $1 \leqslant v \leqslant 2^u$,

$$gww(2, 2^u + v, 1) = (2^u + v - 1)2^{u+1} + 1.$$

Lemma 5. *For $u \geqslant 2$ and $1 \leqslant v \leqslant 2^u$,*

$$gww(2, 2^u + v, 2) \geqslant (2^u + v - 1)2^{u+1} + 5.$$

Proof. Let $t = 2^u + v$ $(t \geqslant 5)$, $n = (2^u + v - 1)2^{u+1} + 4 = (t-1)2^{u+1} + 4$, and $X = \{1, 2, \ldots, n\}$. Consider the colouring $f : X \to \{0, 1\}$ and take the certificate $000(10^{2^u-3}1101^{2^u-3}00)^{t-2}(10^{2^u-3}11)(01^{2^u-3})011$. We show that this colouring of X does not contain a monochromatic set $(2, 2, \ldots, 2; d)$ for any $d > 0$, that is, for each d with $1 \leqslant d \leqslant 2^{u+1}$ and for each $i \in \{0, 1\}$, there exists $j \in \{1, 2, \ldots, t-1\}$ such that $\nu(X_i, d \cdot j) \leqslant 1$.

Note that the largest difference between two integers with colour 0 in the colouring is $(n-2) - 1 = n - 3 = (t-1) \cdot 2^{u+1} + 1 = p$ (say); and the largest difference between two integers with colour 1 in the colouring is $n - 4 = (t-1) \cdot 2^{u+1} = p - 1$.

(a) $d = 2^{u+1}$: Note that $d \cdot (t-1) = (t-1)2^{u+1} = n - 4 = p - 1$. The only pair (x, y) with $f(x) = f(y) = 0$ and $y = x + d \cdot (t-1)$ is $(1, n-3)$ and the only pair (x, y) with $f(x) = f(y) = 1$ and $y = x + d \cdot (t-1)$ is $(4, n)$. Hence, we have $\nu(X_i, d \cdot (t-1)) = 1$ for each $i \in \{0, 1\}$.

(b) $d \equiv 1 \pmod 2$: Write the certificate as

$$000A_0A_1 \ldots A_{2t-5}, A_{2t-4}C11,$$

where $A_i = 10^{2^u-3}11$ if $i \equiv 0 \pmod 2$, $A_i = 01^{2^u-3}00$ if $i \equiv 1 \pmod 2$, and $C = 01^{2^u-3}0$. Take $x, y \in \{4, 5, \ldots, n - 2^u - 2\}$ such that $y = x + d \cdot 2^u$

for some odd $d < 2^{u+1} + 1$. Suppose $x = 3 + i \cdot 2^u + j$, that is, $f(x)$ is the j-th ($1 \leqslant j \leqslant 2^u$) bit in A_i. Then $y = 3 + (i + d) \cdot 2^u + j$, that is, $f(y)$ is the j-th bit in A_{i+d}. If $i \equiv 1 \pmod 2$, then $(i + d) \equiv 0 \pmod 2$ (since d is odd), and vice-versa. Therefore, $f(x) \neq f(y)$. So, two monochromatic integers at distance $d \cdot 2^u$ cannot both be in $\{4, 5, \dots, n - 2^u - 2\}$.

Now, take $x, y \in \{4, 5, \dots, n-2\}$ such that $y = x + d \cdot 2^u$ and $y \in \{n - 2^u - 1, n - 2^u, \dots, n-2\}$. Since $|C| < 2^u$, x must be in $\{4, 5, \dots, n - 2^u - 2\}$. With similar reasoning as above, it can be shown that two monochromatic integers at distance $d \cdot 2^u$ cannot both be in $\{4, 5, \dots, n - 2\}$. Following are the remaining cases:

(b1) If $x = 1$, then $x + d \cdot 2^u = 3 + d \cdot 2^u - 2 = 3 + (d-1) \cdot 2^u + (2^u - 2) = y$ (say). We have $f(x) = 0$. Again $(2^u - 2)$-th bit in A_{d-1} is also zero since d is odd and $A_{d-1} = 10^{2^u-3}11$.

(b2) If $x = 2, 3$ (where $f(x) = 0$), then with similar reasoning as above, $f(x + d \cdot 2^u) = 1$.

(b3) If $y = n - 1$ (where $f(y) = 1$), then

$$\begin{aligned} y - d \cdot 2^u &= (t-1)2^{u+1} + 3 - d \cdot 2^u \\ &= 3 + (2t - 3 - d)2^u + 2^u = x \text{ (say)}. \end{aligned}$$

Since d is odd, $2t - 3 - d$ is even, that is, $A_{2t-3-d} = 10^{2^u-3}11$. The 2^u-th element in A_{2t-3-d} is one, that is $f(x) = 1$.

(b4) If $y = n$ (where $f(y) = 1$), then

$$\begin{aligned} y - d \cdot 2^u &= (t-1)2^{u+1} + 4 - d \cdot 2^u \\ &= 3 + (2t - 2 - d)2^u + 1 = x \text{ (say)}. \end{aligned}$$

Since d is odd, $2t - 2 - d$ is odd, that is, $A_{2t-2-d} = 01^{2^u-3}00$. The 1-st element in A_{2t-2-d} is zero, that is $f(x) = 0$.

Hence $\nu(X_i, d \cdot 2^u) \leqslant 1$ for each $i \in \{0, 1\}$.

(c) Otherwise ($d \neq 2^{u+1}$ and $d \equiv 0 \pmod 2$): Let $d = 2^w \cdot d_o$, with d_o being an odd number and $w \geqslant 1$. Then for each $i \in \{0, 1\}$,

$$\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) \leqslant 1 \text{ (by case } (b)).$$

Therefore, X does not contain a monochromatic set $(2, 2, \dots, 2; d)$ for any $d > 0$. $\square$

Conjecture 3. For $r \geqslant 2$ and $t \geqslant 2^r + 1$,

$$gww(2, t, r) = (t-1)2^{\lfloor \log_2 (t-1) \rfloor + 1} + 2^r + 1.$$

Observation 2. We observe the following experimental results:
(a) Primitive search gives $gww(2, 10, 9) > 186$;
(b) Using the certificate

$$\begin{cases} 01^{t+1}(0^{t-1}1^{t-1})^{t/2+1}, & \text{if } t \equiv 0 \pmod 2; \\ 01^{t+1}(0^{t-1}1^{t-1})^{\lfloor t/2 \rfloor+1}0^{t-1}, & \text{if } t \equiv 1 \pmod 2, \end{cases}$$

we obtain the following lower bounds with $12 \leqslant t \leqslant 48$:

$$\begin{array}{ccc} gww(2, 12, 11) > 168, & gww(2, 14, 13) > 224, & gww(2, 17, 16) > 323, \\ gww(2, 18, 17) > 360, & gww(2, 20, 19) > 440, & gww(2, 24, 23) > 624, \\ gww(2.30, 29) > 960, & gww(2, 32, 31) > 1088, & gww(2, 33, 32) > 1155, \\ gww(2, 38, 37) > 1520, & gww(2, 42, 41) > 1848, & gww(2, 44, 43) > 2024. \end{array}$$

(c) $0^{35}(1^{18}0^{18})^2 1^{20}0^{17}(1^{19}0^{18})^5 1^{19}0^{17}1^{20}0^{10}1^4$ proves $gww(2, 19, 18) > 399$.
(d) $0^{41}1^{21}0^{21}(1^{22}0^{21})^{10}1^{15}$ proves $gww(2, 22, 21) > 528$.

Conjecture 4. For $t \geqslant 4$, $gww(2, t, t-1) \geqslant (t+1)^2$.

Conjecture 5. For $t \geqslant 5$, $gww(2, t, t-1) < 2^{t+1}$.

We do not have enough data to make stronger upper bound conjecture for $gww(2, t, t-1)$, but it may be possible that $gww(2, t, t-1) < t^3$.

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RESEARCH ARTICLE

Indecomposable and irreducible t-monomial matrices over commutative rings

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Abstract. We introduce the notion of the defining sequence of a permutation indecomposable monomial matrix over a commutative ring and obtain necessary conditions for such matrices to be indecomposable or irreducible in terms of this sequence.

Introduction

Let K be a commutative ring (with unity). By a *monomial matrix* $M = (m_{ij})$ over K we mean a quadratic $n \times n$ matrix, in each row and each column of which there is exactly one non-zero element. With such matrix M one can associate the directed graph $\Gamma(M)$ with n vertices numbered from 1 to n and arrows $i \to j$ for all $m_{ij} \neq 0$. Obviously, $\Gamma(M)$ is the disjoint union of cycles, each of which has the same direction of arrows. If there is only one cycle, the monomial matrix M is called *cyclic* (in other words, it is a permutation indecomposable monomial matrix). A cyclic matrix of the form

$$M = \begin{pmatrix} 0 & \dots & 0 & m_{1n} \\ m_{21} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & m_{n,n-1} & 0 \end{pmatrix}$$

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we call *canonically cyclic*. The sequence

$$v = v(M) = (m_{21}, \dots, m_{n-1,n}, m_{1n})$$

we call the *defining sequence of* M, and write

$$M = M(v) = M(m_{21}, \dots, m_{n-1,n}, m_{1n}).$$

The sequence $v^* = v^*(M) = (m_{1n}, m_{n-1,n}, \dots, m_{21})$ is called *dual to* v and the matrix $M^* = M(v^*)$ *dual to* M.

When all elements m_{ij} of a monomial (respectively, cyclic or canonically cyclic) matrix M are of the form $t^{s_{ij}}$ $(t \in K)$, where $s_{ij} \geqslant 0$, the matrix M is called *t-monomial* (respectively, *t-cyclic* or *canonically t-cyclic*); obviously, then $t^{s_i} \neq 0$ for all i.

The most interesting cases are, obviously, those when the element t is non-invertible.

Matrices of such form were studied by the authors in [1], and in this paper we continue our investigation.

1. Defining sequences and indecomposability

Through this section K denotes a commutative local ring with maximal ideal $R = Rad\,K \neq 0$ and $t \in R$. All matrices are considered over K. By E_s one denotes the identity $s \times s$ matrix.

1.1. Permutation similarity. Let

$$M = \begin{pmatrix} 0 & \dots & 0 & t^{s_n} \\ t^{s_1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & t^{s_{n-1}} & 0 \end{pmatrix} \qquad (*)$$

be a canonically t-cyclic matrix. A permutation of the members $x_i = t^{s_i}$ of $v(M)$ of the form $x_i, x_{i+1}, \dots, x_m, x_1, x_2, \dots, x_{i-1}$, is called a *cyclic permutation*. Two matrices $M(v)$ and $M(v')$ is called *cyclically similar* if v can be obtained from v' by a such permutation.

It is easy to prove the following statement[1].

Proposition 1. *a) Two canonically t-cyclic matrices is permutation similar if and only if they are cyclically similar.*

b) The matrix transpose to a canonically t-cyclic matrix is permutation similar to the dual one.

[1]This statement is valid for matrices with elements from any set if one modifies the definitions accordingly.

1.2. Conjecture. Let $M(v)$ be a canonically t-cyclic $n\times n$ matrix with defining sequence $v=(x_1,x_2,\dots,x_n)$, where $x_i=t^{s_i}$ (see $(*)$).

The sequence v is called *periodic* with a period $0<p<n$ if $p|n$ and $x_{s+p}=x_s$ for any $1\leqslant s\leqslant n-p$, and *non-periodic* if otherwise. In the case of v to be periodic, the matrix $M(v)$ can be reduced by a permutation of its rows and column to the following block-monomial form: $N=(N_{ij})_{i,j=1}^{m}$, $m=n/p$, where $N_{21}=x_2E_m, N_{32}=x_3E_m,\dots,N_{n,n-1}=x_nE_m$ and $N_{1m}=x_1M(1,1,\dots,1)$ with 1 to occur m times[2]. The $m\times m$ matrix $M(1,1,\dots,1)$ can be indecomposable or decomposable depending on properties of the ring K, and therefore so can be the matrix $M(v)$.

Conjecture 6 (V. M. Bondarenko, Private Communication)**.** *Any canonically t-cyclic matrix over K with non-periodic defining sequence is indecomposable.*

It is obvious that the idea of a proof of this conjecture basing on decomposition of a given matrix M into a direct sum of others ones, with a final contradiction, is futile. It is most likely the best idea is to use the simple fact that M is indecomposable if any idempotent matrix X such that $MX=XM$ is identity or zero. The main difficulty in this way is that the non-periodicity of the defining sequence of M is not determined by its local properties. In the next subsection we consider a special case in which the condition of the non-periodicity is satisfied automatically.

1.3. 2-homogeneous defining sequences. Let $M(v)$ be again a canonically t-cyclic $n\times n$ matrix with defining sequence $v=(x_1,x_2,\dots,x_n)$, where $x_i=t^{s_i}$. The sequence v is said to be 2-homogeneous if it is translated by a cyclic permutation on that of the form $(a,a,\dots,a,b,b,\dots,b)$, where a and $b\neq a$ both actually occur.

The following theorem proves Conjecture 1 for this case.

Theorem 1. *Any canonically t-cyclic matrix over K with 2-homogeneous defining sequence is indecomposable.*

Proof. Let $M=M(v)$ be a canonically t-cyclic $n\times n$ matrix and let v has s coordinates equal to $a=t^p$ and the other ones equal to $b=t^q$. Assume without loss of generality that $v=(a,a,\dots,a,b,b,\dots,b)$ and $p<q$. So $v=t^pv_0$, where $v_0=(1,1,\dots,1,t^{q-p},t^{q-p},\dots,t^{q-p})$. After replacing v

[2]To do it, one is to arrange the rows and columns in the order 2, 2+p, 2+2p,. . . , 2+(m-1)p, 3, 3+p, 3+2p,. . . , 3+(m-1)p,. . . , p, 2p, 3p,. . . , mp, p+1, (p+1)+p, (p+1)+2p,. . . , (p+1)+(m-2)p,1.

by v_0 and t^{q-p} by t (again without loss of generality), we come to the situation where

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & t \\ 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \end{pmatrix}$$

with s elements to equal 1 and $k = n - s$ elements equal to t.

Arrange the rows and columns of the matrix M in the order $1, 2, \dots,$ $n-k, n, n-1, \dots, n-k+2, n-k+1$, denoting the new matrix by N:

$$N = \left(\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right) = \left(\begin{array}{cccc|cccc} 0 & \dots & 0 & 0 & t & 0 & \dots & 0 \\ 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & 0 & 0 & t & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & t \\ 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right).$$

Prove that there is not a non-trivial idempotent matrix commuting with N (see the previous subsection).

Let C be an $n \times n$ matrix such that $NC = CN$, where

$$C = \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} c_{11} & \cdots & c_{1,n-k} & c_{1,n-k+1} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n-k,1} & \cdots & c_{n-k,n-k} & c_{n-k,n-k+1} & \cdots & c_{n-k,n} \\ \hline c_{n-k+1,1} & \cdots & c_{n-k+1,n-k} & c_{n-k+1,n-k+1} & \cdots & c_{n-k+,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{n,n-k} & c_{n,n-k+1} & \cdots & c_{nn} \end{array}\right)$$

with the same partition as N.

We denote by (i, j) the scalar equality $(NC)_{ij} = (CN)_{ij}$. All comparisons below are considered modulo the maximal ideal $R = Rad\, K$ (that is, they are equalities over the residue field K/R). We use by default the following simple fact: $tx = ty$ implies $x \equiv y$ $(x, y \in K)$.

Since every jth column of CN with $j > n - k$ consists of elements from tK, we have from the equations (i, j) for $i = 2, 3, \ldots, n - k$ and $i = n$ that

$$C \equiv \left(\begin{array}{c|c} C_{11} & 0 \\ \hline C_{21} & C_{22} \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} c_{11} & \cdots & c_{1,n-k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n-k,1} & \cdots & c_{n-k,n-k} & 0 & \cdots & 0 \\ \hline c_{n-k+1,1} & \cdots & c_{n-k+1,n-k} & c_{n-k+1,n-k+1} & \cdots & c_{n-k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{n,n-k} & c_{n,n-k+1} & \cdots & c_{nn} \end{array}\right).$$

The equations (i, j) for $i, j = 1, 2, \ldots, n - k$ mean that the matrix C_{11} commutes modulo R with the lower Jordan block N_{11} and hence the matrix C_{11} is lower triangular modulo R (see, e. g., [2, Chap. VIII] or [3, Theorem 3.2.4.2]).

Further, from the equlities

$$(n - k + i, n - k + j + 1) : tC_{n-k+i+1,n-k+j+1} = tC_{n-k+i,n-k+j}$$

for $1 \leqslant i < j \leqslant k - 1$ it follows that all elements of the matrix C_{22} belonging to its lth upper diagonal[3] are pairwise comparable modulo R, $1 \leqslant l \leqslant k - 2$. But since the equalities

$$(n - k + i, n - k) : tC_{n-k+i+1,n-k} = C_{n-k+i,n},$$

$1 \leqslant i \leqslant k - 1$, imply that the last elements of all (mentioned above) upper diagonals are comparable with 0, we have eventually that the matrix C_{22} is, as well as C_{11} (see above), upper triangular modulo R.

Finally, from the equalities

I. $(2, 1) : c_{11} = c_{22}, \quad (3, 2) : c_{22} = c_{33}, \ldots,$
$(n - k, n - k - 1) : c_{n-k-1,n-k-1} = c_{n-k,n-k},$

II. $(n - k + 1, n - k + 2) : tc_{n-k+1,n-k+1} = tc_{n-k+2,n-k+2},$
$(n - k, n - k + 1) : tc_{n-k,n-k} = tc_{n-k+1,n-k+1}, \ldots,$
$(n - 1, n) : tc_{n-1,n-1} = tc_{n,n},$

[3]The lth upper diagonal of a matrix $M = (m_{ij})$, where $l \geqslant 1$, is the collection of elements $m_{i,i+l}$.

III. $(n, n-k) : c_{n-k,n-k} = c_{n,n}$,
it follows that $c_{11} \equiv c_{22} \equiv \cdots \equiv c_{nn}$.

Thus we prove that the matrix C is comparable to an upper triangular one with the same elements on the main diagonal. It easily follows that if $C^2 \equiv C$, then $C \equiv E_n$ or $C \equiv 0$, and, consequently, because the comparisons are modulo the only maximal ideal of K, $C = E_n$ or $C = 0$, respectively. $\square$

1.4. Applications in the representation theory of groups. Through this subsection K is as above and of characteristic p^s (p is simple, $s \geqslant 1$). All groups G are assumed to be finite of order $|G| > 1$. The number of nonequivalent indecomposable matrix K-representations of degree n of a group G is denoted by $\mathrm{ind}_K(G, n)$.

From [4] it follows that $\mathrm{ind}_K(G, n) \geqslant |K/R|$ for any p-group G of order $|G| > 2$ and $n > 1$. Here we strengthen this result in the case of both cyclic groups and radicals.

Theorem 2. *Let $R = tK \neq 0$ with t being nilpotent. Then, for a cyclic p-group G of some order N (hence of greater order), $\mathrm{ind}_K(G, n) \geqslant n - 1$ for any $n > 1$.*

Proof. Let S be an $n \times n$ matrix over K that is nilpotent modulo R; then $S^n \equiv 0 \, (mod \, R)$ and $S^{2n} \equiv 0 \, (mod \, R^2)$. It is easy to see that the map $\Gamma_S : a \to \Gamma_S(a) = E_n + uS$ with $u = t^{m-2}$ is a K-representation of a cyclic group $G = \langle a \rangle$ of an order $p^r \geqslant 2n$, $r \geqslant 2$. It is indecomposable if and only if so is modulo $Ann \, u := \{x \in K \mid t^{m-2}x = 0\} = R^2$, and representations Γ_S and $\Gamma_{S'}$ are equivalent if and only if the matrices S and S' are similar modulo R^2.

Consider the K-representations Γ_{M_k}, $k = 1, 2, \ldots, n-1$, with the matrices $M_k = M(1, \ldots, 1, t \ldots, t)$, where 1 occurs k times (see the previous subsection). They are non-equivalent, because M_k has rank k modulo R, and indecomposable by Theorem 1 (taking into account that $Rad \, (K/Ann \, u) = R/R^2$ is a principal ideal of K/R^2 generated by $t + R^2 \neq R^2$). $\square$

Theorem 3. *Let the characteristic of K is p and $R = tK \neq 0$ with $t^2 = 0$. Then, for any cyclic p-group G and $n \geqslant |G|$, $\mathrm{ind}_K(G, n) \geqslant |G| - 2$.*

Proof. We use the notation of the proof of Theorem 2. It is easy to see that, for $0 < k < n$, the map $\Lambda_k : a \to \Lambda(a) = E_n + M_k$ is a K-representation of a cyclic group $G = \langle a \rangle$ if $k + 2 \leqslant |G|$; in particular, if $0 < k < |G| - 1 \leqslant n - 1$. By the last condition, their number is equal to

$|G|-2$. The representations Λ_k are indecomposable by Theorem 1, and non-equivalent (see the previous proof). $\square$

2. Defining sequences and irreducibility

Through this section K also denotes a commutative local ring with maximal ideal $R = Rad\, K \neq 0$; $t \neq 0$ is any element of R unless otherwise stated.

For an $n \times n$ matrix A over K, we denote by $[i \overset{a}{\to} j]^+$ (resp. $[i \overset{a}{\to} j]^-$) the following similarity transformation of A: adding ith row (resp. column), multiplied by a, to jth row (resp. column), and then subtracting jth column (resp. rows), multiplied by a, from ith column (resp. row).

2.1. Theorems on irreducible canonically t-cyclic matrices. Let $M = M(v)$ be a canonically t-cyclic $n \times n$ matrix with defining sequence $v = (x_1, x_2, \dots, x_n)$, where $x_i = t^{s_i}$ (see $(*)$). We call the sequence $v_0 = v_0(M) = (s_1, s_2, \dots, x_n)$ the *weighted sequence of* M, and the number $w = w(M) = s_1 + s_2 + \dots + s_n$ the *weight of* M[4].

From the results of [1] it follows the next theorem[5].

Theorem 4. *If the matrix $M(v)$ is irreducible, then its weight is prime to n.*

Corollary 1. *If $t^2 = 0$ and the matrix $M(v)$ is irreducible, then its rank modulo R is prime to n.*

By a *connected subsequence of the length* $1 \leqslant l \leqslant n$ of a defining sequence $v = (x_1, x_2, \dots, x_n)$ we mean any subsequence which maps by a cyclic permutation on one of the form $x_1, x_2, \dots, x_l$. If it is 2-homogeneous i. e., by analogy with the above said, has the form $u = (t^i, t^i, \dots, t^i, t^j, t^j, \dots, t^j), t^i \neq t^j$, where t^i and t^j both actually occur, then the pair (p, q), consisting of the numbers of occurrences of t^i and t^j we call the *type of* u.

We shall obtain the following theorem as a consequence of statements in more general situations.

Theorem 5. *If the matrix $M(v)$ with 2-homogeneous defining sequence v is irreducible and $n > 5$, $t^2 = 0$, then its weight is equal $1, 2$ or $n-1$.*

Concerning the cases when the weight is equal $1, n-1$ see [1, Introduction].

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[4] One can write $M(t, v_0)$ instead of $M(v)$ (as in [1]).

[5] A quadratic matrix is irreducible if it is not similar to a 2×2 upper block triangular matrix with quadratic diagonal blocks.

2.2. Defining sequences with subsequences of type $(2,4)$ and $(4,2)$. We are interested in the cases when a 2-homogeneous subsequence has the form $(t^s, t^s, t^s, t^s, 1, 1)$ or $(1, 1, t^s, t^s, t^s, t^s)$. Since they are mutually dual (see Proposition 1), we consider only the first case.

Proposition 2. *If $t^m = 0$ and a canonically t-cyclic $n \times n$ matrix $M(v)$ is irreducible, then the sequence v does not contain a subsequence of the form $(t^s, t^s, t^s, t^s, 1, 1)$ with $m \leqslant 2s < 2m$.*

We prove a more general statement replacing $(t^s, t^s, t^s, t^s, 1, 1)$ by $(t^i, t^j, t^p, t^q, 1, 1)$ with $0 < i, j, p, q < m$, $i + j \geqslant m, p + q \geqslant m$, assuming (by Proposition 1) that the subsequence is the beginning of v and (by Theorem 4) that $n > 6$.

This follows from the following: if we perform with the reducible matrix

$$N = \left(\begin{array}{cccccc|ccc}
0 & 0 & \dots & 0 & 0 & -t^j & 1 & 0 & 0 \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & t^i & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & t^q \\
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p & 0
\end{array}\right)$$

$(k = n - 6)$ the transformation $[(n-2) \overset{t^j}{\to} (n-3)]^-$, $[1 \overset{-1}{\to} (n-1)]^+$, $[2 \overset{-t^p}{\to} n]^+$, and arrange the rows and columns of the resulting monomial matrix

$$N' = \left(\begin{array}{cccccc|ccc}
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & t^i & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & t^q \\
0 & 0 & \dots & 0 & 0 & t^j & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p & 0
\end{array}\right)$$

in the order $n-4, n-3, n-1, n, n-2, 1, 2, \dots, n-5$, we get the matrix $M(t^i, t^j, t^p, t^q, 1, 1, \alpha_1, \dots, \alpha_k)$.

2.3. Defining sequences with subsequences of type $(3,3)$. We consider the cases, when a 2-homogeneous subsequence has the form $(t^s,t^s,t^s,1,1,1)$ or $(1,1,1,t^s,t^s,t^s)$, in the same way as those in subsection 2.2; therefore we present only the main part and do not repeat similar assumptions and comments.

Proposition 3. *If $t^m=0$ and a canonically t-cyclic $n\times n$ matrix $M(v)$ is irreducible, then the sequence v does not contain a subsequence of the form $(t^s,t^s,t^s,1,1,1)$ with $m\leqslant 2s<2m$.*

We prove a more general statement replacing $(t^s,t^s,t^s,1,1,1)$ by $(t^i,t^j,t^p,1,1,1)$ with $0<p,q,j<m$, $i+j\geqslant m$, $2p\geqslant m$.

This follows from the following: if we perform with the reducible matrix

$$N=\left(\begin{array}{ccccccc|cc}
0 & 0 & 0 & \dots & 0 & 0 & -t^j & 1 & 0\\
1 & 0 & -t^p & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \dots & 0 & t^i & 0 & 0 & 0\\
\hline
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p\\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0
\end{array}\right)$$

($k=n-6$) the transformation $[(n-1)\overset{t^j}{\to}(n-2)]^-$, $[1\overset{-1}{\to}n]^+$, $[2\overset{-t^p}{\to}(n-1)]^+$, $[3\overset{-t^p}{\to}1]^+$, and arrange the rows and columns of the resulting monomial matrix

$$N'=\left(\begin{array}{ccccccc|cc}
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \dots & 0 & t^i & 0 & 0 & 0\\
\hline
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p\\
0 & 0 & 0 & \dots & 0 & 0 & t^j & 0 & 0
\end{array}\right)$$

in the order $n-3,n-2,n,n-1,1,2,\dots,n-4$, we get the matrix $M(t^i,t^j,t^p,1,1,1,\alpha_1,\alpha_2,\dots,\alpha_k)$.

2.4. Proof of Theorem 5. The proof follows from Propositions 2 and 3. Indeed, the first proposition implies at once that $M = M(v)$ is reducible, if $w(M) = n-2$, and the second one that $M = M(v)$ is reducible, if $2 < w(M) < n-2$ (in both the cases it need to take $m = 2, s = 1$).

In conclusion, we note that in the cases $n = 2, 3$ the theorem is trivial, in the case $n = 4$ it follows from Theorem 4 and in the case $n = 5$ there is the only exception, namely the matrix $M(1, 1, t, t, t,)$ of weight 3 is irreducible.

The theorem is proved.

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RESEARCH ARTICLE

The Littlewood-Richardson rule and Gelfand-Tsetlin patterns

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Abstract. We give a survey on the Littlewood-Richardson rule. Using Gelfand-Tsetlin patterns as the main machinery of our analysis, we study the interrelationship of various combinatorial descriptions of the Littlewood-Richardson rule.

1. Introduction

1.1. Let us consider Schur polynomials s_μ, s_ν and s_λ in n variables labelled by partitions μ, ν and λ, respectively. *The Littlewood-Richardson (LR) coefficient* is the multiplicity $c^\lambda_{\mu,\nu}$ of s_ν in the product of s_μ and s_ν:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$$

and its description is called *the LR rule.*

The same number appears in the tensor product decomposition problem in the representation theory of the complex general linear group GL_n and Schubert calculus in the cohomology of the Grassmannians, and is also related to the eigenvalues of the sum of Hermitian matrices. For more details, we refer readers to [8, 15, 27, 29].

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1.2. The LR rule is usually stated in terms of combinatorial objects called *LR tableaux*. Recall that a Young tableau is a filling of the boxes of a Young diagram with positive integers. We shall use the English convention of drawing Young diagrams and tableaux as in [7, 26] and assume a basic knowledge of these objects.

Definition 1. A tableau T on a skew Young diagram is called a LR tableau if it satisfies the following conditions:

1) it is semistandard, that is, the entries in each row of T weakly increase from left to right, and the entries in each column strictly increase from top to bottom; and
2) its reverse reading word is a Yamanouchi word (or lattice permutation). That is, in the word $x_1x_2x_3\dots x_r$ obtained by reading all the entries of T from left to right in each row starting from the bottom one, the sequence $x_rx_{r-1}x_{r-2}\dots x_s$ contains at least as many a's as it does $(a+1)$'s for all $a \geqslant 1$.

For example, the following is a LR tableau on a skew Young diagram $(11, 7, 5, 3)/(5, 3, 1)$

					1	1	1	1	1	1
			1	2	2	2				
	2	3	3	3						
2	4	4								

and its reverse reading word is 24423331222111111.

Remark 1. (1) In this paper we assume each tableau's entries weakly increase from left to right in every row. (2) From the second condition in the above definition, which we will call the *Yamanouchi condition*, the bth row of a LR tableau does not contain any entries strictly bigger than b for all $b \geqslant 1$.

The number of LR tableaux on the skew shape λ/μ with content ν is equal to the LR number $c^{\lambda}_{\mu,\nu}$. Here, the content $\nu = (\nu_1, \dots, \nu_n)$ of a tableau means that the entry k appears ν_k times in the tableau for $k \geqslant 1$. See, for example, [24, §I.9] and [15].

1.3. In this paper, we survey variations of the semistandard and Yamanouchi conditions with an emphasis on dualities in combinatorial descriptions of the LR rule. Although many of the results in this paper can be found in the literature, we will give complete and elementary proofs of our statements.

(1) In Theorem 1 and Theorem 2, we analyse *hives*, introduced by Knutson and Tao along with their honeycomb model [21], in terms of Gelfand-Tsetlin(GT) patterns [10]. We then show how the interlacing conditions in GT patterns are intertwined to form the defining conditions of hives. For the relevant results, see for example [1–4].

(2) In Theorem 3, we show that the semistandard and Yamanouchi conditions in LR tableaux are equivalent to, respectively, the interlacing and exponent conditions in *GZ schemes* introduced by Gelfand and Zelevinsky [11]. As a corollary we obtain a correspondence between LR tableaux and hives equivalent to [18, (3.3)]. We then observe how conditions on LR tableaux, GZ schemes and hives are translated between objects by this bijection. For the relevant results, see, for example, [1, 5, 18, 19].

(3) In Theorem 4, we show that the semistandard and Yamanouchi conditions in LR tableaux are equivalent to, respectively, the exponent and semistandard conditions in their *companion tableaux* introduced by van Leeuwen [29]. Here the correspondence between conditions is obtained by taking the transpose of matrices.

As a consequence, we obtain bijections between the families of combinatorial objects counting the LR number.

1.4. In [16, 17], Howe and his collaborators constructed a polynomial model for the tensor product of representations in terms of two copies of the multi-homogeneous coordinate ring of the flag variety, and then studied its toric degeneration with the SAGBI-Gröbner method. Through the characterization of the leading monomials of highest weight vectors, their toric variety is encoded by the *LR cone* [25]. On the other hand, via toric degenerations, the flag variety may be described in terms of the lattice cone of GT patterns [13, 20, 23]. These results led us to study the LR rule in terms of two sets of interlacing or semistandard conditions and to investigate the interrelationship of various combinatorial descriptions of the LR rule with GT patterns.

2. Hives and GT patterns I

In this section, we define GT patterns, hives, and objects related to them. We also describe hives in terms of pairs of GT patterns.

2.1. We set, once and for all, three polynomial dominant weights of the complex general linear group GL_n, that is, the sequences of nonnegative integers:

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \mu = (\mu_1, \ldots, \mu_n), \quad \nu = (\nu_1, \ldots, \nu_n)$$

such that $\lambda_i \geqslant \lambda_{i+1}$, $\mu_i \geqslant \mu_{i+1}$, and $\nu_i \geqslant \nu_{i+1}$ for all i. We define the dual λ^* of λ to be

$$\lambda^* = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1),$$

and define μ^* and ν^* similarly.

2.2. Let us consider an array of integers, which we will call a *t-array*

$$T = \left(t_1^{(1)}, \ldots, t_j^{(i)}, \ldots, t_n^{(n)}\right) \in \mathbb{Z}^{n(n+1)/2}$$

where $1 \leqslant j \leqslant i \leqslant n$. We are particularly interested in the case when the entries of T are either all non-negative or all non-positive integers.

Definition 2. A t-array $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$ is called a GT pattern for GL_n if it satisfies the *interlacing conditions*:

$$\text{IC}(1)\text{: } t_j^{(i+1)} \geqslant t_j^{(i)}$$
$$\text{IC}(2)\text{: } t_j^{(i)} \geqslant t_{j+1}^{(i+1)}$$

for all i and j.

We shall draw a t-array in the reversed pyramid form. For example, a generic GT pattern for GL_5 is

$$\begin{array}{ccccccccc}
t_1^{(5)} & & t_2^{(5)} & & t_3^{(5)} & & t_4^{(5)} & & t_5^{(5)} \\
& t_1^{(4)} & & t_2^{(4)} & & t_3^{(4)} & & t_4^{(4)} & \\
& & t_1^{(3)} & & t_2^{(3)} & & t_3^{(3)} & & \\
& & & t_1^{(2)} & & t_2^{(2)} & & & \\
& & & & t_1^{(1)} & & & &
\end{array}$$

where the entries are weakly decreasing along the diagonals from left to right.

Then, the *dual array* $T^* = (s_j^{(i)})$ of T is the t-array obtained by reflecting T over a vertical line and then multiplying -1, i.e.,

$$s_j^{(i)} = -t_{i+1-j}^{(i)}$$

for all $1 \leqslant j \leqslant i \leqslant n$.

Definition 3. For a t-array $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$,

1) the kth row of T is $t^{(k)} = (t_1^{(k)}, t_2^{(k)}, \ldots, t_k^{(k)}) \in \mathbb{Z}^k$ for $1 \leqslant k \leqslant n$. The type of T is its nth row;
2) the weight of T is $(w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ where $w_1 = t_1^{(1)}$ and

$$w_i = \sum_{k=1}^{i} t_k^{(i)} - \sum_{k=1}^{i-1} t_k^{(i-1)} \quad \text{for} \quad 2 \leqslant i \leqslant n.$$

Note that if T is of type λ and weight $w \in \mathbb{Z}^n$, then T^* is of type λ^* and weight $-w$.

GT patterns were introduced by Gelfand and Tsetlin in [10] to label the weight basis elements of an irreducible representation of the general linear group. The weight of T is exactly the weight of the basis element labelled by T in the irreducible representation V_n^μ whose highest weight is $\mu = t^{(n)}$. It follows that the dual array T^* of T corresponds to a weight vector in the contragradiant representation of V_n^μ.

2.3. Let us consider an array of nonnegative integers, which we will call a *h-array*,

$$(h_{0,0}, \ldots, h_{a,b}, \ldots, h_{n,n}) \in \mathbb{Z}^{(n+1)(n+2)/2}$$

where $0 \leqslant a \leqslant b \leqslant n$ and $h_{0,0} = 0$.

Definition 4. A hive for GL_n is a h-array $H = (h_{a,b}) \in \mathbb{Z}^{(n+1)(n+2)/2}$ satisfying the *rhombus conditions*:

$$\begin{aligned}
&\text{RC(1)}: (h_{a,b} + h_{a-1,b-1}) \geqslant (h_{a-1,b} + h_{a,b-1}) \quad \text{for} \quad 1 \leqslant a < b \leqslant n,\\
&\text{RC(2)}: (h_{a-1,b} + h_{a,b}) \geqslant (h_{a,b+1} + h_{a-1,b-1}) \quad \text{for} \quad 1 \leqslant a \leqslant b < n,\\
&\text{RC(3)}: (h_{a,b} + h_{a,b+1}) \geqslant (h_{a+1,b+1} + h_{a-1,b}) \quad \text{for} \quad 1 \leqslant a \leqslant b < n.
\end{aligned}$$

We shall draw a h-array in the pyramid form. For example, a generic hive for GL_3 is shown below.

$$\begin{array}{ccccccc}
 & & & h_{0,0} & & & \\
 & & h_{0,1} & & h_{1,1} & & \\
 & h_{0,2} & & h_{1,2} & & h_{2,2} & \\
h_{0,3} & & h_{1,3} & & h_{2,3} & & h_{3,3}
\end{array}$$

The rhombus conditions RC(1), RC(2), and RC(3) then say that, for each fundamental rhombus of one of the following forms,

$$
\begin{array}{cccc}
 & O' & & A' \\
A & & O &
\end{array}
\quad , \quad
\begin{array}{ccc}
 & A & \\
O & & O' \\
 & A' &
\end{array}
\quad , \quad
\begin{array}{cccc}
A' & & O & \\
 & O' & & A
\end{array}
$$

the sum of entries at the obtuse corners is bigger than or equal to the sum of entries at the acute corners, i.e., $O + O' \geqslant A + A'$.

For polynomial dominant weights μ, ν, and λ of GL_n, we let $\mathcal{H}(\mu, \nu, \lambda)$ denote the set of all h-arrays such that

$$
\begin{aligned}
\mu &= (h_{0,1} - h_{0,0}, h_{0,2} - h_{0,1}, \ldots, h_{0,n} - h_{0,n-1}), \\
\nu &= (h_{1,n} - h_{0,n}, h_{2,n} - h_{1,n}, \ldots, h_{n,n} - h_{n-1,n}), \\
\lambda &= (h_{1,1} - h_{0,0}, h_{2,2} - h_{1,1}, \ldots, h_{n,n} - h_{n-1,n-1}).
\end{aligned}
\tag{1}
$$

That is, the three boundary sides of $H \in \mathcal{H}(\mu, \nu, \lambda)$ are fixed:

$$
\begin{aligned}
h_{0,i} &= \mu_1 + \mu_2 + \cdots + \mu_i \\
h_{i,n} &= \sum_{j=1}^{n} \mu_j + \nu_1 + \nu_2 + \cdots + \nu_i \\
h_{i,i} &= \lambda_1 + \lambda_2 + \cdots + \lambda_i
\end{aligned}
$$

for $1 \leqslant i \leqslant n$. Recall that we always set $h_{0,0} = 0$. Let $\mathcal{H}^{\circ}(\mu, \nu, \lambda)$ be the subset of $\mathcal{H}(\mu, \nu, \lambda)$ satisfying the rhombus conditions. This is the set of hives whose boundaries are described by (1).

Hives were introduced by Knutson and Tao in [21] along with their honeycomb model to prove the saturation conjecture. In particular, the number of hives in $\mathcal{H}^{\circ}(\mu, \nu, \lambda)$ is equal to the LR number $c^{\lambda}_{\mu,\nu}$. See also [5, 22, 25].

2.4. For each h-array $H = (h_{a,b}) \in \mathbb{Z}^{(n+1)(n+2)/2}$, let us define its *derived t-arrays*

$$
T_1 = (x^{(i)}_j), \quad T_2 = (y^{(i)}_j), \quad T_3 = (z^{(i)}_j)
$$

whose entries are obtained from the differences of adjacent entries of H.

More specifically, for each fundamental triangle in H,

$$
\begin{array}{ccc}
 & h_{a,b} & \\
h_{a,b+1} & & h_{a+1,b+1}
\end{array}
$$

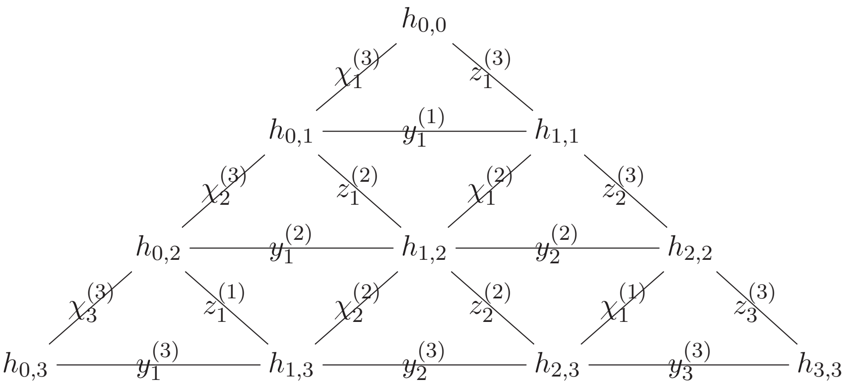


Figure 1. A h-array and its three derived t-arrays.

the entries of the derived t-arrays $(x_j^{(i)})$, $(y_j^{(i)})$, and $(z_j^{(i)})$ are

$$\begin{aligned} x_{b+1-a}^{(n-a)} &= h_{a,b+1} - h_{a,b} \quad \text{(SW–NE direction)} \\ y_{a+1}^{(b+1)} &= h_{a+1,b+1} - h_{a,b+1} \quad \text{(E–W direction)} \\ z_{a+1}^{(n+a-b)} &= h_{a+1,b+1} - h_{a,b} \quad \text{(SE–NW direction)} \end{aligned} \tag{2}$$

for $0 \leqslant a \leqslant b \leqslant n-1$.

