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Mykola Komarnytskyi

(25.05.1948 – 21.04.2016)

Distinguished Professor of Ivan Franko L'viv National University, Doctor of Sciences Mykola Komarnytskyi passed away on April 21, 2016. His sudden death is a great loss for L'viv mathematicians and for the entire Ukrainian mathematics community.

Mykola Komarnytskyi was born in 1948 in the village Komarnyky of the L'viv Region. After graduating of high school, Mykola was enrolled to the Department of Mechanics and Mathematics of Ivan Franko L'viv University. In 1971, after graduating the University he was admitted to the Department of Algebra of Institute of Physics and Mathematics in L'viv.

In February 1979 he was hired in the Department of Mechanics and Mathematics of L'viv University. Since that, his entire life was tightly connected to this Department. In 1993-1995 as a senior researcher he worked hard on many important algebraic problems. His Doctor of Sciences thesis was defended in 1998. Since 1998 he worked a professor of the Department of Algebra and Topology, and in 2002 he became the Chair of this department.

Mykola Komarnytskyi made an impressive contribution in many areas of modern algebra: The theory of rings and modules, the model theory, the categorical logic. He solved the Cozzens-Faith problem on ultrapowers of principal ideal domains. We should also mention his attention-grabbing proof (coauthored by Ivanna Melnyk) of axiomatizability of the class of noncommutative Prüfer rings. Mykola Komarnytskyi has introduced a new class of elementary divisor rings, called the almost invariant elementary divisor rings, and obtained a partial solution for these rings of the Warfield problem (to find an internal characterization of rings such that every finitely presented module decomposes into a direct sum of cyclic submodules). He showed (with Bogdan Zabavsky) that any elementary divisor distributive domain is a duo-domain.

Mykola Komarnytskyi proved that if every ideal of a commutative Bezout domain is transfinite nilpotent, then this domain is adequate, and therefore it is an elementary divisor ring. He also applied the obtained results to simplification of the formulas of the first order theory of modules.

Mykola Komarnytskyi (with Halina Zelisko) considered the lattice of left ideals of a ring which is an ultraproduct of a family of Noetherian V-domains, obtained the formula describing maximal left ideals in such ultraproduct and constructed the spectrum of an ultraproduct of principal ideal V-domains.

In addition to his prolific research work, M. Komarnitskyi was also actively engaged in teaching and especially in finding talented students and engaging them to research. Professor Mykola Komarnytskyi served as an advisor of six Candidate of Sciences theses. He organized international algebraic conferences in L'viv and served as an editor of mathematical journals. In particular, he was a vice-editor of Algebra and Discrete Mathematics.

In 2014, for his many years of productive scientific and educational work, Mykola Komarnytskyi was awarded the title of Distinguished Professor of Ivan Franko L'viv National University.

Professor Komarnytskyi was not only an outstanding mathematician but deeply and widely educated man with very widespread outlook. He had many friends in Ukraine and over the world. The cherished memory of Mykola Komarnytskyi will forever remain in the minds and hearts of his colleagues, students, family and friends.

Yuriy Drozd, Volodymyr Kirichenko, Leonid Kurdachenko, Fedir Lyman, Anatoliy Petravchuk, Vasyl Petrychkovych, Igor Subbotin, Vitaliy Sushchansky, Bogdan Zabavsky, Myhailo Zarichnyi, Anatolii Zhuchok, Yurii Zhuchok

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On a semitopological polycyclic monoid

Serhii Bardyla and Oleg Gutik

Communicated by M. Ya. Komarnytskyj

Abstract. We study algebraic structure of the λ-polycyclic monoid P_λ and its topologizations. We show that the λ-polycyclic monoid for an infinite cardinal $\lambda \geqslant 2$ has similar algebraic properties so has the polycyclic monoid P_n with finitely many $n \geqslant 2$ generators. In particular we prove that for every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free combinatorial 0-bisimple 0-E-unitary inverse semigroup. Also we show that every non-zero element x is an isolated point in (P_λ, τ) for every Hausdorff topology τ on P_λ, such that (P_λ, τ) is a semitopological semigroup, and every locally compact Hausdorff semigroup topology on P_λ is discrete. The last statement extends results of the paper [33] obtaining for topological inverse graph semigroups. We describe all feebly compact topologies τ on P_λ such that (P_λ, τ) is a semitopological semigroup and its Bohr compactification as a topological semigroup. We prove that for every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating and there exists no a Hausdorff feebly compact topological semigroup which contains P_λ as a dense subsemigroup.

2010 MSC: Primary 22A15, 20M18. Secondary 20M05, 22A26, 54A10, 54D30, 54D35, 54D45, 54H11.

Key words and phrases: inverse semigroup, bicyclic monoid, polycyclic monoid, free monoid, semigroup of matrix units, topological semigroup, semitopological semigroup, Bohr compactification, embedding, locally compact, countably compact, feebly compact.

1. Introduction and preliminaries

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [8, 11, 14, 32]. If A is a subset of a topological space X, then we denote the closure of the set A in X by $\operatorname{cl}_X(A)$. By ω we denote the first infinite cardinal.

A semigroup S is called an *inverse semigroup* if every a in S possesses an unique inverse, i.e. if there exists an unique element a^{-1} in S such that

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$

A map which associates to any element of an inverse semigroup its inverse is called the *inversion*.

A *band* is a semigroup of idempotents. If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [36, Definition II.5.12] chain L is called ω-chain if L is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [16] or [11, Section 2.1]):

$$\begin{array}{lcl}
a\mathscr{R}b & \text{if and only if} & aS^1 = bS^1; \\
a\mathscr{L}b & \text{if and only if} & S^1a = S^1b; \\
a\mathscr{J}b & \text{if and only if} & S^1aS^1 = S^1bS^1; \\
\end{array}$$
$$\mathscr{D} = \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L};$$
$$\mathscr{H} = \mathscr{L}\cap\mathscr{R}.$$

A semigroup S is said to be:

- *simple* if S has no proper two-sided ideals, which is equivalent to $\mathscr{J} = S \times S$ in S;
- 0-*simple* if S has a zero and S contains no proper two-sided ideals distinct from the zero;
- *bisimple* if S contains a unique $\mathscr{D}$-class, i.e., $\mathscr{D} = S \times S$ in S;

- 0-*bisimple* if S has a zero and S contains two $\mathscr{D}$-classes: $\{0\}$ and $S\setminus\{0\}$;
- *congruence-free* if S has only identity and universal congruences.

An inverse semigroup S is said to be

- *combinatorial* if $\mathscr{H}$ is the equality relation on S;
- *E-unitary* if for any idempotents $e,f\in S$ the equality $ex=f$ implies that $x\in E(S)$;
- 0-*E-unitary* if S has a zero and for any non-zero idempotents $e,f\in S$ the equality $ex=f$ implies that $x\in E(S)$.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots \\
q & qp & qp^2 & qp^3 & \cdots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

and the semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [11]. Also the nice Andersen Theorem states that *a simple semigroup S with an idempotent is completely simple if and only if S does not contains an isomorphic copy of the bicyclic semigroup* (see [1] and [11, Theorem 2.54]).

Let λ be a non-zero cardinal. On the set $B_\lambda=(\lambda\times\lambda)\cup\{0\}$, where $0\notin\lambda\times\lambda$, we define the semigroup operation "$\cdot$" as follows

$$(a,b)\cdot(c,d)=\begin{cases} (a,d), & \text{if } b=c;\\ 0, & \text{if } b\neq c,\end{cases}$$

and $(a,b)\cdot 0=0\cdot(a,b)=0\cdot 0=0$ for $a,b,c,d\in\lambda$. The semigroup B_λ is called the *semigroup of $\lambda\times\lambda$-matrix units* (see [11]).

In 1970 Nivat and Perrot proposed the following generalization of the bicyclic monoid (see [35] and [32, Section 9.3]). For a non-zero cardinal λ, the polycyclic monoid P_λ on λ generators is the semigroup with zero given by the presentation:

$$P_\lambda = \left\langle \{p_i\}_{i\in\lambda}, \left\{p_i^{-1}\right\}_{i\in\lambda} \mid p_ip_i^{-1} = 1, p_ip_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

It is obvious that in the case when $\lambda = 1$ the semigroup P_1 is isomorphic to the bicyclic semigroup with adjoined zero. For every finite non-zero cardinal $\lambda = n$ the polycyclic monoid P_n is a congruence free, combinatorial, 0-bisimple, 0-E-unitary inverse semigroup (see [32, Section 9.3]).

We recall that a topological space X is said to be:

- *compact* if each open cover of X has a finite subcover;
- *countably compact* if each open countable cover of X has a finite subcover;
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;
- *feebly compact* if each locally finite open cover of X is finite.

According to Theorem 3.10.22 of [14], a Tychonoff topological space X is feebly compact if and only if each continuous real-valued function on X is bounded, i.e., X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space is countably compact, every countably compact space is countably pracompact, and every countably pracompact space is feebly compact (see [3] and [14]).

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S,τ) is a topological (inverse) semigroup, then we shall call τ a (*inverse*) *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space together with a separately continuous semigroup operation.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{C}(p,q)$ is an open subset of S [13]. Bertman and West in [7] extended this result for the case of semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic semigroup [2, 30]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups discussed in [5, 6, 27]. In [13] Eberhart and Selden proved that if the bicyclic monoid $\mathscr{C}(p,q)$ is a dense subsemigroup of a topological monoid S and $I = S \setminus \mathscr{C}(p,q) \neq \varnothing$ then I is a two-sided ideal of the semigroup S. Also, there they described the closure the bicyclic monoid $\mathscr{C}(p,q)$ in a locally compact topological inverse semigroup. The closure of the bicyclic monoid in a countably compact (pseudocompact) topological semigroups was studied in [6].

In [15] Fihel and Gutik showed that any Hausdorff topology τ on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ such that $(\mathscr{C}_{\mathbb{Z}}, \tau)$ is a semitopological semigroup is discrete. Also in [15] studied a closure of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup.

For any Hausdorff topology τ on an infinite semigroup of $\lambda\times\lambda$-matrix units B_λ such that (B_λ, τ) is a semitopological semigroup every non-zero element of B_λ is an isolated point of (B_λ, τ) [22]. Also in [22] was proved that on any infinite semigroup of $\lambda\times\lambda$-matrix units B_λ there exists a unique feebly compact topology τ_A such that (B_λ, τ_A) is a semitopological semigroup and moreover this topology τ_A is compact. A closure of an infinite semigroup of $\lambda\times\lambda$-matrix units in semitopological and topological semigroups and its embeddings into compact-like semigroups were studied in [18, 22, 23].

Semigroup topologizations and closures of inverse semigroups of monotone co-finite partial bijections of some linearly ordered infinite sets, inverse semigroups of almost identity partial bijections and inverse semigroups of partial bijections of a bounded finite rank studied in [9, 10, 17, 20, 23–25, 28, 29].