This rather involved indexing is to make the entries of the derived arrays compatible with those of GT patterns. We may visualize the derived t-arrays by placing their entries between the entries of the h-array used to compute them. For example, if $n = 3$, then a h-array and its three derived t-arrays may be drawn as Figure 1.

2.5. The rhombus conditions for h-arrays are closely related to the interlacing conditions for their derived t-arrays.

Proposition 1. *Let $T_k = T_k(H)$ be a derived t-array of a h-array H for $k = 1, 2, 3$.*

1) *H satisfies RC(1) if and only if T_1 satisfies IC(2) and T_2 satisfies IC(1).*
2) *H satisfies RC(2) if and only if T_1 and T_3 satisfy IC(1).*
3) *H satisfies RC(3) if and only if T_2 and T_3 satisfy IC(2).*
4) *T_3 satisfies IC(1) if and only if T_1 satisfies IC(1).*
5) *T_3 satisfies IC(2) if and only if T_2 satisfies IC(2).*

Proof. Let us consider five adjacent entries of H of the forms

$$\begin{array}{ccccc} & & Z_1 & & \\ & Y_1 & & W_1 & \\ X_1 & & V_1 & & \end{array}, \quad \begin{array}{ccccc} & Y_2 & & W_2 & \\ X_2 & & V_2 & & U_2 \end{array}, \quad \begin{array}{ccccc} & & Z_3 & & \\ Y_3 & & & W_3 & \\ & & V_3 & & U_3 \end{array}.$$

Then, in the first and the third ones, RC(2) says that $Y_i + W_i \geqslant Z_i + V_i$ for $i = 1$ and 3. This is equivalent to $Y_1 - Z_1 \geqslant V_1 - W_1$ and $W_3 - Z_3 \geqslant V_3 - Y_3$, which are IC(1) for T_1 and T_3, respectively. This proves the statement (2). The statements (1) and (3) can be shown similarly.

Next, let us consider fundamental rhombi of the following forms in H

$$\begin{array}{ccc} & K & \\ L & & N \\ & M & \end{array}, \qquad \begin{array}{cccc} P & & S & \\ & Q & & R \end{array}.$$

Note that $N - K \geqslant M - L$ if and only if $L - K \geqslant M - N$, which proves (4). Similarly, $P - Q \geqslant S - R$ if and only if $P - S \geqslant Q - R$, which proves (5). $\square$

Suppose a h-array H satisfies RC(1), RC(2), and RC(3). Then, by the statements (1) and (2) of Proposition 1, $T_1(H)$ satisfies IC(1) and IC(2). Similarly, by the statements (1) and (3), $T_2(H)$ satisfies IC(1) and IC(2). This shows that $T_1(H)$ and $T_2(H)$ are GT patterns. Conversely, if $T_1(H)$ and $T_2(H)$ are GT patterns, then, by the statements (4) and (5), $T_3(H)$ is also a GT pattern. This means all three derived t-arrays satisfy both IC(1) and IC(2), and therefore, from the statements (1), (2), and (3), H is a hive.

Theorem 1. *For a h-array $H \in \mathbb{Z}^{(n+1)(n+2)/2}$ and its derived t-arrays $T_1(H)$ and $T_2(H)$, H is a hive if and only if $T_1(H)$ and $T_2(H)$ are GT patterns for GL_n.*

We remark that, in the above result, $T_1(H)$ and $T_2(H)$ are not independent. Let $T_1 = (x_j^{(i)})$ and $T_2 = (y_j^{(i)})$ be the derived t-arrays of a h-array H. Then, for each rhombus of the form

$$\begin{array}{cccc} & B & & A \\ C & & D & \end{array}$$

we have $(D-C)+(C-B)=(D-A)+(A-B)$, or

$$(C-B)-(D-A)=(A-B)-(D-C)$$

which is, using (2),

$$x_{b-a}^{(n-a-1)}-x_{b+1-a}^{(n-a)}=y_{a+1}^{(b+1)}-y_{a+1}^{(b)} \tag{3}$$

for $0 \leqslant a < b < n$. Note that hives (respectively, GT patterns) for GL_n with non-negative entries form a subsemigroup of $\mathbb{Z}_{\geqslant 0}^{(n+1)(n+2)/2}$ (respectively, $\mathbb{Z}_{\geqslant 0}^{n(n+1)/2}$). Theorem 1 and (3) imply that the semigroup

$$\bigcup_{(\mu,\nu,\lambda)} \mathcal{H}^\circ(\mu,\nu,\lambda)$$

of hives is a fiber product of, over $\mathbb{Z}_{\geqslant 0}^{n(n-1)/2}$, two affine semigroups S_{GT}^1 and S_{GT}^2 of GT patterns with respect to

$$\phi_k : S_{GT}^k \longrightarrow \mathbb{Z}_{\geqslant 0}^{n(n-1)/2}$$

such that, for $0 \leqslant a < b < n$,

$$\begin{aligned}
\phi_1(T_1) &= \left(\ldots, x_{b-a}^{(n-a-1)}-x_{b+1-a}^{(n-a)}, \ldots\right), \\
\phi_2(T_2) &= \left(\ldots, y_{a+1}^{(b+1)}-y_{a+1}^{(b)}, \ldots\right)
\end{aligned}$$

where $T_1=(x_j^{(i)}) \in S_{GT}^1$ and $T_2=(y_j^{(i)}) \in S_{GT}^2$.

We also remark that by exchanging the roles of $T_1(H)$, $T_2(H)$ and $T_3(H)$, one can read the symmetry of the LR rule. See, for example, [28].

3. Hives and GT patterns II

In this section, we study the set $\mathcal{H}^\circ(\mu,\nu,\lambda)$ of hives with a given boundary condition in terms of a single GT pattern.

3.1. Gelfand and Zelevinsky counted the LR number $c_{\mu,\nu}^{\lambda}$ with GT patterns of type μ and weight $\lambda-\nu$ satisfying the following additional condition.

Lemma 1 ([11]). *For a t-array $T=(t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$, we define its exponents as*

$$\varepsilon_j^{(i)}(T)=\sum_{1\leqslant h<j}(t_h^{(i+1)}-2t_h^{(i)}+t_h^{(i-1)})+(t_j^{(i+1)}-t_j^{(i)}).$$

Then the cardinality of the set $GZ(\mu, \lambda - \nu, \nu)$ of all GT patterns T of type μ with weight $\lambda - \nu$ such that, for all i and j,

$$\varepsilon_j^{(i)}(T) \leqslant \nu_i - \nu_{i+1}$$

is equal to the LR number $c^{\lambda}_{\mu,\nu}$.

The elements of $GZ(\mu, \lambda - \nu, \nu)$ will be called *GZ schemes.*

3.2. Note that, for a h-array H, since the derived t-arrays are defined from the differences of the entries in H, if the boundaries of H are fixed, then any one of the derived t-array of H uniquely determines H. Moreover, we can characterize the derived t-arrays as follows.

Theorem 2. *For a h-array H in $\mathcal{H}(\mu, \nu, \lambda)$, consider its derived t-arrays $T_1(H)$ and $T_2(H)$.*

1) *H is a hive if and only if $T_1^*(H) = (T_1(H))^*$ is a GZ scheme in $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$;*
2) *H is a hive if and only if $T_2(H)$ is a GZ scheme in $GZ(\nu, \lambda - \mu, \mu)$.*

Note that this theorem, in particular, gives bijections between hives and GZ schemes:

$$\begin{array}{ccc} \mathcal{H}^\circ(\mu, \nu, \lambda) & \longrightarrow & GZ(\mu^*, \lambda^* - \nu^*, \nu^*) \\ H & \longmapsto & T_1^*(H) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}^\circ(\mu, \nu, \lambda) & \longrightarrow & GZ(\nu, \lambda - \mu, \mu) \\ H & \longmapsto & T_2(H) \end{array}.$$

For the rest of this section, we will prove Theorem 2 by showing the following.

(a) $T_1^*(H)$ satisfies IC(2) if and only if $\varepsilon_j^{(i)}(T_2(H)) \leqslant \mu_i - \mu_{i+1}$;
(b) $T_1^*(H)$ satisfies IC(1) if and only if $T_2(H)$ satisfies IC(1);
(c) $T_1^*(H)$ satisfies $\varepsilon_j^{(i)}(T_1^*(H)) \leqslant \nu_i^* - \nu_{i+1}^*$ if and only if $T_2(H)$ satisfies IC(2).

The weights of the derived t arrays will also be computed.

3.3. Let us first compute the weights of $T_1(H)$ and $T_2(H)$ for $H \in \mathcal{H}(\mu, \nu, \lambda)$.

Lemma 2. *For a h-array $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$,*

1) *the weight of* $T_1(H)$ *is* $\nu^* - \lambda^*$, *i.e.*,

$$(\lambda_n - \nu_n, \lambda_{n-1} - \nu_{n-1}, \dots, \lambda_1 - \nu_1)$$

therefore, the weight of $T_1^*(H)$ *is* $\lambda^* - \nu^*$;

2) *the weight of* $T_2(H)$ *is* $\lambda - \mu$, *i.e.*,

$$(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots, \lambda_n - \mu_n).$$

Proof. We will prove the second statement. The proof of the first case is similar. From Definition 3, (2) and the expressions for λ and μ in terms of the h-array elements it follows

$$w_1 = y_1^{(1)} = h_{1,1} - h_{0,1} = (h_{1,1} - h_{0,0}) + (h_{0,0} - h_{0,1}) = \lambda_1 - \mu_1.$$

Using the same approach for w_i, $i \geqslant 2$, we see

$$\begin{aligned} w_i &= \sum_{k=1}^{i} y_k^{(i)} - \sum_{k=1}^{i-1} y_k^{(i-1)} \\ &= \sum_{k=1}^{i} (h_{k,i} - h_{k-1,i}) - \sum_{k=1}^{i-1} (h_{k,i-1} - h_{k-1,i-1}) \\ &= (h_{i,i} - h_{0,i}) - (h_{i-1,i-1} - h_{0,i-1}) \\ &= \lambda_i - \mu_i. \end{aligned}$$

Therefore $w_i = \lambda_i - \mu_i$ for all i, and the weight of $T_2(H)$ is $\lambda - \mu$. □

3.4. Next, we study the relations between the interlacing conditions and the exponents conditions for derived arrays. Note that, from the definition of dual arrays, a t-array T satisfies IC(1) if and only if T^* satisfies IC(2), and T satisfies IC(2) if and only if T^* satisfies IC(1).

Proposition 2. *For a* h*-array* $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$ *and its derived* t*-arrays* $T_1(H) = (x_j^{(i)})$ *and* $T_2(H) = (y_j^{(i)})$, $T_1(H)$ *satisfies IC(1) if and only if* $\varepsilon_j^{(i)}(T_2(H)) \leqslant \mu_i - \mu_{i+1}$.

Proof. Let us assume $j > 1$. Then the exponent of $T_2(H)$,

$$\varepsilon_j^{(i)}(T_2(H)) = \sum_{1 \leqslant h < j} \Big((y_h^{(i+1)} - y_h^{(i)}) - (y_h^{(i)} - y_h^{(i-1)}) \Big) + \Big(y_j^{(i+1)} - y_j^{(i)} \Big)$$

can be rewritten in terms of the entries of $T_1(H)$. By using (3),

$$\begin{aligned}\varepsilon_j^{(i)}(T_2(H)) &= \sum_{1\leqslant h<j}\Big((x_{i-h+1}^{(n-h)} - x_{i-h+2}^{(n-h+1)}) - (x_{i-h}^{(n-h)} - x_{i-h+1}^{(n-h+1)})\Big)\\ &\quad + \Big(x_{i-j+1}^{(n-j)} - x_{i-j+2}^{(n-j+1)} + y_j^{(i)}\Big) - \Big(x_{i-j}^{(n-j)} - x_{i-j+1}^{(n-j+1)} + y_j^{(i-1)}\Big)\end{aligned}$$

and we see that parts of the consecutive terms cancel to give

$$\varepsilon_j^{(i)}(T_2(H)) = \Big(x_i^{(n)} - x_{i+1}^{(n)}\Big) + \Big(x_{i-j+1}^{(n-j)} - x_{i-j}^{(n-j)} + y_j^{(i)} - y_j^{(i-1)}\Big). \qquad (4)$$

Now note that the interlacing condition IC(1) for $T_1(H)$ implies $x_{i-j+1}^{(n-j+1)} \geqslant x_{i-j+1}^{(n-j)}$ or equivalently, by using (3),

$$x_{i-j}^{(n-j)} \geqslant \Big(x_{i-j+1}^{(n-j)} + y_j^{(i)} - y_j^{(i-1)}\Big)$$

therefore

$$0 \geqslant \Big(x_{i-j+1}^{(n-j)} - x_{i-j}^{(n-j)} + y_j^{(i)} - y_j^{(i-1)}\Big).$$

Hence, from (4), the interlacing condition IC(1) for $T_1(H)$ is equivalent to

$$\varepsilon_j^{(i)}(T_2(H)) \leqslant \Big(x_i^{(n)} - x_{i+1}^{(n)}\Big) = \mu_i - \mu_{i+1}.$$

The case $j = 1$ can be shown similarly for all i. □

Proposition 3. *For a h-array $H = (h_{a,b}) \in \mathcal{H}(\mu,\nu,\lambda)$ and its derived t-arrays $T_1(H) = (x_j^{(i)})$ and $T_2(H) = (y_j^{(i)})$, $T_1(H)$ satisfies IC(2) if and only if $T_2(H)$ satisfies IC(1).*

Proof. Using the equality (3),

$$\Big(x_j^{(i)} \geqslant x_{j+1}^{(i+1)}\Big) \quad \text{if and only if} \quad \Big(y_{n-i}^{(n-i+j)} \geqslant y_{n-i}^{(n-i+j-1)}\Big)$$

and therefore, by setting $i' = n - i + j - 1$ and $j' = n - i$, we have

$$\Big(x_j^{(i)} \geqslant x_{j+1}^{(i+1)}\Big) \quad \text{if and only if} \quad \Big(y_{j'}^{(i'+1)} \geqslant y_{j'}^{(i')}\Big)$$

for $1 \leqslant j \leqslant i \leqslant n-1$ and $1 \leqslant j' \leqslant i' \leqslant n-1$. This shows that IC(2) holds for $T_1(H)$ if and only if IC(1) holds for $T_2(H)$. □

Proposition 4. *For a h-array $H = (h_{a,b}) \in \mathcal{H}(\mu,\nu,\lambda)$ and its derived t-arrays $T_1(H) = (x_j^{(i)})$ and $T_2(H) = (y_j^{(i)})$, $T_1^*(H)$ satisfies $\varepsilon_j^{(i)}(T_1^*(H)) \leqslant \nu_i^* - \nu_{i+1}^*$ if and only if $T_2(H)$ satisfies IC(2).*

Proof. Let us assume $j > 1$. Write the exponents of $T_1^*(H) = (s_j^{(i)})$ using $s_j^{(i)} = -x_{i+1-j}^{(i)}$.

$$\begin{aligned}
\varepsilon_j^{(i)}(T_1^*(H)) &= \sum_{1\leqslant h<j} \left(-x_{i-h+2}^{(i+1)} + 2x_{i-h+1}^{(i)} - x_{i-h}^{(i-1)}\right) \\
&\quad + \left(-x_{i-j+2}^{(i+1)} + x_{i-j+1}^{(i)}\right) \\
&= \sum_{1\leqslant h<j} \left((x_{i-h+1}^{(i)} - x_{i-h+2}^{(i+1)}) - (x_{i-h}^{(i-1)} - x_{i-h+1}^{(i)})\right) \\
&\quad + \left(x_{i-j+1}^{(i)} - x_{i-j+2}^{(i+1)}\right)
\end{aligned}$$

Then, using the identity (3), we can rewrite the exponents in terms of the entries of $T_2(H)$ as

$$\begin{aligned}
\varepsilon_j^{(i)}(T_1^*(H)) &= \sum_{1\leqslant h<j} \left((y_{n-i}^{(n-h+1)} - y_{n-i}^{(n-h)}) - (y_{n-i+1}^{(n-h+1)} - y_{n-i+1}^{(n-h)})\right) \\
&\quad + \left(y_{n-i}^{(n-j+1)} - y_{n-i}^{(n-j)}\right) \\
&\leqslant \sum_{1\leqslant h<j} \left((y_{n-i}^{(n-h+1)} - y_{n-i}^{(n-h)}) - (y_{n-i+1}^{(n-h+1)} - y_{n-i+1}^{(n-h)})\right) \\
&\quad + \left(y_{n-i}^{(n-j+1)} - y_{n-i+1}^{(n-j+1)}\right)
\end{aligned}$$

where the inequality is by IC(2): $y_{n-i}^{(n-j)} \geqslant y_{n-i+1}^{(n-j+1)}$ in $T_2(H)$. Parts of the consecutive terms in the right hand side cancel to give

$$\begin{aligned}
\varepsilon_j^{(i)}(T_1^*(H)) &\leqslant \left((y_{n-i}^{(n)} - y_{n-i}^{(n-j+1)}) - (y_{n-i+1}^{(n)} - y_{n-i+1}^{(n-j+1)})\right) \\
&\quad + \left(y_{n-i}^{(n-j+1)} - y_{n-i+1}^{(n-j+1)}\right) \\
&= \left(y_{n-i}^{(n)} - y_{n-i+1}^{(n)}\right) = \nu_{n-i} - \nu_{n-i+1} = \nu_i^* - \nu_{i+1}^*.
\end{aligned}$$

So the interlacing condition IC(2) for $T_2(H)$ is equivalent to

$$\varepsilon_j^{(i)}(T_1^*(H)) \leqslant \nu_i^* - \nu_{i+1}^*$$

as required. The case $j = 1$ can be shown similarly for all i. □

3.5. Suppose we have a hive H. From Lemma 2, the weights of $T_1^*(H)$ and $T_2(H)$ are $\lambda^* - \nu^*$ and $\lambda - \mu$, respectively. Theorem 1 states that H is a hive if and only if $T_1(H)$ and $T_2(H)$, and hence $T_1^*(H)$ and $T_2(H)$, satisfy both $IC(1)$ and $IC(2)$. Therefore since H is a hive, Proposition 2 and

Proposition 4 imply $T_1^*(H)$ and $T_2(H)$ satisfy the exponent conditions, and consequently they are GZ schemes in $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ and $GZ(\nu, \lambda - \mu, \mu)$, respectively.

Conversely, if $T_1^*(H)$ is a GZ scheme from $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ it satisfies IC(1), IC(2), and the exponent condition, thus from Propositions 2 – 4, $T_2(H)$ is a GZ scheme. In particular, $T_1(H)$ and $T_2(H)$ are GT patterns, meaning H is a hive by Theorem 1. Similarly, if $T_2(H) \in GZ(\nu, \lambda - \mu, \mu)$, then H is a hive. This proves Theorem 2.

4. LR tableaux and GT patterns I

In this section we introduce a bijection between LR tableaux and GZ schemes (Theorem 3). In proving this we will see that the semistandard and Yamanouchi conditions for tableaux are equivalent to, respectively, the interlacing and exponent conditions for t-arrays.

As an interesting consequence we then combine Theorem 3 with Theorem 2 (1) to arrive at a correspondence between LR tableaux and hives (Corollary 1). It turns out that Corollary 1 is equivalent to [18, (3.3)], so we compare the two constructions. The main difference is that our method has an intermediate GZ scheme, which is an artefact of composing Theorem 3 and Theorem 2. To conclude the section we summarise how the conditions on LR tableaux (semistandard and Yamanouchi conditions), GZ schemes (semistandard and exponent conditions) and hives (the rhombus conditions) are translated by the bijections.

The reader may find relevant results and further developments in, for example, [1–4, 6, 18, 19, 25].

4.1. A well-known bijection between semistandard tableaux and GT patterns. Our bijection between LR tableaux and GZ schemes is an extension of a well-known bijection between semistandard tableaux and GT patterns, seen in, for example, [11]. We now review this bijection and state it in the form most useful for our purposes. For this we require some relevant notation.

A non-skew semistandard tableau Y is uniquely determined by its associated matrix $(a_{i,j}(Y))$ where

$$a_{i,j}(Y) = \text{ the number of } i\text{'s in the } j\text{th row} \tag{5}$$

for all $1 \leqslant i, j \leqslant n$. Note that $a_{i,j}(Y) = 0$ for $i < j$. We also note that $\sum_{k=1}^{n} a_{k,j}(Y)$ for $1 \leqslant j \leqslant n$ give the shape of the tableau Y, and $\sum_{k=1}^{n} a_{i,k}(Y)$ for $1 \leqslant i \leqslant n$ give the content of Y. The reader is warned that these are *not* the same a_{ij} as those in [18, (3.4)]. Those label hive entries, not content of a tableau.

We remark that if Y is a semistandard tableau on the skew shape λ/μ, then the $a_{i,j}(Y)$'s are well defined, and the $a_{i,j}(Y)$'s with λ or μ uniquely define Y. It is possible to develop the theory of tableaux exclusively in terms of their associated matrices. See [6] for this direction.

Now consider a semistandard Young tableau. Removing all instances of the largest entry simultaneously yields a tableau with a new shape. Repeating this process, we would achieve a list of successively shrinking shapes, which written downwards would form the rows of a GT pattern. This process is a bijection. See Example 1.

It is easy to symbolically describe the inverse of the bijection. Given a GT pattern $T = (t_j^{(i)})$ of type λ with non-negative entries, it creates a semistandard Young tableau Y_T of shape λ whose entries are elements of $\{1, 2, \ldots, n\}$ and defined by

$$a_{i,j}(Y_T) = t_j^{(i)} - t_j^{(i-1)} \tag{6}$$

for $1 \leqslant i, j \leqslant n$ with the conventions

$$t_j^{(i)} = 0 \quad \text{for} \quad j > i \geqslant 0.$$

Manipulating (6), it follows that the bijection takes a semistandard tableau Y and creates a GT pattern $T_Y = (t_j^{(i)})$ according to the rule

$$t_j^{(i)} = \sum_{k=1}^{i} a_{k,j}(Y) \tag{7}$$

for $1 \leqslant j \leqslant i \leqslant n$. Since $a_{k,j}(Y) = 0$ for $k < j$ in every non-skew semistandard tableau Y, we can in fact write this as

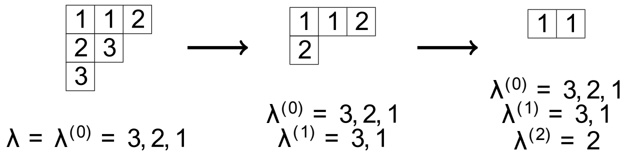
$$t_j^{(i)} = \sum_{k=j}^{i} a_{k,j}(Y). \tag{8}$$

See also, for example, [14, §8.1.2] or [20] for further background on this bijection.

Example 1. As an example we apply the bijection to the tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

and list the successive shapes $\lambda^{(i)}$ as they are created.



Clearly, the shapes form a GT pattern. It is straightforward to check that the expressions (8) and (6) both hold.

Under this bijection, the content of the tableau is equal to the weight of the t-array. We also note that in this bijection, the semistandard condition on the tableau is implied by the interlacing conditions on the t-array and vice versa (cf. Remark 2).

4.2. A well-known bijection between semistandard skew tableaux and truncated GT patterns. The bijection of §4.1 can be extended to act on skew tableaux. Again, this is a well known result included in [11], [12] and [3], among others.

Lemma 3. *There is a bijection between the set of skew semistandard Young tableaux of shape λ/μ with entries from $\{1, 2, \ldots, n\}$ and the set of GT patterns for GL_{2n} whose type is $\lambda' = (\lambda_1, \ldots, \lambda_n, 0, \cdots, 0) \in \mathbb{Z}^{2n}$ and whose kth row is $(\mu_1, \mu_2, \ldots, \mu_k)$ for $1 \leqslant k \leqslant n$.*

Proof. For a given semistandard Young tableau Y of shape λ/μ, replace the i entries with $(n+i)$'s for $1 \leqslant i \leqslant n$, then fill in the empty boxes in the ℓth row of Y with ℓ's for $1 \leqslant \ell \leqslant n$. Then this process uniquely determines a non-skew semistandard Young tableau of shape λ with entries from $\{1, 2, \ldots, 2n\}$, and under the bijection given by (6), its corresponding GT pattern for GL_{2n} is the one described in the statement. □

The first half of Example 2 shows Lemma 3 applied to a skew tableau. We remark that the GT pattern for GL_n whose kth row is $(\mu_1, \mu_2, \ldots, \mu_k)$ for $1 \leqslant k \leqslant n$ corresponds to the highest weight vector of the representation V_n^μ labelled by a Young diagram μ. In fact, the GT patterns described in Lemma 3 encode the weight vectors of $V_{2n}^{\lambda'}$, which are the highest weight vector for V_n^μ under the branching of GL_{2n} down to GL_n.

The bottom $n-1$ rows of a GT pattern described by Lemma 3 hold redundant information because they are determined by μ. It is therefore convention to omit them and achieve what is called a *truncated* GT pattern. It is also common to omit the upper-right portion of this pattern, since the interlacing conditions force those entries to be zero. For example, the first half of the bijection described by [18, (3.3)] uses Lemma 3 with these conventions.

There is an excellent example of Lemma 3 and further explanation in [1, §2].

4.3. Symbolic forms of the semistandard and Yamanouchi conditions. We are almost ready to use Lemma 3 to establish the bijection between LR tableaux and GZ schemes. However, we first need symbolic forms of both the semistandard and Yamanouchi conditions.

Let us express the semistandard condition for a tableau Y in terms of the $a_{i,j}(Y)$ defined in (5). By rearranging the entries in each row if necessary, we can always make the entries of Y weakly increasing along each row from left to right. The strictly increasing condition on the columns of Y can then be rephrased as follows: the number of entries up to ℓ in the $(m+1)$th row is not bigger than the number of entries up to $(\ell-1)$ in the mth row, i.e.,

$$\sum_{k=1}^{\ell-1} a_{k,m}(Y) \geqslant \sum_{k=1}^{\ell} a_{k,m+1}(Y) \tag{9}$$

for $1 \leqslant \ell \leqslant n$ and $1 \leqslant m < n$. Here, if $\ell = 1$, then the left hand side is 0 as an empty sum and the inequality implies that $a_{1,m+1}(Y) = 0$ for $m \geqslant 1$. Inductively, we can obtain $a_{i,m+1}(Y) = 0$ for $m \geqslant i$ from the inequality with $\ell = i$. This shows that for a semistandard Young tableau Y, $a_{i,j}(Y) = 0$ for $j > i$, as we noted after (5).

Remark 2. By using the conversion formula (7), one can directly compute that IC(2) on a GT pattern T is equivalent to the semistandard condition (9) in Y_T corresponding to T. On the other hand, IC(1) in T is equivalent to a rather trivial condition $a_{i,j}(Y_T) \geqslant 0$ for all i, j.

If Y is a skew tableau of shape λ/μ, then, using the same argument as for (9), it is straightforward to see that we can make Y semistandard by rearranging elements along each row if and only if

$$\mu_{m+1} + \sum_{k=1}^{\ell} a_{k,m+1}(Y) \leqslant \mu_m + \sum_{k=1}^{\ell-1} a_{k,m}(Y) \tag{10}$$

for $1 \leqslant \ell \leqslant n$ and $1 \leqslant m < n$. The Yamanouchi condition in a LR tableau Y can be expressed as

$$\sum_{k=1}^{j} a_{i+1,k}(Y) \leqslant \sum_{k=1}^{j-1} a_{i,k}(Y) \tag{11}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant i < n$. Here, if $j = 1$, then the right hand side is 0 as an empty sum and the inequality implies that $a_{i+1,1}(Y) = 0$ for $i \geqslant 1$. Inductively, we can obtain $a_{i+1,\ell}(Y) = 0$ for $i \geqslant \ell$ from the inequality with $j = \ell$. This shows that for an LR tableau Y, $a_{i,j}(Y) = 0$ for $i > j$, as we noted in Remark 1 (2).

4.4. Bijection between LR tableaux and GZ schemes. We now establish a bijection between LR tableaux and GZ schemes using Lemma 3. After applying the lemma to an LR tableau, a center section of the resulting GT pattern is removed. Taking the dual of the removed array we get the desired GZ scheme. In doing this we observe how the conditions on the tableau become those of the scheme.

For a specific example of this bijection, see the first half of Example 2.

Theorem 3. *There is a bijection ϕ between $LR(\lambda/\mu, \nu)$ and $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. In particular, the semistandard and Yamanouchi conditions in $L \in LR(\lambda/\mu, \nu)$ are equivalent to, respectively, the interlacing and exponent conditions in $\phi(L) \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$.*

Proof. Let $L \in LR(\lambda/\mu, \nu)$ be given. By applying Lemma 3 we find its corresponding GT pattern $T = (t_j^{(i)})$ for GL_{2n} and remove the bottom $n-1$ rows to achieve a truncated GT pattern of $n+1$ rows. Furthermore, the truncated pattern for L can be divided into three subtriangular arrays T_X, T_Y and T_Z, as in Figure 2. Note that these are the same size.

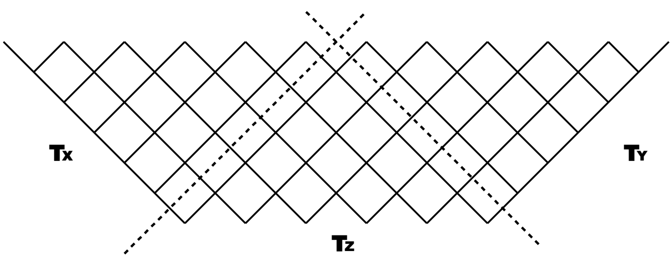


FIGURE 2. Dividing a truncated GT pattern into 3 subpatterns.

The upper left subarray T_X is completely determined by λ because of the Yamanouchi condition (see Remark 1 (2)). The upper right subarray T_Y contains only zeroes. Therefore, given fixed λ, μ, and ν, the LR tableau $L \in LR(\lambda/\mu, \nu)$ is uniquely determined by T_Z. We want to show that the dual array T_Z^* of T_Z is an element of $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$, and from that establish a bijection

$$\begin{array}{rcl} LR(\lambda/\mu,\nu) & \longrightarrow & GZ(\mu^*,\lambda^*-\nu^*,\nu^*) \\ L & \longmapsto & T_Z^* \end{array} .$$

Let us rewrite the middle subarray T_Z as follows by reflecting it over a horizontal line.

$$T_Z = \begin{array}{ccccccccc} t_1^{(n)} & & t_2^{(n)} & & \cdots & & t_{n-1}^{(n)} & & t_n^{(n)} \\ & t_2^{(n+1)} & & t_3^{(n+1)} & & \cdots & & t_n^{(n+1)} & \\ & & t_3^{(n+2)} & & & & t_n^{(n+2)} & & \\ & & & \ddots & & ⋰ & & & \\ & & & & t_n^{(2n-1)} & & & & \end{array}$$

Then $\mu_i = t_i^{(n)}$ for $1 \leqslant i \leqslant n$ and T_Z satisfies the interlacing conditions induced from the truncated GT pattern T, which are assured by the semistandardness of L. Therefore T_Z is a GT pattern of type μ. From the fact that the weights of T_X, T_Y, and T are $(\lambda_1,\dots,\lambda_n)$, $(0,\dots,0)$, and $(\mu_1,\dots,\mu_n,\nu_1,\dots,\nu_n)$ respectively, it is easy to show that the weight of T_Z is $\nu^*-\lambda^*$. Hence its dual T_Z^* is a GT pattern (see §3.4) of type μ^* and weight $\lambda^*-\nu^*$. Next, we want to show that T_Z^* satisfies the exponent conditions.

Let $a_{i,j} = a_{i,j}(L)$, i.e., be the number of i's in the jth row of L for all i and j. Then

$$a_{i,j} = t_j^{(n+i)} - t_j^{(n+i-1)} \quad \text{and } a_{k,k} = \lambda_k - t_k^{(n+k-1)} \tag{12}$$

for $1 \leqslant i < j \leqslant n$ and $1 \leqslant k \leqslant n$. Since the content of L is ν with $\nu_q = \sum_{k=1}^n a_{q,k}$ for $1 \leqslant q \leqslant n$, we can write

$$(-\nu_{i+1}) - (-\nu_i) = \sum_{k=1}^{n} (a_{i,k} - a_{i+1,k}) \tag{13}$$

for $1 \leqslant i < n$.

On the other hand, from the Yamanouchi condition (11) in L, we have

$$\sum_{k=1}^{j} a_{i+1,k} \leqslant \sum_{k=1}^{j-1} a_{i,k} \quad \text{or equivalently,} \quad a_{i+1,j} \leqslant \sum_{k=1}^{j-1} (a_{i,k} - a_{i+1,k}).$$

Then, using this inequality, (13) becomes

$$(-\nu_{i+1}) - (-\nu_i) \geqslant \sum_{k=j+1}^{n} (a_{i,k} - a_{i+1,k}) + a_{i,j}$$

and the right hand side is, via (12), the exponents of T_Z^*. Therefore, $T_Z^* \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. $\square$

4.5. A bijection between LR tableaux and hives. Composing Theorem 3 and Theorem 2 (1) gives a bijection between the set of LR tableaux and the set of hives.

$$\begin{array}{ccccc} & & GZ(\mu^*, \lambda^* - \nu^*, \nu^*) & & \\ & \swarrow\nearrow & & \searrow\nwarrow & \\ LR(\lambda/\mu, \nu) & & & & \mathcal{H}^\circ(\mu, \nu, \lambda) \end{array}$$

Corollary 1. *[18, (3.3)] There is a bijection between $LR(\lambda/\mu, \nu)$ and $\mathcal{H}^\circ(\mu, \nu, \lambda)$.*

Proof. For $L \in LR(\lambda/\mu, \nu)$, we compute the corresponding truncated GT pattern and its middle subarray T_Z. Then, by Theorem 3, its dual T_Z^* belongs to $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. Similarly, for $H \in \mathcal{H}^\circ(\mu, \nu, \lambda)$, its first derived subarray $T_1(H)$ satisfies $T_1^*(H) \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ by Theorem 2. We can therefore identify a H such that $T_1(H) = T_Z$ and this gives us a bijection from $LR(\lambda/\mu, \nu)$ to $\mathcal{H}^\circ(\mu, \nu, \lambda)$. $\square$

We give an example of Corollary 1 below.

Example 2. We start by using Theorem 3 to map the LR tableau below to a GT pattern (whose dual array is a GZ scheme).

Let $\lambda = (11, 7, 5, 3)$, $\mu = (5, 3, 1, 0)$ and $\nu = (7, 5, 3, 2)$. The LR tableau from $LR(\lambda/\mu, \nu)$

					1	1	1	1	1	1
			1	2	2	2				
	2	3	3	3						
2	4	4								

considered as an object for GL_4, corresponds to the following truncated GT pattern.

$$\begin{array}{ccccccccccccccc} 11 & & 7 & & 5 & & 3 & & 0 & & 0 & & 0 & & 0 \\ & 11 & & 7 & & 5 & & 1 & & 0 & & 0 & & 0 & \\ & & 11 & & 7 & & 2 & & 1 & & 0 & & 0 & & \\ & & & 11 & & 4 & & 1 & & 0 & & 0 & & & \\ & & & & 5 & & 3 & & 1 & & 0 & & & & \end{array}$$

Taking out the middle section, we find T_Z is

$$\begin{array}{ccccccc} 5 & & 3 & & 1 & & 0 \\ & 4 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 1 & & & \end{array}$$

with a dual array T_Z^* belonging to $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$.

For the second half of the process, we apply the bijection between hives and GZ schemes (Theorem 2 (1)) to T_Z^*. We know the corresponding hive H will have boundaries given by $\mu = (5, 3, 1, 0), \nu = (7, 5, 3, 2)$ and $\lambda = (11, 7, 5, 3)$ so that it appears as follows

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 5 & & 11 & & & \\ & & 8 & & p & & 18 & & \\ & 9 & & q & & r & & 23 & \\ 9 & & 16 & & 21 & & 24 & & 26 \end{array}$$

with some inner entries p, q and r. Adding $(T_Z^*)^* = T_Z$ along the NE-SW diagonals as if it were $T_1(H)$ we find $p = 15$, $q = 16$ and $r = 20$.

Of course, there are many known bijections between LR tableaux and hives. For example, in the appendix of [5] Fulton gave a bijection between LR tableaux and hives using *contratableaux*. It is interesting to note that his first step is to construct partitions from the hive that are equivalent to the derived t-array T_1. However, that approach diverges from ours once he uses the partitions to form a contratableau.

Our Corollary 1 is simpler than most other bijections between LR tableaux and hives, such as the one by Fulton, but is in fact equivalent to [18, (3.3)]. There, the authors also take an LR tableau and compute the truncated GT pattern via Lemma 3. They then take row sums in the pattern and separate out a bottom section, which becomes the hive. This is simply our process in reverse, since we separate a section of the truncated GT pattern in Theorem 3 by removing T_Z, and then successively add those entries to the boundary of the hive in Theorem 2 (1).

The key difference, however, is that here we establish GZ schemes as an intermediate object in the bijection, which is absent from the simple and elegant treatment in [18]. This provides background as to why their simple bijection works, and also allows us to track the conditions as they move between objects (see tables 1 and 2). We also note that Corollary

1 is not the main focus of our discussion. Rather, it is an interesting consequence that appears when piecing together two sets of combinatorial theory – the derived t-arrays of hives on one hand, and the classical bijection between tableaux and t-arrays on the other.

To complete the section we combine the two approaches for some insights. Though not clear from our presentation, the elegant formula [18, (3.4)] states that the elements of the hive $H = (h_{l,m})$ are given by

$$h_{l,m} = \begin{array}{l}\text{the number of empty boxes and entries} \leqslant l \\ \text{in the first } m \text{ rows of the LR tableau.}\end{array}$$

Finally, in proving [18, Proposition 3.2], King *et al.* show that, under their bijection, conditions[1] on hives correspond to conditions on LR tableaux. Table 1 summarises these equivalences.

Hive	**LR tableau**
RC(1)	trivial
RC(2)	Semistandard condition
RC(3)	Yamanouchi

Table 1. Equivalent LR tableau and hive conditions in [18, (3.3)]

Using our results from Proposition 1, Theorem 2, Remark 2 and Theorem 3 we are able to add a middle column showing the equivalent conditions in the intermediate GZ scheme object. See Table 2.

Hive	**GZ scheme**	**LR tableau**
RC(1)	IC(1)	trivial
RC(2)	IC(2)	Semistandard condition
RC(3)	Exponents	Yamanouchi

Table 2. Equivalent LR tableau, GZ scheme and hive conditions in Corollary 1

5. LR Tableaux and GT Patterns II

In this section, we show that the semistandard and Yamanouchi conditions for tableaux are equivalent to, respectively, the exponent and semistandard conditions for their *companion tableaux*. This correspondence

[1]The conditions RC(1), RC(2) and RC(3) are referred to as R2, R3 and R1 respectively by [18].

is obtained by taking the transpose of matrices describing tableaux. As a result, we show that the companion tableaux of LR tableaux are GZ schemes under the tableau-pattern bijection.

5.1. For a (skew) semistandard tableau Y, as in (5), we let $a_{i,j}(Y)$ denote the number of i's in the jth row.

Definition 5. For a (skew) semistandard tableau Y, its *companion tableau* Y^c is defined as a non-skew tableau whose entries are weakly increasing along each row and whose number of i's in the jth row is equal to $a_{j,i}(Y)$; that is, for $1 \leqslant i, j \leqslant n$,

$$a_{i,j}(Y^c) = a_{j,i}(Y). \tag{14}$$

Example 3. For the LR tableau Y from Example 2, the associated matrix is

$$a_{i,j}(Y) = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Then, from its transpose, we have the following companion tableau Y^c.

1	1	1	1	1	1	2
2	2	2	3	4		
3	3	3				
4	4					

Note that Y is of shape $(11, 7, 5, 3)/(5, 3, 1, 0)$ and content $(7, 5, 3, 2)$ while its companion tableau Y^c is of shape $(7, 5, 3, 2)$ and content $(6, 4, 4, 3)$, which is $(11, 7, 5, 3) - (5, 3, 1, 0)$. The GT pattern T_{Y^c} corresponding to Y^c is

$$\begin{array}{ccccccc} 7 & & 5 & & 3 & & 2 \\ & 7 & & 4 & & 3 & \\ & & 7 & & 3 & & \\ & & & 6 & & & \end{array}.$$

We want to show that this correspondence $Y \mapsto T_{Y^c}$ gives another bijection from the set of LR tableaux to the set of GZ schemes.

In [29], van Leeuwen replaced the Yamanouchi condition in LR tableaux with the semistandard condition in their companion tableaux. Here, we show that the semistandard condition in LR tableaux has a counterpart in the companion tableaux as well, and then we identify the companion tableaux as an independent object equivalent to GZ schemes.