To every directed graph E one can associate a graph inverse semigroup $G(E)$, where elements roughly correspond to possible paths in E. These semigroups generalize polycyclic monoids. In [33] the authors investigated topologies that turn $G(E)$ into a topological semigroup. For instance, they showed that in any such topology that is Hausdorff, $G(E) \setminus \{0\}$ must be discrete for any directed graph E. On the other hand, $G(E)$ need not be discrete in a Hausdorff semigroup topology, and for certain graphs E, $G(E)$ admits a T_1 semigroup topology in which $G(E) \setminus \{0\}$ is not discrete. In [33] the authors also described the algebraic structure and possible cardinality of the closure of $G(E)$ in larger topological semigroups.

In this paper we show that the λ-polycyclic monoid for in infinite cardinal $\lambda \geqslant 2$ has similar algebraic properties so has the polycyclic monoid P_n with finitely many $n \geqslant 2$ generators. In particular we prove that for every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free, combinatorial, 0-bisimple, 0-E-unitary inverse semigroup. Also we show that every non-zero element x is an isolated point in (P_λ, τ) for every Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup, and every locally compact Hausdorff semigroup topology on P_λ is discrete. The last statement extends results of the paper [33] obtaining for topological inverse graph semigroups. We describe all feebly compact topologies τ on P_λ such that (P_λ,τ) is a semitopological semigroup and its Bohr compactification as a topological semigroup. We prove that for every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating and there exists no a Hausdorff feebly compact topological semigroup which contains P_λ as a dense subsemigroup.

2. Algebraic properties of the $\boldsymbol{\lambda}$-polycyclic monoid for an infinite cardinal $\boldsymbol{\lambda}$

In this section we assume that λ is an infinite cardinal.

We repeat the thinking and arguments from [32, Section 9.3].

We shall give a representation for the polycyclic monoid P_λ by means of partial bijections on the free monoid $\mathscr{M}_\lambda$ over the cardinal λ. Put $A = \{x_i \colon i \in \lambda\}$. Then the free monoid $\mathscr{M}_\lambda$ over the cardinal λ is isomorphic to the free monoid $\mathscr{M}_\lambda$ over the set A. Next we define for every $i \in \lambda$ the partial map $\alpha \colon \mathscr{M}_\lambda \to \mathscr{M}_\lambda$ by the formula $(u)\alpha_i = x_i u$ and put that $\mathscr{M}_\lambda$ is the domain and $x_i\mathscr{M}_\lambda$ is the range of α_i. Then for every $i \in \lambda$ we may regard so defined partial map as an element of the symmetric inverse monoid $\mathscr{I}(\mathscr{M}_\lambda)$ on the set $\mathscr{M}_\lambda$. Denote by I_λ the inverse submonoid of $\mathscr{I}(\mathscr{M}_\lambda)$ generated by the set $\{\alpha_i \colon i \in \lambda\}$. We observe that $\alpha_i\alpha_i^{-1}$ is the identity partial map on $\mathscr{M}_\lambda$ for each $i \in \lambda$ and whereas if $i \neq j$ then $\alpha_i\alpha_j^{-1}$ is the empty partial map on the set $\mathscr{M}_\lambda$, $i, j \in \lambda$. Define the map $h \colon P_\lambda \to I_\lambda$ by the formula $(p_i)h = \alpha_i$ and $(p_i^{-1})h = \alpha_i^{-1}$, $i \in \lambda$. Then by Proposition 2.3.5 of [32], I_λ is a homomorphic image of P_λ and by Proposition 9.3.1 from [32] the map $h \colon P_\lambda \to I_\lambda$ is an isomorphism. Since the band of the semigroup I_λ consists of partial identity maps, the identifying the semilattice of idempotents of I_λ with the free monoid $\mathscr{M}_\lambda^0$ with adjoined zero admits the following partial order on $\mathscr{M}_\lambda^0$:

$$\begin{gathered} u \leqslant v \quad \text{if and only if} \quad v \text{ is a prefix of } u \quad \text{for } u, v \in \mathscr{M}_\lambda^0, \\ \text{and} \quad 0 \leqslant u \quad \text{for every } u \in \mathscr{M}_\lambda^0. \end{gathered} \tag{1}$$

This partial order admits the following semilattice operation on $\mathscr{M}_\lambda^0$:

$$u * v = v * u = \begin{cases} u, & \text{if } v \text{ is a prefix of } u; \\ 0, & \text{otherwise,} \end{cases}$$

and $0 * u = u * 0 = 0 * 0 = 0$ for arbitrary words $u, v \in \mathscr{M}_\lambda^0$.

Remark 2.1. We observe that for an arbitrary non-zero cardinal λ the set $\mathscr{M}_\lambda^0 \setminus \{0\}$ with the dual partial order to (1) is order isomorphic to the λ-ary tree T_λ with the countable height.

Hence, we proved the following proposition.

Proposition 2.2. *For every infinite cardinal λ the semigroup P_λ is isomorphic to the inverse semigroup I_λ and the semilattice $E(P_\lambda)$ is isomorphic to $(\mathscr{M}_\lambda^0, *)$.*

Let n be any positive integer and $i_1, \ldots, i_n \in \lambda$. We put

$$P_n^\lambda \langle i_1, \ldots, i_n \rangle = \left\langle p_{i_1}, \ldots, p_{i_n}, p_{i_1}^{-1}, \ldots, p_{i_n}^{-1} \mid p_{i_k} p_{i_k}^{-1} = 1, p_{i_k} p_{i_l}^{-1} = 0 \text{ for } i_k \neq i_l \right\rangle.$$

The statement of the following lemma is trivial.

Lemma 2.3. *Let λ be an infinite cardinal and n be an arbitrary positive integer. Then $P_n^\lambda \langle i_1, \ldots, i_n \rangle$ is a submonoid of the polycyclic monoid P_λ such that $P_n^\lambda \langle i_1, \ldots, i_n \rangle$ is isomorphic to P_n for arbitrary $i_1, \ldots, i_n \in \lambda$.*

Our above representation of the polycyclic monoid P_λ by means of partial bijections on the free monoid $\mathscr{M}_\lambda$ over the cardinal λ implies the following lemma.

Lemma 2.4. *Let λ be an infinite cardinal. Then for any elements $x_1, \ldots, x_k \in P_\lambda$ there exist $i_1, \ldots, i_n \in \lambda$ such that $x_1, \ldots, x_k \in P_n^\lambda \langle i_1, \ldots, i_n \rangle$.*

Theorem 2.5. *For every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free combinatorial 0-bisimple 0-E-unitary inverse semigroup.*

Proof. By Proposition 2.2 the semigroup P_λ is inverse.

First we show that the semigroup P_λ is 0-bisimple. Then by the Munn Lemma (see [34, Lemma 1.1] and [32, Proposition 3.2.5]) it is sufficient to show that for any two non-zero idempotents $e, f \in P_\lambda$ there exists $x \in P_\lambda$ such that $xx^{-1} = e$ and $x^{-1}x = f$. Fix arbitrary two non-zero idempotents $e, f \in P_\lambda$. By Lemma 2.4 there exist $i_1, \ldots, i_n \in \lambda$ such that $e, f \in P_n^\lambda \langle i_1, \ldots, i_n \rangle$. Lemma 2.3, Theorem 9.3.4 of [32] and Proposition 3.2.5 of [32] imply that there exists $x \in P_n^\lambda \langle i_1, \ldots, i_n \rangle \subset P_\lambda$ such that $xx^{-1} = e$ and $x^{-1}x = f$. Hence the semigroup P_λ is 0-bisimple.

The above representation of the polycyclic monoid P_λ by means of partial bijections on the free monoid $\mathscr{M}_\lambda$ over the cardinal λ implies that the $\mathscr{H}$-class in P_λ which contains the unity is a singleton. Then since the polycyclic monoid P_λ is 0-bisimple Theorem 2.20 of [11] implies that every non-zero $\mathscr{H}$-class in P_λ is a singleton. It is obvious that $\mathscr{H}$-class in P_λ which contains zero is a singleton. This implies that the polycyclic monoid P_λ is combinatorial.

Suppose to the contrary that the monoid P_λ is not 0-E-unitary. Then there exist a non-idempotent element $x \in P_\lambda$ and non-zero idempotents $e, f \in P_\lambda$ such that $xe = f$. By Lemma 2.4 there exist $i_1, \ldots, i_n \in \lambda$ such that $x, e, f \in P_n^\lambda \langle i_1, \ldots, i_n \rangle$. Hence the monoid $P_n^\lambda \langle i_1, \ldots, i_n \rangle$ is not 0-E-unitary, which contradicts Lemma 2.3 and Theorem 9.3.4 of [32]. The obtained contradiction implies that the polycyclic monoid P_λ is a 0-E-unitary inverse semigroup.

Suppose the contrary that there exists a congruence $\mathfrak{C}$ on the polycyclic monoid P_λ which is distinct from the identity and the universal congruence on P_λ. Then there exist distinct $x, y \in P_\lambda$ such that $x\mathfrak{C}y$. By Lemma 2.4 there exist $i_1, \ldots, i_n \in \lambda$ such that $x, y \in P_n^\lambda \langle i_1, \ldots, i_n \rangle$. By Lemma 2.3 and Theorem 9.3.4 of [32], since the polycyclic monoid P_n is congruence-free we have that the unity and zero of the polycyclic monoid P_λ are $\mathfrak{C}$-equivalent and hence all elements of P_λ are $\mathfrak{C}$-equivalent. This contradicts our assumption. The obtained contradiction implies that the polycyclic monoid P_λ is a congruence-free semigroup. □

From now for an arbitrary cardinal $\lambda \geqslant 2$ we shall call the semigroup P_λ the λ-*polycyclic monoid*.

Fix an arbitrary cardinal $\lambda \geqslant 2$ and two distinct elements $a, b \in \lambda$. We consider the following subset $A = \{b^i a \colon i = 0, 1, 2, 3, \ldots\}$ of the free monoid $\mathscr{M}_\lambda$. The definition of the above defined partial order $\leqslant$ on $\mathscr{M}_\lambda^0$ implies that two arbitrary distinct elements of the set A are incomparable in $(\mathscr{M}_\lambda^0, \leqslant)$. Let $B(b^i a)$ be a subsemigroup of I_λ generated by the subset

$$\left\{\alpha \in I_\lambda \colon \operatorname{dom}\alpha = b^i a\mathscr{M}_\lambda \text{ and } \operatorname{ran}\alpha = b^j a\mathscr{M}_\lambda \text{ for some } i, j \in \omega\right\}$$

of the semigroup I_λ. Since two arbitrary distinct elements of the set A are incomparable in the partially ordered set $(\mathscr{M}_\lambda^0, \leqslant)$ the semigroup operation of I_λ implies that the following conditions hold:

(i) $\alpha\beta$ is a non-zero element of the semigroup I_λ if and only if $\operatorname{ran}\alpha = \operatorname{dom}\beta$;

(ii) $\alpha\beta = 0$ in I_λ if and only if $\operatorname{ran}\alpha \neq \operatorname{dom}\beta$;

(iii) if $\alpha\beta \neq 0$ in I_λ then $\operatorname{dom}(\alpha\beta) = \operatorname{dom}\alpha$ and $\operatorname{ran}(\alpha\beta) = \operatorname{ran}\beta$;

(iv) $B(b^i a)$ is an inverse subsemigroup of I_λ, for arbitrary $\alpha,\beta\in B(b^ia)$.

Now, if we identify ω with the set of all non-negative integers $\{0,1,2,3,4,\ldots\}$, then simple verifications show that the map $\mathfrak{h}\colon B(b^ia)\to B_\omega$ defined in the following way:

(a) if $\alpha\neq 0$, $\operatorname{dom}\alpha=b^ia\mathscr{M}_\lambda$ and $\operatorname{ran}\alpha=b^ja\mathscr{M}_\lambda$, then $(\alpha)\mathfrak{h}=(i,j)$, for $i,j\in\{0,1,2,3,4,\ldots\}$;

(b) $(0)\mathfrak{h}=0$,

is a semigroup isomorphism.

Hence we proved the following proposition.

Proposition 2.6. *For every cardinal $\lambda\geqslant 2$ the λ-polycyclic monoid P_λ contains an isomorphic copy of the semigroup of $\omega\times\omega$-matrix units B_ω.*

Proposition 2.7. *For every non-zero cardinal λ and any $\alpha,\beta\in P_\lambda\setminus\{0\}$, both sets $\{\chi\in P_\lambda\colon \alpha\cdot\chi=\beta\}$ and $\{\chi\in P_\lambda\colon \chi\cdot\alpha=\beta\}$ are finite.*

Proof. We show that the set $\{\chi\in P_\lambda\colon \alpha\cdot\chi=\beta\}$ is finite. The proof in other case is similar.

It is obvious that

$$\{\chi\in P_\lambda\colon \alpha\cdot\chi=\beta\}\subseteq\left\{\chi\in P_\lambda\colon \alpha^{-1}\cdot\alpha\cdot\chi=\alpha^{-1}\cdot\beta\right\}.$$

Then the definition of the semigroup I_λ implies there exist words $u,v\in\mathscr{M}_\lambda$ such that the partial map $\alpha^{-1}\cdot\beta$ is the map from $u\mathscr{M}_\lambda$ onto $v\mathscr{M}_\lambda$ defined by the formula $(ux)(\alpha^{-1}\cdot\beta)=vx$ for any $x\in\mathscr{M}_\lambda$. Since $\alpha^{-1}\cdot\alpha$ is an identity partial map of $\mathscr{M}_\lambda$ we get that the partial map $\alpha^{-1}\cdot\beta$ is a restriction of the partial map χ on the set $\operatorname{dom}(\alpha^{-1}\cdot\alpha)$. Hence by the definition of the semigroup I_λ there exists words $u_1,v_1\in\mathscr{M}_\lambda$ such that u_1 is a prefix of u, v_1 is a prefix of v and χ is the map from $u_1\mathscr{M}_\lambda$ onto $v_1\mathscr{M}_\lambda$ defined by the formula $(u_1x)(\alpha^{-1}\cdot\beta)=v_1x$ for any $x\in\mathscr{M}_\lambda$. Now, since every word of free monoid $\mathscr{M}_\lambda$ has finitely many prefixes we conclude that the set $\{\chi\in P_\lambda\colon \alpha^{-1}\cdot\alpha\cdot\chi=\alpha^{-1}\cdot\beta\}$ is finite, and hence so is $\{\chi\in P_\lambda\colon \alpha\cdot\chi=\beta\}$. $\square$

Later we need the following lemma.

Lemma 2.8. *Let λ be any cardinal $\geqslant 2$. Then an element x of the λ-polycyclic monoid P_λ is $\mathscr{R}$-equivalent to the identity 1 of P_λ if and only if $x=p_{i_1}\ldots p_{i_n}$ for some generators $p_{i_1},\ldots,p_{i_n}\in\{p_i\}_{i\in\lambda}$.*

Proof. We observe that the definition of the $\mathscr{R}$-relation implies that $x\mathscr{R}1$ if and only if $xx^{-1}=1$ (see [32, Section 3.2]).

($\Rightarrow$) Suppose that an element x of P_λ has a form $x = p_{i_1} \dots p_{i_n}$. Then the definition of the λ-polycyclic monoid P_λ implies that

$$xx^{-1} = (p_{i_1} \dots p_{i_n}) (p_{i_1} \dots p_{i_n})^{-1} = p_{i_1} \dots p_{i_n} p_{i_n}^{-1} \dots p_{i_1}^{-1} = 1,$$

and hence $x\mathscr{R}1$.

($\Leftarrow$) Suppose that some element x of the λ-polycyclic monoid P_λ is $\mathscr{R}$-equivalent to the identity 1 of P_λ. Then the definition of the semigroup P_λ implies that there exist finitely many $p_{i_1}, \dots, p_{i_n} \in \{p_i\}_{i\in\lambda}$ such that x is an element of the submonoid $P_n^\lambda \langle i_1, \dots, i_n \rangle$ of P_λ, which is generated by elements $p_{i_1}, \dots, p_{i_n}$, i.e.,

$$\begin{aligned} &P_n^\lambda \langle i_1, \dots, i_n \rangle \\ &\qquad = \left\langle p_{i_1}, \dots, p_{i_n}, p_{i_1}^{-1}, \dots, p_{i_n}^{-1} \colon p_{i_k} p_{i_k}^{-1} = 1,\ p_{i_k} p_{i_l}^{-1} = 0 \ \text{ for } \ i_k \neq i_l \right\rangle. \end{aligned}$$

Proposition 9.3.1 of [32] implies that the element x is equal to the unique string of the form $u^{-1}v$, where u and v are strings of the free monoid $\mathscr{M}_{\{p_{i_1},\dots,p_{i_n}\}}$ over the set $\{p_{i_1}, \dots, p_{i_n}\}$. Next we shall show that u is the empty string of $\mathscr{M}_{\{p_{i_1},\dots,p_{i_n}\}}$. Suppose that $u = a_1 \dots a_k$ and $v = b_1 \dots b_l$, for some $a_1, \dots, a_k, b_1, \dots, b_l \in \{p_{i_1}, \dots, p_{i_n}\}$ and u is not the empty-string of $\mathscr{M}_{\{p_{i_1},\dots,p_{i_n}\}}$. Then the definition of the λ-polycyclic monoid P_λ implies that

$$\begin{aligned} xx^{-1} &= \left(u^{-1}v\right)\left(u^{-1}v\right)^{-1} = u^{-1}vv^{-1}u \\ &= (a_1 \dots a_k)^{-1} (b_1 \dots b_l) (b_1 \dots b_l)^{-1} (a_1 \dots a_k) \\ &= a_k^{-1} \dots a_1^{-1} b_1 \dots b_l b_l^{-1} \dots b_1^{-1} a_1 \dots a_k \\ &\dots \\ &= a_k^{-1} \dots a_1^{-1} 1 a_1 \dots a_k \\ &= a_k^{-1} \dots a_1^{-1} a_1 \dots a_k \neq 1, \end{aligned}$$

which contradicts the assumption that $x\mathscr{R}1$. The obtained contradiction implies that the element x has the form $x = p_{i_1} \dots p_{i_n}$ for some generators $p_{i_1}, \dots, p_{i_n}$ from the set $\{p_i\}_{i\in\lambda}$. $\square$

3. On semigroup topologizations of the λ-polycyclic monoid

In [13] Eberhart and Selden proved that if τ is a Hausdorff topology on the bicyclic monoid $\mathscr{C}(p, q)$ such that $(\mathscr{C}(p, q), \tau)$ is a topological semigroup then τ is discrete. In [7] Bertman and West extended this results for the case when $(\mathscr{C}(p,q),\tau)$ is a Hausdorff semitopological semigroup. In [33] there proved that for any positive integer $n>1$ every non-zero element in a Hausdorff topological n-polycyclic monoid P_n is an isolated point. The following proposition generalizes the above results.

Proposition 3.1. *Let λ be any cardinal $\geqslant 2$ and τ be any Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup. Then every non-zero element x is an isolated point in (P_λ,τ).*

Proof. We observe that the λ-polycyclic monoid P_λ is a 0-bisimple semigroup, and hence is a 0-simple semigroup. Then the continuity of right and left translations in (P_λ,τ) and Proposition 2.7 imply that it is complete to show that there exists an non-zero element x of P_λ such that x is an isolated point in the topological space (P_λ,τ).

Suppose to the contrary that the unit 1 of the λ-polycyclic monoid P_λ is a non-isolated point of the topological space (P_λ,τ). Then every open neighbourhood $U(1)$ of 1 in (P_λ,τ) is infinite subset.

Fix a singleton word x in the free monoid $\mathscr{M}_\lambda$. Let ε be an idempotent of the λ-polycyclic monoid P_λ which corresponds to the identity partial map of $x\mathscr{M}_\lambda$. Since left and right translation on the idempotent ε are retractions of the topological space (P_λ,τ) the Hausdorffness of (P_λ,τ) implies that εP_λ and $P_\lambda\varepsilon$ are closed subsets of the topological space (P_λ,τ), and hence so is the set $\varepsilon P_\lambda\cup P_\lambda\varepsilon$. The separate continuity of the semigroup operation and Hausdorffness of (P_λ,τ) imply that for every open neighbourhood $U(\varepsilon)\not\ni 0$ of the point ε in (P_λ,τ) there exists an open neighbourhood $U(1)$ of the unit 1 in (P_λ,τ) such that

$$U(1)\subseteq P_\lambda\setminus(\varepsilon P_\lambda\cup P_\lambda\varepsilon), \qquad \varepsilon\cdot U(1)\subseteq U(\varepsilon) \quad \text{and} \quad U(1)\cdot\varepsilon\subseteq U(\varepsilon).$$

We observe that the idempotent ε is maximal in $P_\lambda\setminus\{1\}$. Hence any other idempotent $\iota\in P_\lambda\setminus(\varepsilon P_\lambda\cup P_\lambda\varepsilon)$ is incomparable with ε. Since the set $U(1)$ is infinite there exists an element $\alpha\in U(1)$ such that either $\alpha\cdot\alpha^{-1}$ or $\alpha^{-1}\cdot\alpha$ is an incomparable idempotent with ε. Then we get that either

$$\varepsilon\cdot\alpha=\varepsilon\cdot(\alpha\cdot\alpha^{-1}\cdot\alpha)=(\varepsilon\cdot\alpha\cdot\alpha^{-1})\cdot\alpha=0\cdot\alpha=0\in U(\varepsilon)$$

or

$$\alpha\cdot\varepsilon=(\alpha\cdot\alpha^{-1}\cdot\alpha)\cdot\varepsilon=\alpha\cdot(\alpha^{-1}\cdot\alpha\cdot\varepsilon)=\alpha\cdot 0=0\in U(\varepsilon).$$

The obtained contradiction implies that the unit 1 is an isolated point of the topological space (P_λ,τ), which completes the proof of our proposition. □

A topological space X is called *collectionwise normal* if X is T_1-space and for every discrete family $\{F_\alpha\}_{\alpha\in\mathscr{J}}$ of closed subsets of X there exists a discrete family $\{S_\alpha\}_{\alpha\in\mathscr{J}}$ of open subsets of X such that $F_\alpha \subseteq S_\alpha$ for every $\alpha \in \mathscr{J}$ [14].