Theorem 4. *For a LR tableau Y, we let Y^c denote its companion tableau and let T_{Y^c} denote the GT pattern corresponding to Y^c. The map $\psi(Y) = T_{Y^c}$ gives a bijection from $LR(\lambda/\mu, \nu)$ to $GZ(\nu, \lambda - \mu, \mu)$. In particular, the Yamanouchi and semistandard conditions in Y are equivalent to, respectively, the interlacing condition IC(2) and the exponent condition in T_{Y^c}.*

Proof. From (14), Y is a tableau of shape λ/μ if and only if the content of Y^c is equal to $\lambda - \mu$. The content of Y is equal to the shape of Y^c. The type and weight of T_{Y^c} are therefore ν and $\lambda - \mu$, respectively.

Recall the Yamanouchi condition in Y (11): for $0 \leqslant i < n$ and $1 \leqslant j < n$,

$$\sum_{k=1}^{i} a_{j,k}(Y) \geqslant \sum_{k=1}^{i+1} a_{j+1,k}(Y). \tag{15}$$

Since $a_{i,j}(Y) = a_{j,i}(Y^c)$ for all i and j, this inequality, in terms of the entries in Y^c, is saying that the number of entries less than or equal to $i+1$ in the $(j+1)$th row is not more than the number of entries less than or equal to i in the jth row. It is the semistandard condition for Y^c and therefore the interlacing condition for T_{Y^c}.

To show this, consider expressing the elements of the GT pattern $T_{Y^c} = (t_j^{(i)})$ in terms of $a_{i,j}(Y^c)$. From the standard bijection between semistandard tableaux and GT patterns, (7), we have

$$t_j^{(i)} = \sum_{k=1}^{i} a_{k,j}(Y^c)$$

where $a_{i,j}(Y^c)$ is the number of i entries in the jth row of Y^c.

Consider the interlacing condition IC(2): $t_j^{(i)} \geqslant t_{j+1}^{(i+1)}$ where $0 \leqslant i < n$ and $1 \leqslant j < n$. Writing this with the above relation gives

$$\sum_{k=1}^{i} a_{k,j}(Y^c) \geqslant \sum_{k=1}^{i+1} a_{k,j+1}(Y^c) \quad \Leftrightarrow \quad \sum_{k=1}^{i} a_{j,k}(Y) \geqslant \sum_{k=1}^{i+1} a_{j+1,k}(Y)$$

which is exactly the expression for the Yamanouchi condition (15). It can be similarly shown that, as mentioned in Remark 2, IC(1) is equivalent to $a_{i,j}(Y) \geqslant 0$.

Using (10), the semistandard condition for Y says we have, for all $1 \leqslant \ell \leqslant n$ and $1 \leqslant m < n$,

$$\left(\sum_{k=1}^{\ell} a_{k,m+1}(Y) - \sum_{k=1}^{\ell-1} a_{k,m}(Y) \right) \leqslant (\mu_m - \mu_{m+1}) \tag{16}$$

or

$$\sum_{k=1}^{\ell-1} (a_{m+1,k}(Y^c) - a_{m,k}(Y^c)) + a_{m+1,\ell}(Y^c) \leqslant (\mu_m - \mu_{m+1})\,.$$

To finish our proof, it is enough to show that the left hand side of the above inequality is the exponent $\varepsilon_\ell^{(m)}(T_{Y^c})$. This can be easily seen, by using (8), as

$$\begin{aligned}\varepsilon_\ell^{(m)}(T_{Y^c}) &= \sum_{1\leqslant h<\ell} \left(t_h^{(m+1)} - 2t_h^{(m)} + t_h^{(m-1)}\right) + \left(t_\ell^{(m+1)} - t_\ell^{(m)}\right)\\ &= \sum_{1\leqslant h<\ell} \left(\sum_{k=h}^{m+1} a_{k,h}(Y^c) - 2\sum_{k=h}^{m} a_{k,h}(Y^c) + \sum_{k=h}^{m-1} a_{k,h}(Y^c)\right)\\ &\quad + \left(\sum_{k=\ell}^{m+1} a_{k,\ell}(Y^c) - \sum_{k=\ell}^{m} a_{k,\ell}(Y^c)\right)\\ &= \sum_{1\leqslant k<\ell} (a_{m+1,k}(Y^c) - a_{m,k}(Y^c)) + a_{m+1,\ell}(Y^c).\end{aligned}$$ □

We now have an alternative proof of Corollary 1.

Corollary 2. *There is a bijection between $LR(\lambda/\mu,\nu)$ and $\mathcal{H}^\circ(\mu,\nu,\lambda)$.*

Proof. We can map any $Y \in LR(\lambda/\mu,\nu)$ to $T_{Y^c} \in GZ(\nu,\lambda-\mu,\mu)$ via the bijection in Theorem 4. From Theorem 2 there is a bijection between $\mathcal{H}^\circ(\mu,\nu,\lambda)$ and $GZ(\nu,\lambda-\mu,\mu)$ through the derived t-array T_2 of a hive. The composition of the first bijection with the inverse of the second then gives a bijection which assigns $Y \in LR(\lambda/\mu,\nu)$ to $H \in \mathcal{H}^\circ(\mu,\nu,\lambda)$ if and only if $T_2(H) = T_{Y^c}$. □

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SURVEY ARTICLE

Generalized norms of groups

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Abstract. In this survey paper the authors specify all the known findings related to the norms of the group and their generalizations. Special attention is paid to the analysis of their own study of different generalized norms, particularly the norm of non-cyclic subgroups, the norm of Abelian non-cyclic subgroups, the norm of infinite subgroups, the norm of infinite Abelian subgroups and the norm of other systems of Abelian subgroups.

Introduction

In group theory findings related to the study of characteristic subgroups (in particular, the center, the derived subgroup, Frattini subgroup, etc.) and the impact of properties of these subgroups on the structure of the group are in the focus. Nowadays the list of such characteristic subgroups can be broaden by means of different Σ-norms of a group.

Let Σ be the system of all subgroups of the group which have some theoretical group property. For example, Σ can consist of all subgroups of the group, of all cyclic, all non-cyclic, all Abelian, all non-Abelian, all subnormal, all maximal, all infinite subgroups of the group. The intersection $N_\Sigma(G)$ of the normalizers of all subgroups of the group which

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belong to the system Σ is called Σ-norm of a group G. In the case $\Sigma=\varnothing$ we assume that $G=N_\Sigma(G)$.

By the definition of the Σ-norm it follows that it is a characteristic subgroup of the group and contains the center of the group. Also, $N_\Sigma(G)$ is the maximal subgroup of the group that normalizes all Σ-subgroups of the group. Therefore, all subgroups of the Σ-norm, which belong to the system Σ, are normal in $N_\Sigma(G)$ (although these subgroups may not exist).

Considering the Σ-norm, there are several problems related to the study of the group properties with the given system Σ of subgroups and some restrictions, which the norm satisfies. Many algebraists solved the similar problems but the choice of a system Σ and properties of the Σ-norm varied.

Knowing the structure of Σ-norms and the nature of its attachment to the group, the properties of the group can be characterized in many cases. For example, if the Σ-norm coincides with the group and $\Sigma \neq \varnothing$, then all subgroups of the system Σ are normal in the group. First non-Abelian groups with this property were considered in the XIX century by R. Dedekind [1], who gave a complete description of finite non-Abelian groups, all subgroups of which are normal, and called them Hamiltonian groups. Infinite Hamiltonian groups were described in 1933 by R. Baer [2]. Sets of Abelian and Hamiltonian groups combined are called the set of Dedekind groups.

However, the study of groups with other systems Σ of normal subgroups were continued only in the second part of the XX century, that slowed down the study of Σ-norms. The findings of S. M. Chernikov and his disciples are from the very field of the research. Thus nowadays the structure of groups that coinside with the norm $N_\Sigma(G)$ is known for many systems of subgroup. So the question on the study of the properties of groups, in which the Σ-norm is a proper subgroup, arises naturally.

1. The norm of group and subgroups close to it

For the first time the problem of the study of the properties of groups, which differ from the Σ-norm, was formulated by R. Baer in 30s of the previous century. In [3] he introduced a subgroup $N(G)$, which is the intersection of normalizers of all subgroups of a group, and called it the norm of the group G. It is clear that the norm $N(G)$ is the Σ-norm of the group for the system Σ, which consists of all subgroups of the group. The norm $N(G)$ is contained in all other Σ-norms and they can be considered as its generalizations. It is also clear that Dedekind groups coincide with their norms, so the index of the norm in a group can serve as a certain "degree of Dedekindness" of a group.

The norm of a group was studied by R. Baer [3–10] and several other authors [11–28]. R. Baer noticed that the restrictions that are imposed on the norm of the group influence the structure of the group in a certain way. Thus, there is a proposition.

Proposition 1.1 ([3]). *If a norm $N(G)$ of a group G is Hamiltonian, then the following propositions take place:*

1) *G is a periodic group;*
2) *G contains no elements which orders are divisible by 8;*
3) *all elements of a group G, which have orders multiple of 4, can be represented in the form of za, where $a \in N(G)$, $|a| = 4$, and $z \in G$, more over z is permutable with each element of a norm $N(G)$;*
4) *any element, which order is not divisible by 4, is permutable with every element of the subgroup $N(G)$.*

Studying the relations between the norm and the center of the group R. Baer showed that the norm coincides with the center of the group, if it contains elements of infinite order [3]. Another important result, which specifies the influence of the center of the group on its norm, was offered in [10].

Proposition 1.2 ([10]). *The norm of a group G is identity if and only if G is a group with an identity center.*

Developing the study of the properties of the norm of a group L. Wos [11] found out that the norm $N(G)$ is contained in the third hypercentre of the group, and the group of automorphisms, which are induced on the subgroup *N(G)* by G, is nilpotent of class at most 2. In addition, it was proved that the norm of the group is contained in the second hypercentre if and only if the group of automorphisms induced on $N(G)$ by the group G is Abelian. This result was substantially refined by E. Shenkman in [12].

Proposition 1.3 ([12]). *The norm $N(G)$ of a group G is contained in the second hypercentre of G. The derived subgroup G' is a subgroup of centralizer of a norm $N(G)$ in G.*

So the group of automorphisms induced on $N(G)$ by G is Abelian. Let's note that $N(G) = E$ in groups with the identity center by Proposition 1.3.

In [6,8] the properties of periodic groups with an Abelian norm quotient group were considered. In particular, in [6] it was proved that a periodic group G, which quotient group $G/N(G)$ is Abelian and $N(G) \neq Z(G)$, is a direct product of its primary components, and its norm $N(G)$ is a direct product of norms of these components.

In this regard, let's note that unlike some other characteristic subgroups (the center of the group, the derived subgroup, Fitting subgroup and others) the norm of the direct product of arbitrary subgroups is not equal to the direct product of the norms of the correlative components in general case.

Example 1.1. Let $G = Q \times B$, where B is a non-periodic Abelian group of rank 1, Q is a quaternion group of order 8. In this group

$$N(G) = N(Q \times B) = Q^2 \times B \neq N(Q) \times N(B) = G.$$

The problem of finding the norms of direct products of groups was studied by J. Evan in [13].

Let's also regard the following finding of R. Baer, which characterizes the properties of p-groups with an Abelian norm quotient group.

Proposition 1.4 ([8])**.** *If G is a p-group ($p \neq 2, p \neq 3$) that has an Abelian quotient group for the norm $N(G)$, more over $N(G) \neq Z(G)$ and p^r is the exponent of the group $C_G(N(G))$, then:*

1) *G is a group of finite exponent;*
2) *$N(G)/Z(G)$ is a cyclic group, the order of which is equal to the exponent of the group of automorphisms induced on $N(G)$ by G;*
3) *centralizer $C_G(N(G))$ consists of those and only those elements $x \in G$ for which $x^{p^r} = 1$.*

The restrictions $p \neq 2, p \neq 3$ in Proposition 1.4 are significant, as it is illustrated by the examples of the respective groups (see [6]).

Nowadays the interest to the norm $N(G)$ of a group is not reduced, as research works [13–31], devoted to the study of its properties, are still numerous. Thus, in [17, 18] R. Bryce and J. Cossey considered series of norms

$$1 = N_0(G) \subseteq N_1(G) \subseteq \ldots \subseteq N_i(G) \subseteq \ldots,$$

where $N_i(G)/N_{i-1}(G) = N(G/N_{i-1}(G))$ for $i \geqslant 1$.

It was proved that in the class of 2-groups from the fact that quotient group $N_{i+1}(G)/N_i(G)$ is Hamiltonian it follows that $N_{i+1}(G) = G$. Moreover a finite 2-group, in which the quotient group $G/N(G)$ is Hamiltonian, but any quotient group $N_i(G)/N_{i-1}(G)$ is not Hamiltonian, has order 2^7 and is uniquely determined up to isomorphism [18].

Starting from R. Baer, L. Wos and E. Schenkman studies of the norm $N(G)$ focus on its relation to the centre of the group. In particular, in [19] J. Beidleman, H. Heineken and M. Newell have shown that in an arbitrary p-group G either quotient group $G/Z(G)$ or group $[G, N(G)]$ is cyclic. In this article the problem of the influence of properties of a norm of a group and its center on the capability of a group G is considered.

A group G is called capable, if it is a group of inner automorphisms of some group H that is $G \cong H/Z(H)$. R. Baer [29] studied such groups for the first time. He described capable finitely generated Abelian groups

$$G = Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_k},$$

where $n_{i+1} \vdots n_i$, $n_i \in N \bigcup \{0\}$ and $Z_{n_i} = Z$ is an infinite cyclic group for $n_i = 0$. It was found out that the group G is capable if and only if $k \geqslant 2$ and $n_{k-1} = n_k$. The Baer's characterization remains the only complete one for a certain class of capable groups today.

Developing studies of the norm of a group in capable groups, X. Guo and X. Zhang [20] in 2012 established necessary and sufficient conditions for the coinsiding of the norm of the group with its centre, and also dwelled upon the properties of the norm $N(G)$ in the class of nilpotent groups with a cyclic derived subgroup.

In 2005 N. Gavioli, L. Legarreta, S. Sica, M. Tota [22] considered the relations between the centre $Z(G)$, the norm $N(G)$ and the second hypercentre $Z_2(G)$ depending on the number $v(G)$ of conjugacy classes of non-normal subgroups and the number $w(G)$ of conjugacy classes of subgroups, which are normalizers of some subgroups, in finite p-groups $(p \neq 2)$ of nilpotency class c.

In 2008 F. Russo [23] studied the relations between the centre $Z(G)$, the norm $N(G)$, the quazicenter $Q(G)$ and the hyperquazicenter $Q^*(G)$ of quazicentral-by-finite groups. Let's regard that the quazicenter $Q(G)$ of a group G is the subgroup, generated by all elements x of a group G, such that the subgroup $\langle x \rangle$ is permutable in a group G (with other subgroups). Accordingly, the hyperquazicenter $Q^*(G)$ of a group G is the largest term of the chain of normal subgroups

$$E = Q_0(G) \leqslant Q_1(G) = Q(G) \leqslant \ldots \leqslant Q_\alpha \leqslant Q_{\alpha+1} \leqslant \ldots,$$

where α is an ordinal and $Q_{\alpha+1}(G)/Q_\alpha(G) = Q(G/Q_\alpha(G))$.

Proposition 1.5 ([23], Proposition 3.2). *Let G be a quasicentral-by-finite group, $Q(G)$ be the quasicenter of G, $N(G)$ be the norm of G, then*

1) *if $Q(G)$ contains only elements of prime or infinite order and $Q(G) = N'$, where N is the subgroup generated by the quasicentral elements of infinite order, then G is finite;*
2) *if there is an element $x \in N(G)$ such that the index $|Q(G) : \langle x \rangle_G|$ is finite, then G is central-by-finite;*
3) *G is central-by-finite if and only if the index $|Q(G) : N(G)|$ is finite.*

The relations between the norm $N(G)$ and the center $Z(G)$ in the class of finite groups have also been studied by I. V. Lemeshev in [24]. His findings add much to Baer's results related to finite groups.

The study of finite groups, in which Baer norm has a certain index, is very effective. In particular, in [25] J. Wang and X. Guo studied finite p-groups, in which the norm has a prime index, in [26] they studied finite groups, in which the norm is a subgroup of index p or pq, where p and q are different prime numbers. J. Smith [27] studied groups in which each subgroup of the norm is normal in the group.

Subgroups of an arbitrary group can be considered as elements of some subgroups lattice $L(G)$ relative to the operations of union and intersection, ordered by inclusion. In this sense, the norm $N(G)$ of a group can be defined as following [28]:

$$N(G) = \bigcap_{X \in L(G)} N_G(X).$$

In this context, in [28] the relation between the non-cyclicness of the norm $N(G)$ on the one hand and the subgroup lattice $L(G)$ of the group G and generalized degree of commutativity of the group G on the other hand is under the analysis.

A question naturally arises why this characteristic subgroup, in contrast to the center and the derived subgroup, did not get adequate attention in the early development of group theory in view of the simple definition of the norm and its usefullness in the study of groups. G. Miller [31] explains that at that time other problems were posed in algebra and the main focus of group theory has been directed to the study of solutions of algebraic equations (in this theory simple groups play a fundamental role, while the norm of a simple group of composite order is identity). The norm was also not of high importance in the study of permutation groups of low degrees, which were used in the theory of algebraic equations at that time. The smallest degree of permutation group, which has the norm of prime index, is equal to 8, moreover only one of 200 groups of this order has the norm of prime index. And perhaps R.Baer drew attention to this characteristic subgroup only in 1934 for these reasons.

Considering the intersection of normalizers of subgroups of the group, we can get subgroups associated with Baer norm. These are the intersection of normalizers of all subgroups contained in the given subgroup [32–34] or conversely the intersection of the normalizers which contain the given subgroup [35]. In particular, the concept of invariator $I_G(A)$ of the subgroup A in the group G, which was introduced by I. Ya. Subbotin, is the closest to the concept of the norm $N(G)$ of the group G.

Invariator $I_G(A)$ of subgroup A in the group G [32] (*quazicenralizer* [34]) is the intersection of normalizers of all subgroups of the group A in G. This subgroup can also be called the norm of the subgroup A in the group G [36]. In the case when the subgroup A coincides with the whole group G the invariator $I_G(G)$ is exactly the norm $N(G)$ of the group G.

In 2001 M. De Falco, F. de Giovanni, C. Musella [35] introduced the concept of H-norm of the group G for some subgroup H of the group G. H-norm of a group G is called a subgroup $\ker(G : H)$ that consists of all elements which normalize every subgroup of X in G containing H. Obviously, if $H \leqslant K \leqslant G$, then $H \leqslant \ker(G : H) \leqslant N_G(H)$, $\ker(G : H) \leqslant \ker(G : K)$. Let's note that the E-norm, where E is the identity subgroup of a group G, coincides with the norm $N(G)$ of the group G.

It is clear that the norm $N(G)$ can be defined as the subgroup of a group G consisting of all elements of this group, which normalize every subgroup in G. Replacing the condition of normality to pronormality we get some analogue of the norm of a group for pronormality. It is called *pronorm* $P(G)$.

Let's regard that an element x of the group G *pronormalizes subgroup* H of a group G, if subgroups H and H^x are conjugate in $\langle H, H^x\rangle$. Accordingly, the *pronorm* $P(G)$ of a group G is the set of all elements of a group G which pronormalize every subgroup of a group. For the first time the concept of pronorm $P(G)$ group was introduced by F. de Giovanni, S. Vincenzi [37] in 2000.

In contrast to the norm of a group the pronorm is not always a subgroup of a group. In [38] some classes of groups, in which the set of all elements of a group G that pronormalize every subgroup of a group, forms a subgroup, were studied.

Proposition 1.6 ([37]). *If G is polycyclic group, then its pronorm $P(G)$ is a subgroup.*

In this work a similar statement for the class of locally soluble groups was proved.

Subgroups generated by normalizers of given subgroups are considered in some researches about groups with restrictions on normalizers of given systems of subgroups. In this context, let's consider the research of J. Smith [39], who studied the subgroup $R = R(G)$ generated by all proper normalizers, and called it *conorm of a group*. If the group G has not proper normalizers, then the group G is Dedekind and $R(G) = E$.

In 1990 H. Bell, F. Guzman, L.-Ch. Kappe [40] studied so-called Baer-kernel, which is a ring analogue of the norm of the group. Baer-kernel of the ring K is defined as the set

$$B(K) = \{a \in K | \forall y \in K, \exists r, s \in N(ay = y^r a \wedge ya = ay^s)\}.$$

In 2010 year M. R. Dixon, L. A. Kurdachenko, D. Otal used the so-called norm of subspace in linear groups in the research of linear groups with finite dimensional orbits [41].

Let A be a vector space over a field F, $GL(F, A)$ be a group of all automorphisms of a space A, G be a subgroup of a group $GL(F, A)$, B be a subspace of a space A. The norm of the subspace B in the group G is the intersection of normalizers of all F-subspases in B:

$$Norm_G(B) = \bigcap_{b \in B} N_G(bF).$$

It is known when the group G coincides with the norm $Norm_G(B)$, then the group G is isomorphic to a subgroup of the multiplicative group $U(F)$. If the group G has finite dimensional orbits over A, then A contains a FG-submodule D of finite dimension $\dim_F(D)$. If $K = C_G(D)$, then $K \leqslant Norm_G(A/D)$. When G-orbits of every subspace from A are finite, then A contains a FG-submodule B such that $\dim_F(A/B)$ and $|G : Norm_G(B)|$ are also finite.

Therefore, the research devoted to the study of the norms of the group and related subgroups is a very important and interesting direction in the group theory. At the same time, there are still many questions regarding the structural characteristics of the group depending on the structure of its norms, conditions of coinsiding of the norm of the group and its center, etc. left.

2. Generalized norms of some systems of maximal and subnormal subgroups

As noted above, the norm $N(G)$ is the Σ-norm of the group, in which the system Σ is a system of all subgroups of this group. Narrowing the system of all subgroups, for example, to the system of all Abelian or all maximal subgroups of a group, we will get new Σ-norms, which can be considered as generalizations of the norm $N(G)$.

The first generalizations of this kind were made in the 50-th of the XX century. In particular, in 1953 R. Baer [42] considered the intersection $H(G)$ of normalizers of all Sylow subgroups of a group G and called this intersection as *hypercenter of a group* G. It is clear that hypercenter $H(G)$ is the Σ-norm, where the system Σ consists of all Sylow subgroups of the group. R. Baer proved that $H(G)$ coincides with the intersection of all maximal nilpotent subgroups, and the quotient group $G/H(G)$ is a group with an identity center. Moreover, it was found out that the normal subgroup belongs to a hypercenter if and only if its elements of order p^n generate cyclic subgroups of index p^n.

In 1968 B. Huppert [43] generalized the concept of a hypercenter introducing the concept of $\mathfrak{F}$-*hypercenter*. Let $\mathfrak{F}$ be a class of finite groups which can be represented as direct products of their Hall π-subgroups with respect to some partition of non-empty set π of all primes. This class is a local formation. The chief factor H/K of a group G is called $\mathfrak{F}$-central [44], if $H/K\lambda(G/C_G(H/K)) \in \mathfrak{F}$. The product of all normal subgroups of G which G-chief factors are $\mathfrak{F}$-central in G is called $\mathfrak{F}$-*hypercenter* $Z_{\mathfrak{F}}(G)$ of a group G [45]. In 2013 V. I. Murashka [46] studied the properties of $\mathfrak{F}$-hypercenter and got some Baer's results on the norm of the group as corollaries in some cases.

One of the mentioned generalizations of the norm of the group is a so-called A-norm $N_A(G)$ of the group G. It is the intersection of normalizers of all maximal Abelian subgroups. This norm was introduced by W. Kappe [47] in 1961. As it turned out (see [47]) in finite group A-norm is a subgroup, each element of which is permutable with its conjugate (such groups were studied, in particular, by F. Levi [48]). In addition, it was found that the A-norm is close to a subgroup of right Engel elements of length 2, that allowed to use it in the study of Engel groups.

Let's regard (see e.g. [49]) that the element $x \in G$ is called the *right Engel element of length 2*, if for any element $g \in G$ there is a relation $[[x,\ g],\ g] = 1$.

Let $R(G)$ denote the subgroup of a group G generated by all right Engel elements of length 2 of a group G. The following propositions take place.

Proposition 2.1 ([47]). *A-norm $N_A(G)$ of a group G contains the second hypercenter of a group G and is contained in the subgroup $R(G)$. Moreover the quotient group $R(G)/N_A(G)$ is elementary Abelian group of exponent not exeeding 2.*

Proposition 2.2 ([47]). *For an element $x \in G$ which order is not divisible by 2, the following statements are equivalent:*
1) *$x \in N_A(G)$;*
2) *x is right Engel element of length 2 in G;*
3) *if $\langle x \rangle \lhd G$ and U is the group of automorphisms induced on $\langle x \rangle$ by G, then x belongs to A-norm of the group $\langle x \rangle U$;*
4) *for any elements $g, h \in G$ the equality $[[x, g], h] = [[x, h], g]^{-1}$ takes place.*

The following proposition on a A-norm is a generalization of Wos' [11] and Schenkman's results [12] related to the norm $N(G)$ of the group.

Proposition 2.3 ([47]). *Group G induces on the subgroup $N_A(G)$ a nilpotent group of automorphisms. Its class of nilpotency does not exceed 2.*

Later on W. Kappe [50–52] generalized the concept of the A-norm of the group and introduced a so-called E-norm, which was defined as the intersection of normalizers of all maximal subgroups of the group with the given theoretical group property E. Clearly, E-norm $N_E(G)$ contains the norm $N(G)$. The intersection of an arbitrary subgroup of a group G and the E-norm of the group is contained in the E-norm of this subgroup. Besides $N_E(N_E(G)) = N_E(G)$.

A subgroup $\Delta(G)$ is related to the concept of the E-norm. It was studied by W. Gashutz [53] and was defined as the intersection of normalizers of all maximal subgroups of the group. It is clear that Gashutz subgroup $\Delta(G)$ can be considered as Σ-norm of a group for the system Σ that consists of non-normal in G maximal subgroups. In [53] it was found out that $\Delta(G)$ is nilpotent and $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$, where $\Phi(G)$ is Frattini subgroup.

In 1958 H. Wielandt [54] studied the properties of normalizers of subnormal subgroups and introduced the subgroup $W(G)$. It is the intersection of normalizers of all subnormal subgroups of a group. It is clear that *Wielandt subgroup* $W(G)$ is the *norm of subnormal subgroups* of a group.

It is obvious that a subnormal norm coincides with the norm $N(G)$ in a nilpotent group. In addition, the condition $G = W(G)$ is equivalent to the fact that all subnormal subgroups of a group are normal. By Theorem 13.3.7 [55] Wielandt subgroup $W(G)$ contains every simple non-Abelian subnormal subgroup of G and every minimal normal subgroup of G which satisfies the minimal condition for subnormal subgroups. Therefore, the subgroup $W(G)$ is not identity in a finite group G [54].

D. Robinson [56] and J. Roseblade [57] independently from each other got similar results for some classes of infinite groups.

Proposition 2.4 ([56, 57]). *If a group G satises the minimal condition for subnormal subgroups, then the quotient group $G/W(G)$ is finite.*

These results were summarized by J. Cossey [58] for polycyclic groups. It was found out that these groups have a finite quotient group $G/C_G(W(G))$.

Wieland subgroup and its generalizations were studied intensively by O. Kegel [59], J. Cossey, R. Bryce [60–62], R. Brandl, F. Giovanni, S. Franciosi [63]. A. Camina [64], C. Casolo [65, 66], E. Ormerod [67], C. Wetherell [68, 69], X. Zhang and X. Guo [70, 71].

In [60] it was proved that the subnormal norm $W(G)$ is contained in the FC-centre in a finitely generated soluble-by-finite group of a finite rank. Furthermore, if the norm $W(G)$ coincides with the whole group, then all subnormal subgroups are normal in this group, that is, the normality is transitive relation. Groups with such a property were studied by D. Robinson in [72] and were called T-groups. If G is a finite soluble T-group and G/L is the unique maximal nilpotent quotient group of group G, then the quotient group G/L is Abelian or Hamiltonian and L is Abelian.

In 1989 J. Cossey, R. Bryce [60] introduced *local Wielandt subgroup* $W^p(G)$ that is the intersection of normalizers of all p'-perfect subnormal subgroup of a group G. Let's regard that the p'-perfect group is a group that has no non-identity quotient groups of order coprime with p.

In 1992 C. Casolo [66] studied a special subgroup of a group $W(G)$, which was called *strong Wielandt subgroup* $\overline{W}(G)$, and defined as the intersection of the centralizers of nilpotent subnormal quotient groups of the group G:

$$\overline{W}(G) = \left\{ g \in G | [S, g] \leqslant S^R \text{ for all } S << G \right\},$$

where S^R is nilpotent residual of the subgroup S or the smallest normal subgroup N of S such that the quotient group S/N is nilpotent. C. Casolo proved that strong Wielandt subgroup $\overline{W}(G)$ is non-identity in a finite group. Note that this subgroup was also studied by C. Wetherell [68, 69].

In 1990 R. Bryce [62] introduced one more generalization of Wielandt subgroup, so-called *m-Wielandt subgroup* $U_m(G)$ of a group G that is the intersection of normalizers of all subnormal subgroups of a group G with a defect at most m for integer $m \geqslant 1$. He studied a polynilpotent lattice of finite soluble groups in terms of Wielandt m-length. The concept of m-series of Wielandt group is widely used. It is defined as following: for each natural $m \geqslant 1, U_{m,0}(G) = E$; if $i \geqslant 1$, then $U_{m,i}(G)$ is determined from the condition

$$U_{m,i}(G)/U_{m,i-1}(G) = U_m(G/U_{m,i-1}(G)).$$

If $U_{m,n}(G) = G$ for some integer n, then such a minimal number n is called Wielandt m-length. R. Bryce proved that there are limits of commutator length and Fitting length of finite soluble groups in terms of Wielandt m-length ($m \geqslant 2$), and identified the best such a restriction. Properties of *Wielandt m-subgroup* $U_m(G)$ have also been studied by C. Franchi [76, 77].

In 1995 J. Biedleman, M. Dixon, D. Robinson [73, 74] considered one more Σ-norm of a group – *generalized Wielandt subgroup* $IW(G)$ which is the intersection of normalizers of all infinite subnormal subgroups of a group. It is clear that $IW(G)$ is a characteristic subgroup and contains a subnormal norm $W(G)$. If $G = IW(G)$, then all infinite subnormal subgroups are normal in the group. Such groups have been studied by F. Giovanni, S. Franciosi [75] and were called *IT-groups*. In [73] the structure of the group G with the property $IW(G) \neq W(G)$ and the structure of the quotient group $IW(G)/W(G)$ were studied.

In [78] F. Mari, F. Giovanni introduced a new Σ-norm, in which system Σ consists of all nonsubnormal subgroups of a group. This *norm of nonsubnormal subgroups* was denoted by $W^*(G)$. It is clear that if $W^*(G) = G$, then all subgroups are subnormal in a group. Moreover, if a group G is a group with a finite number of normalizers of subnormal subgroups, then the quotient group $G/W^*(G)$ is finite [78].

Let's also mention the research [79], in which so-called *generalized N-Wielandt subgroup* $W_N(G)$ was introduced. It consists of all elements of the group G, which normalize all subnormal subgroups of N. It is a normal subgroup and, in general, may differ from N.

It is clear that $W(G) \subseteq W_N(G)$, in particular, $W(G) = W_N(G)$, if $N = G$, or $N = W(G)$, or N is the unique maximal normal subgroup. If G is a T-group and N is a normal subgroup of G, then $W_N(G) = G$. The following example proves that the converse is not true.

Example 2.1 ([79])**.** Let $G = D_8 = \langle x, y\rangle$, $x^8 = y^2 = (xy)^2 = 1$, $N_1 = \langle x^2\rangle$, $N_2 = \langle x\rangle$, then $W_{N_1}(G) = W_{N_2}(G) = G$, but G is not a T-group.

3. Generalized norms of characteristic subgroups of a group

Nowadays algebraists direct their attention to a generalization of the norm when the system Σ is selected as a system of some characteristic subgroups. In this context Sh. Lia and Zh. Shen [80, 81] considered the Σ-norm $D(G)$ of a finite group, where the system Σ is chosen as a system of derived subgroups of all subgroups of the group. The authors proved that in the case when $D(G)$ contains all the elements of prime order, the group G is solvable of Fitting length at most 3. In the case when $G = D(G)$, derived subgroup G' is nilpotent and G'' has nilpotency class at most 2.

Recently a number of researches concern the norms of different systems of residuals. In particular, Zh. Shen, W. Shi and G. Qian [82] studied the *norm $S(G)$ of nilpotent residuals of all subgroups of prime order*. It was proved that if all elements of prime order of a finite group G are contained in the norm $S(G)$, then the group G is solvable. L. Gong and X. Guo [83] studied the norm of nilpotent residuals of all subgroups of a finite group. N. Su and Ya. Wang [84] considered the norm $D^{\mathfrak{F}}(G)$ of *$\mathfrak{F}$-residual* $G^{\mathfrak{F}}$ of all subgroups of the group G and the norm $D_p^{\mathfrak{F}}(G)\mathfrak{H}^{\mathfrak{F}}O_{p'}(G)$ of all subgroups H of a finite group G, where $\mathfrak{F}$ is the formation. Recall that $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of a group G is the smallest normal subgroup N of G such that $G/N \in \mathfrak{F}$.

X. Chen and W. Guo [85] introduced the $\mathfrak{h}\mathfrak{F}$-norm $N_{\mathfrak{h}\mathfrak{F}}(G)$ of a group G. It is the intersection of normalizers of products of $\mathfrak{F}$-residuals of all subgroups of a group G and $\mathfrak{h}$-radical of a group G

$$N_{\mathfrak{h},\mathfrak{F}}(G) = \bigcap_{H \leqslant G} N_G(H^{\mathfrak{F}} G_{\mathfrak{h}}),$$

where $\mathfrak{h}$ is Fitting class, $\mathfrak{F}$ is formation. Let's regard that $\mathfrak{h}$-radical $G_{\mathfrak{h}}$ of a group G is maximal normal $\mathfrak{h}$-subgroup of a group G.

If $\mathfrak{h} = 1$, then the subgroup $N_{1,\mathfrak{F}}(G)$ is called $\mathfrak{F}$-norm $N_{\mathfrak{F}}(G)$ of a group G and defined as

$$N_{\mathfrak{F}}(G) = \bigcap_{H \leqslant G} N_G(H^{\mathfrak{F}}).$$

If $\mathfrak{h} = \mathfrak{G}_\pi$, where $\mathfrak{G}_\pi$ is the class of finite π-solvable groups, then the subgroup $N_{\mathfrak{G}_\pi,\mathfrak{F}}(G)$ is called $\pi\mathfrak{F}$-norm $N_{\pi\mathfrak{F}}(G)$ of a group G and defined as

$$N_{\pi\mathfrak{F}}(G) = \bigcap_{H \leqslant G} N_G(H^{\mathfrak{F}} O_\pi(G)).$$

X. Chen and W. Guo studied the properties of $\mathfrak{h}\mathfrak{F}$-norm, in particular, $\pi\mathfrak{F}$-norm of a finite group G and the relations between $\pi'\mathfrak{F}$-norm and $\pi\mathfrak{F}$-hypercentre of a group G.

In 2014 A. Ballester-Bolinches, J. Cossey, L. Zhang [86] proposed to generalize the structure of Σ-norms which had appeared recently. The authors defined the *C-norm $kC(G)$* of a finite group G as the intersection of the normalizers of all subgroups of the group G which *do not belong to the class C*

$$k_C(G) = \bigcap_{H \notin C} N_G(H)$$

provided that $k_C(G) = G$, if $G \in C$. With this approach Baer norm $N(G)$ can be considered as the norm $k_C(G)$, where C is the class of groups of order 1. Groups with $k_C(G) = G$ are called C-Dedekind. In [86] the structure of non-nilpotent C-Dedekind groups for the class of nilpotent groups is described. It is also shown that the groups, which C-norm is not hypercentral, have a very restricted structure. The authors gave the classification of nilpotent classes closed under subgroups, quotient groups and direct products of groups of coprime orders, and showed that the known classifications can be deduced from this one.

Proposition 3.1 ([86]). *If $k_C(G)$ contains a non-central chief factor of G, then $k_C(G)$ contains exactly one non-central chief factor (in any chief series through $k_C(G)$ of a group G) and if p is a prime divisor of the order of this chief factor, then Hall p'-subgroup of G is C-group and G has nilpotency class at most 3.*

Consider also R. Laue's research [87]. He dealt with a subgroup close to the Σ-norm

$$A(\Sigma) = \bigcap_{X \in \Sigma} N_{Aut(G)}(X),$$

which consists of automorphisms that normalize every Σ-subgroup of a group G.

4. Generalized norms of different systems of Abelian and non-cyclic subgroups

The narrowing of a system Σ of all subgroups of the group G to the system of all Abelian and all cyclic subgroups does not lead to extension of the concept of the norm $N(G)$. However, when the system Σ is the system of all non-cyclic subgroups (provided that such subgroups exist in the group), then such Σ-norm (let's call it the *norm of non-cyclic subgroups*) differs from the norm $N(G)$ in a general case. The opportunity to study the norm of non-cyclic subgroups was provided by F. M. Lyman's research [88–90]. He received a description of some classes of non-Abelian groups in which all non-cyclic subgroups are normal. These groups were called $\overline{H}$-*groups* ($\overline{H_p}$-*groups* in the case of p-groups).

The concept of the *non-cyclic norm* N_G of a group as the intersection of the normalizers of all non-cyclic subgroups of the group was introduced by F. M. Lyman in 1997 [91], where he studied infinite groups, in which a non-cyclic norm is locally-graded and has a finite index.

Proposition 4.1 ([91]). *In the group G a non-cyclic norm is locally-graded and has a finite index if and only if the group G is central-by-finite.*

In addition, it was proved that for the condition $1 < |G/N_G| < \infty$ in the class of infinite locally finite groups the non-cyclic norm N_G is Dedekind, and in the class of non-periodic locally soluble-by-finite groups it is Abelian [91].

The study of the non-cyclic norm was continued by F. M. Lyman and T. D. Lukashova in [92–96], where the authors characterized the structure of wide classes of groups, which non-cyclic norm is non-Dedekind. Since O. Yu. Olshansky infinite groups [97] exist, periodic groups were considered by the authors provided their local finiteness. O. Yu. Olshansky infinite groups are groups, all subgroups of which are cyclic and which are the norms of their non-cyclic subgroups. Thus, in [98] it was proved that the class of infinite locally finite p-groups ($p \neq 2$), in which a non-cyclic norm N_G is non-Abelian, coincides with the class of non-Abelian p-groups, all non-cyclic subgroups of which are normal. At the same time, there are infinite locally finite 2-groups which have a proper non-Dedekind norm of non-cyclic subgroups. The structure of locally finite p-groups (p is prime), which non-cyclic norm is non-Dedekind, is described in [92–94].