Proposition 3.2. *Every Hausdorff topological space X with a unique non-isoloated point is collectionwise normal.*

Proof. Suppose that a is a non-isolated point of X. Fix an arbitrary discrete family $\{F_\alpha\}_{\alpha\in\mathscr{J}}$ of closed subsets of the topological space X. Then there exists an open neighbourhood $U(a)$ of the point a in X which intersects at most one element of the family $\{F_\alpha\}_{\alpha\in\mathscr{J}}$. In the case when $U(a) \cap F_\alpha = \varnothing$ for every $\alpha \in \mathscr{J}$ we put $S_\alpha = F_\alpha$ for all $\alpha \in \mathscr{J}$. If $U(a) \cap F_{\alpha_0} \neq \varnothing$ for some $\alpha_0 \in \mathscr{J}$ we put $S_{\alpha_0} = U(a) \cup F_{\alpha_0}$ and $S_\alpha = F_\alpha$ for all $\alpha \in \mathscr{J} \setminus \{\alpha_0\}$. Then $\{S_\alpha\}_{\alpha\in\mathscr{J}}$ is a discrete family of open subsets of X such that $F_\alpha \subseteq S_\alpha$ for every $\alpha \in \mathscr{J}$. □

Propositions 3.1 and 3.2 imply the following corollary.

Corollary 3.3. *Let λ be any cardinal $\geqslant 2$ and τ be any Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup. Then the topological space (P_λ, τ) is collectionwise normal.*

In [33] there proved that for arbitrary finite cardinal $\geqslant 2$ every Hausdorff locally compact topology τ on P_λ such that (P_λ, τ) is a topological semigroup, is discrete. The following proposition extends this result for any infinite cardinal λ.

Proposition 3.4. *Let λ be an infinite cardinal and τ be a locally compact Hausdorff topology on P_λ such that (P_λ, τ) is a topological semigroup. Then τ is discrete.*

Proof. Suppose to the contrary that there exist a Hausdorff locally compact non-discrete semigroup topology τ on P_λ. Then by Proposition 3.1 every non-zero element the semigroup P_λ is an isolated point in (P_λ, τ). This implies that for any compact open neighbourhoods $U(0)$ and $V(0)$ of zero 0 in (P_λ, τ) the set $U(0) \setminus V(0)$ is finite. Hence zero 0 of P_λ is an accumulation point of any infinite subset of an arbitrary open compact neighbourhood $U(0)$ of zero in (P_λ, τ).

Put R_1 is the $\mathscr{R}$-class of the semigroup P_λ which contains the identity 1 of P_λ. Then only one of the following conditions holds:

(1) there exists a compact open neighbourhood $U(0)$ of zero 0 in (P_λ,τ) such that $U(0)\cap R_1=\varnothing$;

(2) $U(0)\cap R_1$ is an infinite set for every compact open neighbourhood $U(0)$ of zero 0 in (P_λ,τ).

Suppose that case (1) holds. For arbitrary $x\in R_1$ we put

$$R[x]=\left\{a\in R_1\colon x^{-1}a\in U(0)\right\}.$$

Next we shall show that the set $R[x]$ is finite for any $x\in R_1$. Suppose to the contrary that $R[x]$ is infinite for some $x\in R_1$. Then Lemma 2.8 implies that $x^{-1}a$ is non-zero element of P_λ for every $a\in R[x]$, and hence by Proposition 2.7,

$$B=\left\{x^{-1}a\colon a\in R[x]\right\}$$

is an infinite subset of the neighbourhood $U(0)$. Therefore, the above arguments imply that $0\in \operatorname{cl}_{P_\lambda}(B)$. Now, the continuity of the semigroup operation in (P_λ,τ) implies that

$$0=x\cdot 0\in x\cdot\operatorname{cl}_{P_\lambda}(B)\subseteq \operatorname{cl}_{P_\lambda}(x\cdot B).$$

Then Lemma 2.8 implies that $xx^{-1}=1$ for any $x\in R_1$ and hence we have that

$$x\cdot B=\left\{xx^{-1}a\colon a\in R[x]\right\}=\{a\colon a\in R[x]\}=R[x]\subseteq R_1.$$

This implies that every open neighbourhood $U(0)$ of zero 0 in (P_λ,τ) contains infinitely many elements from the class R_1, which contradicts our assumption.

Suppose that case (2) holds. Then the set $\{0\}$ is a compact minimal ideal of the topological semigroup (P_λ,τ). Now, by Lemma 1 of [31] (also see [8, Vol. 1, Lemma 3,12]) for every open neighbourhood $W(0)$ of zero 0 in (P_λ,τ) there exists an open neighbourhood $O(0)$ of zero 0 in (P_λ,τ) such that $O(0)\subseteq W(0)$ and $O(0)$ is an ideal of $\operatorname{cl}_{P_\lambda}(O(0))$, i.e., $O(0)\cdot\operatorname{cl}_{P_\lambda}(O(0))\cup\operatorname{cl}_{P_\lambda}(O(0))\cdot O(0)\subseteq O(0)$. But by Proposition 3.1 all non-zero elements of P_λ are isolated points in (P_λ,τ), and hence we have that $\operatorname{cl}_{P_\lambda}(O(0))=O(0)$. This implies that $O(0)$ is an open-and-closed subsemigroup of the topological semigroup (P_λ,τ). Therefore, the topological λ-polycyclic monoid (P_λ,τ) has a base $\mathscr{B}(0)$ at zero 0 which consists of open-and-closed subsemigroups of (P_λ,τ). Fix an arbitrary $S\in\mathscr{B}(0)$. Then our assumption implies that there exists $x\in S\cap R_1$. Since $x\in R_1$, Lemma 2.8 implies that $xx^{-1}=1$. Without loss of generality we may assume that $x^{-1}x \neq 1$, because S is a proper ideal of P_λ. Put $\mathbb{B}(x) = \langle x, x^{-1}\rangle$. Then Lemma 1.31 of [11] implies that $\mathbb{B}(x)$ is isomorphic to the bicyclic monoid, and since by Proposition 3.1 all non-zero elements of P_λ are isolated points in (P_λ, τ), $\mathbb{B}^0(x) = \mathbb{B}(x) \sqcup \{0\}$ is a closed subsemigroup of the topological semigroup (P_λ, τ), and hence by Corollary 3.3.10 of [14], $\mathbb{B}^0(x)$ with the induced topology $\tau_{\mathbb{B}}$ from (P_λ, τ) is a Hausdorff locally compact topological semigroup. Also, the above presented arguments imply that $\langle x\rangle \cup \{0\}$ with the induced topology from (P_λ, τ) is a compact topological semigroup, which is contained in $\mathbb{B}^0(x)$ as a subsemigroup. But by Corollary 1 from [19], $(\mathbb{B}^0(x), \tau_{\mathbb{B}})$ is the discrete space, which contains a compact infinite subspace $\langle x\rangle \cup \{0\}$. Hence case (2) does not hold.

The presented above arguments imply that there exists no non-discrete Hausdorff locally compact semigroup topology on the λ-polycyclic monoid P_λ. □

The following example shows that the statements of Proposition 3.4 does not extend in the case when (P_λ, τ) is a semitopological semigroup with continuous inversion. Moreover there exists a compact Hausdorff topology $\tau_{\mathsf{A\text{-}c}}$ on P_λ such that $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$ is semitopological inverse semigroup with continuous inversion.

Example 3.5. Let λ is any cardinal $\geqslant 2$. Put $\tau_{\mathsf{A\text{-}c}}$ is the topology of the one-point Alexandroff compactification of the discrete space $P_\lambda \setminus \{0\}$ with the narrow $\{0\}$, where 0 is the zero of the λ-polycyclic monoid P_λ. Since $P_\lambda \setminus \{0\}$ is a discrete open subspace of $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$, it is complete to show that the semigroup operation is separately continuous in $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$ in the following two cases:

$$x \cdot 0 \quad \text{and} \quad 0 \cdot x,$$

where x is an arbitrary non-zero element of the semigroup P_λ. Fix an arbitrary open neighbourhood $U_A(0)$ of the zero in $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$ such that $A = P_\lambda \setminus U_A(0)$ is a finite subset of P_λ. By Proposition 2.7,

$$R_x^A = \{a \in P_\lambda \colon x \cdot a \in A\} \quad \text{and} \quad L_x^A = \{a \in P_\lambda \colon a \cdot x \in A\}$$

are finite not necessary non-empty subsets of the semigroup P_λ. Put $U_{R_x^A}(0) = P_\lambda \setminus R_x^A$, $U_{L_x^A}(0) = P_\lambda \setminus L_x^A$ and $U_{A^{-1}} = P_\lambda \setminus \{a \colon a^{-1} \in A\}$. Then we get that

$$x \cdot U_{R_x^A}(0) \subseteq U_A(0), \quad U_{L_x^A}(0) \cdot x \subseteq U_A(0) \quad \text{and} \quad (U_{A^{-1}})^{-1} \subseteq U_A(0),$$

and hence the semigroup operation is separately continuous and the inversion is continuous in $(P_\lambda, \tau_{\mathsf{A\text{-}c}})$.

Proposition 3.6. *Let λ is any cardinal $\geqslant 2$ and τ be a Hausdorff topology on P_λ such that (P_λ, τ) is a semitopological semigroup. Then the following conditions are equivalent:*

(i) $\tau = \tau_{\mathsf{A\text{-}c}}$*;*
(ii) (P_λ, τ) *is a compact semitopological semigroup;*
(iii) (P_λ, τ) *is a feebly compact semitopological semigroup.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial and implication $(ii) \Rightarrow (i)$ follows from Proposition 3.1.

$(iii) \Rightarrow (ii)$ Suppose there exists a feebly compact Hausdorff topology τ on P_λ such that (P_λ, τ) is a non-compact semitopological semigroup. Then there exists an open cover $\{U_\alpha\}_{\alpha\in\mathscr{J}}$ which does not contain a finite subcover. Let U_{α_0} be an arbitrary element of the family $\{U_\alpha\}_{\alpha\in\mathscr{J}}$ which contains zero 0 of the semigroup P_λ. Then $P_\lambda \setminus U_{\alpha_0} = A_{U_{\alpha_0}}$ is an infinite subset of P_λ. By Proposition 3.1, $\{U_{\alpha_0}\} \cup \left\{\{x\} \colon x \in A_{U_{\alpha_0}}\right\}$ is an infinite locally finite family of open subset of the topological space (P_λ, τ), which contradicts that the space (P_λ, τ) is feebly compact. The obtained contradiction implies the requested implication. □

It is well known that the closure $\mathrm{cl}_S(T)$ of an arbitrary subsemigroup T in a semitopological semigroup S again is a subsemigroup of S (see [37, Proposition I.1.8(ii)]). The following proposition describes the structure of a narrow of the λ-polycyclic monoid P_λ in a semitopological semigroup.

Proposition 3.7. *Let λ is any cardinal $\geqslant 2$, S be a Hausdorff semitopological semigroup and P_λ is a dense subsemigroup of S. Then $S \setminus P_\lambda \cup \{0\}$ is a closed ideal of S.*

Proof. First we observe by Proposition I.1.8(iii) from [37] the zero 0 of the λ-polycyclic monoid P_λ is a zero of the semitopological semigroup S. Hence the statement of the proposition is trivial when $S \setminus P_\lambda = \varnothing$.

Assume that $S \setminus P_\lambda \neq \varnothing$. Put $I = S \setminus P_\lambda \cup \{0\}$. By Theorem 3.3.9 of [14], I is a closed subspace of S. Suppose to the contrary that I is not an ideal of S. If $I \cdot S \nsubseteq I$ then there exist $x \in I \setminus \{0\}$ and $y \in P_\lambda \setminus \{0\}$ such that $x \cdot y = z \in P_\lambda \setminus \{0\}$. By Theorem 3.3.9 of [14], y and z are isolated points of the topological space S. Then the separate continuity of the semigroup operation in S implies that there exists an open neighbourhood $U(x)$ of the point x in S such that $U(x) \cdot \{y\} = \{z\}$. Then we get that $|U(x) \cap P_\lambda| \geqslant \omega$

which contradicts Proposition 2.7. The obtained contradiction implies the inclusion $I \cdot S \subseteq I$. The proof of the inclusion $S \cdot I \subseteq I$ is similar.

Now we shall show that $I \cdot I \subseteq I$. Suppose to the contrary that there exist $x, y \in I \setminus \{0\}$ such that $x \cdot y = z \in P_\lambda \setminus \{0\}$. By Theorem 3.3.9 of [14], z is an isolated point of the topological space S. Then the separate continuity of the semigroup operation in S implies that there exists an open neighbourhood $U(x)$ of the point x in S such that $U(x) \cdot \{y\} = \{z\}$. Since $|U(x) \cap P_\lambda| \geqslant \omega$ there exists $a \in P_\lambda \setminus \{0\}$ such that $a \cdot y \in a \cdot I \nsubseteq I$ which contradicts the above part of our proof. The obtained contradiction implies the statement of the proposition. □

4. Embeddings of the λ-polycyclic monoid into compact-like topological semigroups

By Theorem 5 of [23] the semigroup of $\omega\times\omega$-matrix units does not embed into any countably compact topological semigroup. Then by Proposition 2.6 we have that for every cardinal $\lambda \geqslant 2$ the λ-polycyclic monoid P_λ does not embed into any countably compact topological semigroup too.

A homomorphism $\mathfrak{h}$ from a semigroup S into a semigroup T is called *annihilating* if there exists $c \in T$ such that $(s)\mathfrak{h} = c$ for all $s \in S$. By Theorem 6 of [23] every continuous homomorphism from the semigroup of $\omega\times\omega$-matrix units into an arbitrary countably compact topological semigroup is annihilating. Then since by Theorem 2.5 the semigroup P_λ is congruence-free Theorem 6 of [23] and Theorem 2.5 imply the following corollary.

Corollary 4.1. *For every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating.*

Proposition 4.2. *For every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into a topological semigroup S such that $S \times S$ is a Tychonoff pseudocompact space is annihilating, and hence S does not contain the λ-polycyclic monoid P_λ.*

Proof. First we shall show that S does not contain the λ-polycyclic monoid P_λ. By [4, Theorem 1.3] for any topological semigroup S with the pseudocompact square $S \times S$ the semigroup operation $\mu\colon S \times S \to S$ extends to a continuous semigroup operation $\beta\mu\colon \beta S \times \beta S \to \beta S$, so S is a subsemigroup of the compact topological semigroup βS. Therefore the λ-polycyclic monoid P_λ is a subsemigroup of compact topological semigroup βS which contradicts Corollary 4.1. The first statement of the proposition implies from the statement that P_λ is a congruence-free semigroup. $\square$

Recall [12] that a *Bohr compactification of a topological semigroup S* is a pair $(\beta, B(S))$ such that $B(S)$ is a compact topological semigroup, $\beta\colon S \to B(S)$ is a continuous homomorphism, and if $g\colon S \to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f\colon B(S) \to T$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[d, "g"'] & B(S) \arrow[dl, "f"] \\ T & \end{tikzcd}$$

commutes.

By Theorem 2.5 for every infinite cardinal λ the polycyclic monoid P_λ is a congruence-free inverse semigroup and hence Corollary 4.1 implies the following corollary.

Corollary 4.3. *For every cardinal $\lambda \geqslant 2$ the Bohr compactification of a topological λ-polycyclic monoid P_λ is a trivial semigroup.*

The following theorem generalized Theorem 5 from [23].

Theorem 4.4. *For every infinite cardinal λ the semigroup of $\lambda\times\lambda$-matrix units B_λ does not densely embed into a Hausdorff feebly compact topological semigroup.*

Proof. Suppose to the contrary that there exists a Hausdorff feebly compact topological semigroup S which contains the semigroup of $\lambda\times\lambda$-matrix units B_λ as a dense subsemigroup.

First we shall show that the subsemigroup of idempotents $E(B_\lambda)$ of the semigroup $\lambda\times\lambda$-matrix units B_λ with the induced topology from S is compact. Suppose to the contrary that $E(B_\lambda)$ is not a compact subspace of S. Then there exists an open neighbourhood $U(0)$ of the zero 0 of S such that $E(B_\lambda) \setminus U(0)$ is an infinite subset of $E(B_\lambda)$. Since the closure of semilattice in a topological semigroup is subsemilattice (see [21, Corollary 19]) and every maximal chain of $E(B_\lambda)$ is finite, Theorem 9 of [38] implies that the band $E(B_\lambda)$ is a closed subsemigroup of S. Now, by Lemma 2 from [22] every non-zero element of the semigroup B_λ is an isolated point in the space S, and hence by Theorem 3.3.9 of [14], $B_\lambda \setminus \{0\}$ is an open discrete subspace of the topological space S. Therefore we get that $E(B_\lambda) \setminus U(0)$ is an infinite open-and-closed discrete subspace of S. This contradicts the condition that S is a feebly compact space.

If the subsemigroup of idempotents $E(B_\lambda)$ is compact then by Theorem 1 from [23] the semigroup of $\lambda\times\lambda$-matrix units B_λ is closed subsemigroup of S and since B_λ is dense in S, the semigroup B_λ coincides with the topological semigroup S. This contradicts Theorem 2 of [22] which states that there exists no a feebly compact Hausdorff topology τ on the semigroup of $\lambda\times\lambda$-matrix units B_λ such that (B_λ, τ) is a topological semigroup. The obtained contradiction implies the statement of the theorem. □

Lemma 4.5. *Every Hausdorff feebly compact topological space with a dense discrete subspace is countably pracompact.*

Proof. Suppose to the contrary that there exists a feebly compact topological space X with a dense discrete subspace D such that X is not countably pracompact. Then every dense subset A in the topological space X contains an infinite subset B_A such that B_A hasn't an accumulation point in X. Hence the dense discrete subspace D of X contains an infinite subset B_D such that B_D hasn't an accumulation point in the topological space X. Then B_D is a closed subset of X. By Theorem 3.3.9 of [14], D is an open subspace of X, and hence we have that B_D is a closed-and-open discrete subspace of the space X, which contradicts the feeble compactness of the space S. The obtained contradiction implies the statement of the lemma. □

Theorem 4.6. *For arbitrary cardinal $\lambda \geqslant 2$ there exists no Hausdorff feebly compact topological semigroup which contains the λ-polycyclic monoid P_λ as a dense subsemigroup.*

Proof. By Proposition 3.1 and Lemma 4.5 it is suffices to show that there does not exist a Hausdorff countably pracompact topological semigroup which contains the λ-polycyclic monoid P_λ as a dense subsemigroup.

Suppose to the contrary that there exists a Hausdorff countably pracompact topological semigroup S which contains the λ-polycyclic monoid P_λ as a dense subsemigroup. Then there exists a dense subset A in S such that every infinite subset $B \subseteq A$ has an accumulation point in the topological space S. By Proposition 3.1, $P_\lambda \setminus \{0\}$ is a discrete dense subspace of S and hence Theorem 3.3.9 of [14] implies that $P_\lambda \setminus \{0\}$

is an open subspace of S. Therefore we have that $P_\lambda \setminus \{0\} \subseteq A$. Now, by Proposition 2.6 the λ-polycyclic monoid P_λ contains an isomorphic copy of the semigroup of $\omega\times\omega$-matrix units B_ω. Then the countable pracompactness of the space S implies that every infinite subset C of the set $B_\omega\{0\}$ has an accumulating point in X, and hence the closure $\mathrm{cl}_S(B_\omega)$ is a countably pracompact subsemigroup of the topological semigroup S. This contradicts Theorem 4.4. The obtained contradiction implies the statement of the theorem. □

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Representation of Steinitz's lattice in lattices of substructures of relational structures

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Abstract. General conditions under which certain relational structure contains a lattice of substructures isomorphic to Steinitz's lattice are formulated. Under some natural restrictions we consider relational structures with the lattice containing a sublattice isomorphic to the lattice of positive integers with respect to divisibility. We apply to this sublattice a construction that could be called "lattice completion". This construction can be used for different types of relational structures, in particular for universal algebras, graphs, metric spaces etc. Some examples are considered.

1. Introduction

Steinitz's lattice was introduced at the beginning of the XX century by German mathematician A.Steinitz for describing the structure of subfields of algebraically closed field of prime characteristic [1]. It can be determined as the lattice of supernatural numbers with a relation of the divisibility. Steinitz's lattice is complete, i.e. for an arbitrary subset of its elements exists the exact lower and the exact upper bounds. It contains a various sublattices including Boolean algebras. So, an existence of such lattice in the lattice of substructures of mathematical structure shows its inner richness that may be a basis for the use of the structure as an universal object for corresponding class of structures of the same type.

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In the paper we formulate the general conditions in which certain relational structure has some lattice of substructures, which is isomorphic to Steinitz's lattice. If these conditions are satisfied, then the isomorphic representation of Steinitz's lattice in a lattice of substructures of this relational structure is obtained. Under certain natural restrictions it is enough to view structures with the lattice containing sublattice, that is isomorphic to the lattice of positive integers with divisibility. We apply to this sublattice a construction that could be called "lattice completion". This construction can be used to different types of relational structures, in particular — universal algebra, graphs, metric spaces etc. For example, Steinitz's lattice is isomorphic to:

(i) the lattice of all subfields of algebraic closure of finite field (see, e.g., [2]);
(ii) the lattice of Glimm's subalgebras of limited C^*-algebra (see, e.g., [3], [4]);
(iii) lattices of so-called homogeneously symmetric and homogeneously alternating subgroups of symmetric groups of permutations of natural numbers (see, e.g., [5], [6]).