Proposition 4.2 ([92]). *Locally finite p-groups ($p \neq 2$), which have non-Abelian non-cyclic norm N_G, are groups of the following types:*

1) *G is an $\overline{H}_p$-group, $N_G = G$;*
2) *$G = (\langle x\rangle \times \langle b\rangle)\lambda\langle c\rangle$, $|x| = p^n$, $n > 1$, $|b| = |c| = p$, $[b,c] = x^{p^{n-1}}$, $[x,c] = x^{p^{n-1}\alpha}b^{\beta}$, $(\beta,p) = 1$; $N_G = (\langle x^p\rangle \times \langle b\rangle)\lambda\langle c\rangle$;*
3) *$G = \langle x\rangle\langle b\rangle, |x| = p^k, |b| = p^m, m > 1, k \geqslant m+r, Z(G) = \langle x^{p^{r+1}}\rangle \times \langle b^{p^{r+1}}\rangle, 1 \leqslant r \leqslant m-1$, $[x,b] = x^{p^{k-r-1}s}b^{p^{m-1}t}, (s,p) = 1, N_G = \langle x^{p^r}\rangle\lambda\langle b\rangle$.*

Proposition 4.3 ([93,94]). *Locally finite 2-groups G with a non-Dedekind non-cyclic norm N_G are groups of the following types:*

1) *G is a non-Hamiltonian $\bar{H}_2$-group, $G = N_G$;*
2) *$G = (A \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle$, A is a quasicyclic 2-group, $[A,\langle c\rangle] = 1$, $|b| = |c| = |d| = 2$, $d^{-1}ad = a^{-1}$ for any element $a \in A$, $[b,c] = [d,b] = [d,c] = a_1$, $a_1 \in A, |a_1| = 2$; $N_G = (\langle a\rangle \times \langle b\rangle)\lambda\langle c\rangle$, where $a \in A, |a| = 4$;*
3) *$G = (A \times H)\langle d\rangle$ A is a quasicyclic 2-group, $d^2 = a_1 \in A, \mid a_1 \mid = 2, d^{-1}ad = a^{-1}$ for any element $a \in A$, $H = \langle h_1,h_2\rangle, |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1,h_2]$, $[d,h_1] = a_1, [d,h_2] = 1$; $N_G = \langle h_2\rangle\lambda\langle h_1 a\rangle, |a| = 4, a \in A$, $\mid a \mid = 4$;*
4) *$G = (\langle x\rangle \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle, |x| = 2^n, n > 2, |b| = |c| = |d| = 2, [x,c] = 1, d^{-1}xd = x^{-1}$, $[b,c] = [d,b] = [d,c] = x^{2^{n-1}}$; $N_G = (\langle x^{2^{n-2}}\rangle \times \langle b\rangle)\lambda\langle c\rangle$;*
5) *$G = (\langle x\rangle\lambda\langle b\rangle)\lambda\langle c\rangle$, $|x| = 2^n$, $n > 3$, $|b| = |c| = 2$, $[x,c] = x^{\pm 2^{n-2}}b$, $[x,b] = x^{2^{n-1}}$; $N_G = (\langle x^2\rangle \times \langle b\rangle)\lambda\langle c\rangle$;*

6) $G = \langle x\rangle\lambda H, |x| = 2^n$, $n > 2, H = \langle h_1, h_2\rangle, |h_1| = |h_1| = 4$, $h_1^2 = h_2^2 = [h_1, h_2], [\langle x\rangle, H] = \langle x^{2^{n-1}}\rangle; N_G = \langle x^2\rangle \times H$;
7) $G = (\langle x\rangle \times H)\langle y\rangle, |x| = 2^n, n \geqslant 2, H = \langle h_1, h_2\rangle, |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2], y^2 = x^{2^{n-1}}, [y, h_2] = 1, [y, h_1] = y^2, y^{-1}xy = x^{-1}$; $N_G = \langle h_2\rangle\lambda\langle h_1x^{2^{n-2}}\rangle$;
8) $G = \langle x\rangle\langle b\rangle, |x| = 2^k, |b| = 2^m, m \geqslant 1$; *if* $m = 1$, *then* $k = 3$, $[x, b] = x^2$ *and* $N_G = \langle x^2\rangle\lambda\langle b\rangle$; *if* $m > 1$, *then* $k \geqslant m + r, 1 \leqslant r \leqslant m - 1, Z(G) = \langle x^{2^{r+1}}\rangle \times \langle b^{2^{r+1}}\rangle, [x, b] = x^{2^{k-r-1}s}b^{2^{m-1}t}$, $0 < s < 2, 0 \leqslant t < 2$, *(*$k > 3$ *and* $t = 0$ *if* $m = 2$*)*; $N_G = \langle x^{2^r}\rangle\lambda\langle b\rangle$.

Non-primary locally finite groups with a non-Dedekind non-cyclic norm were studied in [93, 105]. It was found out that infinite locally finite non-primary groups with such restrictions on the norm N_G are locally nilpotent.

Proposition 4.4 ([93]). *Infinite locally finite non-primary groups with a non-Dedekind non-cyclic norm N_G, are locally nilpotent and are groups of the following types:*

1) *G is an infinite non-primary non-Hamiltonian $\bar{H}$–group, $G = N_G$;*
2) *$G = G_2 \times \langle y\rangle, G_2$ is a group of one type 2) or 3) of proposition 4.3, $(|y|, 2) = 1; N_G = N_{G_2} \times \langle y\rangle$.*

Thus, a locally finite group, which non-cyclic norm N_G is non-nilpotent, is finite.

Developing the study of locally finite groups with a non-Dedekind non-cyclic norm, in [95] it was proved that finite nilpotent groups with such restrictions are groups of the type

$$G = G_p \times \langle y\rangle,$$

where G_p is a Sylow p-subgroup of a group G and a finite group with a non-trivial norm N_{G_p}, $(|y|, p) = 1$. In addition, if the non-cyclic norm N_G is non-nilpotent in the class of locally finite groups, then all non-cyclic subgroups in a group are normal.

Non-periodic locally soluble-by-finite groups with a non-Dedekind non-cyclic norm are considered in [96].

Proposition 4.5 ([96]). *Any non-periodic locally soluble-by-finite group G that has a non-Dedekind non-cyclic norm N_G is $\overline{H}$-group and $G = N_G$.*

Note that locally finite or non-periodic locally soluble-by-finite groups with a non-Dedekind norm of non-cyclic subgroups are soluble and their degree of solvability does not exceed 3 according to the results of [92–96].

Zh. Shen, W. Shi, J. Zhang [98, 99] studied the properties of the norm N_G of non-cyclic subgroups in the class of finite groups and its influence on the group. The authors proved that the norm of non-cyclic subgroups of a finite group is soluble. Note that this proposition is a direct corollary from the description of finite $\overline{H}$-groups (see [88–90]). It was also proved that a finite group is soluble if all its elements of prime order are contained in norm N_G of non-cyclic subgroups. In addition, it was found out that the derived subgroup is nilpotent if all elements of prime order or of order 4 of a group are contained in N_G [98].

Proposition 4.6 ([98])**.** *A finite group has a nilpotent derived subgroup if and only if a derived subgroup of a quotient group through norm N_G is also nilpotent.*

The study of infinite groups with given restrictions on normalizers of different systems Σ of infinite subgroups have been the subject matter of many theoretical-group researches for a long time. Therefore, when considering infinite groups with restrictions on Σ-norm, it is naturally to choose one of systems of infinite subgroups as a system Σ.

In this context, in the study of Σ-norms of infinite groups F. M. Lyman and T. D. Lukashova [96, 100–102] considered systems of all infinite, all infinite Abelian and all infinite cyclic subgroups, provided that these systems are non-empty. These Σ-norms were denoted as follows: $N_G(\infty)$ is *the norm of infinite subgroups of a group G*; $N_G(A_\infty)$ is *the norm of infinite Abelian subgroups of a group G*; $N_G(C_\infty)$ *is the norm of infinite cyclic subgroups of a group G*.

If the group G coincides with one of these Σ-norms, then all Σ-subgroups are normal in it. Infinite non-Abelian groups with the property $N_G(\infty) = G$ and $N_G(A_\infty) = G$ (if such subgroups exist in them) were studied by S. M. Chernikov [103, 104] and called *INH-groups* and *IH-groups* respectively.

Restrictions, which these Σ-norms satisfied, were non-Dedekindness of Σ-norm or finiteness of its index in the group. The following proposition gives sufficient conditions of Dedekindness of each of these norms.

Proposition 4.7 ([100])**.** *In non-periodic groups the norm $N_G(\infty)$ of infinite subgroups, the norm $N_G(A_\infty)$ of Abelian infinite subgroups, the norm $N_G(C_\infty)$ of infinite cyclic subgroups are Dedekind in each of the following cases:*

1) *G is a torsion free group or a mixed group without involution;*

2) *the center of a group G contains elements of infinite order;*
3) *G is central-by-finite;*
4) *these norms are finite;*
5) *G contains a subgroup M from the system Σ such that $M \bigcap N_G(\Sigma) = E$.*

The problem of the relations between these norms in non-periodic groups is quite interesting. The following relation is derived from the above definitions

$$Z(G) \subseteq N(G) \subseteq N_G(\infty) \subseteq N_G(A_\infty) \subseteq N_G(C_\infty).$$

So the natural question is: under what conditions do these norms coincide? The following proposition gives the answer (in terms of sufficient conditions).

Proposition 4.8 ([100]). *In a non-periodic group G the equality takes place*

$$N(G) = N_G(\infty) = N_G(A_\infty) = N_G(C_\infty)$$

provided that at least one of the statements takes place:
1) *the center of a group G contains elements of infinite order;*
2) *G is a torsion free group;*
3) *G is central-by-finite.*

Infinite groups with restrictions on the norm $N_G(\infty)$ of infinite subgroups were studied in [100]. It turned out that non-periodic groups, which norm $N_G(\infty)$ has a finite index, are mixed and are finite extensions of their centres.

It was also proved that the norm $N_G(\infty)$ of infinite subgroups of the non-periodic group is Abelian and coincides with the center of the group, if it contains elements of infinite order. This result generalizes Baer's theorem [10] on the coincidence of the norm $N(G)$ of the group and its center in the case of a non-periodic norm $N(G)$. Infinite locally finite groups, which norm $N_G(\infty)$ is non-Dedekind, are a finite extension of a quasicyclic subgroup, which is a divisible part of the norm $N_G(\infty)$ [101].

The structure of non-periodic groups, which norm $N_G(A_\infty)$ of infinite Abelian subgroups is IH-group, are characterized by the following proposition.

Proposition 4.9 ([96]). *A non-periodic group G has non-Abelian norm $N_G(A_\infty)$ of infinite Abelian subgroups, if and only if all elements of infinite* *order of the group G generate Abelian normal subgroup D that contains every infinite Abelian subgroup of a group G and there is an element b of order 2 or 4, such that $b^{-1}db = d^{-1}$ for an arbitrary element $d \in D$. Moreover $N_G(A_\infty) = D\langle b\rangle$.*

A natural generalization of Baer norm for non-periodic groups is the norm $N_G(C_\infty)$ of infinite cyclic subgroups. The study of this norm and its influence on properties of the group was started by F. M. Lyman and T. D. Lukashova in [102]. It was proved that the norm $N_G(C_\infty)$ coincides with the center of the group in torsion free groups, and any finite over the norm $N_G(C_\infty)$ torsion free group is Abelian. The following proposition characterizes the properties of the group that has non-Abelian norm $N_G(C_\infty)$.

Proposition 4.10 ([102]). *A non-periodic group G has non-Abelian norm $N_G(C_\infty)$, if and only if all elements of infinite order of the group G generate an Abelian normal subgroup A and there is an element b of order 2 or 4, such that $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$. Moreover $N_G(C_\infty) = A\langle b\rangle$.*

Let's note, if the norm $N_G(A_\infty)$ is non-Abelian in a non-periodic group, then the norm $N_G(C_\infty)$ of infinite cyclic subgroups is non-Abelian. Moreover, in this case $N_G(C_\infty) = N_G(A_\infty)$. The following example shows that non-periodic groups, which norm $N_G(C_\infty)$ is non-Abelian and norm $N_G(A_\infty)$ is Abelian, exist.

Example 4.1. $G = (\langle a\rangle \leftthreetimes \langle b\rangle) \times C, |a| = \infty, |b| = 2, C$ is an infinite elementary Abelian 2-group, $b^{-1}ab = a^{-1}$.

It is easy to prove that $N_G(A_\infty) = C$ is Abelian, $N_G(C_\infty) = G$ and $N_G(A_\infty) \neq N_G(C_\infty)$.

The following proposition characterizes the conditions when the norm $N_G(A_\infty)$ coinsides with the norm $N_G(C_\infty)$ in a non-periodic group G (provided that the subgroup $N_G(C_\infty)$ is non-Abelian).

Proposition 4.11. *Let G be a non-periodic group, which norm $N_G(C_\infty)$ of infinite cyclic subgroups is non-Abelian. Subgroups $N_G(C_\infty)$ and $N_G(A_\infty)$ coincide, if and only if $N_G(C_\infty)$ is central-by-finite and contains every infinite Abelian subgroup of G.*

In connection with the existence of O. Yu. Olshansky groups, periodic groups with non-Dedekind norm of infinite Abelian subgroups were studied under the condition of their local finiteness. In [100] it was proved that such groups satisfy the minimal condition for subgroups, if and only if subgroup $N_G(A_\infty)$ satisfies this condition. Moreover, if $N_G(A_\infty)$ is a group with minimal condition for subgroups, then G is a finite extension of its divisible part and therefore $[G : N_G(A_\infty)] < \infty$.

Note that the norm $N(G)$ can be considered as the intersection of the normalizers of all cyclic subgroups. In this connection it is naturally to consider Σ-norm, where Σ consists of all cyclic subgroups of non-prime order of this group. Such a norm was studied by T. D. Lukashova and M. G. Drushlyak [105] in the class of non-periodic groups and was called *the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order of the group G.*

It is clear that all cyclic subgroups of compound or infinite order are normal in non-periodic group G, which coincides with the norm $N_G(C_{\bar{p}})$. Such non-Dedekind groups were studied by T. G. Lelechenko, F. M. Lyman [106] and were called *almost Dedekind groups*.

Since the norm $N_G(C_{\bar{p}})$ normalizes each infinite cyclic subgroup of a group G, $N_G(C_{\bar{p}}) \subseteq N_G(C_\infty)$ in non-periodic groups. It turns out that these norms coincide, if the norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is non-Abelian.

Proposition 4.12 ([105])**.** *Any non-periodic group G that has non-Abelian norm $N_G(C_{\bar{p}})$ of cyclic subgroups of non-prime order is almost Dedekind and coincides with this norm. Moreover $G = A\lambda\langle b\rangle$, where A is a normal Abelian subgroup which contains all elements of prime order of group G, $|b| = 2$, $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$.*

In 2004 in [107,108] F. M. Lyman and T. D. Lukashova introduced one more Σ-norm, where Σ is a system of all Abelian non-cyclic subgroups of a group. This Σ-norm was called *the norm of Abelian non-cyclic subgroups* of a group G and denoted by N_G^A. It is clear if the group G coincides with the norm N_G^A then all Abelian non-cyclic subgroups are normal in it (assuming the existence of at least one of such a subgroup). Non-Abelian groups with this property were fully described in [107, 109, 110] and called $\overline{HA}$-groups ($\overline{HA}_p$-groups in the case of p-groups).

In [92,107,114] infinite locally finite $p-$groups (p is an arbitrary prime), which norm N_G^A is non-Dedekind, are considered. The authors obtained a complete description of such groups and proved that if the norm N_G^A is infinite and non-Dedekind, then all Abelian non-cyclic subgroups are normal in a group, that is in this case $G = N_G^A$. It was also proved that locally finite p-groups with non-Dedekind norm N_G^A are finite extensions of a quazicyclic group. In particular, the following propositions take place.

Proposition 4.13 ([107]). *Infinite 2-groups with non-Dedekind norm N_G^A of Abelian non-cyclic subgroups are groups of one of the following types:*

1) *G is an infinite $\overline{HA}_2$-group, $N_G^A = G$;*
2) *$G = (A \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle$, where A is a quasicyclic 2-group, $|b| = |c| = |d| = 2, [A, \langle c\rangle] = 1, [b, c] = [b, d] = [c, d] = a_1 \in A, |a_1| = 2, d^{-1}ad = a^{-1}$ for any element $a \in A$; $N_G^A = N_G = (\langle a_2\rangle \times \langle b\rangle)\lambda\langle c\rangle, a_2 \in A, |a_2| = 4$;*
3) *$G = A\langle y\rangle H$, where A is a quasicyclic 2-group, $[A, H] = E, H = \langle h_1, h_2\rangle, |h_1| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $|y| = 4, y^2 = a_1 \in A, y^{-1}ay = a^{-1}$ for any element $a \in A, [\langle y\rangle, H] \subseteq \langle a_1\rangle \times \langle h^2\rangle$; $N_G^A = \langle a_2\rangle \times H, a_2 \in A, |a_2| = 4, N_G = \langle h_2\rangle\lambda\langle h_1 a_2\rangle$.*

Proposition 4.14 ([108]). *Infinite locally finite p-groups ($p \neq 2$), which norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind, are $\overline{HA}_p$-groups and $G = N_G^A = N_G$.*

Let's note that Proposition 4.14 fails in the case of infinite locally finite 2-groups: there are infinite 2-groups with finite non-Dedekind norm N_G^A of Abelian non-cyclic subgroups, which may not coincide with the norm N_G.

The study of finite p-groups for certain restrictions on the norm N_G^A of Abelian non-cyclic subgroups were continued by M. G. Drushlyak, T. D. Lukashova and F. M. Lyman in [112,113]. In particular, in [112] the structure of finite p-groups ($p \neq 2$) with a non-Abelian norm of Abelian non-cyclic subgroups was completely described, in [113] the structure of finite 2-groups with a non-cyclic centre and non-Dedekind norm N_G^A of Abelian non-cyclic subgroups was described. It is also proved that an arbitrary 2-group with a non-cyclic centre and a non-Dedekind norm N_G^A does not contain a quaternion subgroup, if and only if the norm N_G^A does not contain such a subgroup. In this case the norm N_G^A coinsides with the norm N_G [113].

The following proposition clarifies the result of [92] on the coincidence of norms N_G^A and N_G for infinite locally finite p-groups ($p \neq 2$) under the condition that the subgroup N_G is non-Abelian.

Proposition 4.15 ([112]). *If either norm N_G^A or N_G is non-Abelian, then $N_G = N_G^A$ in the class of locally finite p-groups ($p \neq 2$).*

The Proposition 4.15 leads to the conclusion that any finite p-group ($p \neq 2$) with non-Abelian norm N_G^A is a group of one of the types of 1)-3) of Proposition 4.2.

Proposition 4.16 ([113]). *Finite 2-groups with a non-cyclic centre and a non-Dedekind norm N_G^A of Abelian non-cyclic subgroups is a group of the following types:*

1) *G is a non-Dedekind non-metacyclic $\overline{HA_2}$–group with a non-cyclic center, $G = N_G^A$;*
2) *$G = H \cdot Q$ is a product of a quaternion group of order 8 and a generalized quaternion group; $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $Q = \langle y, x\rangle$, $|y| = 2^n$, $n \geqslant 3, |x| = 4, y^{2^{n-1}} = x^2, x^{-1}yx = y^{-1}, [Q, H] \subseteq \langle x^2, h_1^2\rangle; N_G^A = \langle y^{2^{n-2}}\rangle \times H;$*
3) *$G = \langle x\rangle\langle b\rangle, |x| = 2^k$, $|b| = 2^m, m > 2, k \geqslant m + r, 1 \leqslant r < m - 1, Z(G) = \langle x^{2^{r+1}}\rangle \times \langle b^{2^{r+1}}\rangle, [x, b] = x^{2^{k-r-1}s}b^{2^{m-1}t}, (s, 2) = 1, 0 \leqslant t < 2; N_G^A = N_G = \langle x^{2^{m-1}}\rangle\lambda\langle b\rangle.$*

Developing the study of finite 2-groups T. D. Lukashova, F. M. Lyman and M. G. Drushlyak obtained a structural description of groups with a cyclic center and a non-metacyclic non-Dedekind norm N_G^A.

Proposition 4.17. *Finite 2-groups with a non-metacyclic non-Dedekind norm N_G^A of Abelian non-cyclic subgroups and a cyclic centre are groups of the following types:*

1) *G is a non-metacyclic non-Hamiltonian $\overline{HA}_2$-group with a cyclic center, $G = N_G^A$;*
2) *$G = (\langle x\rangle\lambda\langle c\rangle)\lambda\langle b\rangle$, $|x| = 2^n$, $n > 3$, $|b| = |c| = 2$, $[x, b] = x^{\pm 2^{n-2}}c$, $[b, c] = [x, c] = x^{2^{n-1}}$, $N_G^A = N_G = (\langle x^2\rangle \times \langle c\rangle)\lambda\langle b\rangle$;*
3) *$G = (\langle x\rangle \times \langle b\rangle)\lambda\langle c\rangle\lambda\langle d\rangle$, $|x| = 2^n, n > 2, |b| = |c| = |d| = 2$, $[x, c] = [x, b] = 1$, $[b, c] = [c, d] = [b, d] = x^{2^{n-1}}$, $d^{-1}xd = x^{-1}$, $N_G^A = N_G = (\langle x^{2^{n-2}}\rangle \times \langle b\rangle)\lambda\langle c\rangle$;*
4) *$G = (\langle c\rangle\lambda H)\langle y\rangle$, $H = \langle h_1, h_2\rangle, |h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $|c| = 4, [c, h_2] = 1$, $[c, h_1] = c^2$, $y^2 = h_1, [y, h_2] = c^2h_1^2, [y, c] = h_2^{\pm 1}; N_G^A = \langle c\rangle\lambda H.$*

The question of the structure of finite 2-groups with a cyclic center, in which the norm N_G^A is a metacyclic non-Dedekind group, is still open.

Study of the influence of the properties of the norm of Abelian non-cyclic subgroups on the properties of the group was continued in [114], where infinite periodic groups, which norm N_G^A is non-Dedekind and locally nilpotent, were considered. It was proved that such groups satisfy the minimal condition for Abelian subgroups and are Chernikov groups.

Proposition 4.18 ([114]). *An infinite periodic locally nilpotent group G has a non-Dedekind norm of Abelian non-cyclic subgroups, if and only if*

$$G = G_p \times G_{p'},$$

where G_p is an infinite Sylow p-subgroup of a group G with a non-Dedekind norm $N^A_{G_p}$ of Abelian non-cyclic subgroups (where $p \in \pi(G)$) and $G_{p'}$ is a finite cyclic or finite Hamiltonian p'-subgroup, all Abelian subgroups of which are cyclic, and $N^A_G = N^A_{G_p} \times G_{p'}$.

If G is a locally finite, not locally nilpotent group which has an infinite locally nilpotent non-Dedekind norm N^A_G, then $G = G_p \leftthreetimes H$, where G_p is an infinite $\overline{HA}_p$-group, which coincides with a Sylow p-subgroup of a norm N^A_G, and H is a finite group, all Abelian subgroups of which are cyclic, $(|H|, p) = 1$. In addition, the structure of infinite locally finite non-nilpotent groups, which norm N^A_G is finite non-Dedekind nilpotent subgroup, was described.

Study of the norm N^A_G of Abelian non-cyclic subgroups in the class of non-periodic groups were continued by M. G Drushlyak and F. M. Lyman. In [115, 116] non-periodic groups with non-Dedekind norm of Abelian non-cyclic subgroups depending on the presence [115] or the absence [116] of a free Abelian subgroup of rank 2 were considered.

Proposition 4.19 ([115]). *If a non-periodic group G contains a free Abelian subgroup of rank 2, its norm N^A_G of Abelian non-cyclic subgroups is non-Dedekind, contains an Abelian non-cyclic subgroup and a finite Abelian, normal in G, subgroup F and the centralizer $C_G(F)$ contains all elements of infinite order of a group, then $N^A_G = N_G(C_\infty) = B\langle d\rangle$, where B is the Abelian subgroup generated by all elements of infinite order of the group G, $|d| = 2$ or $|d| = 4$, $d^2 \in B$, d^2 is a unique involution in G and $d^{-1}bd = b^{-1}$ for an arbitrary element $b \in B$.*

It was also proved that the non-periodic group G does not contain free Abelian subgroups of rank 2, if its norm N^A_G is non-Hamiltonian $\overline{HA}$-group and does not contain such subgroups.

In 2015 F. M. Lyman and T. D. Lukashova [117] considered one more generalization of the concept of the norm of the group – *the norm N^d_G of decomposable subgroups*, which is defined as the intersection of the normalizers of all decomposable subgroups of the group. In the case when the group does not contain any decomposable subgroups, we can assume that $N_G^d = G$. The structure of locally soluble groups, in which a system of decomposable subgroups is empty, as well as groups, in which each decomposable subgroup is normal (groups with the condition $N_G^d = G$), was described in [118].

It is clear that the group contains decomposable subgroups, if and only if it contains decomposable Abelian subgroups. Therefore, the study of the norm N_G^d of decomposable subgroups was conducted, regarding on the existence of systems of decomposable Abelian subgroups in the group. Thus the norm N_G^d of decomposable subgroups is closely related to the norm N_G^A of Abelian non-cyclic subgroups. In particular, in [117] it was proved that these norms coinside in the class of locally finite p-groups. The inclusion $N_G^A \supseteq N_G^d$ takes place and the case $N_G^A \neq N_G^d$ is achieved in classes of finite non-primary groups, as well as in classes of infinite periodic locally nilpotent non-primary groups.

Proposition 4.20 ([117]). *A periodic locally nilpotent group G which contains an Abelian non-cyclic subgroup has a non-Dedekind norm N_G^d of decomposable subgroups, if and only if G is a locally finite p-group with a non-Dedekind norm N_G^A of Abelian non-cyclic subgroups.*

Study of the influence of the norm N_G^d of decomposable subgroups on the properties of the group were extended by the authors in the class of non-periodic groups. In particular, in [119] the following was established.

Proposition 4.21 ([119]). *Let G be a non-periodic group that has a non-Dedekind norm N_G^d of decomposable subgroups. Then the following propositions take place:*

1) *G does not contain decomposable subgroups if and only if the norm N_G^d of the group does not contain them;*
2) *G contains a free Abelian subgroup of rank $r \geqslant 2$, if and only if the norm N_G^d contains a free Abelian subgroup of such a rank;*
3) *G contains a non-primary Abelian subgroup, if and only if the norm N_G^d of the group contains subgroups with this property;*
4) *any decomposable Abelian subgroup of a group G is mixed, if and only if any decomposable Abelian subgroup of its norm N_G^d is mixed.*

It was also proved that in the class of non-periodic locally soluble groups only one of the inclusions $N_G^A \supseteq N_G^d$ or $N_G^A \subseteq N_G^d$ takes place, provided that at least one of these norms is non-Dedekind and the norm

N_G^d is infinite. The following examples confirm that the condition of the infiniteness of the norm N_G^d is essential.

Example 4.2 ([119])**.** $G = (\langle a\rangle \leftthreetimes B) \leftthreetimes \langle c\rangle$, where $|a| = p, p$ is a prime $(p \neq 2)$, B is a group isomorphic to an additive group of q-adic numbers, $q \notin \{2, p\}, B = B_1\langle x\rangle, x^2 \in B_1,\ x^{-1}ax = a^{-1}, [B_1, \langle a\rangle] = E, |c| = 2, [c, a] = 1, c^{-1}bc = b^{-1}$ for any element $b \in B$.

In this group all periodic decomposable subgroups are of order $2p$ and are groups of the type $\langle a^m cb_1^k\rangle$, where $b_1 \in B_1,\ k \in \{0, 1\}, (m, p) = 1$. Accordingly, all non-periodic decomposable subgroups are mixed and contained in the group $B_1 \times \langle a\rangle$ and therefore are normal in G. Since $N_G(\langle a^m cb_1^k\rangle) = \langle a^m cb_1^k\rangle$, $N_G^d = \langle a\rangle$.

On the other hand, G does not contain periodic Abelian non-cyclic subgroups and all mixed Abelian subgroups contain $\langle a\rangle$ and are subgroups of the group $(B_1 \times \langle a\rangle)$ and therefore are normal in G. Moreover, all Abelian non-cyclic subgroups of rank 1 are contained either in the subgroup B, or in subgroups conjugated with it $g^{-1}Bg, g \in G$, or in the group $(B_1 \times \langle a\rangle)$.

Let's consider an infinite sequence of subgroups in B_1:

$$\langle b_1\rangle \subset \langle b_2\rangle \subset \cdots \subset \langle b_n\rangle \subset \cdots,$$

$|b_1| = \infty, b_{n+1}^{\alpha_{n+1}} = b_n,\ \alpha_{n+1} \in \mathbb{N}, (\alpha_{n+1}, p) = 1$ for $n = 1, 2, \ldots$ Since the isolator A of the subgroup $\langle ab_1\rangle$ is non-cyclic (because the root of an arbitrary degree coprime with p can be taken from the element a), $N_G(A) = \langle a, B_1\rangle$. $N_G^A = B_1$ and $N_G^d \cap N_G^A = E$ by $N_G(B) = B \leftthreetimes \langle c\rangle$.

Example 4.3 ([119])**.** $G = (\langle a\rangle \leftthreetimes B) \leftthreetimes \langle c\rangle$, where $|a| = p, p$ is a prime $(p \neq 2)$, B is a group isomorphic to an additive group of p-adic numbers, $B = B_1\langle x\rangle, x^2 \in B_1, x^{-1}ax = a^{-1}, [B_1, \langle a\rangle] = E, |c| = 2, [c, a] = 1, c^{-1}bc = b^{-1}$ for any element $b \in B$.

As in Example 4.2 in this group the norm of decomposable subgroups is $N_G^d = \langle a\rangle$. However, the norm of Abelian non-cyclic subgroups is $N_G^A = (B_1 \leftthreetimes \langle c\rangle)$. This follows from the fact that for any non-identity element $y_1 \in B_1$ the isolator of a subgroup $\langle ay_1\rangle$ is cyclic, and therefore the element c normalizes each Abelian non-cyclic subgroup of a group G. In this case, the norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind and $N_G^d \cap N_G^A = E$.

In 2005 F. Mari, F. de Giovanni [78] considered the concept of *the non-Abelian norm* $N^*(G)$ that is the intersection of normalizers of all non-Abelian subgroups of the group. If $N^*(G) = G$, then all non-Abelian subgroups are normal in the group. These groups were studied by G. M. Romalis and N. F. Sesekin [120–122] and were called *metahamiltonian*. Further metahamiltonian groups were studied by V. T. Nagrebezkiy [123], O. A. Makhnev [124], S. M. Chernikov [125], M. M. Semko and M. F. Kuzennyi [126].

In [78] the results that generalize Schur theorem [127] on finiteness of derived subgroups in central-by-finite groups were offered.

Proposition 4.22 ([78]). *If G is a locally graded group and the quotient group $G/N^*(G)$ is finite, then a derived subgroup G' is finite.*

Conclusion

The authors make a conclusion that the study of different Σ-norms and properties of groups with respect on properties of their Σ-norms is a very important field in the group theory. Nowadays the research of groups that differ from their Σ-norms as well as groups that have a non-Dedekind Σ-norm becomes possible, because the structure of groups that coincide with Σ-norms is well known in many cases. Therefore it will give the opportunity to extend the known classes of generalized Dedekind groups and will allow to study groups with restrictions on the normalizers of different systems of subgroups more effectively.

There are still a number of problems in the study of groups with generalized norms:

– the study of groups that coincide with their Σ-norms;

– the study of groups that have identity Σ-norms or their Σ-norms coinside with the center;

– the study of groups that have non-central Dedekind Σ-norms;

– the study of groups that have non-Dedekind Σ-norms;

– the study of infinite groups that have Σ-norms of finite index.

The solution of these problems will significantly expand the base of the modern group theory.

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Hamming distance between the strings generated by adjacency matrix of a graph and their sum

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Abstract. Let $A(G)$ be the adjacency matrix of a graph G. Denote by $s(v)$ the row of the adjacency matrix corresponding to the vertex v of G. It is a string in the set $\mathbb{Z}_2^n$ of all n-tuples over the field of order two. The Hamming distance between the strings $s(u)$ and $s(v)$ is the number of positions in which $s(u)$ and $s(v)$ differ. In this paper the Hamming distance between the strings generated by the adjacency matrix is obtained. Also $H_A(G)$, the sum of the Hamming distances between all pairs of strings generated by the adjacency matrix is obtained for some graphs.

1. Introduction

Let $\mathbb{Z}_2 = \{0, 1\}$ and $(\mathbb{Z}_2, +)$ be the additive group, where $+$ denotes addition modulo 2. For any positive integer n,

$$\begin{aligned} \mathbb{Z}_2^n &= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \ (n \ \text{ factors}) \\ &= \{(x_1, x_2, \ldots, x_n) | x_1, x_2, \ldots, x_n \in \mathbb{Z}_2\}. \end{aligned}$$

Thus every element of $\mathbb{Z}_2^n$ is an n-tuple $(x_1, x_2, \ldots, x_n)$ written as $x = x_1x_2 \ldots x_n$ where every x_i is either 0 or 1 and is called a *string* or *word*. The number of 1's in $x = x_1x_2 \ldots x_n$ is called the *weight* of x and is denoted by $wt(x)$.

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Let $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ be the elements of $\mathbb{Z}_2^n$. Then the sum $x + y$ is computed by adding the corresponding components of x and y under addition modulo 2. That is, $x_i + y_i = 0$ if $x_i = y_i$ and $x_i + y_i = 1$ if $x_i \neq y_i$, $i = 1, 2, \ldots, n$.

The *Hamming distance* $H_d(x, y)$ between the strings $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ is the number of i's such that $x_i \neq y_i$, $1 \leqslant i \leqslant n$. Thus $H_d(x, y) =$ Number of positions in which x and y differ $= wt(x + y)$.

Example 1. Let $x = 01001$ and $y = 11010$. Therefore $x + y = 10011$. Hence $H_d(x, y) = wt(x + y) = 3$.

Lemma 1 ([8])**.** *For all $x, y, z \in \mathbb{Z}_2^n$, the following conditions are satisfied:*
(i) $H_d(x, y) = H_d(y, x)$;
(ii) $H_d(x, y) \geqslant 0$;
(iii) $H_d(x, y) = 0$ *if and only if* $x = y$;
(iv) $H_d(x, z) \leqslant H_d(x, y) + H_d(y, z)$.

A graph G with vertex set $V(G)$ is called a *Hamming graph* [4, 7] if each vertex $v \in V(G)$ can be labeled by a string $s(v)$ of a fixed length such that $H_d(s(u), s(v)) = d_G(u, v)$ for all $u, v \in V(G)$, where $d_G(u, v)$ is the length of shortest path joining u and v in G.

Hamming graphs are known as an interesting graph family in connection with the error-correcting codes and association schemes. For more details see [1, 2, 4 - 7, 9 - 11, 13, 14].

Motivated by the work on Hamming graphs, in this paper we study the Hamming distance between the strings generated by the adjacency matrix of a graph. Also we obtain the sum of the Hamming distances between all pairs of strings for certain graphs.

2. Preliminaries

Let G be an undirected graph without loops and multiple edges with n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The vertices adjacent to the vertex v are called the *neighbours* of v. The *degree* of a vertex v, denoted by $deg_G(v)$ is the number of neighbours of v. A graph is said to be *r-regular* if the degree of each vertex is equal to r. The vertices which are adjacent to both u and v simultaneously are called the *common neghbours* of u and v and the vertices which are neither adjacent to u nor adjacent to v are called *non-common neighbours* of u and v. The *distance* between two vertices u and v in G is the length of shortest path joining u and v and is denoted by $d_G(u, v)$.

The *adjacency matrix* of G is a square matrix $A(G) = [a_{ij}]$ of order n, in which $a_{ij} = 1$ if the vertex v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. Denote by $s(v)$ the row of the adjacency matrix corresponding to the vertex v. It is a string in the set $\mathbb{Z}_2^n$ of all n-tuples over the field of order two.

Sum of Hamming distances between all pairs of strings generated by the adjacency matrix of a graph G is denoted by $H_A(G)$. That is,

$$H_A(G) = \sum_{1 \leqslant i < j \leqslant n} H_d(s(v_i), s(v_j)).$$

$H_A(G)$ is a graph invariant. For graph theoretic terminology we refer to the books [3, 12].

Example 2.

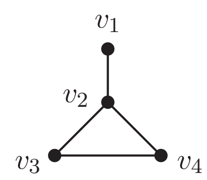

FIGURE 1. Graph G.

Adjacency matrix of a graph G of Fig. 1 is

$$A(G) = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and the strings are $s(v_1) = 0100$, $s(v_2) = 1011$, $s(v_3) = 0101$, $s(v_4) = 0110$.

$$H_d(s(v_1), s(v_2)) = 4, \quad H_d(s(v_1), s(v_3)) = 1, \quad H_d(s(v_1), s(v_4)) = 1,$$
$$H_d(s(v_2), s(v_3)) = 3, \quad H_d(s(v_2), s(v_4)) = 3, \quad H_d(s(v_3), s(v_4)) = 2.$$

Therefore $H_A(G) = 4 + 1 + 1 + 3 + 3 + 2 = 14$.

3. Hamming distance between strings

In this section we obtain the Hamming distance between a given pair of strings generated by the adjacency matrix of a graph.

Theorem 1. *Let G be a graph with n vertices. Let the vertices u and v of G have k common neighbours and l non-common neighbours.*

(i) *If u and v are adjacent vertices, then*

$$H_d(s(u), s(v)) = n - k - l.$$

(ii) *If u and v are nonadjacent vertices, then*

$$H_d(s(u), s(v)) = n - k - l - 2.$$

Proof. (i) Let u and v be the adjacent vertices of G. Let u and v have k common neighbours and l non-common neighbours. Therefore remaining $n - k - l - 2$ vertices other than u and v are adjacent to either u or v but not to both. Therefore the strings of u and v from $A(G)$ will be in the form

$$s(u) = x_1 x_2 x_3 \ldots x_{k+1} x_{k+2} x_{k+3} \ldots x_{k+l+2} x_{k+l+3} \ldots x_n$$

and

$$s(v) = y_1 y_2 y_3 \cdots y_{k+1} y_{k+2} y_{k+3} \cdots y_{k+l+2} y_{k+l+3} \cdots y_n$$

where $x_1 = 0$, $x_2 = 1$, $y_1 = 1$, $y_2 = 0$, $x_i = y_i = 1$ for $i = 3, 4, \ldots, k+2$, $x_i = y_i = 0$ for $i = k+3, k+4, \ldots, k+l+2$ and $x_i \neq y_i$ for $i = k+l+3, k+l+4, \ldots, n$.

Therefore $s(u)$ and $s(v)$ differ at $n - k - l - 2 + 2 = n - k - l$ places. Hence $H_d(s(u), s(v)) = n - k - l$.

(ii) Let u and v be the nonadjacent vertices of G. Let u and v have k common neighbours and l non-common neighbours. Therefore remaining $n - k - l - 2$ vertices other than u and v are adjacent to either u or v but not to both. Therefore the strings of u and v from $A(G)$ will be in the form

$$s(u) = x_1 x_2 x_3 \ldots x_{k+1} x_{k+2} x_{k+3} \ldots x_{k+l+2} x_{k+l+3} \ldots x_n$$

and

$$s(v) = y_1 y_2 y_3 \cdots y_{k+1} y_{k+2} y_{k+3} \cdots y_{k+l+2} y_{k+l+3} \cdots y_n$$

where $x_1 = 0$, $x_2 = 0$, $y_1 = 0$, $y_2 = 0$, $x_i = y_i = 1$ for $i = 3, 4, \ldots, k+2$, $x_i = y_i = 0$ for $i = k+3, k+4, \ldots, k+l+2$ and $x_i \neq y_i$ for $i = k+l+3, k+l+4, \ldots, n$.

Therefore $s(u)$ and $s(v)$ differ at $n - k - l - 2$ places. Hence $H_d(s(u), s(v)) = n - k - l - 2$. $\square$

Lemma 2. *Let G be a graph with n vertices. Let the vertices u and v of G have k common neighbours and l non-common neighbours.*

(i) *If u and v are adjacent vertices, then*

$$deg_G(u) + deg_G(v) = n + k - l.$$

(ii) *If u and v are nonadjacent vertices, then*

$$deg_G(u) + deg_G(v) = n + k - l - 2.$$

Proof. (i) Let u and v be the adjacent vertices of G. Let u and v have k common neighbours and l non-common neighbours. Therefore remaining $n - k - l - 2$ vertices other than u and v are adjacent to either u or v but not to both. Therefore

$$deg_G(u) + deg_G(v) = k + k + (n - k - l - 2) + 2 = n + k - l.$$

(ii) Let u and v be the nonadjacent vertices of G. Let u and v have k common neighbours and l non-common neighbours. Therefore remaining $n - k - l - 2$ vertices other than u and v are adjacent to either u or v but not to both. Therefore

$$deg_G(u) + deg_G(v) = k + k + (n - k - l - 2) = n + k - l - 2. \qquad \square$$

Theorem 2. *Let G be an r-regular graph with n vertices. Let u and v be the distinct vertices of G. If u and v have k common neighbours, then $H_d(s(u), s(v)) = 2r - 2k$.*

Proof. We consider here two cases.

Case 1. Let u and v be the adjacent vertices of G. Let u and v have k common neighbours and l non-common neighbours. Then by Theorem 1 (i),

$$H_d(s(u), s(v)) = n - k - l. \tag{1}$$

But from Lemma 2 (i), $2r = n + k - l$. Which implies $l = n + k - 2r$. Substituting this in (1),

$$H_d(s(u), s(v)) = n - k - (n + k - 2r) = 2r - 2k.$$

Case 2. Let u and v be the nonadjacent vertices of G. Let u and v have k common neighbours and l non-common neighbours. Then by Theorem 1 (ii),

$$H_d(s(u), s(v)) = n - k - l - 2. \tag{2}$$

But from Lemma 2 (ii), $2r = n+k-l-2$. Which implies $l+2 = n+k-2r$. Substituting this in (2),

$$H_d(s(u), s(v)) = n - k - (n + k - 2r) = 2r - 2k. \qquad \square$$

A connected acyclic graph is called *tree*.

Theorem 3. *Let G be a tree with n vertices. Let u and v be the distinct vertices of G.*

(i) *If $d_G(u, v) \neq 2$, then $H_d(s(u), s(v)) = deg_G(u) + deg_G(v)$.*

(ii) *If $d_G(u, v) = 2$, then $H_d(s(u), s(v)) = deg_G(u) + deg_G(v) - 2$.*

Proof. (i) Let $d_G(u, v) \neq 2$. We consider here two cases.