Furthermore, some elements of Steinitz's lattice of substructures in certain relational structures were studied in many works of various authors (see, e.g., [7]–[10]).

The paper is organized as follows. In Section 2 we give the definition of Steinitz's lattice and its basic properties. In Section 3 we present basic information on the theory of relational structures. In Section 4 basic construction of "lattice completion" of relational structures is described. It is shown how to build the certain isomorphism between Steinitz's lattice and a lattice that occurs as a result of "lattice completion". In Section 5 we give examples of an application of the construction of "lattice completion" in the theory of infinite groups and semigroups transformations. This makes it possible to introduce new objects namely, groups and semigroups of periodically defined transformations of natural numbers. Last section describes lattices of subspaces of Besicovitch's space which are constructed by the use of "lattice completion", and therefore are isomorphic to Steinitz's lattice.

Results of the article were earlier partially announced in the article [10] of third author. All symbols are commonly used in the paper. For determination of indefinite terms we refer readers to [11]–[13].

2. Steinitz's lattice

2.1. Let $\mathbb{N}$ be a set of natural numbers and let $\mathbb{P}$ be its subset of primes.

Definition 1. Supernatural number (or Steinitz's number) is called a formal product

$$\prod_{p\in\mathbb{P}} p^{k_p}, \quad k_p \in \mathbb{N} \cup \{0, \infty\}. \tag{1}$$

Denote by the symbol $\mathbb{SN}$ the set of all supernatural numbers. Every natural number is a supernatural number, so that $\mathbb{N} \subset \mathbb{SN}$. Numbers from $\mathbb{SN} \setminus \mathbb{N}$ will be called infinite supernatural numbers. Divisible relation $|$ on $\mathbb{N}$ in natural way is extended to $\mathbb{SN}$. Namely, for arbitrary supernatural numbers

$$u = \prod_{p\in\mathbb{P}} p^{k_p}, \quad v = \prod_{p\in\mathbb{P}} p^{l_p}, \quad k_p, l_p \in \mathbb{N} \cup \{0, \infty\}, \tag{2}$$

we get $u \mid v$ if and only if for all $p \in \mathbb{P}$ inequalities $k_p \leqslant l_p$ hold (it is assumed that ∞ is more than zero and all natural numbers). Main properties of set $\mathbb{SN}$, ordered by the divisible relation $|$, are characterized by the following lemma.

Lemma 1. *The ordered set* $(\mathbb{SN}, |)$ *is a lattice. The lattice* $(\mathbb{SN}, |)$ *is a complete one and contains the largest and the smallest elements, that accordingly are such supernatural numbers*

$$\mathbb{I} = \prod_{p\in\mathbb{P}} p^{\infty} \qquad 1 = \prod_{p\in\mathbb{P}} p^{0}. \tag{3}$$

The proof of this statement is not difficult.

The exact lower and the exact upper bounds of supernatural numbers u, v, that are given by their decompositions (2), are defined by the equalities

$$u \vee v = \prod_{p\in\mathbb{P}} p^{\max(k_p, l_p)} \tag{4}$$

$$u \wedge v = \prod_{p\in\mathbb{P}} p^{\min(k_p, l_p)} \tag{5}$$

where $\max(k, \infty) = \infty$, $\min(k, \infty) = k$ for $k \in \mathbb{N} \cup \{0\}$.

Definition 2. Lattice $(\mathbb{SN}, \wedge, \vee)$ will be called Steinitz's lattice.

The following lemma follows from equations (4), (5).

Lemma 2. *Steinitz's lattice is a complete distributive lattice.*

In the set of supernatural numbers we select two subsets.

Definition 3. Supernatural number $u = \prod_{p\in\mathbb{P}} p^{k_p}$ is called complete, if for each $p \in \mathbb{P}$ there is inclusion $k_p \in \{0, \infty\}$.

According to the definition a complete supernatural number u is uniquely determined by the subset $\mathcal{O}(u)$ of that primes p from $\mathbb{P}$, for that $k_p = \infty$. The set $\mathcal{C}$ of complete supernatural numbers is closed on the operations $\vee$, $\wedge$ and contains the numbers $\mathbb{I}$ and 1 that is defined by (3). Moreover, on the set $\mathcal{C}$ one can define the operation of addition, namely the addition $\bar{u}$ of the number $u \in \mathcal{C}$ is called complete supernatural number that is determined by subset $\mathcal{O}(\bar{u}) = \mathbb{P} \setminus \mathcal{O}(u)$. It is obvious, that $u \vee \bar{u} = \mathbb{I}$, $u \wedge \bar{u} = 1$.

Lemma 3. *The set* $\mathcal{C}$ *with the operations* $\vee, \wedge, {}^{-}$ *forms a Boolean algebra with* 1 *as a zero element and* $\mathbb{I}$ *as an unit element. The Boolean algebra* $(\mathcal{C}, \vee, \wedge, {}^{-})$ *is isomorphic to the algebra of subsets of a countable set.*

Proof. Define the mapping φ from the algebra of subsets of the set $\mathbb{P}$ to the algebra $\mathcal{C}$ in such way. For any subset $\mathcal{X} \subset \mathbb{P}$ put $\varphi(\mathcal{X}) = u$, were $\mathcal{O}(u) = \mathcal{X}$. From mentioned above it follows that φ is a bijection. In addition, for any $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{P}$ for which $\varphi(\mathcal{X}_1) = u_1$, $\varphi(\mathcal{X}_2) = u_2$ we have

$$\varphi(\mathcal{X}_1 \cup \mathcal{X}_2) = u_1 \vee u_2, \quad \varphi(\mathcal{X}_1 \cap \mathcal{X}_2) = u_1 \wedge u_2, \quad \varphi(\overline{\mathcal{X}}_1) = \overline{u}_1.$$

So, the mapping φ is an isomorphism, and this, in particular, means that $(\mathcal{C}, \vee, \wedge, {}^{-})$ is a Boolean algebra. $\square$

Definition 4. A supernatural number is called notsquare one, if indicators of powers in its canonical decomposition (1) takes on only two values $0, 1$.

Let $\mathcal{B}$ be a set of all notsquare supernatural numbers. It is clear, that $\mathcal{B}$ is closed regarding to the operations $\vee$ and $\wedge$. Moreover, on $\mathcal{B}$ can also be defined the complement operation by the rule: for the number $u = \prod_{p\in\mathcal{X}} p$, we define

$$\overline{u} = \prod_{p\in\mathbb{P}\setminus\mathcal{X}} p.$$

The number $\mathrm{J} = \prod_{p\in\mathbb{P}} p$ is the largest element in the set $\mathcal{B}$.

Lemma 4. *Sublattice* $\mathcal{B}$ *of the lattice* $(\mathbb{SN}, \wedge, \vee)$ *with the additional operation, namely complement one, forms a Boolean algebra, the largest element of which is number* J*, and the least element is* 1*. The algebra* $(\mathcal{B}, \wedge, \vee, {}^{-})$ *is isomorphic to the subalgebra of subsets of the set* $\mathbb{P}$*.*

Proof. Define the mapping, which puts the subset $\mathcal{X}$ in correspondence to the number $\prod_{p\in\mathcal{X}} p$. This mapping is an isomorphism. $\square$

2.2. The sequence of positive integers $\chi = \langle k_1, k_2, \ldots\rangle$ will be called divisible if $k_i \mid k_{i+1}$ for $i = 1, 2, \ldots$. Let DS be a set of the most possible of divisible sequences over $\mathbb{N}$.

Definition 5. The sequence $\chi = \langle k_i\rangle_{i\in\mathbb{N}}$ divides the sequence $\chi' = \langle k'_i\rangle_{i\in\mathbb{N}}$, if for any $i \in \mathbb{N}$ there are $j \in \mathbb{N}$ for which $k_i \mid k'_j$.

Let $\mid$ denote the divisibility of sequences. The relation $\mid$ on DS is:

(i) reflexive, i.e. $\chi|\chi$ for arbitrary sequence $\chi \in DS$;

(ii) transitive, i.e. from $\chi_1|\chi_2$ and $\chi_2|\chi_3$ follows $\chi_1|\chi_3$ for any $\chi_1, \chi_2, \chi_3 \in DS$.

But the relation of divisibility is not symmetric or antisymmetric relation.

Definition 6. Sequences χ and χ' are called exactly divisible if at the same time $\chi|\chi'$ and $\chi'|\chi$.

The exactly divisible relation is equivalence on DS, which we denote by the symbol $\sim$. An arbitrary sequence $\chi \in DS$ determine a supernatural number $\operatorname{char}\chi$ (characteristic χ), which is defined thus

(i) each member of the sequence χ be a divisor of $\operatorname{char}\chi$;

(ii) every natural divisor of $\operatorname{char}\chi$ be a divisor of some member of the sequence χ.

For example, if $\chi = \langle 1, p, p^2, \ldots\rangle$, $p \in \mathbb{P}$, then $\operatorname{char}\chi = p^\infty$, and when $\chi = \langle 1, 2!, 3!, 4!, \ldots\rangle$, then $\operatorname{char}\chi = \mathbb{I}$. From the definition of characteristic we get easy

Lemma 5. 1) *For arbitrary $\chi_1, \chi_2 \in DS$ the divisibility $\chi_1|\chi_2$ holds if and only if* $\operatorname{char}\chi_1|\operatorname{char}\chi_2$.

2) *The sequences $\chi_1, \chi_2 \in DS$ are exactly divisible if and only if when* $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$.

So, sets of exactly divisible sequences are characterized by supernatural numbers, moreover the correspondence between these classes of objects is a bijective.

3. Relational structures

3.1. Recall that n-arity relation over the set A is called an arbitrary subset of Cartesian degree A^n. A relational structure over the set A is called an a pair of the type

$$\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I}\rangle, \tag{6}$$

where I is some set of indices, and for every $\alpha \in I$ $\Phi_\alpha^{k_\alpha}$ is some relation of the arity k_α over A ($k_\alpha \in \mathbb{N} \cup \{0\}$). The set A is called a support of the relational structure $\Re$, the set $\{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I}$ is called its signature, and the set $(k_\alpha)_{\alpha \in I}$ is called its type.

Examples

1) Every directed graph without multiple edges with the set of vertices V and the set of edges $E \subset V \times V$ is a relational structure (V, E) with the support V and the signature E of the type (2).

2) Every colored graph with the set of vertices V, whose edges are colored in k colors, is a relational structure with the support V and the signature E_1, …, E_k of the type $(\underbrace{2, \ldots, 2}_{k})$.