Case 1. Let $d_G(u, v) = 1$. Then u and v have zero common neighbours and $n - (deg_G(u) + deg_G(v))$ non-common neighbours. Therefore from Theorem 1 (i),

$$\begin{aligned} H_d(s(u), s(v)) &= n - 0 - [n - (deg_G(u) + deg_G(v))] \\ &= deg_G(u) + deg_G(v). \end{aligned}$$

Case 2. Let $d_G(u, v) > 2$. Then u and v have zero common neighbours and $n - (deg_G(u) + deg_G(v) + 2)$ non-common neighbours. Therefore from Theorem 1 (ii),

$$\begin{aligned} H_d(s(u), s(v)) &= n - 0 - [n - (deg_G(u) + deg_G(v) + 2)] - 2 \\ &= deg_G(u) + deg_G(v). \end{aligned}$$

(ii) Let $d_G(u, v) = 2$. Then u and v have 1 common neighbour and $n - (deg_G(u) + deg_G(v) + 1)$ non-common neighbours. Therefore from Theorem 1 (ii),

$$\begin{aligned} H_d(s(u), s(v)) &= n - 1 - [n - (deg_G(u) + deg_G(v) + 1)] - 2 \\ &= deg_G(u) + deg_G(v) - 2. \end{aligned} \qquad \square$$

4. Sum of Hamming distances of some graphs

As usual, by K_n, P_n, C_n and $K_{a,n-a}$ we denote respectively the complete graph, the path, the cycle and the complete bipartite graph on n vertices.

Theorem 4. *For a complete graph K_n, $H_A(K_n) = n(n - 1)$.*

Proof. Complete graph K_n on n vertices is a regular graph of degree $n-1$. In a complete graph every pair of adjacent vertices has $n-2$ common neighbours and zero non-common neighbours. Also there is no pair of nonadjacent vertices.

Therefore from Theorem 2, $H_d(s(u), s(v)) = 2(n-1) - 2(n-2) = 2$ for every pair of vertices u and v of K_n and there are $\binom{n}{2}$ such pairs in K_n.

Therefore

$$H_A(K_n) = \sum_{\{u,v\} \subseteq V(K_n)} H_d(s(u), s(v)) = \sum_{\binom{n}{2}} 2 = n(n-1). \qquad \square$$

Theorem 5. *For a complete bipartite graph $K_{p,q}$, $H_A(K_{p,q}) = pq(p+q)$.*

Proof. The graph $K_{p,q}$ has $n = p+q$ vertices and $m = pq$ edges. Every pair of adjacent vertices of $K_{p,q}$ has zero common neighbours and zero non-common neighbours. Therefore from Theorem 1 (i), $H_d(s(u), s(v)) = p+q$ for adjacent vertices u and v.

Let V_1 and V_2 be the partite sets of the vertices of a graph $K_{p,q}$, where $|V_1| = p$ and $|V_2| = q$. Let u and v be the nonadjacent vertices. If $u, v \in V_1$ then u and v has q common neighbours and $p-2$ non-common neighbours. Therefore from Theorem 1 (ii), $H_d(s(u), s(v)) = (p+q)-(q)-(p-2)-2 = 0$. Similarly, if $u, v \in V_2$, then $H_d(s(u), s(v)) = 0$. Therefore

$$\begin{aligned} H_A(K_{p,q}) &= \sum_{d_G(u,v)=1} H_d(s(u), s(v)) + \sum_{d_G(u,v)\neq 1} H_d(s(u), s(v)) \\ &= \sum_{m} (p+q) = m(p+q) = pq(p+q). \qquad \square \end{aligned}$$

Theorem 6. *For a cycle C_n on $n \geqslant 3$ vertices, $H_A(C_n) = 2n(n-2)$.*

Proof. For $n = 3$ and $n = 4$, the result follows from the Theorems 4 and 5 respectively. Now we prove for $n \geqslant 5$. Cycle C_n is a regular graph of degree $r = 2$. In C_n, $n \geqslant 5$, every pair of adjacent vertices has zero common neighbours. Therefore from Theorem 2, $H_d(s(u), s(v)) = 2r - 0 = 4$ for every pair of adjacent vertices u and v of C_n.

Let $d_{C_n}(u, v) = 2$. Then u and v have 1 common neighbour. Therefore from Theorem 2, $H_d(s(u), s(v)) = 2r - 2 = 2$.

Let $d_{C_n}(u, v) \geqslant 3$. Then u and v have zero common neighbours. Therefore from Theorem 2, $H_d(s(u), s(v)) = 2r - 0 = 4$.

Therefore

$$\begin{aligned}
H_A(C_n) &= \sum_{d_{C_n}(u,v)=1} H_d(s(u),s(v)) + \sum_{d_{C_n}(u,v)=2} H_d(s(u),s(v)) \\
&\quad + \sum_{d_{C_n}(u,v)>2} H_d(s(u),s(v)) \\
&= m(4) + n(2) + \left(\binom{n}{2} - m - n\right)(4) = 2n(n-2).
\end{aligned}$$ □

A graph G is said to be *strongly regular* with parameters (n, r, a, b) if it is non complete r-regular graph with n vertices in which every pair of adjacent vertices has a common neighbours and every pair of nonadjacent vertices has b common neighbours.

Theorem 7. *Let G be a strongly regular graph with parameters (n, r, a, b). Then $H_A(G) = n(n-1)(r-b) + nr(b-a)$.*

Proof. Let G be a strongly regular graph with parameters (n, r, a, b) and m edges.

From Theorem 2, $H_d(s(u), s(v)) = 2r - 2a$, if u and v are adjacent vertices and $H_d(s(u), s(v)) = 2r - 2b$, if u and v are nonadjacent vertices. Therefore

$$\begin{aligned}
H_A(G) &= \sum_{d_G(u,v)=1} H_d(s(u),s(v)) + \sum_{d_G(u,v)\neq 1} H_d(s(u),s(v)) \\
&= \sum_{m}(2r-2a) + \sum_{\binom{n}{2}-m}(2r-2b) \\
&= m(2r-2a) + \left(\binom{n}{2} - m\right)(2r-2b) \\
&= \frac{nr}{2}(2r-2a) + \left(\binom{n}{2} - \frac{nr}{2}\right)(2r-2b) \\
&= n(n-1)(r-b) + nr(b-a).
\end{aligned}$$ □

Theorem 8. *Let G be a graph with n vertices and m edges and $\overline{G}$ be the complement of G. Then*

$$H_A(\overline{G}) = H_A(G) + n(n-1) - 4m. \tag{3}$$

Proof. Let u and v be the vertices of G. If the adjacent vertices u and v have k_1 common neighbours and l_1 non-common neighbours in G, then from Theorem 1 (i)

$$H_d(s(u), s(v)) = n - k_1 - l_1. \tag{4}$$

Further, if the nonadjacent vertices u and v have k_2 common neighbours and l_2 non-common neighbours in G, then from Theorem 1 (ii)

$$H_d(s(u), s(v)) = n - k_2 - l_2 - 2. \tag{5}$$

Therefore

$$\begin{aligned} H_A(G) &= \sum_{d_G(u,v)=1} H_d(s(u), s(v)) + \sum_{d_G(u,v)\neq 1} H_d(s(u), s(v)) \\ &= \sum_{d_G(u,v)=1} (n - k_1 - l_1) + \sum_{d_G(u,v)\neq 1} (n - k_2 - l_2 - 2) \\ &= \sum_{d_G(u,v)=1} (n-k_1-l_1) + \sum_{d_G(u,v)\neq 1} (n-k_2-l_2) - 2\left(\binom{n}{2} - m\right). \qquad (6) \end{aligned}$$

If the vertices u and v are adjacent (nonadjacent) in G, then they are nonadjacent (adjacent) in $\overline{G}$. Therefore from (4) and (5), in $\overline{G}$, $H_d(s(u), s(v)) = n - k_1 - l_1 - 2$ for nonadjacent pairs of vertices and $H_d(s(u), s(v)) = n - k_2 - l_2$ for adjacent pairs of vertices. Therefore

$$\begin{aligned} H_A(\overline{G}) &= \sum_{d_{\overline{G}}(u,v)\neq 1} H_d(s(u), s(v)) + \sum_{d_{\overline{G}}(u,v)=1} H_d(s(u), s(v)) \\ &= \sum_{d_{\overline{G}}(u,v)\neq 1} (n - k_1 - l_1 - 2) + \sum_{d_{\overline{G}}(u,v)=1} (n - k_2 - l_2) \\ &= \sum_{d_G(u,v)=1} (n - k_1 - l_1 - 2) + \sum_{d_G(u,v)\neq 1} (n - k_2 - l_2) \\ &= \sum_{d_G(u,v)=1} (n - k_1 - l_1) - 2m + \sum_{d_G(u,v)\neq 1} (n - k_2 - l_2). \qquad (7) \end{aligned}$$

From (6) and (7) the result follows. □

A graph is said to be *selfcomplementary* if it is isomorphic to its complement.

Theorem 9. *Let G be a graph with n vertices and m edges. Let $\overline{G}$ be the complement of G. Then $H_A(G) = H_A(\overline{G})$ if and only if G is selfcomplementary graph.*

Proof. If G is a selfcomplementary graph, then $G \cong \overline{G}$. Therefore $H_A(G) = H_A(\overline{G})$.

Conversely, let $H_A(G) = H_A(\overline{G})$. Therefore from (3), $n(n-1)-4m = 0$. This gives that $m = (n(n-1))/4$ implying G is a selfcomplementary graph [12]. $\square$

Theorem 10. *Let G be a tree on n vertices. Then*

$$H_A(G) = \sum_{d_G(u,v)\neq 2} [deg(u) + deg(v)] + \sum_{d_G(u,v)=2} [deg(u) + deg(v) - 2].$$

Proof. Follows from the Theorem 3. $\square$

Theorem 11. *For a path P_n on $n \geqslant 2$ vertices, $H_A(P_n) = 2n^2 - 6n + 6$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n where v_i is adjacent to v_{i+1}, $i = 1, 2, \ldots, n-1$. There are $n-2$ pairs of vertices which are at distance two in P_n. Out of these $n-2$ pairs, two pairs (v_1, v_3) and (v_{n-2}, v_n) have Hamming distance equal to one and the remaining pairs have Hamming distance 2.

There are $\binom{n}{2} - (n-2)$ pairs of vertices in P_n, which are at distance 1 or at distance greater than 2. Out of these pairs, the one pair (v_1, v_n) has Hamming distance 2, $2n-6$ pairs of vertices, in which exactly one vertex is end vertex, have Hamming distance 3 and the remaining $\binom{n}{2} - 3n + 7$ pairs of vertices have Hamming distance 4. Therefore from Theorem 3,

$$\begin{aligned}
H_A(P_n) &= \sum_{d_{P_n}(u,v)=2} H_d(s(u), s(v)) + \sum_{d_{P_n}(u,v)\neq 2} H_d(s(u), s(v)) \\
&= \sum_{d_{P_n}(u,v)=2} (deg_{P_n}(u) + deg_{P_n}(v) - 2) \\
&\quad + \sum_{d_{P_n}(u,v)\neq 2} (deg_{P_n}(u) + deg_{P_n}(v)) \\
&= [2(3-2) + (n-4)(4-2)] \\
&\quad + \left\{[(1)(2) + (2n-6)(3)] + \left[\tbinom{n}{2} - (n-2) - (2n-6) - 1\right](4)\right\} \\
&= 2n^2 - 6n + 6.
\end{aligned}$$
$\square$

5. Conclusion

Theorems 1 and 2 gives the Hamming distance between the strings generated by adjacency matrix of a graph in terms of number of common neighbours and non-common neighbours. Results of Section 4 gives the sum of Hamming distances between all pairs of strings generated by the adjacency matrix for some standard graphs like complete graph, complete bipartite graph, tree, cycle, path, strongly regular graph. Further there is a scope to extend these reults to graph valued functions such as line graph, total graph, product graphs.

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Amply (weakly) Goldie-Rad-supplemented modules

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Abstract. Let R be a ring and M be a right R-module. We say a submodule S of M is a *(weak) Goldie-Rad-supplement* of a submodule N in M, if $M = N + S$, ($N \cap S \leqslant Rad(M)$) $N \cap S \leqslant Rad(S)$ and $N\beta^{**}S$, and M is called *amply (weakly) Goldie-Rad-supplemented* if every submodule of M has ample (weak) Goldie-Rad-supplements in M. In this paper we study various properties of such modules. We show that every distributive projective weakly Goldie-Rad-Supplemented module is amply weakly Goldie-Rad-Supplemented. We also show that if M is amply (weakly) Goldie-Rad-supplemented and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then M is Artinian.

Introduction

Throughout this article, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules unless indicated otherwise. Let M be an R-module. $N \leqslant M$ will mean N is a submodule of M. $End(M)$ and $Rad(M)$ will denote the ring of endomorphisms of M and the Jacobson radical of M, respectively. The notions which are not explained here will be found in [6].

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Recall that a submodule S of M is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. A module H is called *hollow* if every proper submodule of H is small in H. Let N and L be submodules of M. Then N is called a *supplement* of L in M if $N + L = M$ and N is minimal with respect to this property, or equivalently, N is a supplement of L in M if $M = N + L$ and $N \cap L \ll N$. N is said to be a *supplement submodule* of M if N is a supplement of some submodule of M. Recall from [3] that M is called a *supplemented module* if any submodule of M has a supplement in M. M is called an *amply supplemented module* if for any two submodule A and B of M with $A + B = M$, B contains a supplement of A. M is called a *weakly supplemented module* if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. Let $K, N \leqslant M$. K is a *(weak) Rad-supplement* of N in M, if $M = N + K$ and $(N \cap K \leqslant Rad(M))$ $N \cap K \leqslant Rad(K)$ (in this case K is a *(weak) generalized supplement* of N (see, [5])). K is said to be a *(weak) Rad-supplement submodule* of M if K is a (weak) Rad-supplement of some submodule of M (in this case K is a generalized (weakly) supplement submodule (see, [5])). A module M is called *(weakly) Rad-supplemented* if every submodule of M has a (weak) Rad-supplement (in this case M is a generalized (weakly) supplemented module (see, [5])).

In [2], the authors introduced a new class of modules namely Goldie*-Supplemented by defining and studying the β^* relation as the following: Let $X, Y \leqslant M$. X and Y are β^* equivalent, $X\beta^*Y$, provided $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$. After this work, Talebi et. al. [4] defined and studied the β^{**} relation and investigated some properties of this relation. In [4], this β^{**} relation was defined as the following:
Let $X, Y \leqslant M$. X and Y are β^{**} equivalent, $X\beta^{**}Y$, provided $\frac{X+Y}{X} \leqslant \frac{Rad(M)+X}{X}$ and $\frac{X+Y}{Y} \leqslant \frac{Rad(M)+Y}{Y}$.
Based on definition of β^{**} relation they introduced a new class of modules namely Goldie-Rad-supplemented. A module M is called *Goldie-Rad-supplemented* if for any submodule N of M, there exists a Rad-supplement submodule D of M such that $N\beta^{**}D$.

Let M be an R-module. We say a submodule S is a *(weak) Goldie-Rad-supplement* of a submodule N in M, if $M = N + S$, $(N \cap S \leqslant Rad(M))$ $N \cap S \leqslant Rad(S)$ and $N\beta^{**}S$. We say that M is *weakly Goldie-Rad-supplemented* if every submodule of M has a weak Goldie-Rad-supplement in M. We say that a submodule N of M has *ample (weak) Goldie-Rad-supplements* in M if, for every $L \leqslant M$ with $N + L = M$, there exists a (weak) Goldie-Rad-supplement S of N with $S \leqslant L$. We say that M is *amply (weakly) Goldie-Rad-supplemented* if every submodule of M has ample (weak) Goldie-Rad-supplements in M.

We prove that every distributive projective weakly Goldie-Rad-supplemented module is amply weakly Goldie-Rad-supplemented. We show that if M is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then M is Artinian. In addition, let M be a radical module ($Rad(M) = M$). Then M is Artinian if and only if M is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules. Moreover, we also show that the class of amply (weakly) Goldie-Rad-supplemented modules is closed under supplement submodules and homomorphic images.

Lemma 1. ([6, 41.1]) *Let M be a module and K be a supplement submodule of M. Then $K \cap Rad(M) = Rad(K)$.*

Theorem 1. ([1, Theorem 5]) *Let R be any ring and M be a module. Then $Rad(M)$ is Artinian if and only if M satisfies DCC on small submodules.*

1. Amply (weakly) Goldie-Rad-supplemented modules

In this section, we discuss the concept of amply (weakly) Goldie-Rad-supplemented modules and we give some properties of such modules.

Proposition 1. *Every amply (weakly) Goldie-Rad-supplemented module is a (weakly) Goldie-Rad-supplemented module.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module and N be a submodule of M. Then $N + M = M$. Since M is amply (weakly) Goldie-Rad-supplemented, M contains a (weak) Goldie-Rad-supplement S of N. So S is a (weak) Goldie-Rad-supplement of N in M. Hence M is (weakly) Goldie-Rad-supplemented. □

Example 1. An hollow radical module M ($Rad(M) = M$) is amply Goldie-Rad-supplemented.

Lemma 2. *Let M be an R-module and $L \leqslant N \leqslant M$. If S is a (weak) Goldie-Rad-supplement of N in M, then $(S + L)/L$ is a (weak) Goldie-Rad-supplement of N/L in M/L.*

Proof. By the proof of [5, Proposition 2.6 (1)], $(S+L)/L$ is a (weak) Rad-supplement of N/L in M/L. By [4, Proposition 2.3 (1)], $\frac{N}{L}\beta^{**}\left(\frac{S+L}{L}\right)$. Hence $(S+L)/L$ is a (weak) Goldie-Rad-supplement of N/L in M/L. □

Proposition 2. *Every factor module of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module and M/K be any factor module of M. Let $N/K \leqslant M/K$. For $L/K \leqslant M/K$, let $N/K + L/K = M/K$. Then $N + L = M$. Since M is an amply (weakly) Goldie-Rad-supplemented module, there exists a (weak) Goldie-Rad-supplement S of N with $S \leqslant L$. By Lemma 2, $(S+K)/K$ is a (weak) Goldie-Rad-supplement of N/K in M/K. Since $(S+K)/K \leqslant L/K$, N/K has ample (weak) Goldie-Rad-supplements in M/K. Thus M/K is amply (weakly) Goldie-Rad-supplemented. □

Corollary 1. *Every direct summand of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module. Since every direct summand of M is isomorphic to a factor module of M, then by Proposition 2, every direct summand of M is amply (weakly) Goldie-Rad-supplemented. □

Corollary 2. *Every homomorphic image of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module. Since every homomorphic image of M is isomorphic to a factor module of M, every homomorphic image of M is amply (weakly) Goldie-Rad-supplemented by Proposition 2. □

Let M be a module. Then M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N+(K\cap L) = (N+K)\cap(N+L)$ or $N\cap(K+L) = (N\cap K)+(N\cap L)$.

Proposition 3. *Every supplement submodule of a distributive amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module and S be any supplement submodule of M. Then there exists a submodule N of M such that S is a supplement of N. Let $L \leqslant S$ and $L + S' = S$

for $S' \leqslant S$. Then $N + L + S' = M$. Since M is amply (weakly) Goldie-Rad-supplemented, $N + L$ has a (weak) Goldie-Rad-supplement S'' in M with $S'' \leqslant S'$.

In this case $(N+L)+S'' = M$, $((N+L)\cap S^{''} \leqslant Rad(M))$ $(N+L)\cap S^{''} \leqslant Rad(S^{''})$ and $(N+L)\beta^{**}S^{''}$. Since $L+S'' \leqslant S$ and S is a supplement of N in M, $L+S'' = S$. On the other hand, $L\cap S'' \leqslant (N+L)\cap S'' \leqslant Rad(S^{''})$. Now, we show that $L\beta^{**}S^{''}$ in S. By Lemma 1, $S \cap Rad(M) = Rad(S)$. Therefore, since $(N + L)\beta^{**}S^{''}$,

$$\begin{aligned}\frac{L+S^{''}}{S^{''}} = \frac{S\cap(L+S^{''})}{S^{''}} \leqslant \frac{S\cap(N+L+S^{''})}{S^{''}} \leqslant \frac{S\cap(Rad(M)+S^{''})}{S^{''}} &= \frac{S^{''}+(S\cap Rad(M))}{S^{''}} \\ &= \frac{S^{''}+Rad(S)}{S^{''}},\end{aligned}$$

and since $N \cap S \ll S$, $N + L + S^{''} \leqslant Rad(M) + N + L$,

$$\begin{aligned}\frac{L+S^{''}}{L} = \frac{S\cap(L+S^{''})}{L} \leqslant \frac{S\cap(L+S^{''}+N))}{L} &\leqslant \frac{S\cap(Rad(M)+N+L)}{L} \\ &= \frac{L+(S\cap(Rad(M)+N))}{L} \leqslant \frac{L+Rad(S)}{L}.\end{aligned}$$

Hence $S^{''}$ is a (weak) Goldie-Rad-supplement of L in S. Since $S^{''} \leqslant S^{'}$, L has ample (weak) Goldie-Rad-supplements in S. Thus S is amply (weakly) Goldie-Rad-supplemented. □

A module M is said to be *π-projective* if, for every two submodules N, L of M with $L + N = M$, there exists $f \in End(M)$ with $Imf \leqslant L$ and $Im(1 - f) \leqslant N$ (see, [6]).

Theorem 2. *Let M be a distributive weakly Goldie-Rad-supplemented and π-projective module. Then M is an amply weakly Goldie-Rad-supplemented module.*

Proof. Let $N \leqslant M$ and $L + N = M$ for $L \leqslant M$. Since M is weakly Goldie-Rad-supplemented, there exists a weak Goldie-Rad-supplement S of N in M. Then $S + N = M$, $S \cap N \leqslant Rad(M)$ and $S\beta^{**}N$. Since M is π-projective, there exists $f \in End(M)$ such that $f(M) \leqslant L$ and $(1 - f)(M) \leqslant N$. Note that $f(N) \leqslant N$ and $(1 - f)(L) \leqslant L$. Then

$$M = f(M) + (1 - f)(M) \leqslant f(S + N) + N = f(S) + N.$$

Let $n \in N \cap f(S)$. Then there exists $s \in S$ with $n = f(s)$. In this case $s - n = s - f(s) = (1 - f)(s) \in N$ and then $s \in N$. Hence $s \in N \cap S$ and

$N \cap f(S) \leqslant f(N \cap S)$. Since $N \cap S \leqslant Rad(M)$, $f(N \cap S) \leqslant f(Rad(M))$. Then

$$N \cap f(S) \leqslant f(N \cap S) \leqslant f(Rad(M)) \leqslant Rad(f(M)) \leqslant Rad(M)$$

Next we show that $f(S)\beta^{**}N$. Since $S\beta^{**}N$, $S + N \leqslant Rad(M) + N$ and $S + N \leqslant Rad(M) + S$. Hence

$$f(S) + N = M = S + N \leqslant Rad(M) + N,$$

and since $S \cap N \leqslant Rad(M)$,

$$\begin{aligned} f(S) + N = f(S) + (N \cap M) &= f(S) + (N \cap (Rad(M) + S)) \\ &\leqslant f(S) + Rad(M). \end{aligned}$$

Hence $f(S)$ is a weak Goldie-Rad-supplement of N in M. Since $f(S) \leqslant L$, N has ample weak Goldie-Rad-supplements in M. Thus M is amply weakly Goldie-Rad-supplemented. □

Corollary 3. *Every projective distributive weakly Goldie-Rad-supplemented module is an amply weakly Goldie-Rad-supplemented module.*

Proof. Since every projective module is π-projective, every projective and distributive weakly Goldie-Rad-supplemented module is an amply weakly Goldie-Rad-supplemented module by Theorem 2. □

Corollary 4. *Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be a distributive module and $M_1, M_2, \cdots$, M_n be projective modules. Then $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ is amply weakly Goldie-Rad-supplemented if and only if for every $1 \leqslant i \leqslant n$, M_i is amply weakly Goldie-Rad-supplemented.*

Proof. "$\Longrightarrow$" is clear from Corollary 1.
"$\Longleftarrow$" Since M_i is amply weakly Goldie-Rad-supplemented, M_i is weakly Goldie-Rad-supplemented. Let $U \leqslant M$ and $U_i = M_i \cap U$. There exists $S_i \leqslant M_i$ such that $S_i\beta^{**}U_i$, $S_i + U_i = M_i$, $S_i \cap U_i \leqslant Rad(M_i)$ for $i = 1, \cdots n$. By [4, Proposition 2.5], $U\beta^{**}(\Sigma_{i=1}^n S_i)$. Moreover, $U + (\Sigma_{i=1}^n S_i) = M$ and

$$U \cap (\Sigma_{i=1}^n S_i) = \Sigma_{i=1}^n (S_i \cap U_i) \leqslant \Sigma_{i=1}^n Rad(M_i) \leqslant Rad(\Sigma_{i=1}^n M_i) = Rad(M).$$

This means that, $(\Sigma_{i=1}^n S_i)$ is a weak Goldie-Rad-supplement of U in M. Hence M is weakly Goldie-Rad-supplemented. Since, for every $1 \leqslant i \leqslant n$, M_i is projective, $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ is also projective. Then M is amply weakly Goldie-Rad-supplemented by Corollary 3.

□

Proposition 4. *Let M be an amply (weakly) Goldie-Rad-supplemented module. If M satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then M is Artinian.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module which satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules. Then $Rad(M)$ is Artinian by Theorem 1. It suffices to show that $M/Rad(M)$ is Artinian. Let N be any submodule of M containing $Rad(M)$. Then there exists a (weak) Goldie-Rad-supplement S of N in M, i.e, $M = N+S$, $N \cap S \leqslant Rad(S) \leqslant Rad(M)$ and $N\beta^{**}S$. Thus $M/Rad(M) = (N/Rad(M)) \oplus ((S + Rad(M))/Rad(M))$ and so every submodule of $M/Rad(M)$ is a direct summand. Therefore $M/Rad(M)$ is semisimple.

Now suppose that $Rad(M) \leqslant N_1 \leqslant N_2 \leqslant N_3 \leqslant \cdots$ is an ascending chain of submodules of M. Because M is amply (weakly) Goldie-Rad-supplemented, there exists a descending chain of submodules $S_1 \geqslant S_2 \geqslant S_3 \geqslant \cdots$ such that S_i is a (weak) Goldie-Rad-supplement of N_i in M for each $i \geqslant 1$. By hypothesis, there exists a positive integer t such that $S_t = S_{t+1} = S_{t+2} = \cdots$. Because $M/Rad(M) = N_i/Rad(M) \oplus (S_i + Rad(M))/Rad(M)$ for all $i \geqslant t$, it follows that $N_t = N_{t+1} = \cdots$. Thus $M/Rad(M)$ is Noetherian and since $M/Rad(M)$ is semisimple, by [6, 31.3] $M/Rad(M)$ is Artinian, as desired. □

Corollary 5. *Let M be a finitely generated amply (weakly) Goldie-Rad-supplemented module. If M satisfies DCC on small submodules, then M is Artinian.*

Proof. Since $M/Rad(M)$ is semisimple and M is finitely generated, then by [6, 31.3] $M/Rad(M)$ is Artinian. Now that M satisfies DCC on small submodules, $Rad(M)$ is Artinian by Theorem 1. Thus M is Artinian. □

Corollary 6. *Let M be a radical module (Rad(M)=M). Then M is Artinian if and only if M is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules.*

Proof. "$\Longleftarrow$" is clear by Proposition 4.
"$\Longrightarrow$" It suffices to prove that M is amply (weakly) Goldie-Rad-supplemented. It is well known that a module M is Artinian if and only if M is an amply supplemented module and satisfies DCC on supplement submodules and on small submodules. Since an amply supplemented

module is amply Rad-supplemented and for every submodules N, S of M, $N\beta^{**}S$, M is amply (weakly) Goldie-Rad-supplemented, as desired. $\square$

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RESEARCH ARTICLE

Transformations of $(0,1]$ preserving tails of Δ^μ-representation of numbers

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Abstract. Let $\mu \in (0,1)$ be a given parameter, $\nu \equiv 1-\mu$. We consider Δ^μ-representation of numbers $x = \Delta^\mu_{a_1a_2\ldots a_n\ldots}$ belonging to $(0,1]$ based on their expansion in alternating series or finite sum in the form:

$$x = \sum_n (B_n - B'_n) \equiv \Delta^\mu_{a_1a_2\ldots a_n\ldots},$$

where $B_n = \nu^{a_1+a_3+\ldots+a_{2n-1}-1}\mu^{a_2+a_4+\ldots+a_{2n-2}}$,
$B'_n = \nu^{a_1+a_3+\ldots+a_{2n-1}-1}\mu^{a_2+a_4+\ldots+a_{2n}}$, $a_i \in \mathbb{N}$.
This representation has an infinite alphabet $\{1,2,\ldots\}$, zero redundancy and N-self-similar geometry.

In the paper, classes of continuous strictly increasing functions preserving "tails" of Δ^μ-representation of numbers are constructed. Using these functions we construct also continuous transformations of $(0,1]$. We prove that the set of all such transformations is infinite and forms non-commutative group together with an composition operation.

Introduction

We consider representation of real numbers belonging to half-interval $(0,1]$. It depends on real parameter $\mu \in (0,1)$ and has an infinite alphabet $\mathbb{N} = \{1,2,3,\ldots\}$. This representation is based on the following theorem.

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Theorem 1 ([19]). *Let* $(0,1)\ni\mu$ *be a fixed real number,* $\nu\equiv1-\mu$. *For any* $x\in(0,1]$, *there exists a finite tuple of positive integers* $(a_1,a_2,\ldots,a_m)$ *or a sequence of positive integers* (a_n) *such that*

$$x=\nu^{a_1-1}-\nu^{a_1-1}\mu^{a_2}+\nu^{a_1+a_3-1}\mu^{a_2}-\nu^{a_1+a_3-1}\mu^{a_2+a_4}+\ldots=$$

$$=\sum_n(B_n-B'_n), \tag{1}$$

where $B_n=\nu^{a_1+a_3+\ldots+a_{2n-1}-1}\mu^{a_2+a_4+\ldots+a_{2n-2}}$, $B'_n=B_n\cdot\mu^{a_{2n}}$.

We call expansion of the number x in the form of alternating series (1) the Δ^μ-*expansion* and its symbolic notation $\Delta^\mu_{a_1a_2\ldots a_m(\varnothing)}$ for finite expansion of number x or $\Delta^\mu_{a_1a_2\ldots a_n\ldots}$ for infinite sum the Δ^μ-*representation.*

Remark that expansion of a number in the form of alternating series (1) first appeared in papers [23, 24] in an expression of strictly increasing singular function φ_μ being an unique continuous solution of a system of functional equations:

$$\begin{cases} \varphi_\mu\left(\dfrac{x}{1+x}\right)=(1-\mu)\varphi_\mu(x), \\ \varphi_\mu(1-x)=1-\varphi_{1-\mu}(x). \end{cases}$$

This function generalizes the well-known singular Minkowski function [1–8, 10–16, 25] and coincides with it for $\mu=1/2$. In this case the Δ^μ-representation is the $\Delta^\sharp$-representation studied in papers [20, 21].

There exists a countable everywhere dense in $[0,1]$ set of numbers having two Δ^μ-representation. These numbers have a form: $\Delta^\mu_{a_1\ldots[a_m+1](\varnothing)}=$ $=\Delta^\mu_{a_1\ldots a_m1(\varnothing)}$. We call these numbers Δ^μ-*finite*. Other numbers belonging to $(0,1]$ have a unique Δ^μ-representation, their expansions are infinite, so we call them Δ^μ-*infinite* numbers. That is, Δ^μ-representation has a zero redundancy. We denote the set of all Δ^μ-infinite numbers by H and the set of Δ^μ-finite numbers by S.

The Δ^μ-representation of number is called the *rational* Δ^μ-*representation* if $\mu\in(0,1)$ is rational. In this case irrational numbers belonging to $(0,1]$ have infinite non-periodic Δ^μ-representation and rational numbers have either finite or infinite periodic or infinite non-periodic Δ^μ-representation [19]. So the set H contains all irrational numbers and everywhere dense in $[0,1]$ subset of rational numbers.

Remark that Δ^μ-representation has much in common with encoding of real numbers by regular continued fraction [9, 17], namely, they have the same topology, rules for comparing numbers etc. However, Δ^μ-representation generates other metric relations, that is, it has own original metric theory [19].

In the paper, we construct an infinite non-commutative group of continuous strictly increasing piecewise linear transformations of $(0,1]$ preserving tails of Δ^μ-representation of numbers. Analogous objects for E-representation based on expansions of numbers in the form of positive Engel series are discussed in paper [18]. This representation has fundamental distinctions from E-representation in topological as well as metric aspects.

1. Geometry of Δ^μ-representation of numbers

Geometric meaning of digits of Δ^μ-representation of numbers and essence of related positional and metric problems are disclosed by the following important notion.

Definition 1. Let $(c_1, c_2, \ldots, c_m)$ be a tuple of positive integers. *Cylinder of rank m with base $c_1c_2\ldots c_m$* is a set $\Delta^\mu_{c_1c_2\ldots c_m}$ of numbers $x \in (0,1]$ having Δ^μ-representation such that $a_i(x) = c_i$, $i = \overline{1,m}$.

Cylinders have the following properties.

1. $\bigcup\limits_{a_1\in\mathbb{N}} \bigcup\limits_{a_2\in\mathbb{N}} \ldots \bigcup\limits_{a_m\in\mathbb{N}} \Delta^\mu_{a_1a_2\ldots a_m} = (0,1]$; 2. $\Delta^\mu_{c_1c_2\ldots c_m} = \bigcup\limits_{i=1}^{\infty} \Delta^\mu_{c_1c_2\ldots c_m i}$;
3. Cylinder $\Delta^\mu_{c_1c_2\ldots c_m}$ is a closed interval, moreover, if m is odd, then $\Delta^\mu_{c_1c_2\ldots c_{2k-1}} = [a-\delta, a]$, where
$$\delta = \nu^{c_1+c_3+\ldots+c_{2k-1}-1} \cdot \mu^{c_2+c_4+\ldots+c_{2k-2}+1};$$
$$a = \nu^{c_1-1} - \nu^{c_1-1}\mu^{c_2} + \ldots + \nu^{c_1+c_3+\ldots+c_{2k-1}-1}\mu^{c_2+c_4+\ldots+c_{2k-2}},$$
if m is even, then $\Delta^\mu_{c_1c_2\ldots c_{2k}} = [a, a+\delta]$, where
$$\delta = \nu^{c_1+c_3+\ldots+c_{2k-1}} \cdot \mu^{c_2+c_4+\ldots+c_{2k}}.$$
$$a = \nu^{c_1-1} - \nu^{c_1-1}\mu^{c_2} + \ldots +$$
$$+\nu^{c_1+c_3+\ldots+c_{2k-1}-1}\mu^{c_2+c_4+\ldots+c_{2k-2}} - \nu^{c_1+c_3+\ldots+c_{2k-1}-1}\mu^{c_2+c_4+\ldots+c_{2k}},$$
4. The length of cylinder of rank m is calculated by the formulae:
$$|\Delta^\mu_{c_1\ldots c_m}| = \begin{cases} \nu^{c_1+c_3+\ldots+c_{2k-1}-1} \cdot \mu^{c_2+c_4+\ldots+c_{2k-2}+1} & \text{if } m=2k-1, \\ \nu^{c_1+c_3+\ldots+c_{2k-1}} \cdot \mu^{c_2+c_4+\ldots+c_{2k}} & \text{if } m=2k. \end{cases}$$

5. If $\Delta^{\mu}_{c_1c_2\ldots c_m}$ is a fixed cylinder, then the following equality (basic metric relation) holds:

$$\frac{|\Delta^{\mu}_{c_1c_2\ldots c_m i}|}{|\Delta^{\mu}_{c_1c_2\ldots c_m}|} = \begin{cases} \nu\mu^{i-1} & \text{if} \quad m = 2k-1, \\ \mu\nu^{i-1} & \text{if} \quad m = 2k. \end{cases}$$

6. $\min\Delta^{\mu}_{c_1\ldots c_{2k-1}i} = \max\Delta^{\mu}_{c_1\ldots c_{2k-1}(i+1)}$; $\max\Delta^{\mu}_{c_1\ldots c_{2k}} = \min\Delta^{\mu}_{c_1\ldots c_{2k}(i+1)}$;
7. Cylinders of the same rank do not intersect or coincide. Moreover,

$$\Delta^{\mu}_{c_1c_2\ldots c_m} = \Delta^{\mu}_{c'_1c'_2\ldots c'_m} \iff c_i = c'_i \quad i = \overline{1, m};$$

8. For any sequence (c_m), $c_m \in \mathbb{N}$, intersection

$$\bigcap_{m=1}^{\infty} \Delta^{\mu}_{c_1c_2\ldots c_m} = x \equiv \Delta^{\mu}_{c_1c_2\ldots c_m\ldots}$$

is a point belonging to half-interval $(0, 1]$.

In paper [19], it is proved that geometry of Δ^{μ}-representation of numbers is N-self-similar and foundations of metric theory are laid. In paper [22], functions with fractal properties defined in terms of Δ^{μ}-representation are considered. Geometry plays an essential role in studies of such functions.

2. Tail sets and functions preserving tails of $\boldsymbol{\Delta^{\mu}}$-representation of numbers

Let $\mathcal{Z}^{\mu}_{H}$ be the set of all Δ^{μ}-representations of numbers belonging to set H. We introduce binary relation "has the same tail" (symbolically: $\sim$) on the set $\mathcal{Z}^{\mu}_{H}$.

Two Δ^{μ}-representations $\Delta^{\mu}_{a_1a_2\ldots a_n\ldots}$ and $\Delta^{\mu}_{b_1b_2\ldots b_n\ldots}$ are said *to have the same tail* (or they are *$\sim$-related*) if there exist positive integers k and m such that $a_{k+j} = b_{m+j}$ for any $j \in \mathbb{N}$.

It is evident that binary relation $\sim$ is an equivalence relation (i.e., it is reflexive, symmetric and transitive) and provides a partition of the set $\mathcal{Z}^{\mu}_{H}$ into equivalence classes. Any equivalence class is said to be a *tail set*. Any tail set is uniquely determined by its arbitrary element (representative).

We say that two numbers x and y belonging to set H have the same tail of Δ^{μ}-representation (or they are $\sim$-related) if their Δ^{μ}-representations are $\sim$-related. We denote this symbolically as $x \sim y$.

Theorem 2. *Any tail set is countable and dense in* $(0, 1]$*; quotient set* $F \equiv (0, 1]/\sim$ *is a continuum set.*

Proof. Suppose K is an arbitrary equivalence class, $x_0 = \Delta^{\mu}_{c_1 c_2 \ldots c_n \ldots}$ is its representative. Then it is evident that, for any $m \in \mathbb{Z}_0$, there exists set $K_m = \Big\{x : \ x = \Delta^{\mu}_{a_1 \ldots a_k c_{m+1} c_{m+2} \ldots}, \quad a_i \in \mathbb{N}, \ k = 0, 1, 2, \ldots\Big\}$ of numbers x such that for some $k \in \mathbb{Z}_0$

$$a_{k+j}(x) = c_{m+j} \quad \text{for any } j \in \mathbb{N} \quad \text{and} \quad K = \bigcup_{m \in \mathbb{Z}_0} K_m \, .$$

The set K is countable because it is a countable union of countable sets.

Now we prove that K is a dense in $(0, 1]$ set. Since number x belongs or does not belong to the set K irrespective of any finite amount of first digits of its Δ^{μ}-representation, we have that any cylinder of arbitrary rank m contains point belonging to K. Thus K is an everywhere dense in half-interval $(0, 1]$ set.

To prove that quotient set $F \equiv (0, 1]/\sim$ is continuum set, we assume the converse. Suppose that F is a countable set. Then half-interval $(0, 1]$ is a countable set as a countable union of countable sets (equivalence classes of quotient set F). This contradiction proves the theorem. $\square$

Remark that it is easy to introduce a distance function (metric) in the quotient set F.

Definition 2. Suppose function f is defined on the set H and takes values from this set. We say that function f *preserves tails* of Δ^{μ}-representations of numbers if for any $x \in (0, 1]$ there exist positive integers $k = k(x)$ and $m = m(x)$ such that

$$a_{k+n}(x) = a_{m+n}\left(f(x)\right) \quad \text{for all } n \in \mathbb{N}.$$

It is clear that functions preserving tails of Δ^{μ}-representations of numbers form an infinite set. However, only continuous functions are interested for us. Identity transformation $y = e(x)$ is a simplest example of such function.

By X we denote the set of all functions satisfying Definition 2. In the sequel, we consider some representatives of this class.

3. Function $\sigma_1(x)$

We consider function defined on the set H by equality

$$y = \sigma_1(x) = \sigma_1\left(\Delta^{\mu}_{a_1(x) a_2(x) a_3(x) a_4(x) \ldots a_n(x) \ldots}\right) = \Delta^{\mu}_{[a_1 + a_2 + a_3] a_4 \ldots a_n \ldots} \, .$$

This function is well-defined due to uniqueness of Δ^{μ}-representation of numbers belonging to the set H. It is evident that it preserves tails of Δ^{μ}-representation of numbers.

Lemma 1. *Analytic expression for function $y = \sigma_1(x)$ is given by formula*

$$\sigma_1(x) = \left(\frac{\nu}{\mu}\right)^{a_2(x)} \cdot x + \nu^{a_1(x)+a_2(x)-1}\left(1 - \frac{1}{\mu^{a_2(x)}}\right), \tag{2}$$

this function is linear on every cylinder of rank 2 and has the following properties:

1) it is continuous strictly increasing function;

2) $\sup\limits_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = \nu^{i+j}$, $\inf\limits_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = 0$*;*

3) $\int\limits_{\Delta^{\mu}_{ij}} \sigma_1(x)dx = \frac{1}{2}\nu^{2i+j}\mu^j$*;* *4)* $\int\limits_0^1 \sigma_1(x)dx = \frac{1}{2} \cdot \frac{\nu^3}{1+\nu^3}$.

Proof. 1. Indeed, if $x = \Delta^{\mu}_{a_1a_2a_3a_4a_5\ldots a_n\ldots}$, then

$$x = \nu^{a_1-1} - \nu^{a_1-1}\mu^{a_2} + \nu^{a_1+a_3-1}\mu^{a_2} - \nu^{a_1+a_3-1}\mu^{a_2+a_4} + \ldots =$$

$$= \nu^{a_1-1} - \nu^{a_1-1}\mu^{a_2} + \frac{\mu^{a_2}}{\nu^{a_2}} \cdot \sigma_1(x).$$

Whence it follows that

$$\sigma_1(x) = \left(\frac{\nu}{\mu}\right)^{a_2(x)} \cdot x + \nu^{a_1(x)+a_2(x)-1}\left(1 - \frac{1}{\mu^{a_2(x)}}\right).$$

It is evident that function $\sigma_1(x)$ is linear. Therefore it is continuous strictly increasing on the set $H \cap \Delta^{\mu}_{a_1a_2}$. Extending by continuity in Δ^{μ}-finite points we obtain continuous function on the whole cylinder $\Delta^{\mu}_{a_1a_2}$.

2. Boundary values of function $\sigma_1(x)$ on cylinder Δ^{μ}_{ij} can be calculated by formulae:

$$\sup_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = \lim_{k\to\infty} \sigma_1\left(\Delta^{\mu}_{ij1(k)}\right) = \Delta^{\mu}_{[i+j+1](\varnothing)} = \nu^{i+j}.$$

$$\inf_{x\in\Delta^{\mu}_{ij}} \sigma_1(x) = \lim_{k\to\infty} \sigma_1\left(\Delta^{\mu}_{ij(k)}\right) = \lim_{k\to\infty} \Delta^{\mu}_{[i+j+k](k)} = 0.$$

3. Calculate integral on cylinder Δ^{μ}_{ij}:

$$\int\limits_{\Delta^{\mu}_{ij}} \sigma_1(x)dx = \int\limits_{\Delta^{\mu}_{ij(\varnothing)}}^{\Delta^{\mu}_{i[j+1](\varnothing)}} \sigma_1(x)dx = \int\limits_{\nu^{i-1}(1-\mu^j)}^{\nu^{i-1}\left(1-\mu^{j+1}\right)} \sigma_1(x)dx = \frac{1}{2}\nu^{2i+j}\mu^j.$$

4. Calculate integral on the unit interval:

$$\int_0^1 \sigma_1(x)dx = \frac{1}{2}\sum_{i=1}^{\infty} \nu^{2i} \sum_{j=1}^{\infty} \nu^j \mu^j = \frac{1}{2} \cdot \frac{\nu^2}{1-\nu^2} \cdot \frac{\nu\mu}{1-\nu\mu} = \frac{1}{2} \cdot \frac{\nu^3}{1+\nu^3}. \quad \square$$

4. Function $d_s(x)$

Let s be a fixed positive integer. We consider function depending on parameter s, well-defined on half-interval $(0,1]$ by equality

$$y = d_s(x) = d_s\left(\Delta^{\mu}_{a_1(x)a_2(x)a_3(x)\ldots}\right) = \Delta^{\mu}_{[s+a_1]a_2a_3\ldots}.$$

Since s is an arbitrary positive integer, we have a countable class of functions $y = d_s(x)$.

Theorem 3. *Function d_s is analytically expressed by formula:*

$$d_s(x) = \nu^s \cdot x$$

and has the following properties:

1) it is linear strictly increasing, 2) $\inf\limits_{x\in(0,1]} d_s(x) = 0$, $\sup\limits_{x\in(0,1]} d_s(x) = \nu^s$.

Moreover, equation $\sigma_1(x) = d_s(x)$ does not have solutions if $a_2 \geqslant s$, and has a countable set of solutions:

$$E = \left\{x : x = \Delta^{\mu}_{a_1(a_2[s-a_2])}, \quad \text{where } a_1 \in \mathbb{N},\ a_2 \in \{1,2,\ldots,s-1\}\right\}$$

if $a_2 < s$.

Proof. By definition of function d_s, we have

$$d_s(x) = \Delta^{\mu}_{[s+a_1]a_2a_3\ldots} = \nu^{s+a_1-1} - \nu^{s+a_1-1}\mu^{a_2} + \ldots = \nu^s \cdot x,$$

Thus $d_s(x) = \nu^s \cdot x$. It is evident that function d_s is linear strictly increasing on half-interval $(0,1]$. Moreover,

$$\inf_{x\in(0,1]} d_s(x) = \lim_{x\to 0+0} d_s(x) = \lim_{k\to\infty} d_s\left(\Delta^{\mu}_{(k)}\right) = \lim_{k\to\infty} \Delta^{\mu}_{[s+k](k)} = 0;$$

$$\sup_{x\in(0,1]} d_s(x) = \lim_{x\to 1-0} d_s(x) = \lim_{k\to\infty} d_s\left(\Delta^{\mu}_{1(k)}\right) = \Delta^{\mu}_{[s+1](\varnothing)} = \nu^s.$$

We can write equation $\sigma_1(x) = d_s(x)$ in the form

$$\Delta^{\mu}_{[a_1(x)+a_2(x)+a_3(x)]a_4(x)\ldots} = \Delta^{\mu}_{[s+a_1(x)]a_2(x)a_3(x)a_4(x)\ldots}.$$

From uniqueness of Δ^{μ}-representation of numbers belonging to set H it follows that following equalities hold simultaneously:

$$\begin{aligned}
&a_1(x)+a_2(x)+a_3(x)=s+a_1(x), \quad a_4(x)=a_2(x),\\
&a_5(x)=a_3(x)=s-a_2(x), \quad \ldots \quad a_{2k}(x)=a_2(x),\\
&a_{2k+1}(x)=s-a_2(x), \quad k\in\mathbb{N}.
\end{aligned}$$

It is evident that this system is inconsistent if $a_2 \geqslant s$. However, for $a_2 < s$, equation has a countable set of solutions $x=\Delta^{\mu}_{a_1(a_2[s-a_2])}$, where a_1, a_2 are independent positive integer parameters. □

5. Left shift operator on digits of Δ^{μ}-representation of number

Let $\mathcal{Z}^{\mu}_{H}$ be the set of all Δ^{μ}-representations of numbers belonging to set H. We consider shift operator ω_2 on digits defined by equality

$$\omega_2\left(\Delta^{\mu}_{a_1a_2a_3a_4\ldots a_n\ldots}\right)=\Delta^{\mu}_{a_3a_4\ldots a_n\ldots}.$$

This operator generates function $y=\omega_2(x)=\Delta^{\mu}_{a_3(x)a_4(x)\ldots a_n(x)\ldots}$ on the set H. It is evident that operator ω_2 is surjective but not injective.

Any point $\Delta^{\mu}_{(ij)}=\dfrac{\nu^{i-1}(1-\mu^j)}{1-\nu^i\mu^j}$, where (i,j) is any pair of positive integers, is an invariant point of the mapping ω_2.

Lemma 2. *Function $y=\omega_2(x)$ is analytically expressed by formula*

$$\omega_2(x)=\frac{x}{\nu^{a_1(x)}\mu^{a_2(x)}}-\frac{1-\mu^{a_2(x)}}{\nu\mu^{a_2(x)}} \tag{3}$$

and is continuous monotonically increasing on any cylinder of rank 2.

Proof. Let $x\in\Delta^{\mu}_{ij}$. Then $x=\Delta^{\mu}_{ija_3a_4\ldots}$ and

$$\begin{aligned}
x=\nu^{i-1}-\nu^{i-1}\mu^j+\nu^{i+a_3-1}\mu^j-\nu^{i+a_3-1}\mu^{j+a_4}+\ldots=\\
=\nu^{i-1}-\nu^{i-1}\mu^j+\nu^i\mu^j\cdot\omega_2(x).
\end{aligned}$$

Whence, $\omega_2(x)=\dfrac{x}{\nu^i\mu^j}-\dfrac{1-\mu^j}{\nu\mu^j}$.

Since function ω_2 is linear, we have that this function is continuous strictly increasing on the set $H\cap\Delta^{\mu}_{a_1a_2}$. Extending by continuity in the points of the set S we obtain continuous function on the whole cylinder $\Delta^{\mu}_{a_1a_2}$. □

Lemma 3. *Equation $d_s(x)=\omega_2(x)$ has a countable set of solutions having the form $x=\Delta^{\mu}_{a_1(a_2[s+a_1])}$, where a_1, a_2 are arbitrary positive integers.*

Proof. We can write equation $d_s(x)=\omega_2(x)$ in the form

$$\Delta^{\mu}_{[s+a_1(x)]a_2(x)a_3(x)a_4(x)\ldots}=\Delta^{\mu}_{a_3(x)a_4(x)\ldots}.$$

From uniqueness of Δ^{μ}-representation of numbers belonging to set H it follows that the following equalities hold simultaneously:

$$\begin{array}{l} s+a_1(x)=a_3(x), \quad a_2(x)=a_4(x), \quad a_3(x)=a_5(x)=s+a_1(x), \\ a_4(x)=a_6(x)=a_2(x), \quad \ldots, \quad a_{2k+1}(x)=s+a_1(x), \\ a_{2k}(x)=a_2(x), \quad k\in\mathbb{N}. \end{array}$$

Then solutions of equation are numbers having the form $x=\Delta^{\mu}_{a_1(a_2[s+a_1])}$, where a_1, $a_2\in\mathbb{N}$. □

6. Right shift operator on digits of Δ^{μ}-representation of number

Let i,j be fixed positive integers. We consider operator depending on parameters i,j, well-defined on half-interval $(0,1]$ by equality

$$\delta_{ij}(x)=\delta_{ij}\left(\Delta^{\mu}_{a_1(x)a_2(x)\ldots}\right)=\Delta^{\mu}_{ija_1a_2\ldots}.$$

This operator defines a countable set of functions $y=\delta_{ij}(x)$, $i\in\mathbb{N}$, $j\in\mathbb{N}$.

Lemma 4. *Function $y=\delta_{ij}(x)$ is analytically expressed by formula*

$$y=\delta_{ij}(x)=\nu^i\mu^j\cdot x+\nu^{i-1}\left(1-\mu^j\right)$$

and is linear strictly increasing on half-interval $(0,1]$, moreover,

$$\inf_{x\in(0,1]}\delta_{ij}(x)=\Delta^{\mu}_{ij(\varnothing)}=\nu^{i-1}\left(1-\mu^j\right),$$

$$\sup_{x\in(0,1]}\delta_{ij}(x)=\Delta^{\mu}_{ij1(\varnothing)}=\nu^{i-1}\left(1-\mu^{j+1}\right).$$

Proof. In fact, by definition of function δ_{ij}, we have:

$$y=\delta_{ij}(\Delta^{\mu}_{a_1a_2\ldots})=\Delta^{\mu}_{ija_1a_2\ldots}=\nu^{i-1}-\nu^{i-1}\mu^j+\nu^{i+a_1-1}\mu^j-\nu^{i+a_1-1}\mu^{j+a_2}+\ldots=$$

$$= \nu^{i-1} - \nu^{i-1}\mu^j + \nu^i\mu^j \underbrace{\left(\nu^{a_1-1} - \nu^{a_1-1}\mu^{a_2} + \ldots\right)}_{x} = \nu^{i-1} - \nu^{i-1}\mu^j + \nu^i\mu^j \cdot x.$$

Therefore, $y = \delta_{ij}(x) = \nu^i\mu^j \cdot x + \nu^{i-1}\left(1 - \mu^j\right)$.

From linearity of function δ_{ij} it follows that it is a continuous strictly increasing function on $(0, 1]$ for any pair of positive integers (i, j). Moreover,

$$\inf_{x\in(0,1]} \delta_{ij}(x) = \lim_{x\to 0+0} \delta_{ij}(x) = \lim_{k\to\infty} \delta_{ij}\left(\Delta^{\mu}_{(k)}\right) = \lim_{k\to\infty} \Delta^{\mu}_{ij(k)} =$$

$$= \Delta^{\mu}_{ij(\varnothing)} = \nu^{i-1}\left(1 - \mu^j\right);$$

$$\sup_{x\in(0,1]} \delta_{ij}(x) = \lim_{x\to 1-0} \delta_{ij}(x) = \lim_{k\to\infty} \delta_{ij}\left(\Delta^{\mu}_{1(k)}\right) =$$

$$= \Delta^{\mu}_{ij1(\varnothing)} = \nu^{i-1}\left(1 - \mu^{j+1}\right). \qquad \square$$

For functions ω_2 and δ_{ij}, the following equalities are obvious:

$$\omega_2\left(\delta_{ij}\right) = x, \qquad \delta_{a_1(x)a_2(x)}\left(\omega_2(x)\right) = x.$$

Theorem 4. *For function δ_{ij}, the following propositions are true.*
1. *Equation $\sigma_1(x) = \delta_{ij}(x)$ does not have any solution if $a_1 + a_2 \geqslant i$ and has a countable set of solutions*

$$E = \left\{x : \ x = \Delta^{\mu}_{(a_1a_2[i-a_1-a_2]j)},\ a_1 \in \mathbb{N},\ a_2 \in \mathbb{N},\ a_1 + a_2 \in \{1, 2, \ldots, i-1\}\right\}$$

if $a_1 + a_2 < i$.

2. *Equation $d_s(x) = \delta_{ij}(x)$ does not have any solution if $s \geqslant i$ and has a countable set of solutions*

$$E = \left\{x : \ x = \Delta^{\mu}_{([i-s]j)},\ s \in \mathbb{N},\ s \in \{1, 2, \ldots, i-1\}\right\}$$

if $s < i$.

3. *Equation $\omega_2(x) = \delta_{ij}(x)$ has infinitely many solutions having a general form*

$$x = \Delta^{\mu}_{(a_1a_2ij)}, \quad \text{where } (a_1, a_2) \text{ is an arbitrary pair of positive integers.}$$

Proof. 1. We can write equation $\sigma_1(x) = \delta_{ij}(x)$ in the form

$$\Delta^{\mu}_{[a_1(x)+a_2(x)+a_3(x)]a_4(x)a_5(x)\ldots} = \Delta^{\mu}_{ija_1(x)a_2(x)a_3(x)a_4(x)\ldots}.$$

From uniqueness of Δ^{μ}-representation of numbers belonging to H it follows that following equalities holds simultaneously:

$$\begin{aligned}&a_1(x)+a_2(x)+a_3(x)=i,\ \ a_4(x)=j,\ \ a_5(x)=a_1(x),\ \ a_6(x)=a_2(x),\\ &a_7(x)=a_3=i-(a_1+a_2),\ a_8(x)=a_4=j,\ \ldots,\ a_{4k-1}(x)=i-(a_1+a_2),\\ &a_{4k}(x)=j,\ a_{4k+1}(x)=a_1,\ a_{4k+2}(x)=a_2,\quad k\in\mathbb{N}.\end{aligned}$$

Then this system does not have any solution if $a_1+a_2\geqslant i$ and have a countable set of solutions $E=\left\{x:\ x=\Delta^{\mu}_{(a_1a_2[i-a_1-a_2]j)}\right\}$, where a_1, a_2 are independent positive integer parameters, if $a_1+a_2<i$.

Similarly, we can prove statements 2 and 3 of the theorem. $\square$

7. Transformations preserving tails of Δ^{μ}-representation of numbers

Recall that *transformation* of non-empty set E is any bijective (i.e., both injective and surjective) mapping of this set onto itself.

It is clear that continuous transformations of $[0,1]$ are strictly monotonic (increasing or decreasing) functions such that $f(0)=0$ and $f(1)=1$ or $f(0)=1$ and $f(1)=0$.

If f is a transformation of $[0,1]$, then $\varphi(x)=1-f(x)$ is also transformation of this set. Therefore, to study continuous transformations of $[0,1]$, we can consider only strictly increasing functions, i.e., continuous probability distribution functions.

Simple examples of continuous strictly increasing transformations preserving tails of Δ^{μ}-representation of numbers are the following functions:

$$\varphi_\tau(x)=\begin{cases} d_i(x) & \text{if}\quad 0<x\leqslant x_1\equiv\Delta^{\mu}_{a_1(a_2[i+a_1])},\\ \omega_2(x) & \text{if}\quad x_1<x\leqslant x_2\equiv\Delta^{\mu}_{(a_1a_2)},\\ e(x) & \text{if}\quad x_2<x\leqslant 1,\end{cases}$$

where $\tau=(i,a_1,a_2)$ is an arbitrary triplet of positive integers;

$$\psi(x)=\begin{cases} d_1(x) & \text{if}\quad 0<x\leqslant x_1\equiv\Delta^{\mu}_{1(12)},\\ \omega_2(x) & \text{if}\quad x_1<x\leqslant x_2\equiv\Delta^{\mu}_{(1112)},\\ \delta_{12}(x) & \text{if}\quad x_2<x\leqslant x_3\equiv\Delta^{\mu}_{(12)},\\ e(x) & \text{if}\quad x_3<x\leqslant 1;\end{cases}$$

$$\gamma(x)=\begin{cases} d_3(x) & \text{if} \quad 0<x\leqslant x_1\equiv\Delta^{\mu}_{1(12)},\\ \sigma_1(x) & \text{if} \quad x_1<x\leqslant x_2\equiv\Delta^{\mu}_{(1111)},\\ \delta_{31}(x) & \text{if} \quad x_2<x\leqslant x_3\equiv\Delta^{\mu}_{(1231)},\\ \omega_2(x) & \text{if} \quad x_3<x\leqslant x_4\equiv\Delta^{\mu}_{(1212)},\\ e(x) & \text{if} \quad x_4<x\leqslant 1. \end{cases}$$

Theorem 5. *The set G of all continuous strictly increasing transformations of half-interval $(0,1]$ preserving tails of Δ^{μ}-representation of numbers together with an operation $\circ$ (function composition) form an infinite non-commutative group.*

Proof. The set of continuous transformations of $(0,1]$ is a subset of all transformations of $(0,1]$ forming a group. Thus we use a subgroup test. It is evident that set G is closed under the composition operation. For continuous strictly increasing function, inverse function is continuous and strictly increasing too. If transformation f preserves "tails" of Δ^{μ}-representations, then inverse transformation preserves them too. Therefore, for transformation $f\in G$, inverse transformation belongs to G too.

Since set of transformations φ_{τ}, $\tau\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, is countable, we see that set G is infinite.

To prove that group $(G,\circ)$ is non-commutative, we provide an example of two transformations f_1 and f_2 such that they are not commute, i.e., $f_2\circ f_1\neq f_1\circ f_2$. Consider two transformations $\varphi_{\tau_1}(x)$ and $\varphi_{\tau_2}(x)$, where $\tau_1=(1,2,3)$, $\tau_2=(1,1,2)$, i.e.,

$$\varphi_{\tau_1}(x)=\begin{cases} d_1(x) & \text{if} \quad 0<x\leqslant x_1\equiv\Delta^{\mu}_{2(33)},\\ \omega_2(x) & \text{if} \quad x_1<x\leqslant x_2\equiv\Delta^{\mu}_{(23)},\\ e(x) & \text{if} \quad x_2<x\leqslant 1; \end{cases}$$

$$\varphi_{\tau_2}(x)=\begin{cases} d_1(x) & \text{if} \quad 0<x\leqslant x_3\equiv\Delta^{\mu}_{1(22)},\\ \omega_2(x) & \text{if} \quad x_3<x\leqslant x_4\equiv\Delta^{\mu}_{(12)},\\ e(x) & \text{if} \quad x_4<x\leqslant 1. \end{cases}$$

Then, for $x_0=\Delta^{\mu}_{12(3)}$, tacking into account inequalities $x_0>x_2=\Delta^{\mu}_{(23)}$ but $\varphi_{\tau_1}(x_0)<x_3=\Delta^{\mu}_{1(22)}$ and $x_0<x_3=\Delta^{\mu}_{1(22)}$ but $\varphi_{\tau_2}(x_0)<x_1=\Delta^{\mu}_{2(33)}$, we obtain

$$\varphi_{\tau_2}\left(\varphi_{\tau_1}\left(\Delta^{\mu}_{12(3)}\right)\right)=\varphi_{\tau_2}\left(\Delta^{\mu}_{12(3)}\right)=\Delta^{\mu}_{22(3)};$$

$$\varphi_{\tau_1}\left(\varphi_{\tau_2}\left(\Delta^{\mu}_{12(3)}\right)\right)=\varphi_{\tau_1}\left(\Delta^{\mu}_{22(3)}\right)=\Delta^{\mu}_{32(3)}\neq\Delta^{\mu}_{22(3)}.$$

Therefore $\varphi_{\tau_2}\circ\varphi_{\tau_1}\neq\varphi_{\tau_1}\circ\varphi_{\tau_2}$ and $(G,\circ)$ is a non-commutative group. $\square$

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RESEARCH ARTICLE

On nilpotent Lie algebras of derivations of fraction fields

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Abstract. Let $\mathbb{K}$ be an arbitrary field of characteristic zero and A an integral $\mathbb{K}$-domain. Denote by R the fraction field of A and by $W(A) = RDer_{\mathbb{K}}A$, the Lie algebra of $\mathbb{K}$-derivations on R obtained from $Der_{\mathbb{K}}A$ via multiplication by elements of R. If $L \subseteq W(A)$ is a subalgebra of $W(A)$ denote by $rk_R L$ the dimension of the vector space RL over the field R and by $F = R^L$ the field of constants of L in R. Let L be a nilpotent subalgebra $L \subseteq W(A)$ with $rk_R L \leqslant 3$. It is proven that the Lie algebra FL (as a Lie algebra over the field F) is isomorphic to a finite dimensional subalgebra of the triangular Lie subalgebra $u_3(F)$ of the Lie algebra $DerF[x_1, x_2, x_3]$, where $u_3(F) = \{f(x_2, x_3)\frac{\partial}{\partial x_1} + g(x_3)\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3}\}$ with $f \in F[x_2, x_3], g \in F[x_3]$, $c \in F$.

Introduction

Let $\mathbb{K}$ be an arbitrary field of characteristic zero and A an associative commutative $\mathbb{K}$-algebra that is a domain. The set $\mathrm{Der}_{\mathbb{K}}A$ of all $\mathbb{K}$-derivations of A is a Lie algebra over $\mathbb{K}$ and an A-module in a natural way: given $a \in A, D \in \mathrm{Der}_{\mathbb{K}}A$, the derivation aD sends any element $x \in A$ to $a \cdot D(x)$. The structure of the Lie algebra $\mathrm{Der}_{\mathbb{K}}A$ is of great interest because in case $\mathbb{K} = \mathbb{R}$ and $A = \mathbb{R}[[x_1, \ldots, x_n]]$, the ring of formal power series, the Lie algebra of all $\mathbb{K}$-derivations of the form

$$D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n},\ f_i \in \mathbb{R}[[x_1, \ldots, x_n]]$$

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can be considered as the Lie algebra of vector fields on $\mathbb{R}^n$ with formal power series coefficients. Such Lie algebras with polynomial, formal power series, or analytical coefficients were studied by many authors. Main results for fields $\mathbb{K}=\mathbb{C}$ and $\mathbb{K}=\mathbb{R}$ in case $n=1$ and $n=2$ were obtained in [7] [4], [5] (see also [1], [3], [9], [10]).

One of the important problems in Lie theory is to describe finite dimensional subalgebras of the Lie algebra $\overline{W_3}(\mathbb{C})$ consisting of all derivations on the ring $\mathbb{C}[[x_1,x_2,x_3]]$ of the form

$$a_1\frac{\partial}{\partial x_1}+a_2\frac{\partial}{\partial x_2}+a_3\frac{\partial}{\partial x_3}, a_i\in\mathbb{C}[[x_1,\dots,x_n]].$$

In order to characterize nilpotent subalgebras of the Lie algebra $\overline{W_3}(\mathbb{C})$ we consider more general situation. Let $R=\mathrm{Frac}(A)$ be the field of fractions of an integral domain A and $W(A)=R\mathrm{Der}_{\mathbb{K}}(A)$ the Lie algebra of derivations of the field R obtained from derivations on A by multiplying by elements of the field R (obviously $\mathrm{Der}_{\mathbb{K}}A\subseteq W(A)$). For a subalgebra L of the Lie algebra $W(A)$ let us define $\mathrm{rk}_R(L)=\dim_R RL$ and denote by $F=R^L=\{r\in R \mid D(r)=0,\forall D\in L\}$ the field of constants of the Lie algebra L. The $\mathbb{K}$-space FL is a vector space over the field F and a Lie algebra over F. If L is a nilpotent subalgebra of $W(A)$, then FL is finite dimensional over F (by Lemma 5).

The main result of the paper: If L is a nilpotent subalgebra of rank $k\leqslant 3$ over R from the Lie algebra $W(A)$, then FL is isomorphic to a finite dimensional subalgebra of the triangular Lie algebra $u_k(F)$ (Theorem 2). Triangular Lie algebras were studied in [1] and [2], they are locally nilpotent but not nilpotent, the structure of their ideals was described in these papers.

We use standard notation, the ground field is arbitrary of characteristic zero. The quotient field of the integral domain A under consideration is denoted by R. Any derivation D of A can be uniquely extended to a derivation of R by the rule: $D(a/b)=(D(a)b-aD(b))/b^2$. If F is a subfield of the field R and $r_1,\dots,r_k\in R$, then the set of all linear combinations of these elements with coefficients in F is denoted by $F\langle r_1,\dots,r_k\rangle$, it is a subspace of the F-space R. The triangular subalgebra $u_n(\mathbb{K})$ of the Lie algebra $W_n(\mathbb{K})=\mathrm{Der}(\mathbb{K}[x_1,\dots,x_n])$ consists of all the derivations on the ring $\mathbb{K}[x_1,\dots,x_n]$ of the form $D=f_1(x_2,\dots x_n)\frac{\partial}{\partial x_1}+\dots+f_{n-1}(x_n)\frac{\partial}{\partial x_{n-1}}+f_n\frac{\partial}{\partial x_n}$, where $f_i\in\mathbb{K}[x_{i+1},\dots x_n]$, $f_n\in\mathbb{K}$.

1. Some properties of nilpotent subalgebras of $W(A)$

We will use some statements about derivations and nilpotent Lie algebras of derivations from the paper [8]. The next statement can be immediately checked.

Lemma 1. *Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then it holds:*

1. $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.
2. *If $a, b \in R^{D_1} \cap R^{D_2}$, then $[aD_1, bD_2] = ab[D_1, D_2]$.*

Let L be a subalgebra of rank k over R of the Lie algebra $W(A)$ and $F = R^L$ its field of constants. Denote by RL the set of all linear combinations over $\mathbb{K}$ of elements aD, where $a \in R$ and $D \in L$. The set FL is defined analogously.

Lemma 2 ([8], Lemma 2)**.** *Let L be a nonzero subalgebra of $W(A)$ and let FL, RL be $\mathbb{K}$-spaces defined as above. Then:*

1. *FL and RL are $\mathbb{K}$-subalgebras of the Lie algebra $W(A)$. Moreover, FL is a Lie algebra over the field F.*
2. *If the algebra L is abelian, nilpotent, or solvable then the Lie algebra FL has the same property, respectively.*

Lemma 3 ([8], Lemma 3)**.** *Let L be a subalgebra of finite rank over R of the Lie algebra $W(A)$, $Z = Z(L)$ the center of L, and $F = R^L$ the field of constants of L. Then $\mathrm{rk}_R Z = \dim_F FZ$ and FZ is a subalgebra of the center $Z(FL)$. In particular, if L is abelian, then FL is an abelian subalgebra of $W(A)$ and $\mathrm{rk}_R L = \dim_F FL$.*

Lemma 4 ([8], Lemma 4)**.** *Let L be a subalgebra of the Lie algebra $W(A)$ and I be an ideal of L. Then the vector space $RI \cap L$ (over $\mathbb{K}$) is also an ideal of L.*

Lemma 5 ([8], Proposition 1, Theorem 1)**.** *Let L be a nilpotent subalgebra of $W(A)$ and $F = R^L$ be its field of constants. Then:*

1. *If $\mathrm{rk}_R L < \infty$, then $\dim_F FL < \infty$.*
2. *If $\mathrm{rk}_R L = 1$, then L is abelian and $\dim_F FL = 1$.*
3. *If $\mathrm{rk}_R L = 2$, then there exist elements $D_1, D_2 \in FL$ and $a \in R$ such that*

$$FL = F\langle D_1, aD_1, \ldots, \tfrac{a^k}{k!}D_1, D_2\rangle, \ k \geqslant 0$$

$$(\text{if } k = 0, \ \text{then put } FL = F\langle D_1, D_2\rangle),$$

where $[D_1, D_2] = 0$, $D_1(a) = 0$, $D_2(a) = 1$.

Lemma 6. *Let $D_1, D_2, D_3 \in W(A)$ and $a \in R$ be such elements that $D_1(a) = D_2(a) = 0$, $D_3(a) = 1$ and let $F = \cap_{i=1}^3 R^{D_i}$. If there exists an element $b \in R$ such that $D_1(b) = D_2(b) = 0$, $D_3(b) \in F\langle 1, a, \ldots, a^s/s!\rangle$ for some $s \geqslant 0$, then $b \in F\langle 1, a, \ldots, a^{s+1}/(s+1)!\rangle$.*

Proof. Write down $D_3(b) = \sum_{i=0}^s \beta_i a^i/i!$ with $\beta_i \in F$ and take the element $c = \sum_{i=0}^s \beta_i a^{i+1}/(i+1)!$ of the field R. It holds obviously $D_3(b-c) = 0$ and (by the conditions of Lemma) $D_1(b-c) = 0$ and $D_2(b-c) = 0$. Then we have $b - c \in \cap_{i=1}^3 R^{D_i} = F$, and therefore $b = \gamma + \sum_{i=0}^s \beta_i a^{i+1}/(i+1)!$ for some element $\gamma \in F$. The latter means that $b \in F\langle 1, a, \ldots, a^{s+1}/(s+1)!\rangle$. □

Lemma 7. *Let $D_1, D_2, D_3 \in W(A)$ and $a, b \in R$ be such elements that $D_1(a) = D_1(b) = 0$, $D_2(a) = 1$ $D_2(b) = 0$, $D_3(a) = 0$, $D_3(b) = 1$ and let $F = \cap_{i=1}^3 R^{D_i}$. If there exists an element $c \in R$ such that*

$$D_1(c) = 0, \quad [D_2, D_3](c) = 0,$$

$$D_2(c) \in F\langle\{\tfrac{a^i b^j}{i!j!}\}, 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant k\rangle,$$

$$D_3(c) \in F\langle\{\tfrac{a^i b^j}{i!j!}\}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k-1\rangle,$$

then $c \in F\langle\{\frac{a^i b^j}{i!j!}\}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k\rangle$.

Proof. The elements $D_2(c)$ and $D_3(c)$ can be written (by conditions of the lemma) in the form $D_2(c) = f(a,b), D_3(c) = g(a,b)$ where $f, g \in F[u,v]$ are some polynomials of u, v. Since $[D_2, D_3](c) = 0$, it holds $D_2(g) = D_3(f)$. It follows from the relations $D_2(g) = \frac{\partial}{\partial a} g(a,b), D_3(f) = \frac{\partial}{\partial b} f(a,b)$ that $\frac{\partial}{\partial a} g(a,b) = \frac{\partial}{\partial b} f(a,b)$. Hence there exists a polynomial $h(a,b) \in F[a,b]$ (the potential of the vector field $f(a,b)\frac{\partial}{\partial a} + g(a,b)\frac{\partial}{\partial b}$) such that $D_3(h(a,b)) = g, D_2(h(a,b)) = f$. The polynomial $h(a,b)$ is obtained from the polynomials f, g by formal integration on a and on b, so we have $h(a,b) \in F\langle\{\frac{a^i b^j}{i!j!}\}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k\rangle$. Further, using properties of the element $h(a,b)$ we get $D_2(h-c) = D_3(h-c) = 0$. Besides, it holds $D_1(h-c)$ =0. The latter means that $h - c \in F = \cap_{i=1}^3 R^{D_i}$. But then $c = \gamma + h$ for some $\gamma \in F$ and therefore $c \in F\langle\{\frac{a^i b^j}{i!j!}\}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k\rangle$. □

2. On nilpotent subalgebras of small rank of $W(A)$

Lemma 8. *Let L be a nilpotent subalgebra of rank 3 over R from the Lie algebra $W(A)$ and $F = R^L$ be its field of constants. If the center $Z(L)$*

of the algebra L is of rank 2 *over R and* $\dim_F FL \geqslant 4$, *then there exist* $D_1, D_2, D_3 \in L, a \in R$ *such that the Lie algebra FL is contained in a nilpotent Lie algebra $\widetilde{L}$ of the Lie algebra $W(A)$ of the form*

$$\widetilde{L} = F\langle D_3, D_1, aD_1, \ldots, (a^n/n!)D_1, D_2, aD_2, \ldots, (a^n/n!)D_2\rangle$$

for some $n \geqslant 1$, *with* $[D_i, D_j] = 0$, $i, j = 1, 2, 3$, $D_1(a) = D_2(a) = 0$, $D_3(a) = 1$.

Proof. Take any elements $D_1, D_2 \in Z(L)$ that are linearly independent over R and denote $I = (RD_1 + RD_2) \cap L$. Then I is an ideal of the Lie algebra L (by Lemma 4). Take an arbitrary element $D \in I$ and write down $D = a_1D_1 + a_2D_2$ for some elements $a_i \in R$. Since $[D_i, D] = 0 = D_i(a_1)D_1 + D_i(a_2)D_2, i = 1, 2$ we get $D_i(a_j) = 0, i, j = 1, 2$. It follows easily that for any element $D' \in I$ it holds the equality $[D, D'] = 0$, so the ideal I is abelian. The Lie algebra FL is finite dimensional over F and $\dim_F FL/FI = 1$ by Lemma 5. Take any element $D_3 \in L \setminus I$. Then $FL = FI + FD_3$ and D_1, D_2, D_3 are linearly independent over R.

Since $\mathrm{rk}_R Z(L) = 2$, (by conditions of the lemma) we have $\dim_F FZ(L) = 2$ by Lemma 3. The ideal I of the Lie algebra L is abelian by the above proven, so the ideal FI of the Lie algebra FL over the field F is also abelian. Since $FL = FI + FD_3$, there exists a basis of the F-space FI in which the nilpotent linear operator $\mathrm{ad}D_3$ has a matrix consisting of two Jordan blocks. Let J_1 and J_2 be the correspondent Jordan bases; without loss of generality one can assume that $D_1 \in J_1, D_2 \in J_2$ and the elements D_1, D_2 are the first members of the bases J_1 and J_2 respectively.

If $\dim_F F\langle J_1\rangle = \dim_F F\langle J_2\rangle = 1$ then $FL = F\langle D_3, D_1, D_2\rangle$ is of dimension 3 over F which contradicts the conditions of the lemma. So, we may assume that $\dim_F F\langle J_1\rangle \geqslant \dim_F F\langle J_2\rangle$ and $\dim_F F\langle J_1\rangle = n+1, n \geqslant 1$. Denote the elements of the basis J_1 by $D_1, a_1D_1 + b_1D_2, \ldots, a_nD_1 + b_nD_2$, where the elements a_i, b_i belong to R and put for convenience $a = a_1$. Let us prove by induction on i that $a_i, b_i \in F\langle 1, a, \ldots, a^i/i!\rangle$. If $i = 1$, then $a_1 = a \in F\langle 1, a\rangle$ by definition. It follows from the relation $[D_3, a_1D_1 + b_1D_2] = D_1 = D_3(a_1)D_1 + D_3(b_1)D_2$ that $D_3(b_1) = 0$. Since FI is abelian (by the above proven), we have $D_1(b_1) = D_2(b_1) = 0$. The latter means that $b_1 \in F \subset F\langle 1, a\rangle$.

Further, the relation

$$[D_3, a_iD_1 + b_iD_2] = a_{i-1}D_1 + b_{i-1}D_2 = D_3(a_i)D_1 + D_3(b_i)D_2$$

gives the equalities $D_3(a_i) = a_{i-1}$ and $D_3(b_i) = b_{i-1}$. By the inductive assumption, $a_{i-1}, b_{i-1} \in F\langle 1, a, \ldots, a^{i-1}/(i-1)!\rangle$ and taking into account the relations $D_j(a_i) = D_j(b_i) = 0, j = 1, 2$ (they hold because FI is abelian) we get by Lemma 6 that $a_i, b_i \in F\langle 1, \ldots, a^i/i!\rangle$. The latter relation means that the F-subspace $F\langle J_1\rangle$ of FI lies in the subalgebra $\widetilde{L}$ from the conditions of the lemma.

Now let

$$J_2 = \{D_2, c_1D_1 + d_1D_2, \ldots, c_kD_1 + d_kD_2\}$$

be a basis corresponding to the second Jordan block. The relation $[D_3, c_1D_1 + d_1D_2] = D_2$ implies the equality $D_3(d_1) = 1$ and therefore $D_3(a - d_1) = 0$. Since $D_1(a - d_1) = D_2(a - d_1) = 0$, we get $a - d_1 \in F$, i.e. $d_1 = a + \gamma$ for some $\gamma \in F$. Applying the above considerations to the Jordan basis J_2 we obtain that $F\langle J_2\rangle \subset \widetilde{L}$. But then the Lie algebra L is entirely contained in $\widetilde{L}$. □

Lemma 9. *Let L be a nilpotent subalgebra of rank 3 over R from the Lie algebra $W(A)$ and $F = R^L$ be its field of constants. If the center $Z(L)$ of the algebra L is of rank 1 over R and $\dim_F FL \geqslant 4$, then the Lie algebra FL is contained either in the nilpotent Lie algebra $\widetilde{L}$ from the conditions of Lemma 8 or in a subalgebra $\overline{L}$ of $W(A)$ of the form*

$$\overline{L} = F\langle D_3, D_2, aD_2, \ldots, (a^n/n!)D_2, \{\tfrac{a^ib^j}{i!j!}D_1\}, 0 \leqslant i, j \leqslant m\rangle$$

where $n \geqslant 0, m \geqslant 1, D_i \in L, [D_i, D_j] = 0$, for $i, j = 1, 2, 3$, and $a, b \in R$ such that $D_1(a) = D_2(a) = 0, D_3(a) = 1, D_1(b) = D_3(b) = 0, D_2(b) = 1$.

Proof. Take any nonzero element $D_1 \in Z(L)$ and denote $I_1 = RD_1 \cap L$. Then I_1 is an abelian ideal of the algebra Lie L and $\mathrm{rk}_R I_1 = 1$ by Lemma 3. Choose any nonzero element $D_2 + I_1$ in the center of the quotient Lie algebra L/I_1 and denote $I_2 = (RD_1 + RD_2) \cap L$. By the same Lemma 3, I_2 is an ideal of the Lie algebra L and $\mathrm{rk}_R I_2 = 2$. Further, take any element $D_3 \in L \setminus I_2$. Since $\dim_F FL/FI_2 = 1$ by Lemma 5 from the paper [8], we have $FL = FI_2 + FD_3$.

Case 1. The ideal I_2 is abelian. Let us show that FL is contained in the Lie algebra $\widetilde{L}$ from the conditions of Lemma 8. It is obvious that FI_2 is an abelian ideal of codimension 1 of the Lie algebra FL over the field F. By Lemma 3, $\mathrm{rk}_R Z(L) = \dim_F FZ(L)$ and by the conditions of the lemma, we see that $\dim_F FZ(L) = 1$. The linear operator $\mathrm{ad}D_3$ acts on the F-space FI_2 and $\dim_F \mathrm{Ker}(\mathrm{ad}D_3) = \dim_F FZ(L)$. Therefore $\dim_F \mathrm{Ker}(\mathrm{ad}D_3) = 1$ and there exists a basis of FI_2 in which $\mathrm{ad}D_3$ has a matrix in the form of a single Jordan block. The same is true for the action of $\mathrm{ad}D_3$ on the vector space FI_1 (since $[D_3, I_1] \subseteq I_1$, the ideal FI_1 is invariant under $\mathrm{ad}D_3$). The subalgebra $FI_1 + FD_3$ is of rank 2 over R. If $\dim_F FI_1 > 1$, then the center of the Lie algebra $F_1 + FD_3$ is of dimension 1 over F. By Lemma 5, there exists a Jordan basis in FI_1 of the form

$$\{D_1, aD_1, \ldots, (a^s/s!)D_1\}, \text{ where } s \geqslant 0, D_3(a) = 1, [D_3, D_1] = 0.$$

If $\dim_F FI_1 = 1$, then $s = 0$ and the desired basis of F_1 is of the form $\{D_1\}$.

Let first $s > 0$. Since $(FD_2 + FI_1)/FI_1$ is a central ideal of the quotient algebra FL/FI_1, we have $[D_3, D_2] \in FI_1$ and hence one can write $[D_3, D_2] = \gamma_0 D_1 + \ldots + (\gamma_s a^s/s!)D_1$ for some $\gamma_i \in F$. Taking $D_2 - \sum_{i=0}^{s-1}(\gamma_i a^{i+1}/(i+1)!)D_1$ instead D_2 we may assume that $[D_3, D_2] = \gamma_s a^s/s!D_1$. Note that $\gamma_s \neq 0$. Really, in the opposite case $[D_3, D_2] = 0$ and therefore $D_2 \in Z(L)$. Then $\mathrm{rk}_R Z(L) = 2$ which is impossible because of the conditions of the lemma. After changing D_3 by $\gamma_s^{-1} D_3$ we may assume that $[D_3, D_2] = (a^s/s!)D_1$.