3) Every metric space (X, d) with the set I of values of the metric is a relational structure $\langle X, \{D_\alpha\}_{\alpha \in I}\rangle$, where

$$D_\alpha = \{(x, y) \mid x, y \in X, d(x, y) = d(y, x) = \alpha\}.$$

This relational structure has the type $(2_\alpha)_{\alpha \in I}$.

4) Every universal algebra

$$\pounds = \langle A, \{\varphi_\alpha^{k_\alpha}\}_{\alpha \in \mathrm{I}}\rangle, \quad \text{where} \quad \varphi_\alpha^{k_\alpha} : A^{k_\alpha} \to A$$

is an operation of the arity k_α on A, can seen as a relational structure of the form (6) of the type $(k_\alpha + 1)_{\alpha \in I}$, where

$$\Phi_\alpha^{k_\alpha + 1}(x_1, \ldots, x_{k_\alpha}, y)$$

occurs if and only if, then $\varphi_\alpha^{k_\alpha}(x_1, \ldots, x_{k_\alpha}) = y$.

The relational structures

$$\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I}\rangle \quad \text{and} \quad \Re' = \langle B, \{\Psi_\beta^{l_\beta}\}_{\beta \in J}\rangle$$

have the same type, if the sets I and J have the same cardinality and there is a bijection $f : I \leftrightarrow J$ so that for every $\alpha \in I$ the equality $k_\alpha = l_{f(\alpha)}$ holds.

Let the relational structures $\Re$ and $\Re'$ have the same type. The bijection $F : A \to B$ is called an isomorphism of these structures, if for any $\alpha \in I$ the relations

$$\Phi_\alpha^{k_\alpha}(x_1, x_2, \ldots, x_{k_\alpha}), \quad x_1, x_2, \ \ldots, x_{k_\alpha} \in A,$$
$$\text{and} \quad \Psi_{f(\alpha)}^{l_{f(\alpha)}}(F(x_1), F(x_2), \ldots, F(x_{k_\alpha}))$$

are isomorphic.

In particular, an isomorphism of universal algebras or graphs as relational structures is their isomorphism in the usual sense, but an isomorphism of relational structures related to metric spaces means that these spaces are isometric.

An isomorphism of relational structure itself is called an automorphism. All automorphisms of relational structure $\Re$ form a group with the operation of superposition of automorphisms, which denoted by the symbol $\mathrm{Aut}\Re$ and named the group of automorphisms of the structure $\Re$.

Let A' be an arbitrary nonempty subset of the set A. For the subset A' we can consider restriction $\Phi_\alpha^{k_\alpha}|_{A'}$ of the relation $\Phi_\alpha^{k_\alpha}$ $(\alpha \in I)$ on the set A':

$$\Phi_\alpha^{k_\alpha}|_{A'}(x_1, x_2, \ldots, x_{k_\alpha}) = \Phi_\alpha^{k_\alpha}(x_1, x_2, \ldots, x_{k_\alpha}) \cap (A')^{k_\alpha}.$$

Note, that in this case some restrictions $\Phi_\alpha^{k_\alpha}|_{A'}$ may be equal to empty relations.

The relational structure $\langle A', \{\Phi_\alpha^{k_\alpha}|_{A'}\}_{\alpha\in I}\rangle$ is called *a substructure* of the relational structure $\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha\in I}\rangle$.

An isomorphism of $\Re$ onto some substructure of the structure $\Re'$ is called isomorphic emmbeding of the structure $\Re$ in the structure $\Re'$.

3.2. The partially ordered set $(I, \geqslant)$ is called a directed to the right, if for any $a, b \in I$ there exist such element $c \in I$, that $a \leqslant c$, $b \leqslant c$.

Definition 7. The family of structures $\{\Re_i\}_{i\in I}$ and embedding $f_{i,k} : \Re_i \to \Re_k$, $i, k \in I$, $i \leqslant k$, satisfying the following requirements:

1) I be a directed to the right partially ordered set;
2) for arbitrary indices $i, k \in I$, $i \leqslant k$, there exist emmbedings $f_{i,k} : \Re_i \to \Re_k$, where $f_{i,i}$ be identity isomorphism;
3) if $i, j, k \in I$ and $i \leqslant j \leqslant k$, then $f_{i,j} \cdot f_{j,k} = f_{i,k}$,

is called an inductive family Σ of relational structures over a set of indices I.

Every inductive family of relational structures

$$\Sigma = \langle \Re_i, f_{i,k} \rangle_{i,k \in I}$$

defines the limit structure which is called inductive limit of family Σ and is denoted by the symbol

$$\Re(\Sigma) = \varinjlim_{i} (\Re_i, f_{i,k}), \quad i, k \in I. \tag{7}$$

Elements of the structure (7) are the so-called strings, the relation $\Phi_\alpha^{k_\alpha}$ extends to their by the standard way ([13], pp. 151–156).

We will apply the construction of an inductive limit in a special case when the index set I be the set of positive integers with the natural order. In this case, the inductive family is the sequence $\Re_1$, $\Re_2$, …, and morphisms $f_{i,k}$ will be defined as compositions of morphisms $f_i = f_{i,i+1}$ $(i, k \in \mathbb{N})$. Moreover, sequences of the type $u = a_k a_{k+1} \ldots$ $(k \in \mathbb{N})$ are strings, if the following conditions hold:

(i) $a_i \in A_i$ $(i \geqslant k)$;
(ii) $f_i(a_i) = a_{i+1}$ $(i \geqslant k)$;
(iii) there is no an element $a_{k-1} \in A_{k-1}$, for which $f_{k-1}(a_{k-1}) = a_k$.

Let $(u_i = a_{l_i}^{(i)} a_{l_i+1}^{(i)} \ldots,\ 1 \leqslant i \leqslant k_\alpha)$, and let $\Phi_\alpha^{k_\alpha}$ be a relation from the signature of relational structures $\Re_i$, $i \in \mathbb{N}$. From the definition follows that the tuple of strings $(u_1, u_2, \ldots, u_{k_\alpha})$ is in the relation $\Phi_\alpha^{k_\alpha}$ if and only if for $l \geqslant \max\{l_1, l_2, \ldots, l_{k_\alpha}\}$ the tuples $(a_l^{(1)}, a_l^{(2)}, \ldots, a_l^{(k_\alpha)})$ is in this relation. Let the subset $\Re^{(k)}$ of the support of the structure $\Re(\Sigma)$, be the set of all strings that was began with the elements from A_l, $l \leqslant k$. Then subset $\Re^{(k)}$ determines a substructure of the structure $\Re(\Sigma)$, and the equality

$$\Re(\Sigma) = \bigcup_{k=1}^{\infty} \Re^{(k)}$$

takes place.

4. Construction of "lattice completion"

Let $\Re = \langle A, \{\Phi_\alpha^{k_\alpha}\}_{\alpha \in I} \rangle$ be relational structures with the signature $(k_\alpha)_{\alpha \in I}$. Suppose that for every natural number n the structure $\Re$ has the only one its own substructure $\Re(n)$, moreover for the family of substructures $\langle \Re(n), n \in \mathbb{N} \rangle$ following conditions hold:

(A) if $n_1 \neq n_2$, then $\Re(n_1) \neq \Re(n_2)$;

(B) the inclusion $\Re(n_1) \subseteq \Re(n_2)$ is satisfied if and only if, when $n_1|n_2$;

It follows from properties (A), (B) that family of substructures $\Re(n)$, $n \in \mathbb{N}$, forms a lattice under the inclusion. A correspondence $n \leftrightarrow \Re(n)$, $n \in \mathbb{N}$, is an isomorphism between this lattice and lattice $(\mathbb{N}, |)$ of natural numbers.

Using the family $\Re(n)$, $n \in \mathbb{N}$ of substructures we define a new family from $\Re$ as follows. Let $\chi = \langle n_1, n_2, \ldots\rangle \in DS$ be an arbitrary divisible sequence of natural number. Construct now by sequence χ a growing chain of substructures of structure $\Re$ of the form

$$\Re(n_1) \subseteq \Re(n_2) \subseteq \cdots .$$

If sequence χ is bounded, then

$$\Re(\chi) = \bigcup_{i=1}^{\infty} \Re(n_i)$$

coincides with one of substructures $\Re(n_k)$, $k \in \mathbb{N}$. Therefore it suffices to consider only divisible unbounded sequences. Suppose that for any χ as unbounded divisible sequences the family of substructures $\Re(\chi)$ satisfies the following conditions:

(C) union $\Re(\chi)$ is its own substructure of structure $\Re$;

(D) $\Re(\chi)$ does not coincide with any substructures $\Re(n)$, $n \in \mathbb{N}$.

Definition 8. The family of substructures $\Re(\chi)$, $\chi \in DS$, of structure $\Re$ will be called the lattice completion of lattice $\Re(n)$, $n \in \mathbb{N}$.

Since DS contains arbitrary bounded divisible sequences, the family $\Re(\chi)$, $\chi \in DS$, contains sublattice $\Re(n)$, $n \in \mathbb{N}$.

Theorem 1. *Suppose, that the family of substructures $\Re(n)$, $n \in \mathbb{N}$, of relational structure $\Re$ satisfies properties (A)–(D). Then*

(i) *substructures $\Re(\chi_1)$ and $\Re(\chi_2)$, χ_1, $\chi_2 \in DS$, coincide if and only if* $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$;

(ii) *the family of substructures $\Re(\chi)$, $\chi \in DS$, forms a lattice under the inclusion, which is isomorphic to Steinitz's lattice.*

Proof. (i) Suppose $\Re(\chi_1)$ and $\Re(\chi_2)$ are determined by divisible sequences of natural numbers $\chi_1 = \langle n_i^{(1)}\rangle_{i\in\mathbb{N}}$, $\chi_2 = \langle n_i^{(2)}\rangle_{i\in\mathbb{N}}$. If $\Re(\chi_1) = \Re(\chi_2)$, then $\Re(\chi_1) \subseteq \Re(\chi_2)$ and $\Re(\chi_2) \subseteq \Re(\chi_1)$. The inclusion $\Re(\chi_1) \subseteq \Re(\chi_2)$ holds if and only if for any natural i there exists a number j, such that $\Re(n_i^{(1)}) \subseteq \Re(n_j^{(2)})$. According to the property (B) it means that

$n_i^{(1)}|n_j^{(2)}$, that is the sequence χ_1 is a divisor of the sequence χ_2. On the other hand, the inclusion $\Re(\chi_2) \subseteq \Re(\chi_1)$ means that for any $j \in \mathbb{N}$ substructures $\Re(n_j^{(2)})$ is contained into some substructures $\Re(n_i^{(1)})$, namely for an arbitrary $j \in \mathbb{N}$ there exists such $i \in \mathbb{N}$, that $n_j^{(2)}|n_i^{(1)}$. This means that the relation $\chi_2|\chi_1$ holds. Thus, sequences χ_2 and χ_1 are exactly divisible. So, $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$ by lemma 5.