Since the linear operator $\mathrm{ad}D_3$ has in a basis of the F-space FI_2 a matrix, consisting of a single Jordan block, the same is true for the linear operator $\mathrm{ad}D_3$ on the vector space FI_2/FI_1. Let $\dim_F FI_2/FI_1 = k$ and $\{\overline{S}_1, \ldots, \overline{S}_k\}$ be a Jordan basis for $\mathrm{ad}D_3$ on FI_2/FI_1, where $\overline{S}_i = (c_i D_1 + d_i D_2) + FI_1,\ i = 1, \ldots, k, c_i, d_i \in R$. The representatives $c_i D_1 + d_i D_2$ of the cosets $\overline{S}_i$ can be chosen in such a way that $[D_3, c_i D_1 + d_i D_2] = c_{i-1} D_1 + d_{i-1} D_2, i = 2, \ldots, k$ and

$$[D_3, c_1 D_1 + d_1 D_2] = \sum_{i=0}^{s} \beta_i (a^i/i!) D_1 \tag{1}$$

for some $\beta_i \in F$. Let us show by induction on i that the relations hold:

$$d_i \in F\langle 1, \ldots, a^{i-1}/(i-1)!\rangle, \quad c_i \in F\langle 1, \ldots, a^{s+i}/(s+i)!\rangle. \tag{2}$$

Really, for $i = 1$ it follows from the relation (1) that

$$[D_3, c_1 D_1 + d_1 D_2] = \sum_{i=0}^{s} \beta_i a^i/i! D_1 =$$

$$= D_3(c_1)D_1 + (d_1 a^s/s!)D_1 + D_3(d_1)D_2. \tag{3}$$

It follows from (3) that $D_3(d_1) = 0$, and since the ideal FI_2 is abelian, it holds $D_1(d_1) = D_2(d_1) = 0$. The latter means that $d_1 \in F = F\langle 1\rangle$.

We also get from (3) that $D_3(c_1) \in F\langle 1, \ldots, a^s/s!\rangle$ and obviously it holds $D_1(c_1) = D_2(c_1) = 0$. Then, by Lemma 6, $c_1 \in F\langle 1, a, \ldots, a^{s+1}/(s+1)!\rangle$ and the relations (2) hold for $i = 1$. Assume they hold for $i-1$. Let us prove that the relations (2) hold for i. Using the equalities $[D_3, c_iD_1 + d_iD_2] = c_{i-1}D_1 + d_{i-1}D_2$ and $[D_3, D_2] = a^s/s!D_1$ we get $D_3(d_i) = d_{i-1}, D_3(c_i) = d_ia^s/s! + c_{i-1}$. By the inductive assumption, we have $d_{i-1} \in F\langle 1, a, \ldots, a^{i-2}/(i-2)!\rangle$, hence $d_i \in F\langle 1, a, \ldots, a^{i-1}/(i-1)!\rangle$ by Lemma 6. Analogously, by the inductive assumption it holds

$$c_{i-1} \in F\langle 1, a, \ldots, a^{s+i-1}/(s+i-1)!\rangle$$

and therefore $D_3(c_i) \in F\langle 1, a, \ldots, a^{s+i-1}/(s+i-1)!\rangle$. Since $D_1(c_i) = D_2(c_i) = 0$ we get by Lemma 6 that $c_i \in F\langle 1, a, \ldots, a^{s+i}/(s+i)!\rangle$. But then we have inclusion

$$FI_2 \subseteq F\langle D_1, aD_1, \ldots, (a^{s+k}/(s+k)!)D_1, D_2, aD_2, \ldots, (a^k/k!)D_2\rangle.$$

The last subalgebra of the Lie algebra $W(R)$ is contained in the subalgebra of the form

$$F\langle D_1, aD_1, \ldots, (a^{s+k}/(s+k)!)D_1, D_2, aD_2, \ldots, (a^{s+k}/(s+k)!)D_2\rangle.$$

But then the Lie algebra L is contained in the subalgebra $\widetilde{L}$ from the conditions Lemma 8.

Let now $s = 0$. Then $FI_1 = FD_1$ and without loss of generality we may assume that $[D_3, D_2] = D_1$. Repeating the above considerations we can build a Jordan basis $\{(c_iD_1 + d_iD_2) + FI_1, i = 1, \ldots, k\}$ of the quotient algebra FI_2/FI_1 with $[D_3, c_iD_1 + d_iD_2] = c_{i-1}D_1 + d_{i-1}D_2, i = 2, \ldots, k$ and $[D_3, c_1D_1 + d_1D_2] = \alpha D_1$ for some $\alpha \in F$. It follows from the last equality that $D_3(d_1) = 0$ and taking into account the equalities $D_1(d_1) = 0$ and $D_2(d_1) = 0$ we see that $d_1 \in F$. Since $a_1D_1 + d_1D_2 \notin FI_1$, we have $d_1 \neq 0$. By conditions of the lemma, $\dim_F FL > 3$, so we have $k \geqslant 2$ and the relation $[D_3, c_2D_1 + d_2D_2] = c_1D_1 + d_1D_2$ implies the equality $D_3(d_2) = d_1$. But then $D_3(d_2d_1^{-1}) = 1$ and multiplying all the elements of the Jordan basis considered above by d_1^{-1} we may assume that $D_3(d_2) = 1$. Denoting $a = d_2$ and repeating the considerations from the subcase $s > 0$ we see that the Lie algebra L is contained in the subalgebra $\widetilde{L}$ from the conditions of Lemma 8.

Case 2. The ideal I_2 is nonabelian. We may assume without loss of generality that I_1 coincides with its centralizer in L, i.e. $C_L(I_1) = I_1$. Really, let $C_L(I_1) \supset I_1$ with strong containment. Choose a one-dimensional (central) ideal $(D_4 + I_1)/I_1$ in the ideal $C_L(I_1)/I_1$ of the quotient algebra L/I_1. Then $I_4 := (RD_1 + RD_4) \cap L$ is an abelian ideal of rank 2 of the algebra L and $\dim_F FL/FI_4 = 1$ by Lemma 5 from [8]. Thus the problem is reduced to the case 1 (one should take FI_4 instead of FI_2). So, we assume that $C_L(I_1) = I_1$. It follows from this equality that $C_{FL}(FI_1) = FI_1$.

As in the case 1 we write $FL = FI_2 + FD_3$ and $[D_3, D_2] = rD_1$ for some $r \in R$. Since the ideal FI_1 is abelian, the linear operator $\mathrm{ad}[D_3, D_2] = \mathrm{ad}(rD_1)$ acts trivially on the vector space FI_1, and therefore the linear operators $\mathrm{ad}D_2$ and $\mathrm{ad}D_3$ commute on FI_1. Denote by M_2 the kernel $\mathrm{Ker}(\mathrm{ad}D_2)$ on the F-space FI_1. It is obvious that M_2 is an abelian subalgebra of FI_1 and M_2 is invariant under the action of $\mathrm{ad}D_3$. Since $[D_1, M_2] = [D_2, M_2] = 0$ the linear operator $\mathrm{ad}D_3$ has on the F-space FI_1 the kernel of dimension 1 (in other case the center of the Lie algebra FL would have dimension $\geqslant 2$ over F which contradicts our assumption). Using Lemma 5 one can easily show that

$$M_2 = F\langle D_1, aD_1, \ldots, (a^k/k!)D_1\rangle$$

for some $a \in R$ with $D_1(a) = 0, D_2(a) = 0, D_3(a) = 1$ (if $k = 0$, then put $M_2 = FD_1$). Further denote $M_3 = \mathrm{Ker}(\mathrm{ad}D_3)$ on the vector space FI_1. As above one can prove that M_3 is invariant under action of $\mathrm{ad}D_2$, this linear operator has one-dimensional kernel on M_3, and

$$M_3 = F\langle D_1, bD_1, \ldots, (b^m/m!)D_1\rangle$$

for some $b \in R$ with $D_1(b) = D_3(b) = 0$ and $D_2(b) = 1$ (if $m = 0$ put $M_3 = FD_1$).

Take now any element cD_1 of the ideal $FI_1, c \in R$. Since the linear operators $\mathrm{ad}D_2$ and $\mathrm{ad}D_3$ act nilpotently on FI_1, there exist the least positive integers k_0 and m_0 (depending on the element cD_1) such that $(\mathrm{ad}D_2)^{k_0}(cD_1) = 0, (\mathrm{ad}D_3)^{m_0}(cD_1) = 0$. Let us show by induction on $s = m_0 + k_0$ that the element cD_1 is a linear combination (with coefficients from F) of elements of the form $\frac{a^i b^j}{i!j!} D_1 \in W(A)$ for some $0 \leqslant i \leqslant k_0 - 1, 0 \leqslant j \leqslant m_0 - 1$ (note that the elements $\frac{a^i b^j}{i!j!} D_1$ can be outside of FI_1). If $s = 2$ (obviously $s \geqslant 2$), then we must only consider the case $m_0 = 1, k_0 = 1$. In this case, we have $[D_3, cD_1] = 0, [D_2, cD_1] = 0$. These equalities imply that $cD_1 \in Z(FL) = FD_1$ and all is done. Let $s \geqslant 3$. The element $[D_2, cD_1]$ can be written by the inductive assumption in the form

$$[D_2, cD_1] = \sum_{i=0}^{k_0-2} \sum_{j=0}^{m_0-1} \gamma_{ij} \frac{a^i b^j}{i!j!} D_1 \text{ for some } \gamma_{ij} \in F.$$

Analogously we get

$$[D_3, cD_1] = \sum_{i=0}^{k_0-1} \sum_{j=0}^{m_0-2} \delta_{ij} \frac{a^i b^j}{i!j!} D_1 \text{ for some } \ \delta_{ij} \in F.$$

It follows from the previous two equalities that

$$D_2(c) = \sum_{i=0}^{k_0-2} \sum_{j=0}^{m_0-1} \gamma_{ij} \frac{a^i b^j}{i!j!}, \quad D_3(c) = \sum_{i=0}^{k_0-1} \sum_{j=0}^{m_0-2} \delta_{ij} \frac{a^i b^j}{i!j!}.$$

Note that $[D_3, D_2](c) = rD_1(c) = 0$ and therefore by Lemma 7 $c \in F\langle \frac{a^i b^j}{i!j!},\ 0 \leqslant i \leqslant k_0 - 1, 0 \leqslant j \leqslant m_0 - 1\rangle$. Since cD_1 is arbitrarily chosen we have $FI_1 \subseteq F\langle \frac{a^i b^j}{i!j!} D_1, 0 \leqslant i \leqslant k_0 - 1, 0 \leqslant j \leqslant m_0 - 1\rangle$. One can straightforwardly check that $k_0 \leqslant k$, where $k = \dim M_2 - 1$ and analogously $m_0 \leqslant m = \dim M_3 - 1$. Let, for example, $m \geqslant n$. Then $FI_1 \subseteq F\langle \frac{a^i b^j}{i!j!} D_1, 0 \leqslant i, j \leqslant m\rangle$.

Further, by the above proven, the linear operator $\mathrm{ad} D_3$ on the vector space FI_2/FI_1 has a matrix in a basis in the form of a single Jordan block. This basis can be chosen in the form $(u_1 D_1 + v_1 D_2) + FI_1, \ldots, (u_t D_1 + v_t D_2) + FI_1$ such that

$$[D_3, u_i D_1 + v_i D_2] = u_{i-1} D_1 + v_{i-1} D_2, i \geqslant 2,$$

$$[D_3, u_1 D_1 + v_1 D_2] = f D_1 \tag{4}$$

for some element f, $\ f \in F\langle \frac{a^i b^j}{i!j!}, 0 \leqslant i, j \leqslant m\rangle$. Let us show by induction on s that

$$u_s \in F\langle \tfrac{a^i b^j}{i!j!},\ 0 \leqslant i, j \leqslant m + s\rangle, v_s \in F\langle 1, \ldots, a^{s-1}/(s-1)!\rangle.$$

If $s = 1$, then the equalities

$$[D_3, u_1 D_1 + v_1 D_2] = f D_1 = D_3(u_1) D_1 + D_3(v_1) D_2 + v_1 r D_1 \tag{5}$$

imply $D_3(v_1) = 0$ (let us recall here that $[D_3, D_2] = rD_1$). Taking into account the relations $[D_1, u_1 D_1 + v_1 D_2] = 0$ and $[D_2, u_1 D_1 + v_1 D_2] \in FI_1$ we obtain that $v_1 \in \cap_{i=1}^3 R^{D_i} = F$, that is $v_1 \in F\langle 1\rangle$. It follows from the relations (4) that $D_3(u_1) + v_1 r \in F\langle \{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i, j \leqslant m.\rangle$. Analogously the inclusion $[D_2, u_1 D_1 + v_1 D_2] \in FI_1$ implies the relation

$$D_2(u_1) \in F\langle \{\tfrac{a^i b^j}{i!j!}\},\ 0 \leqslant i, j \leqslant m.\rangle.$$

Since $[D_3, D_2] = rD_1$ and $rD_1(u_1) = 0$, we see (using Lemma 7) that $u_1 \in F\langle\{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m+1.\rangle$. By inductive assumption, we have

$$u_{s-1} \in F\langle\{\tfrac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m+s-1\rangle, \quad v_{s-1} \in F\langle 1, \ldots, \tfrac{a^{s-2}}{(s-2)!}\rangle.$$

Note that the relations (4) imply the equalities

$$D_3(u_s) = u_{s-1} - rv_s,\ D_3(v_s) = v_{s-1}$$

(here $[D_3, D_2] = rD_1$). Analogously it follows from the relation

$$[D_2, u_s D_1 + v_s D_2] \in FI_1$$

that

$$D_2(v_s) = 0, D_2(u_s) \in F\langle\{\tfrac{a^i b^j}{i!j!} D_1\},\ 0 \leqslant i,j \leqslant m\rangle.$$

Since $D_1 \in Z(L)$, we have the equalities $D_1(u_s) = D_1(v_s) = 0$. Therefore we get by Lemma 6 that $v_s \in F\langle 1, \ldots, a^{s-1}/(s-1)!\rangle$. By Lemma 7, $u_s \in F\langle\{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m+s\rangle$ (since $D_3(u_s) \in F\langle\{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m+s-1\rangle$ by the relations (4)). Since $rD_1 \in FI_1$, we have $rv_s \in F\langle\{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m+s-1\rangle$. But then by Lemma 7 $u_s \in F\langle\{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m+s\rangle$. So, we have proved that the Lie algebra L is contained in the subalgebra $\overline{L}$ from the conditions of the lemma. To finish with the proof we must prove that the element D_2 can be chosen in $W(A)$ in such a way that $[D_3, D_2] = 0$. Take the element $D_2 - r_0 D_1$ instead D_2, where the element r_0 is obtained from r by formal integration on variable a (recall that $r \in F\langle\{\frac{a^i b^j}{i!j!}\},\ 0 \leqslant i,j \leqslant m\rangle$). Then $[D_3, D_2] = 0$. The proof is complete. □

Collect now the results about nilpotent Lie algebras into the next statement.

Theorem 1. *Let $\mathbb{K}$ be a field of characteristic zero, A an integral $\mathbb{K}$-domain with fraction field R. Denote by $W(A)$ the subalgebra $R\mathrm{Der}_{\mathbb{K}}A$ of the Lie algebra $\mathrm{Der}_{\mathbb{K}}R$. Let L be a nilpotent subalgebra of rank 3 over R from $W(A)$ and $F = R^L$ its field of constants. If $\dim_F FL \geqslant 4$, then there exist integers $n \geqslant 0, m \geqslant 0$, elements $D_1, D_2, D_3 \in FL$ such that $[D_i, D_j] = 0, i,j = 1,2,3$ and the Lie algebra FL is contained in one of the following subalgebras of the Lie algebra $W(A)$:*

1) $L_1 = F\langle D_3, D_1, aD_1, \ldots, (a^n/n!)D_1, D_2, aD_2, \ldots, (a^n/n!)D_2\rangle$, where $a \in R$ is such that $D_1(a) = D_2(a) = 0, D_3(a) = 1$.

2) $L_2 = F\langle D_3, D_2, aD_2, \ldots, (a^n/n!)D_2, \{\frac{a^i b^j}{i!j!} D_1\}, 0 \leqslant i,j \leqslant m\rangle$ where $a, b \in R$ are such that $D_1(a) = D_2(a) = 0$, $D_3(a) = 1$, $D_1(b) = D_3(b) = 0$, $D_2(b) = 1$.

As a corollary we get the next characterization of nilpotent Lie algebras of rank $\leqslant 3$ from the Lie algebra $W(A)$.

Theorem 2. *Under conditions of Theorem 1, every nilpotent subalgebra L of rank $k \leqslant 3$ over R from the Lie algebra $W(A)$ is isomorphic to a finite dimensional subalgebra of the triangular Lie algebra $u_k(F)$.*

Proof. If $k = 1$ then the Lie algebra FL is one-dimensional over F and therefore is isomorphic to $u_1(F) = F\frac{\partial}{\partial x_1}$. In the case $k = 2$, the Lie algebra FL is (by Lemma 4) of the form

$$FL = F\langle D_1, aD_1, \ldots, \tfrac{a^k}{k!}D_1, D_2\rangle,\ k \geqslant 0$$

$$(\text{if } k = 0,\ \text{then put } FL = F\langle D_1, D_2\rangle),$$

where $[D_1, D_2] = 0$, $D_1(a) = 0$, $D_2(a) = 1$. The Lie algebra FL is isomorphic to a suitable subalgebra of the triangular Lie algebra $u_2(F) = \{f(x_2)\frac{\partial}{\partial x_1} + F\frac{\partial}{\partial x_2}\}$: the correspondence $D_i \mapsto \frac{\partial}{\partial x_i}$, $i = 1, 2$ and $a \mapsto x_2$ can be extended to an isomorphism between FL and a subalgebra of $u_2(F)$. Let now $k = 3$. If $\dim_F FL = 3$, then FL is either abelian or has a basis D_1, D_2, D_3 with multiplication rule $[D_3, D_2] = D_1, [D_2, D_1] = [D_3, D_1] = 0$. In the first case, FL is isomorphic to the subalgebra $F\langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\rangle$, in the second case it is isomorphic to the subalgebra $F\langle\frac{\partial}{\partial x_1}, x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\rangle$ of the triangular Lie algebra $u_3(F)$.

Let now $\dim_F FL \geqslant 4$. The Lie algebra FL is contained (by Theorem 1) in one of the Lie algebras L_1 or L_2 from the statement of that theorem. Note that the Lie algebra L_1 is isomorphic to the subalgebra $\overline{L}_1$ of the Lie algebra $u_3(F)$ of the form

$$\overline{L}_1 = F\langle\tfrac{\partial}{\partial x_3}, \tfrac{\partial}{\partial x_1}, \ldots, (x_3^n/n!)\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, (x_3^n/n!)\tfrac{\partial}{\partial x_2}\rangle,$$

Analogously the Lie algebra L_2 is isomorphic the the subalgebra $\overline{L}_2$ of $u_3(F)$ of the form

$$\overline{L}_2 = F\langle\tfrac{\partial}{\partial x_3}, \tfrac{\partial}{\partial x_2}, \ldots, (x_3^n/n!)\tfrac{\partial}{\partial x_2}, \{\tfrac{x_2^i x_3^j}{i!j!}\tfrac{\partial}{\partial x_1}\}, 0 \leqslant i, j \leqslant m\rangle. \qquad \square$$

Corollary 1. *Let L be a nilpotent subalgebra of the Lie algebra $W_3(\mathbb{K}) =$ $\mathrm{Der}(\mathbb{K}[x_1, x_2, x_3])$ and F the field of constants for the Lie algebra L in the field $\mathbb{K}(x_, x_2, x_3)$. Then the Lie algebra FL (over the field F) is isomorphic to a finite dimensional subalgebra of the triangular Lie algebra $u_3(F)$.*

Remark 1. If L is a nilpotent subalgebra of rank 3 over R from the Lie algebra $W_3(\mathbb{K})$, then L being isomorphic to a subalgebra of the triangular Lie algebra $u_3(\mathbb{K})$ can be not conjugated (by an automorphism of $W_3(\mathbb{K})$) with any subalgebra of $u_3(\mathbb{K})$. Indeed, the subalgebra $L = \mathbb{K}\langle x_1\frac{\partial}{\partial x_1}, x_2\frac{\partial}{\partial x_2}, x_3\frac{\partial}{\partial x_3}\rangle$ is nilpotent but not conjugated with any subalgebra of $u_3(\mathbb{K})$ (L is selfnormalized in $W_3(\mathbb{K})$, but any finite dimensional subalgebra of $u_3(\mathbb{K})$ is not, because of locally nilpotency of the Lie algebra $u_3(\mathbb{K})$).

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Finite local nearrings with split metacyclic additive group

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Abstract. In the paper the split metacyclic groups which are the additive groups of finite local nearrings are classified.

Introduction

Nearrings are generalized rings in the sense that the addition need not be commutative and only one distributive law is assumed. For a detailed account of basic concepts concerning the nearrings we refer the reader to the books [12] or [13]. A nearring R with an identity is called local if the set of all non-invertible elements of R forms a subgroup of the additive group of R.

Maxson [9] described all non-isomorphic zero-symmetric local nearrings with non-cyclic additive group of order p^2 which are not nearfields. He also shown in [10] that every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring. This result was extended to infinite abelian p-groups of finite exponent [5].

However in the case of finite non-abelian p-groups the situation is different. For instance, neither a generalized quaternion group nor a non-abelian group of order 8 can be the additive group of a local nearring [11] (see also [10]).

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In [14] all minimal non-abelian groups (the Miller–Moreno groups in other words) which are the additive groups of finite nearrings with identity are classified. In this paper the split metacyclic groups which appear as the additive groups of finite local nearrings are considered and their full classification is given.

1. Preliminaries

First we recall some notions and facts concerning nearrings and metacyclic groups.

Definition 1. A (left) nearring is a set $R = (R, +, \cdot)$ with two binary operations, addition " + " and multiplication " · ", such that

1) $(R, +)$ is a group with neutral element 0,
2) $(R, \cdot)$ is a semigroup, and
3) $x(y + z) = xy + xz$ for all x, y, $z \in R$.

The group $(R, +)$ of a nearring R is denoted by R^+ and called the *additive group* of R. It is easy to see that for each subgroup M of R^+ and for each element $x \in R$ the set $xM = \{x \cdot y | y \in M\}$ is a subgroup of R^+ and in particular $x \cdot 0 = 0$. If in addition $0 \cdot x = 0$ for all $x \in R$, then the nearring R is called *zero-symmetric*. In general, the set of all $y \in R$ with $0 \cdot y = 0$ is a subnearring called the *zero-symmetric part* of R. Furthermore, R is a *nearring with an identity* i if the semigroup $(R, \cdot)$ is a monoid with identity element i. In the latter case the group of all invertible elements of the monoid $(R, \cdot)$ is denoted by R^* and called the *multiplicative group* of R. A subgroup M of R^+ is called R^*-*invariant*, if $rM \leqslant M$ for each $r \in R^*$, and (R, R)-*subgroup*, if $xMy \subseteq M$ for arbitrary x, $y \in R$.

As usual, for every element $r \in R$ and each integer $n \in \mathbb{Z}$ we define the element rn of R as follows:

$$rn = \begin{cases} \underbrace{r + \cdots + r}_{n \text{ times}} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \underbrace{(-r) + \cdots + (-r)}_{-n \text{ times}} & \text{if } n < 0. \end{cases}$$

Then $r(m + n) = rm + rn$ for any integers m and n, so that we can identify the neutral element 0 with integer 0. On the other hand, if i is an identity of R, then we will not identify i with integer 1, because in general $(in)r \neq rn = r(in)$ for $n \neq 1$. Thus, to avoid a confusion, we do not use a notation nr with an integer n.

The following two simple assertions are well-known.

Lemma 1. *Let R be a finite nearring R with identity i. Then the exponent of the additive group R^+ is equal to the additive order of i which coincides with additive order of every element of the multiplicative group R^*.*

Proof. Indeed, if $ik = 0$ for some positive integer k, then for each $x \in R$ we have $xk = (xi)k = x(ik) = x0 = 0$. On the other hand, if $y \in R^*$ and $yl = 0$ for a positive integer l, then $il = y^{-1}(yl) = 0$, so that the additive orders of r and i coincide. □

Lemma 2. *Let R be a nearring with identity i and $a \in R^*$. For any elements x, $y \in R$ we put $x \circ y = xa^{-1}y$. Then with respect to the operations "$+$" and "$\circ$" the set $(R, +, \circ)$ is a nearring with identity a which is isomorphic to R.*

Proof. It can be easily verified that the operation "$\circ$" is associative and left distributive with respect to the addition and the mapping $r \mapsto ar$ determines an isomorphism of the nearring R onto $(R, +, \circ)$. □

Definition 2. **[8]** A nearring R with identity is said to be local if the set $L = R \setminus R^*$ of all non-invertible elements of R is a subgroup of R^+.

As it was shown in [8], Theorem 7.4, the additive group of a finite local nearring is a p-group for a prime p.

The following lemma characterizes the main properties of local nearrings (see [1], Lemma 3.2).

Lemma 3. *Let R be a local nearring with an identity i and L the subgroup of all non-invertible elements of R. Then the following statements hold:*

1) *L is an (R, R)-subgroup of R^+;*
2) *each proper R^*-invariant subgroup of R^+ is contained in L;*
3) *the set $i + L$ forms a subgroup of the multiplicative group R^*.*

Recall that a group G is called *metacyclic* if there exists a cyclic normal subgroup $\langle a \rangle$ such that the factor-group $G/\langle a \rangle$ is cyclic. For a prime p, a metacyclic p-group G is *split* if and only if it is decomposed in a semidirect product $G = \langle a \rangle \rtimes \langle b \rangle$ of the cyclic normal subgroup $\langle a \rangle$ and a cyclic subgroup $\langle b \rangle$.

The following useful characterization of non-abelian split metacyclic p-groups is due to B. King (see [7], Theorem 3.2 and Proposition 4.10).

Proposition 1. *Let $G = \langle a\rangle \rtimes \langle b\rangle$ be a non-abelian split metacyclic p-group with $a^{p^m} = b^{p^n} = 1$ for some positive integers m and n. Then the exponent of G is equal to $\max\{p^m, p^n\}$ and one of the following statements holds*:

I. $b^{-1}ab = a^{1+p^{m-r}}$ *with* $1 \leqslant r < \min\{m, n+1\}$ *and* $r < m-1$ *for* $p = 2$;

II. $p = 2$ *and* $b^{-1}ab = a^{-1+2^{m-r}}$ *with* $0 \leqslant r < \min\{m-1, n+1\}$.

Henceforth, a group G satisfying one of statements I or II of Proposition 1 will be denoted by $G(p^m, p^n, r)$ or $G(2^m, 2^n, -r)$, respectively. Furthermore, for any integers v and $w \geqslant 0$ we put $j(v, 0) = 0$ and $j(v, w) = 1 + v + \ldots + v^{w-1}$ for $w \geqslant 1$.

Lemma 4. *Let p be a prime and t, u positive integers. If d, k and l are non-negative integers, then the following statements hold*:

1) $j(t^d, k) + j(t^d, l)t^{dk} = j(t^d, k+l)$;

2) *if* $t \equiv 1 \pmod{p^u}$, *then*

$$t^d \equiv t^{dt} \equiv 1 + d(t-1) \pmod{p^{2u}}$$

and

$$j(t^d, k) \equiv k + \binom{k}{2} d(t-1) \pmod{p^{2u}};$$

3) *if* $t \equiv -1 \pmod{2^u}$, *then*

$$t^{d2^k} \equiv \begin{cases} (-1)^d(1 - d(t+1)) \pmod{2^{2u}} & \text{if } k = 0, \\ 1 - d(t+1)2^k \pmod{2^{2u+k-1}} & \text{if } k > 0, \end{cases}$$

and

$$j(t^d, k) \equiv \begin{cases} \frac{1-(-1)^k}{2} + \frac{(2k-1)(-1)^k+1}{4} d(t+1) & \pmod{2^{2u}} \\ & \text{if } d \equiv 1 \pmod 2, \\ k - \binom{k}{2} d(t-1) & \pmod{2^{2u}} \\ & \text{if } d \equiv 0 \pmod 2. \end{cases}$$

Proof. Since all statements are obvious for $d = 0$, we assume that $d > 0$. Clearly statement 1) is trivial if $kl = 0$. In the other case we have

$$\begin{aligned} j(t^d, k) + j(t^d, l)t^{dk} &= (1 + t^d + \cdots + t^{d(k-1)}) + (1 + t^d + \cdots + t^{d(l-1)})t^{dk} \\ &= 1 + t^d + \cdots + t^{d(k+l-1)} = j(t^d, k+l), \end{aligned}$$

as desired.

Statement 2) is easily proved by induction on d. Indeed, we have

$$t^{d-1} \equiv 1 + (d-1)(t-1) \pmod{p^{2u}}$$

and so $t^{d-1}(t-1) \equiv t-1 \pmod{p^{2u}}$. Therefore

$$t^d \equiv t^{d-1} + t - 1 \equiv 1 + d(t-1) \pmod{p^{2u}}.$$

This implies

$$t^{dt} - t^d \equiv 1 + dt(t-1) - (1 + d(t-1)) = d(t-1)^2 \equiv 0 \pmod{p^{2u}}$$

and thus $t^{dt} \equiv t^d \pmod{p^{2u}}$. Furthermore,

$$j(t^d,k) = 1+t^d+\cdots+t^{d(k-1)} \equiv 1+(1+d(t-1))+\cdots+(1+d(k-1)(t-1))$$

$$= k + (1+\cdots+(k-1))d(t-1) = k + \binom{k}{2}d(t-1) \pmod{p^{2u}},$$

which proves statement 2).

For proving statement 3), we put $v = t+1$. Then $v \equiv 0 \pmod{2^u}$ and

$$t^{d2^k} = (-1+v)^{d2^k} = (-1)^{d2^k} + (-1)^{d2^k-1}\binom{d2^k}{1}v$$

$$+(-1)^{d2^k-2}\binom{d2^k}{2}v^2 + \cdots + v^{d2^k}.$$

Since $\binom{d2^k}{2} \equiv 0 \pmod{2^{k-1}}$, the congruence for t^{d2^k} follows from this equality. Therefore

$$j(t^d,k) = 1 + t^d + \cdots + t^{(k-1)d} \equiv 1 + (-1)^d(1-v)$$

$$+(-1)^{2d}(1-2dv) + \cdots + (-1)^{(k-1)d}(1-(k-1)dv) \pmod{2^{2u}}.$$

In particular, for odd d we have

$$j(t^d,k) \equiv 1 + (-1+dv) + (1-2dv) + \cdots + ((-1)^{k-1} + (-1)^{k-2}(k-1)dv)$$

$$= \frac{1-(-1)^k}{2} + (1-2+3-\cdots+(-1)^{k-2}(k-1))dv$$

$$= \frac{1-(-1)^k}{2} + \frac{(2k-1)(-1)^k+1}{4}dv \pmod{2^{2u}}.$$

If d is even, then

$$j(t^d, k) \equiv 1 + (1 - dv) + (1 - 2dv) + \cdots + (1 - (k-1)dv)$$

$$= k - (1 + 2 + \cdots + (k-1))dv = k - \binom{k}{2} dv \pmod{2^{2u}},$$

as claimed. □

Lemma 5. *Let G be an additively written group whose elements a and b satisfy the relation $a + b = b + as$ for some natural number s. If t is the least natural number such that $ast = 0$, then for any non-negative integers d, k and u the equalities $au + bd = bd + as^d u$, $bd + au = aut^d + bd$, $(au + bd)k = auj(t^d, k) + bdk$ and $(bd + au)k = bdk + auj(s^d, k)$ hold.*

Proof. Since $-b + a + b = as$ and $-b + at + b = (-b + a + b)t = ast = a$, we have $b + a = at + b$ and so $b + au = atu + b$. By induction on d, we derive $au + bd = bd + as^d u$ and $bd + au = at^d u + bd$. Therefore

$$(au + bd)k = au(1 + t^d + \ldots + t^{d(k-1)}) + bdk = auj(t^d, k) + bdk$$

and hence

$$(bd + au)k = bdk + au(1 + s^d + \ldots + s^{d(k-1)}) = bdk + auj(s^d, k).$$

□

The following proposition on the automorphism group of a non-abelian split metacyclic p-group can be found in [2], Theorem 3.1, for $p > 2$ and in [4], Theorem 3.5, for $p = 2$.

Proposition 2. *Let G be a split non-abelian metacyclic p-group and let S be a Sylow p-subgroup of the automorphism group $Aut(G)$. Then S is a normal subgroup of index $p - 1$ in $Aut(G)$. In particular, if $p = 2$, then $Aut(G)$ is a 2-group.*

An information about orbits of the group G under the action of its automorphism group $\mathrm{Aut}(G)$ is given by the following lemma.

Lemma 6. *Let $G = G(p^m, p^n, r)$ with $m \leqslant n + r$, $A = Aut(G)$ and let $\langle x \rangle$ be a cyclic subgroup of G. Then the following statements hold:*

1) *if $\langle x \rangle$ is a normal subgroup of order p^m in G, then*

$$|x^A| \leqslant p^{2m-r-1}(p-1);$$

2) *if* $p > 2$, $m \leqslant n$ *and* $\langle x\rangle$ *is a non-normal subgroup of order* p^n, *then* $x^{-1} \notin x^A$.

Proof. If $G = \langle a\rangle \rtimes \langle b\rangle$ with $b^{-1}ab = a^{1+p^{m-r}}$ and $\langle x\rangle$ is a normal subgroup of order p^m in G, then either $\langle a\rangle \cap \langle x\rangle = 1$ and so $\langle x\rangle$ centralizes the subgroup $\langle a\rangle$, or $a^{p^{m-1}} \in \langle x\rangle$. Since $G' = \langle a^{p^{m-r}}\rangle$ is a characteristic subgroup of G, it follows that in the first case $\langle a\rangle \cap \langle x^\alpha\rangle = 1$ for each $\alpha \in A$. Hence $x^A \subseteq C_G(a) = \langle a\rangle \times \langle b^{p^r}\rangle$ and so $|x^A| \leqslant p^{2m-r-1}(p-1)$. In the second case $G = \langle x\rangle \rtimes \langle b\rangle$ and so $G' = \langle a^{p^{m-r}}\rangle \leqslant \langle x\rangle$. Then $|\langle x\rangle\langle x^\alpha\rangle| = \frac{|x||x^\alpha|}{|\langle x\rangle \cap \langle x\rangle^\alpha|} \leqslant p^{2m-r}$, whence $\langle x\rangle\langle x^\alpha\rangle \leqslant \langle x\rangle \rtimes \langle b^{p^{n+r-m}}\rangle$ and in particular $x^\alpha \in \langle x\rangle \rtimes \langle b^{p^{n+r-m}}\rangle$. Taking into account that the number of elements of order p^m in $\langle x\rangle$ is equal to $p^{m-1}(p-1)$, we have $|x^A| \leqslant p^{2m-r-1}(p-1)$, which proves statement 1).

Now let $p > 2$, $m \leqslant n$ and let $\langle x\rangle$ be a non-normal subgroup of order p^n in G. Since $G' = \langle a^{p^{m-r}}\rangle$, it follows that $\langle a\rangle \cap \langle x\rangle = \langle a^{p^s}\rangle$ for some integer s such that $m \geqslant s > m-r$ and so $\langle a^{p^s}\rangle = \langle x^{p^{n-m+s}}\rangle$. Therefore $x = a^u b^{vp^{m-s}}$ for some integers u and v with $(v,p) = 1$ and hence $[a,x] = [a, b^{vp^{m-s}}] = a^{wp^{2m-r-s}}$, where

$$w = \frac{(1+p^{m-r})^{vp^{m-s}-1}}{p^{2m-r-s}}$$

and in particular $(w,p) = 1$.

Assume that $x^\alpha = x^{-1}$ for some automorphism $\alpha \in A$. As it was shown above, $a^\alpha \in \langle a\rangle \rtimes \langle b^{p^{n+r-m}}\rangle$, whence $a^\alpha = a^k b^{lp^{n+r-m}}$ for some integers k and l with $(k,p) = 1$. Furthermore, $\langle a^{p^{m-r}}\rangle^\alpha = \langle a^{p^{m-r}}\rangle$ and so $(a^{p^{m-r}})^\alpha = (a^k b^{lp^{n+r-m}})^{p^{m-r}} = a^{kp^{m-r}} b^{lp^n} = a^{kp^{m-r}}$. Thus $(a^{p^{m-r}})^\alpha = a^{kp^{m-r}}$. On the other hand, because of $m \leqslant n$ it follows that $b^{lp^{n+r-m}} \in \langle b^r\rangle \leqslant Z(G)$. Therefore $a^{kwp^{m-r}} = (a^{wp^{m-r}})^\alpha = [a,x]^\alpha = [a^\alpha, x^{-1}] = [a^k b^{lp^{n+r-m}}, x^{-1}] = [a^k, x^{-1}] = [a, x^{-1}]^k = ([a,x]^{-k})^{x^{-1}} = (a^{-kwp^{m-r}})^{x^{-1}}$ and hence $(a^{kwp^{m-r}})^x = a^{-kwp^{m-r}}$. However for $p > 2$ the last equality holds only in the case where $a^{kwp^{m-r}} = 1$. Since $(kw,p) = 1$, this means that $a^{p^{m-r}} = 1$, contrary to the hypothesis of the lemma. Therefore, $x^{-1} \notin x^A$, as claimed in statement 2). $\square$

Lemma 7. *Let* R *be a local nearring whose additive group* R^+ *is a split non-abelian metacyclic* p*-group and let* L *be the subgroup of all non-invertible elements of* R*. Then* L *is a subgroup of index* p *in* R^+.

Proof. Indeed, we have the index $|R^+ : L| = p^k$ for some $k \geqslant 1$ and so $|R| = p^k|L|$. Since $R = R^* \cup L$ with $R^* \cap L = \varnothing$, it follows that

$|R^*| = p^k|L| - |L| = (p^k - 1)|L|$ and thus the order of R^* is divisible by $p^k - 1$. On the other hand, for each element $r \in R^*$ the mapping $x \mapsto rx$ with $x \in R$ is an automorphism of R^+, because of $r(x + y) = rx + ry$ for all $x, y \in R$. Therefore R^* can be viewed as a subgroup of $\mathrm{Aut}(R^+)$. Furthermore, it follows from Proposition 2 that the order of $\mathrm{Aut}(R^+)$ is divisible by $p^k - 1$ only if $k = 1$. Hence $|R^+ : L| = p$, as desired. □

As a direct consequence of Lemmas 1, 2 and 7 we have the following assertion.

Corollary 1. *Let R be a local nearring whose additive group R^+ is a non-abelian split metacyclic p-group. Then the group R^+ is generated by elements a and b of orders p^m and p^n, respectively, one of which coincides with identity element of R and $a+b = b+a(1+p^{m-r})$, if R^+ is isomorphic to the group $G(p^m, p^n, r)$, and $a+b = b+a(-1+2^{m-r})$, if R^+ is isomorphic to the group $G(2^m, 2^n, -r)$.*

2. Nearrings with identity on non-abelian split metacyclic p-groups

Let R be a nearring with identity whose additive group R^+ is a split non-abelian metacyclic p-group with $p \geqslant 2$. Then $R^+ = \langle a\rangle + \langle b\rangle$ for some elements a and b of R satisfying the relations $ap^m = bp^n = 0$ and $b+a = at+b$ with $(p, t) = 1$. In particular, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \leqslant x_1 < p^m$ and $0 \leqslant x_2 < p^n$. In this section we will consider the cases when at least one of the elements a or b is invertible in R, i. e. it belongs to the multiplicative group R^*.

Assume first that $a \in R^*$. Then R^+ is a group of exponent p^m by Lemma 1 and so $m \geqslant n$. Furthermore, according to Lemma 2, without loss of generality we can assume that a is an identity of R, i. e. $ax = xa = x$ for each $x \in R$. Moreover, for each $x \in R$ there exist coefficients $\alpha(x)$ and $\beta(x)$ such that $xb = a\alpha(x) + b\beta(x)$. It is clear that they are uniquely defined modulo p^m and p^n, respectively, so that some mappings $\alpha : R \to Z_{p^m}$ and $\beta : R \to Z_{p^n}$ are determined.

Lemma 8. *Let $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ be elements of the nearring R. If a is an identity of R, then $m \geqslant n$ and the following statements hold*:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric*;
(1) $\alpha(a) = 0$ *and* $\beta(a) = 1$;

(2) $xy = a(x_1 j(t^{x_2}, y_1) + \alpha(x) j(t^{\beta(x)}, y_2) t^{x_2 y_1}) + b(x_2 y_1 + \beta(x) y_2)$;
(3) $\alpha(x)(t^{x_2 t} - 1) \equiv x_1(t^{\beta(x)} - j(t^{x_2}, t)) \pmod{p^m}$;
(4) $x_2(t-1) \equiv 0 \pmod{p^n}$.

Proof. Since $0 \cdot a = a \cdot 0 = 0$, it follows that R is a zero-symmetric nearring if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0)$ or equivalently $\alpha(0) = \beta(0) = 0$. Moreover, since $b = ab = a\alpha(a) + b\beta(a)$, we have $\alpha(a) = 0$ and $\beta(a) = 1$, so that statements (0) and (1) hold.

Further, using the left distributive law, we derive

$$xy = (xa)y_1 + (xb)y_2 = (ax_1 + bx_2)y_1 + (a\alpha(x) + b\beta(x))y_2.$$

Applying Lemma 5, we have also

$$(ax_1 + bx_2)y_1 = ax_1 j(t^{x_2}, y_1) + bx_2 y_1,$$

$$(a\alpha(x) + b\beta(x))y_2 = a\alpha(x) j(t^{\beta(x)}, y_2) + b\beta(x) y_2$$

and

$$bx_2 y_1 + a\alpha(x) j(t^{\beta(x)}, y_2) = a\alpha(x) j(t^{\beta(x)}, y_2) t^{x_2 y_1} + bx_2 y_1.$$

Thus

$$xy = a(x_1 j(t^{x_2}, y_1) + \alpha(x) j(t^{\beta(x)}, y_2) t^{x_2 y_1}) + b(x_2 y_1 + \beta(x) y_2)$$

and so statement (2) holds. Setting in this formula $y = at + b$, we derive

$$xy = a(x_1 j(t^{x_2}, t) + \alpha(x) t^{x_2 t}) + b(x_2 t + \beta(x)).$$

On the other hand, $y = b + a$ and so

$$xy = xb + x = a(\alpha(x) + x_1 t^{\beta(x)}) + b(x_2 + \beta(x))$$

by Lemma 5. Comparing the coefficients under a and b in the latter two expressions for xy, we get for each $x \in R$ the equalities

$$\alpha(x)(t^{x_2 t} - 1) \equiv x_1(t^{\beta(x)} - j(t^{x_2}, t)) \pmod{p^m}$$

and

$$x_2(t-1) \equiv 0 \pmod{p^n},$$

i. e. statements (3) and (4), as desired. □

Consider now the case when $b \in R^*$.

Lemma 9. *If $b \in R^*$, then $m \leqslant n$, $a \notin R^*$ and $p = 2$.*

Proof. Since b is of order p^n, the group R^+ is of exponent p^n by Lemma 1 and so $m \leqslant n$. Let A denote the automorphism group $\mathrm{Aut}(R^+)$ of R^+. Considering R^* as a subgroup of A, we have $R^*x \subseteq x^A$ for each $x \in R$ and in particular $R^* = R^*b \subseteq b^A$. If $a \in b^A$, then $a = b^\phi$ for some automorphism $\phi \in A$ and so $\langle a\rangle^\phi = \langle b\rangle$. Since the subgroup $\langle a\rangle$ is normal in R^+ and the subgroup $\langle b\rangle$ is not, the latter equality is impossible. Therefore $a \notin b^A$ and hence $a \notin R^*$.

Assume that $p > 2$. Then $-b \notin b^A$ by Lemma 6 and so $-b \notin R^*$. On the other hand, if i is an identity of R, then $b^{-1}(-b) = -(b^{-1}b) = -i$. Since $(-i)^2 = -(-i) = i$, this implies $b^{-1}(-b) = -i \in R^*$ and so $-b \in bR^* = R^*$. This contradiction shows that $p = 2$ and completes the proof. □

As above, according to Lemma 2, in the case $b \in R^*$ we can assume that b is an identity of R and for each $x \in R$ there exist the coefficients $\alpha(x)$ and $\beta(x)$ which are uniquely determined modulo 2^m and 2^n, respectively, such that $xa = a\alpha(x) + b\beta(x)$.

Lemma 10. *Let $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ be elements of the nearring R. If b is an identity of R, then $p = 2$, $m \leqslant n$ and the following statements hold*:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric*;
(1) $\alpha(b) = 1$ *and* $\beta(b) = 0$;
(2) $xy = a(\alpha(x)j(t^{\beta(x)}, y_1) + x_1 j(t^{x_2}, y_2)t^{\beta(x)y_1}) + b(\beta(x)y_1 + x_2y_2)$;
(3) $\alpha(x)(j(t^{\beta(x)}, t) - t^{x_2}) \equiv x_1(1 - t^{\beta(x)t}) \pmod{2^m}$;
(4) $\beta(x)(t - 1) \equiv 0 \pmod{2^n}$.

Proof. Observe first that $p = 2$ and $m \leqslant n$ by Lemma 9. Since $0\cdot b = b\cdot 0 = 0$, the nearring R is zero-symmetric if and only if $0 = 0\cdot a = a\alpha(0) + b\beta(0)$, whence $\alpha(0) = \beta(0) = 0$. Similarly, the equality $a = ba = a\alpha(b) + b\beta(b)$ implies that $\alpha(b) = 1$ and $\beta(b) = 0$, i. e. statements (0) and (1) hold. Further, applying the left distributive law, we obtain

$$xy = (xa)y_1 + (xb)y_2 = (a\alpha(x) + b\beta(x))y_1 + (ax_1 + bx_2)y_2.$$

Using Lemma 5, we have also

$$(a\alpha(x) + b\beta(x))y_1 = a\alpha(x)j(t^{\beta(x)}, y_1) + b\beta(x)y_1,$$

$$(ax_1 + bx_2)y_2 = ax_1 j(t^{x_2}, y_2) + bx_2y_2$$

and

$$b\beta(x)y_1 + ax_1 j(t^{x_2}, y_2) = ax_1 j(t^{x_2}, y_2)t^{\beta(x)y_1} + b\beta(x)y_1.$$

Therefore

$$\begin{aligned} xy = a(\alpha(x)j(t^{\beta(x)}, y_1) + x_1 j(t^{x_2}, y_2)t^{\beta(x)y_1}) \\ + b(\beta(x)y_1 + x_2 y_2), \end{aligned}$$

which proves statement (2). Substituting $y = at + b$ in this equality, we get

$$xy = a(\alpha(x)j(t^{\beta(x)}, t) + x_1 t^{\beta(x)t}) + b(x_2 + \beta(x)t).$$

On the other hand, $y = b + a$ and thus

$$xy = x + xa = a(x_1 + \alpha(x)t^{x_2}) + b(x_2 + \beta(x)).$$

Comparing the coefficients under a and b in the latter two expressions for xy, we obtain the congruences

$$\alpha(x)j(t^{\beta(x)}, t) + x_1 t^{\beta(x)t} \equiv x_1 + \alpha(x)t^{x_2} \pmod{2^m}$$

and

$$x_2 + \beta(x)t \equiv x_2 + \beta(x) \pmod{2^n},$$

from which statements (3) and (4) follow directly. □

2.1. Nearrings with identity on the group $G(p^m, p^n, r)$

Assume now that m, n and r are positive integers satisfying statement I of Proposition 1, and let t be the least natural number such that $(1 + p^{m-r})t \equiv 1 \pmod{p^m}$. It is easy to see that $t = 1 + hp^{m-r}$ for some h with $0 < h < p^r$ and $(h, p) = 1$.

The following two lemmas describe the multiplication in a nearring R whose additive group R^+ is isomorphic to the group $G(p^m, p^n, r)$, i. e. R^+ is generated by elements a and b satisfying the relations $ap^m = bp^n = 0$ and $b + a = at + b$. As it was mentioned above, we restrict ourselves to the cases when one of the generators a or b is an identity of R. In what follows $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ are arbitrary elements of R.

Lemma 11. *If a is an identity of R, then $m \geqslant n + r \geqslant 2r$ and*

$$xy = a(x_1 y_1 + \alpha(x)y_2 - x_1 x_2 \binom{y_1}{2} p^{m-r}) + b(x_2 y_1 + \beta(x)y_2).$$

Moreover, the following statements hold:

(1) $\alpha(x) \equiv 0 \pmod{p^{m-n}}$;
(2) *either* $x_1(\beta(x)-1) \equiv 0 \pmod{p^r}$ *or* $p=2$, $m>2r$ *and* $x_1(\beta(x)-1) \equiv 0 \pmod{2^r}$;
(3) $\alpha(xy) = x_1\alpha(y) + \alpha(x)\beta(y) - x_1x_2\binom{\alpha(y)}{2}p^{m-r}$;
(4) $\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y)$.

Proof. Since $x_2(t-1) \equiv 0 \pmod{p^n}$ by statement (4) of Lemma 8 and $t-1 = hp^{m-r}$ with $(h,p)=1$, we have $m-r \geqslant n$. Therefore

$$\text{(i)} \qquad m \geqslant n+r \geqslant 2r$$

and in particular $2(m-r) \geqslant m$. Furthermore, since $(1+p^{m-r})t \equiv 1 \pmod{p^m}$, it follows that

$$\text{(ii)} \qquad t-1 \equiv -p^{m-r} \pmod{p^m}.$$

Using this and statement 2) of Lemma 4, we obtain the congruences

$$\text{(iii)} \qquad j(t^{x_2}, y_1) \equiv y_1 - x_2\binom{y_1}{2}p^{m-r} \pmod{p^m},$$

$$\text{(iv)} \qquad j(t^{\beta(x)}, y_2) \equiv y_2 - \beta(x)\binom{y_2}{2}p^{m-r} \pmod{p^m}$$

and

$$\text{(v)} \qquad t^{x_2y_1} \equiv 1 - x_2y_1p^{m-r} \pmod{p^m}.$$

Substituting now in formula (2) of Lemma 8 instead of the left parts of congruences (iii)–(v) their right parts, we derive the equality

$$(*) \qquad \begin{aligned} xy = a((x_1y_1 + \alpha(x)y_2) - (x_1x_2\tbinom{y_1}{2} + \alpha(x)\beta(x)\tbinom{y_2}{2} \\ + \alpha(x)x_2y_1y_2)p^{m-r}) + b(x_2y_1 + \beta(x)y_2). \end{aligned}$$

Setting in this equality $y = bp^n = 0$, we have

$$0 = x(bp^n) = a(\alpha(x)p^n - \alpha(x)\beta(x)\binom{p^n}{2}p^{m-r})$$

$$= a\alpha(x)p^n(1 - \beta(x)\binom{p^n}{2}p^{m-r-n}).$$

As $m-r \geqslant 1$ for $p>2$ and $m-r \geqslant 2$ for $p=2$, it follows that $1-\beta(x)\binom{p^n}{2}p^{m-r-n} \equiv 1 \pmod{p}$ and so $a\alpha(x)p^n = 0$. Therefore

$$\text{(vi)} \qquad \alpha(x) \equiv 0 \pmod{p^{m-n}},$$

i.e. statement (1) holds. Moreover, since $m-n \geqslant r$ by (i), it follows that $a\alpha(x)p^{m-r} = 0$ and hence equality (*) can be rewritten in the form

$$xy = a(x_1y_1 + \alpha(x)y_2 - x_1x_2\binom{y_1}{2}p^{m-r}) + b(x_2y_1 + \beta(x)y_2),$$

as claimed.

Replacing in this equality y by $yb = a\alpha(y) + b\beta(y)$ and taking into account that $x(yb) = (xy)b = a\alpha(xy) + b\beta(xy)$, we obtain two expressions for the element $x(yb)$. Comparing the coefficients at a and b in these expressions, we derive the equalities

$$\alpha(xy) = x_1\alpha(y) + \alpha(x)\beta(y) - x_1x_2\binom{\alpha(y)}{2}p^{m-r}$$

and

$$\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y)$$

of statements (3) and (4) of the lemma.

Furthermore, using statement 2) of Lemma 4, we have also

$$\text{(vii)} \qquad t^{x_2t} \equiv 1 - x_2p^{m-r} \pmod{p^m},$$

$$\text{(viii)} \qquad t^{\beta(x)} \equiv 1 - \beta(x)p^{m-r} \pmod{p^m}$$

and

$$\text{(ix)} \quad j(t^{x_2}, t) \equiv \begin{cases} 1 - p^{m-r} \pmod{p^m} & \text{if} \quad p > 2, \\ 1 - 2^{m-r}(1 - x_22^{m-r-1}) \pmod{2^m} & \text{if} \quad p = 2. \end{cases}$$

Substituting the right parts of congruences (vi)–(viii) in congruence (3) of Lemma 8, we get the congruences

$$\text{(x)} \qquad \alpha(x)x_2 \equiv x_1(\beta(x) - 1) \pmod{p^r}$$

for $p > 2$ and

$$\text{(xi)} \qquad \alpha(x)x_2 \equiv x_1(\beta(x) - 1 + x_22^{m-r-1}) \pmod{2^r}$$

for $p=2$. Since $m-n \geqslant r$ by (i), it follows from conditions (vi), (x) and (xi) that $x_1(\beta(x)-1) \equiv 0 \pmod{p^r}$ for $p>2$ and $x_1(\beta(x)-1+x_22^{m-r-1}) \equiv 0 \pmod{2^r}$ for $p=2$. In the latter case $m > 2r$ and this implies $x_1(\beta(x)-1) \equiv 0 \pmod{2^r}$, so that statement (2) holds. $\square$

Lemma 12. *If b is an identity of R, then $p = 2 < m \leqslant n$, $r = 1$ and*

$$xy = a(\alpha(x)y_1 + x_1 j(t^{x_2}, y_2)) + b(\beta(x)y_1 + x_2 y_2).$$

Moreover, the following statements hold:
(0) $\alpha(0) = \beta(0) = 0$;
(1) $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$;
(2) $\alpha(x)(1 - x_2) \equiv 0 \pmod{2}$;
(3) $\alpha(xy) = \alpha(x)\alpha(y) + x_1 j(t^{x_2}, \beta(y))$;
(4) $\beta(xy) = \beta(x)\alpha(y) + x_2\beta(y)$.

Proof. It follows from Lemma 10 that $p = 2$ and $m \leqslant n$. Furthermore, statement (4) of this lemma and the equality $t-1 = h2^{m-r}$ with $(h, 2) = 1$ imply that

$$\beta(x) \equiv 0 \pmod{2^{n-m+r}}.$$

Therefore it follows from statement 2) of Lemma 4 that for each integer $k \geqslant 0$ the congruences

$$t^{\beta(x)k} \equiv 1 \pmod{2^n} \tag{i}$$

and

$$j(t^{\beta(x)}, k) \equiv k \pmod{2^n} \tag{ii}$$

hold. In particular, taking $k = y_1$ and applying these congruences to formula (2) of Lemma 10, we get for R the multiplication formula

$$xy = a(\alpha(x)y_1 + x_1 j(t^{x_2}, y_2)) + b(\beta(x)y_1 + x_2 y_2), \tag{**}$$

as claimed. Furthermore, expressing the left part of the equality $x(ya) = (xy)a$ by formula (**) and taking into consideration that $ya = a\alpha(y) + b\beta(y)$ and $(xy)a = a\alpha(xy) + b\beta(xy)$, we derive the formulas for $\alpha(xy)$ and $\beta(xy)$, i. e. statements (3) and (4) of the lemma.

Next, setting $k = t$ in congruences (i) and (ii), we have

$$1 - t^{\beta(x)t} \equiv 0 \pmod{2^n} \tag{iii}$$

and

$$j(t^{\beta(x)}, t) - t^{x_2} \equiv t - t^{x_2} \pmod{2^n}. \tag{iv}$$

Since $m \leqslant n$, it follows from congruences (iii), (iv) and statement (3) of Lemma 10 that

$$\alpha(x)(t - t^{x_2}) \equiv 0 \pmod{2^m}. \tag{v}$$

On the other hand,

$$t^{x_2} \equiv 1 + x_2 h 2^{m-r} \pmod{2^{2(m-r)}}$$

by statement 2) of Lemma 4 and hence

$$\text{(vi)} \qquad t - t^{x_2} \equiv (1 - x_2) h 2^{m-r} \pmod{2^{2(m-r)}}.$$

Therefore congruences (v) and (vi) imply that

$$\alpha(x)(1 - x_2) \equiv 0 \pmod{2^{\min\{r, m-r\}}}.$$

In particular, $\alpha(-b)(1+1) \equiv 0 \pmod{2^{\min\{r,m-r\}}}$ and hence

$$\text{(vii)} \qquad \alpha(-b) \equiv 0 \pmod{2^{\min\{r,m-r\}-1}}.$$

Finally, since $b = (-b)^2$ and $\alpha(b) = 1$ by statement (1) of Lemma 10, it follows that $\alpha((-b)^2) = 1$. However, $\alpha((-b)^2) = \alpha(-b)^2$ by statement (3) of the lemma, so that $\alpha(-b) \equiv \pm 1 \pmod 2$. Comparing this congruence with congruence (vii), we conclude that $\min\{r, m-r\} = 1$ and

$$\alpha(x)(1 - x_2) \equiv 0 \pmod 2,$$

i. e. statement (2) of the lemma holds. Moreover, as $r < m - 1$ by Proposition 1, it follows that $r = 1$ and thus $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$. In particular, if $x = 0$, then both $\alpha(0)$ and $\beta(0)$ are even integers. Since $\alpha(0) = \alpha(0)^2$ and $\beta(0) = \beta(0)\alpha(0)$ by statements (3) and (4) of the lemma, we get $\alpha(0) = \beta(0) = 0$. This proves statements (0) and (1) of the lemma and completes the proof. $\square$

2.2. Nearrings with identity on the group $G(2^m, 2^n, -r)$

In this subsection the integers m, n and r satisfy statement II of Proposition 1 and t is the least natural number satisfying the congruence $(-1 + 2^{m-r})t \equiv 1 \pmod{2^m}$. It is easy to check that $t = -1 + h2^{m-r}$ for some odd h with $0 < h < 2^r$.

We describe the multiplication in a nearring R whose additive group R^+ is isomorphic to the group $G(2^m, 2^n, -r)$ and one of two generators a and b of this group is an identity of R. Recall that the generators a and b of R^+ satisfy the relations $a2^m = b2^n = 0$ and $b + a = at + b$. As before, $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ denote arbitrary elements of R.

Lemma 13. *If a is an identity of R, then $m = 2$, $n = 1$ and $r = 0$, i. e., R^+ is the dihedral group of order 8.*

Proof. Since $x_2(-2+h2^{m-r}) \equiv 0(\bmod 2^n)$ by statement (4) of Lemma 8, it follows that $-1+h2^{m-r-1} \equiv 0 \pmod{2^{n-1}}$ and so $n = 1$. Hence $0 \leqslant r \leqslant 1$ and thus either $r = 0$ and $t = -1+2^m$ or $r = 1$ and $t = -1+2^{m-1}$. But if $t = -1+2^m$, then $R^+ = \langle a\rangle + \langle b\rangle$ is isomorphic to the dihedral group of order 2^{m+1} and this is possible only if $m = 2$ by [6], Proposition 4.4.

Let $t = -1+2^{m-1}$. Then $m \geqslant 3$ and statement (3) of Lemma 8 implies that

$$\alpha(x)(t^{x_2 t}-1) \equiv x_1(t^{\beta(x)} - j(t^{x_2}, t)) \pmod{2^m}$$

for each $x = ax_1 + bx_2$ of R. In particular, if $x = a+b$, then

$$\alpha(x)(t^t - 1) \equiv t^{\beta(x)} - j(t,t) \pmod{2^m}.$$

Moreover, $t^t \equiv -1+2^{m-1} \pmod{2^m}$, $j(t,t) \equiv 1+2^{m-1} \pmod{2^m}$ and $t^{\beta(x)} \equiv (-1)^{\beta(x)}(1-\beta(x)2^{m-1}) \pmod{2^m}$ by statement 3) of Lemma 4. Therefore

$$\alpha(x)(-2-2^{m-1}) \equiv ((-1)^{\beta(x)}-1) - ((-1)^{\beta(x)}\beta(x)-1)2^{m-1} \pmod{2^m}$$

and hence either

(i) $$\alpha(x) \equiv 1 \pmod{2^{m-1}}$$

if $\beta(x) \equiv 1 \pmod 2$ or

(ii) $$\alpha(x) \equiv 2^{m-2} \pmod{2^{m-1}}$$

if $\beta(x) \equiv 0 \pmod 2$.

On the other hand, we have $b2 = 0$ and $xb = a\alpha(x) + b\beta(x)$, so that $0 = (xb)2 = a\alpha(x)j(t^{\beta(x)}, 2)$ by Lemma 5. Since

$$j(t^{\beta(x)}, 2) = 1 + t^{\beta(x)} \equiv \begin{cases} 2^{m-1} \pmod{2^m} \text{ if } \beta(x) \equiv 1 \pmod 2, \\ 2 \pmod{2^m} \text{ if } \beta(x) \equiv 0 \pmod 2, \end{cases}$$

it follows that $a\alpha(x)2^{m-1} = 0$ in the case (i) and $a\alpha(x)2 = 0$ in the case (ii). But then in both cases $a2^{m-1} = 0$ and this contradiction completes the proof. □

It should be noted that the nearrings with identity on the dihedral group of order 8 were firstly classified by J. Clay in [3]. He shown in particular that there exist exactly 7 non-isomorphic such nearrings.

Lemma 14. *If b is an identity of R, then $r+1 < m \leqslant n$, $0 \leqslant r \leqslant 1$ and*

$$xy = a(\alpha(x)y_1 + x_1 j(t^{x_2}, y_2)) + b(\beta(x)y_1 + x_2y_2).$$

Moreover, the following statements hold:

(0) $\alpha(0) = \beta(0) = 0$;

(1) $\beta(x) \equiv 0 \pmod{2^{n-1}}$;

(2) $\alpha(xy) = \begin{cases} \alpha(x)\alpha(y) + x_1\beta(y), \textit{ if } m = n \textit{ and } x_2 \equiv 0 \pmod 2, \textit{ and} \\ \alpha(x)\alpha(y), \textit{ in the other cases}; \end{cases}$

(3) $\beta(xy) = \beta(x)\alpha(y) + x_2\beta(y)$.

Proof. Note first that $r+1 < m$ by Proposition 1, $m \leqslant n$ by Lemma 9 and $\beta(x)(t-1) \equiv 0 \pmod{2^n}$ by statement (4) of Lemma 10. Since $t = -1 + h2^{m-r}$ for some odd integer h, we have $\beta(x)(-2 + h2^{m-r}) \equiv 0 \pmod{2^n}$ and so $\beta(x) \equiv 0 \pmod{2^{n-1}}$, i. e., statement (1) of the lemma holds. As $2(m-r) + n - 2 \geqslant m + n - r$ and $t^{\beta(x)} \equiv 1 + h2^{m+n-r-1} \pmod{2^{m+n-r}}$ by statement 3) of Lemma 4, it follows that $t^{\beta(x)k} \equiv 1 \pmod{2^m}$ and so $j(t^{\beta(x)}, k) \equiv k \pmod{2^m}$ for every integer $k \geqslant 0$. In particular, setting $k = y_1$ and using the latter two congruences in statement (2) of Lemma 10, we can rewrite the formula for xy in the form

$$(***) \qquad xy = a(\alpha(x)y_1 + x_1 j(t^{x_2}, y_2)) + b(\beta(x)y_1 + x_2y_2),$$

as claimed.

Next, if $k = t$, then the above-mentioned congruences and statement (3) of Lemma 10 imply that $\alpha(x)(t - t^{x_2}) \equiv 0 \pmod{2^m}$. In particular, if $x = -b = b(2^n - 1)$, then $x_2 = 2^n - 1$ and $t - t^{x_2} = t - t^{2^n-1} \equiv t^2 - t^{2^n} \pmod{2^m}$. Since $t^{2^{m-1}} \equiv 1 \pmod{2^m}$ and $m \leqslant n$, it follows that $t - t^{x_2} \equiv t^2 - 1 = h2^{m-r+1}(-1 + h2^{m-r-1}) \pmod{2^m}$. Thus $\alpha(-b)2^{m-r+1} \equiv 0 \pmod{2^m}$, so that either $r = 0$ or $r \geqslant 1$ and

$$\text{(i)} \qquad \alpha(-b) \equiv 0 \pmod{2^{r-1}}.$$

Now, expressing both parts of the equality $x(ya) = (xy)a$ by formula (***) and comparing the coefficients at a and b, we derive

$$\text{(ii)} \qquad \alpha(xy) = \alpha(x)\alpha(y) + x_1 j(t^{x_2}, \beta(y))$$

and

$$\text{(iii)} \qquad \beta(xy) = \beta(x)\alpha(y) + x_2\beta(y).$$

In particular, if $x = y = -b$, then $x_1 = 0$ and from equality (ii) it follows that $\alpha((-b)^2) = \alpha(-b)^2$. As $(-b)^2 = -(-b) = b$ and $\alpha(b) = 1$ by statement (1) of Lemma 10, this implies $\alpha(-b) \equiv \pm 1 \pmod{2^m}$ and hence congruence (i) holds if and only if $r = 1$.

Finally, it follows from statement 3) of Lemma 4 that $j(t^{x_2}, \beta(x)) \equiv 0 \pmod{2^{n-1}}$ for $x_2 \equiv 0 \pmod 2$ and $j(t^{x_2}, \beta(x)) \equiv 0 \pmod{2^n}$ for $x_2 \equiv 1 \pmod 2$. Therefore statements (2) and (3) of the lemma follow directly from equalities (ii) and (iii). Furthermore, if $x = y = 0$, then $\alpha(0) = \alpha(0)^2$ by equality (ii) and $\beta(0) = \beta(0)\alpha(0)$ by equality (iii), so that either $\alpha(0) = \beta(0) = 0$ or $\alpha(0) = 1$. Since in the latter case $0 \cdot y = ay_1 + b\beta(0)y_1$ by formula (***), it follows that $0 \cdot y = 0$ if and only if $y_1 = 0$ and hence $y \in \langle b \rangle$. But then, as the zero-symmetric part of R, the subgroup $\langle b \rangle$ is normal in R^+ by [12], Theorem 1.15, and thus the group R^+ is abelian, contrary to the assumption. This proves statement (0) of the lemma and completes the proof. □

3. Local nearrings on the groups $G(p^m, p^n, r)$ and $G(2^m, 2^n, -r)$

Now we apply the results of the previous section for describing local nearrings whose additive groups are non-abelian split metacyclic. Recall that if R is such a local nearring, then the additive group R^+ is a p-group for some prime number p and so it is isomorphic to one of the groups $G(p^m, p^n, r)$ or $G(2^m, 2^n, -r)$ by Proposition 1. Furthermore, the set L of all non-invertible elements of R is a subgroup of index p in R^+ by Lemma 7.

Our first theorem concerns local nearrings on the group $G(p^m, p^n, r)$.

Theorem 1. *Let R be a local nearring whose additive group R^+ is isomorphic to the group $G(p^m, p^n, r)$. Then $R^+ = \langle a \rangle + \langle b \rangle$, one of the elements a or b coincides with an identity of R and the following statements hold:*

1) *$ap^m = bp^n = 0$ and $a + b = b + a(1 + p^{m-r})$ with $1 \leqslant r < \min\{m, n+1\}$ and $r < m - 1$ for $p = 2$;*
2) *if a is an identity of R, then $m \geqslant n + r \geqslant 2r + [\frac{2}{p}]$, $L = \langle ap \rangle + \langle b \rangle$ and $R^* = \{ax_1 + bx_2 \mid x_1 \not\equiv 0 \pmod p\}$;*
3) *if b is an identity of R, then $p = 2 < m \leqslant n$, $r = 1$, $L = \langle a \rangle + \langle b2 \rangle$ and $R^* = \{ax_1 + bx_2 \mid x_2 \equiv 1 \pmod 2\}$.*

Proof. It follows from Corollary 1 that $R^+ = \langle a \rangle + \langle b \rangle$ for some elements a and b one of which coincides with an identity of R and that statement 1) of the theorem holds.

If a is an identity of R, then $m \geqslant n + r \geqslant 2r + [\frac{2}{p}]$ by Lemma 11. In particular, $m > n$ and so $b \in L$ by Lemma 1. Therefore $L = \langle ap\rangle + \langle b\rangle$. Since $R^* = R \setminus L$, an element $x = ax_1 + bx_2$ belongs to R^* if and only if $x_1 \not\equiv 0 \pmod{p}$.

Similarly, if b is an identity of R, then Lemmas 9 and 12 imply that $a \in L$, $p = 2 < m \leqslant n$ and $r = 1$. Hence $L = \langle a\rangle + \langle b2\rangle$ and so an element $x = ax_1 + bx_2$ belongs to R^* if and only if $x_2 \equiv 1 \pmod{2}$. $\square$

Applying now statements 2) and 3) of Theorem 1 to Lemmas 11 and 12, respectively, we obtain the following formulas for multiplying any two elements $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ in a local nearring R whose additive group is isomorphic to $G(p^m, p^n, r)$.

Corollary 2. *If a is an identity of R and $xb = a\alpha(x) + b\beta(x)$, then $m \geqslant n + r \geqslant 2r > 0$ and*

$$xy = a(x_1y_1 + \alpha(x)y_2 - x_1x_2\binom{y_1}{2}p^{m-r}) + b(x_2y_1 + \beta(x)y_2)$$

with coefficients $\alpha(x)$ and $\beta(x)$ satisfying the following conditions:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric*;
(1) $\alpha(a) = 0$ *and* $\beta(a) = 1$;
(2) $\alpha(x) \equiv 0 \pmod{p^{m-n}}$;
(3) $x_1(\beta(x) - 1) \equiv 0 \pmod{p^r}$ *and* $m \geqslant 2r + [\frac{2}{p}]$;
(4) $\alpha(xy) = x_1\alpha(y) + \alpha(x)\beta(y) - x_1x_2\binom{\alpha(y)}{2}p^{m-r}$;
(5) $\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y)$.

Corollary 3. *If b is an identity of R and $xa = a\alpha(x) + b\beta(x)$, then $p = 2 < m \leqslant n$, $r = 1$ and*

$$xy = a(\alpha(x)y_1 + x_1j(t^{x_2}, y_2)) + b(\beta(x)y_1 + x_2y_2)$$

with coefficients $\alpha(x)$ and $\beta(x)$ satisfying the following conditions:

(0) $\alpha(0) = \beta(0) = 0$;
(1) $\alpha(b) = 1$ *and* $\beta(b) = 0$;
(2) $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$;
(3) $\alpha(x)(1 - x_2) \equiv 0 \pmod{2}$;
(4) $\alpha(xy) = \alpha(x)\alpha(y) + x_1j(t^{x_2}, \beta(y))$;
(5) $\beta(xy) = \beta(x)\alpha(y) + x_2\beta(y)$.

We now turn to local nearrings on the group $G(2^m, 2^n, -r)$.

Theorem 2. *Let R be a local nearring whose additive group R^+ is isomorphic to the group $G(2^m, 2^n, -r)$. Then $R^+ = \langle a\rangle + \langle b\rangle$, the element b is an identity of R and the following statements hold*:

1) $r + 1 < m \leqslant n$ *and* $0 \leqslant r \leqslant 1$;
2) $a2^m = b2^n = 0$ *and* $a + b = b + a(-1 + 2^{m-r})$;
3) $L = \langle a\rangle + \langle b2\rangle$ *and* $R^* = \{ax_1 + bx_2 \mid x_2 \equiv 1 \pmod 2\}$.

Proof. As in the proof of Theorem 1, it follows from Corollary 1 that there exists a decomposition $R^+ = \langle a\rangle + \langle b\rangle$ in which one of the elements a or b is an identity of R and that statement 2) of the theorem holds. But if a is an identity of R, then the group R^+ is dihedral of order 8 by Lemma 13 and so it cannot be the additive group of a local nearring by [11]. Hence the element b is an identity of R. Then $r + 1 < m \leqslant n$ and $0 \leqslant r \leqslant 1$ by Lemma 14 and $a \in L$ by Lemma 9. Therefore $L = \langle a\rangle + \langle b2\rangle$ by Lemma 1 and thus $R^* = \{ax_1 + bx_2 \mid x_2 \equiv 1 \pmod 2\}$, as claimed. □

As a consequence of Lemmas 10, 14 and Theorem 2, we have the following formula for multiplying any two elements in a local nearring R whose additive group is isomorphic to $G(2^m, 2^n, -r)$.

Corollary 4. *If $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ are elements of R, then*

$$xy = a(\alpha(x)y_1 + x_1 j(t^{x_2}, y_2)) + b(\beta(x)y_1 + x_2y_2)$$

with coefficients $\alpha(x)$ and $\beta(x)$ satisfying the following conditions:

(0) $\alpha(0) = \beta(0) = 0$;
(1) $\alpha(b) = 1$ *and* $\beta(b) = 0$;
(2) $\beta(x) \equiv 0 \pmod{2^{n-1}}$;
(3) $\alpha(xy) = \begin{cases} \alpha(x)\alpha(y) + x_1\beta(y), \text{ if } m = n \text{ and } x_2 \equiv 0 \pmod 2, \text{ and} \\ \alpha(x)\alpha(y), \text{ in the other cases}; \end{cases}$
(4) $\beta(xy) = \beta(x)\alpha(y) + x_2\beta(y)$.

4. Groups $G(p^m, p^n, r)$ and $G(2^m, 2^n, -r)$ as the additive groups of local nearrings

The following two theorems show that the conditions given in Theorems 1 and 2 are also sufficient for existing finite local nearrings on groups $G(p^m, p^n, r)$ and $G(2^m, 2^n, -r)$. Therefore this completes our classification of all non-abelian split metacyclic p-groups which are the additive groups of local nearrings.

Theorem 3. *For each prime p and positive integers m, n and r such that either $m \geqslant n + r \geqslant 2r + [\frac{2}{p}]$ or $p = 2$, $2 < m \leqslant n$ and $r = 1$ there exists a local nearring R whose additive group R^+ is isomorphic to the group $G(p^m, p^n, r)$.*

Proof. Let G be an additively written group $G(p^m, p^n, r)$ with generators a, b satisfying the relations $ap^m = 0$, $bp^n = 0$ and $a + b = b + a(1 + p^{m-r})$. Then $G = \langle a \rangle + \langle b \rangle$ and each element $x \in G$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \leqslant x_1 < p^m$ and $0 \leqslant x_2 < p^n$.

We assume first that $m \geqslant n + r \geqslant 2r > 0$ and put $x \cdot b = b$ for each $x \in G$. Then the coefficients $\alpha(x) = 0$ and $\beta(x) = 1$ satisfy the conditions (1) - (5) of Corollary 2 and so the formula

$$x \cdot y = a(x_1 y_1 - x_1 x_2 \binom{y_1}{2} p^{m-r}) + b(x_2 y_1 + y_2)$$

determines a multiplication "$\cdot$" on G such that the system $R = (G, +, \cdot)$ is a nearring with identity element a. Furthermore, it is easy to check that an element $x = ax_1 + bx_2 \in G$ is invertible in R if and only if $x_1 \equiv 1 \pmod{p}$. Therefore the set of all non-invertible elements of R coincides with the subgroup $L = \langle ap \rangle + \langle b \rangle$ of index p in G, so that the nearring R is local. Moreover, it is also easily verified that the zero-symmetric part of R coincides with the subgroup $\langle a \rangle$ and the constant part $0 \cdot R = \langle b \rangle$.

In the other case, if $p = 2, 2 < m \leqslant n$ and $r = 1$, then G is a metacyclic Miller-Moreno p-group, so that G is the additive group of a zero-symmetric local nearring with identity element b by [15], Theorem 2. □

Theorem 4. *If m, n and r are integers such that $r + 1 < m \leqslant n$ and $0 \leqslant r \leqslant 1$, then there exists a local nearring R whose additive group R^+ is isomorphic to the group $G(2^m, 2^n, -r)$.*

Proof. Let G be an additively written group $G(2^m, 2^n, -r)$ with generators a, b satisfying the relations $a2^m = 0$, $b2^n = 0$ and $a + b = b + at$ with $t = -1 + 2^{m-r}$. Then $G = \langle a \rangle + \langle b \rangle$ and each element $x \in G$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \leqslant x_1 < 2^m$ and $0 \leqslant x_2 < 2^n$.

In order to define a required multiplication "$\cdot$" on G, for each $x \in G$ we put $x \cdot a = a\alpha(x)$ with

$$\alpha(x) = \begin{cases} 1, \text{if } x_2 \equiv 1 \pmod{2}, \text{and} \\ 0, \text{ if } x_2 \equiv 0 \pmod{2}. \end{cases}$$

Then the coefficients $\alpha(x)$ and $\beta(x) = 0$ satisfy the conditions (0) - (4) of Corollary 4 and so the formula

$$x \cdot y = a(\alpha(x)y_1 + x_1 j(t^{x_2}, y_2)) + b(x_2 y_2)$$

determines multiplication "$\cdot$" on G such that the system $R = (G, +, \cdot)$ is a nearring with identity element b.

Indeed, it is easy to see that $x \cdot b = a(\alpha(x) \cdot 0 + x_1 j(t^{x_2}, 1)) + bx_2 = ax_1 + bx_2 = x = b \cdot x$, so that b is the identity of R.

We show further that $x \cdot (y+z) = x \cdot y + x \cdot z$ for arbitrary $y = ay_1 + by_2$ and $z = az_1 + bz_2$ of G. Since $y + z = a(y_1 + z_1 t^{y_2}) + b(y_2 + z_2)$ by Lemma 5, we have

$$(i) \quad x \cdot (y+z) = a(\alpha(x)(y_1 + z_1 t^{y_2}) + x_1 j(t^{x_2}, y_2 + z_2)) + bx_2(y_2 + z_2).$$

On the other hand,

$$x \cdot z = a(\alpha(x)z_1 + x_1 j(t^{x_2}, z_2)) + b(x_2 z_2)$$

and

$$b(x_2 y_2) + a(\alpha(x)z_1 + x_1 j(t^{x_2}, z_2)) = a(\alpha(x)z_1 + x_1 j(t^{x_2}, z_2))t^{x_2 y_2}) + b(x_2 y_2)$$

by Lemma 5. Therefore

$$(ii) \qquad x \cdot y + x \cdot z = a(\alpha(x)(y_1 + z_1 t^{x_2 y_2})$$
$$+ x_1(j(t^{x_2}, y_2) + j(t^{x_2}, z_2)t^{x_2 y_2})) + bx_2(y_2 + z_2).$$

Subtracting equality (ii) from (i), we obtain

$$x \cdot (y+z) - (x \cdot y + x \cdot z) = a(\alpha(x)(y_1 + z_1 t^{y_2}) + x_1 j(t^{x_2}, y_2 + z_2))$$
$$- a(\alpha(x)(y_1 + z_1 t^{x_2 y_2}) + x_1(j(t^{x_2}, y_2) + j(t^{x_2}, z_2)t^{x_2 y_2})$$
$$= a(\alpha(x)(y_1 + z_1 t^{y_2}) + x_1 j(t^{x_2}, y_2 + z_2) - x_1(j(t^{x_2}, y_2) + j(t^{x_2}, z_2)t^{x_2 y_2})$$
$$-(\alpha(x)(y_1 + z_1 t^{x_2 y_2})) = a(\alpha(x)(y_1 + z_1 t^{y_2} - z_1 t^{x_2 y_2} - y_1)),$$

because

$$j(t^{x_2}, y_2 + z_2) = j(t^{x_2}, y_2) + j(t^{x_2}, z_2)t^{x_2 y_2})$$

by statement 1) of Lemma 4. Thus

$$x \cdot (y+z) - (x \cdot y + x \cdot z) = a(\alpha(x)(y_1 + z_1 t^{y_2} - z_1 t^{x_2 y_2} - y_1))$$

and since $\alpha(x) = 0$ for $x_2 \equiv 0 \pmod 2$, it remains to consider the case $\alpha(x) = 1$ in which $x_2 \equiv 1 \pmod 2$. But then $t^{x_2} \equiv t \pmod{2^m}$ by statement 3) of Lemma 4 and so $t^{x_2 y_2} \equiv t^{y_2} \pmod{2^m}$. Therefore $(y_1 + z_1 t^{y_2} - z_1 t^{x_2 y_2} - y_1) \equiv 0 \pmod{2^m}$ and hence $a(y_1 + z_1 t^{y_2} - z_1 t^{x_2 y_2} - y_1) = 0$, as claimed.

It is also clear that the associativity of multiplication "$\cdot$" follows from its left distributivity and the equality $x \cdot (y \cdot a) = (x \cdot y) \cdot a$. Indeed, since $y \cdot a = a\alpha(y)$ and $(x \cdot y) \cdot a = a\alpha(x \cdot y)$ by definition, we have $x \cdot (y \cdot a) = x \cdot (a\alpha(y)) = (x \cdot a)\alpha(y) = (a\alpha(x))\alpha(y) = a(\alpha(x)\alpha(y)) = a\alpha(x \cdot y)$.

Finally, we show that an element $x = ax_1 + bx_2 \in G$ is invertible if and only if $x_2 \equiv 1 \pmod 2$. This means that we need to find an element $y = ay_1 + by_2$ such that $x \cdot y = y \cdot x = b$. Clearly there exists an odd integer y_2 such that $x_2 y_2 \equiv 1 \pmod{2^n}$. Thus if we put $y_1 = -x_1 j(t^{x_2}, y_2)$, then it easy to see that $x \cdot y = y \cdot x = b$. Therefore $R^* = \{ax_1 + bx_2 \mid x_2 \equiv 1 \pmod 2\}$ and hence the set of all non-invertible elements of R coincides with the subgroup $L = \langle a \rangle + \langle b2 \rangle$ of G. Thus $R = (G, +, \cdot)$ is a local nearring, as desired. $\square$

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