Now suppose $\operatorname{char}\chi_1 = \operatorname{char}\chi_2$. Then sequences χ_1 and χ_2 are exactly divisible, that is $\chi_1|\chi_2$ and $\chi_2|\chi_1$. As properties (A) and (B) hold for the family $\Re(n)$, $n \in \mathbb{N}$, using the considerations similar to above we obtain that $\Re(\chi_1) \subseteq \Re(\chi_2)$ and $\Re(\chi_2) \subseteq \Re(\chi_1)$. In other words, these substructures coincide.

(ii) Let $DS^{(0)}$ be a set of fixed representatives of each class of exactly divisible sequences from DS. Then

$$\{\Re(\chi) \mid \chi \in DS\} = \{\Re(\chi) \mid \chi \in DS^{(0)}\}.$$

We shall show, that the family of substructures on the right side of this equality satisfies the condition (ii) of this theorem. Then the subset of $DS^{(0)}$ is determined by classes of exactly divisible sequences on DS. Hence, the mapping $\lambda : \mathbb{SN} \to DS^{(0)}$, such that $\lambda(u) = \chi \quad \text{iff} \quad \operatorname{char}\chi = u$, is a bijective by using lemma 5. So, properties (C), (D) of the family $\Re(\chi)$, $\chi \in DS$, imply that a mapping $\bar{\lambda} : DS^{(0)} \to \{\Re(\chi) \mid \chi \in DS\}$ defined by

$$\bar{\lambda}(\chi) = \bigcup_{n\in\chi} \Re(n) = \Re(\chi), \quad \chi \in DS^{(0)},$$

is a bijective too. Thus the mapping $\mu = \lambda \cdot \bar{\lambda} : \mathbb{SN} \to \{\Re(\chi) \mid \chi \in DS^{(0)}\}$ will also be a bijective. So that the image of arbitrary supernatural number will be own substructure of the structure $\Re$. It is also clear that $\mu(1) = \Re(1)$, $\mu(\mathbb{I}) = \cup_{n\in\chi}\Re(n)$, where χ be such divisible sequence, that $\operatorname{char}\chi = \mathbb{I}$. It remains to show that the mapping μ is consistent with the lattice operations $\vee$ and $\wedge$. To this end, we are going to check that μ is consistent with the divisibility $|$ on the set $\mathbb{SN}$ and the inclusion of $\subseteq$ on substructures $\{\Re(\chi) \mid \chi \in DS^{(0)}\}$. Indeed, let the condition $u_1|u_2$ hold for supernatural numbers u_1, u_2. Then $\lambda(u_1)|\lambda(u_2)$. It follows from (i) that $\bar{\lambda}(\lambda(u_1)) \subseteq \bar{\lambda}(\lambda(u_2))$. So $(\lambda \cdot \bar{\lambda})(u_1) \subseteq (\lambda \cdot \bar{\lambda})(u_2)$. The theorem is proved. □

Theorem 1 can be reformulated in terms of inductive limits as follows. Let $\Re(n)$, $n \in \mathbb{N}$ be a family of structures. This family satisfies the condition (A). There is an embedding φ_{n_1,n_2} of the structure $\Re(n_1)$ into the structure $\Re(n_2)$ for arbitrary $n_1, n_2 \in \mathbb{N}$, such that $n_1|n_2$. Assume that for monomorphisms φ_{n_1,n_2} following standard requirements:

(a) $\varphi_{n,n} = Id$ for any $n \in \mathbb{N}$;

(b) $\varphi_{n_1,n_3} = \varphi_{n_1,n_2} \cdot \varphi_{n_2,n_3}$ for arbitrary n_1, n_2, $n_3 \in \mathbb{N}$ such, that $n_1|n_2$ and $n_2|n_3$;

hold.

Then we have the direct spectrum $\langle \Re(n),\ \varphi_{n,m} \rangle_{n,m \in \mathbb{N}}$ of relational structures, and let $\Re$ be a limit of this spectrum. For arbitrary sequence $\chi = \langle n_i \rangle_{i \in \mathbb{N}}$ it can be considered a direct spectrum $\langle \Re(n_i),\ \varphi_{n_i,n_{i+1}} \rangle_{i \in \mathbb{N}}$ and its direct limit $\Re(\chi)$. This structure can considered as a substructure of structure $\Re$ in an obvious way.

Theorem 2. 1) *Direct limits, defined by divisible sequences χ_1 and χ_2, coincide as substructures of $\Re$ if and only if* $\operatorname{char} \chi_1 = \operatorname{char} \chi_2$.

2) *All boundary structures considered as substructures of $\Re$ and defined by divisible sequences form a lattice with respect to inclusion. This lattice is isomorphic to the Steinitz's lattice.*

We shall show examples, described how such construction works in two different situations: for an universal algebras and metric spaces.

5. Groups and semigroups of periodically defined transformations of natural numbers

Let PT_n be a semigroup of all partial defined transformations of set the $\{1, 2, \ldots, n\}$, let $PT(N)$ be a semigroup of all partial defined transformations of the set of natural numbers $\mathbb{N}$.

Definition 9. A transformation $\widehat{\pi} \in PT(N)$ is called periodically defined expansion of the transformation $\pi \in PT_n$ on the set $\mathbb{N}$, if its act on natural numbers is defined by the following table

$$\left(\begin{array}{cccc|cccc|c} 1 & 2 & \ldots & n & n+1 & n+2 & \ldots & 2n & \ldots\ldots \\ 1^\pi & 2^\pi & \ldots & n^\pi & n+1^\pi & n+2^\pi & \ldots & n+n^\pi & \ldots\ldots \end{array} \right).$$

A transformation $\alpha \in PT(N)$ is called periodically defined with the period of definition n, if there is a transformation $\pi \in PT_n$ such, that $\alpha = \widehat{\pi}$.

Every periodically defined transformation has infinitely many of different periods that are multiples of the minimal period. Let T_n, IS_n, S_n be, respectively, the complete semigroup of everywhere defined transformations of the set $\{1, 2, \ldots, n\}$, the complete inverse semigroup of partial permutations and the complete symmetric group over this set. Thus let $T(N)$, $IS(N)$, $S(N)$ denote these semigroups over the set of natural numbers.

Lemma 6. *For an arbitrary $n \in \mathbb{N}$ the mapping $\varphi_n : \pi \to \widehat{\pi}$, $\pi \in PT_n$, is a homomorphic emmbeding of the semigroup PT_n into $PT(N)$ and its restrictions on T_n, IS_n, S_n, respectively, is an emmbeding into $T(N)$, $IS(N)$ and $S(N)$.*

Proof. The injection of the mapping φ_n follows directly from its definition and its consistency with the operation of the multiplication of permutations is obvious. Moreover, for any partial permutation $\pi \in IS_n$ and its inverse permutation π^* we have $\varphi_n(\pi^*) = \widehat{\pi^*} = \widehat{\pi}^*$. In particular, for any everywhere defined permutation $\pi \in S_n$ we shall get $\varphi_n(\pi^{-1}) = \widehat{\pi}^{-1}$. So, φ_n will be a monomorphism IS_n into $IS(N)$ and S_n into $S(N)$. $\square$

We will denote by the symbol G_n one of semigroups PT_n, T_n, IS_n, S_n, $n \in \mathbb{N}$, and will denote by the symbol G one of the corresponding semigroups $PT(N)$, $T(N)$, $IS(N)$ or $S(N)$. Thus, all formulated statements will take place simultaneously for all four series of semigroups.

Let $\widehat{G}_n$ be the image $\varphi_n(G_n)$ of the semigroup G_n. Then $\widehat{G}_n$ be a subsemigroup in G, which is isomorphic to G_n. A family of semigroups $\widehat{G}_n$, $n \in \mathbb{N}$, in G is partially ordered by the inclusion. The main property of this partially ordered set will be given in following lemma.

Lemma 7. *A partially ordered set $(\{\widehat{G}_n, n \in \mathbb{N}\}, \subseteq)$ is a lattice, which is isomorphic to the lattice of positive integers with a relation of the divisibility.*

Proof. We shall verify that the correspondence $n \leftrightarrow \widehat{G}_n$, $n \in \mathbb{N}$, will be an isomorphism of partially ordered sets $(\mathbb{N}, |)$ and $(\{\widehat{G}_n, n \in \mathbb{N}\}, \subseteq)$. This correspondence is bijective, hence it is enough to show that for any $m, n \in \mathbb{N}$ the condition $m|n$ holds if and only if $\widehat{G}_m \subseteq \widehat{G}_n$. Let $m|n$ and $n = km$. So that the permutation $\widehat{\pi}^{(k)}$ given by the equality

$$\widehat{\pi}^{(k)} = \left(\begin{array}{cccc|c|ccc} 1 & 2 & \ldots & m & \ldots & (k-1)m+1 & \ldots & km \\ 1^\pi & 2^\pi & \ldots & m^\pi & \ldots & (k-1)m+1^\pi & \ldots & (k-1)m+m^\pi \end{array}\right)$$

contains into G_n, and the equality holds:

$$\varphi_m(\pi) = \varphi(\widehat{\pi}^{(k)}) = \widehat{\pi}.$$

So, for any transformation $\alpha \in \widehat{G}$ such, that $\alpha \in \widehat{G}_m$, we obtain $\alpha \in \widehat{G}_n$. Hence, $\widehat{G}_m \subseteq \widehat{G}_n$.

On the other hand, let $\widehat{G}_m \subseteq \widehat{G}_n$. In semigroup $\widehat{G}_m$ there are permutations with the minimal period of a definition m. Any such permutation has the period of a definition n because it belongs to $\widehat{G}_n$. It follows that $m|n$. □

Since the conditions (A)–(B) from section 4 for the family $\{G_n, n \in \mathbb{N}\}$ hold, then it is possible to apply the construction of lattice completion. Namely, we introduce $G(\chi)$ by setting

$$G(\chi) = \bigcup_{i=1}^{\infty} \widehat{G}_{n_i}$$

for an arbitrary sequence $\chi = \langle n_1, n_2, \ldots \rangle \in DS$. Hence, for so determined subsemigroups of the semigroup G the conditions (C) and (D) from section 4 hold. This allows us to get the following result.

Theorem 3. *The family of subsemigroups $G(\chi)$, $\chi \in DS$, forms a lattice regarding to an inclusion in the semigroup G, which is a Steinitz's lattice. Different elements of this lattice are pairwise non-isomorphic semigroups.*

Proof. The first part of the statement is a direct consequence of Theorem 1. So, we have to show its second part. Let first $G = S(N)$ be the group of permutations on the set $\mathbb{N}$, and let $G(u) = S(u)$ be the corresponding group of periodically defined permutations ($u \in \mathbb{SN}$). Due to [6] groups $G(u_1)$ and $G(u_2)$ for $u_1, u_2 \in \mathbb{SN}$, $u_1 \neq u_2$, are non-isomorphic, because one of them contains a permutation a centralizator that can be non isomorphic to a centralizator of any permutation from another group. Suppose now that $G(u)$ is a subsemigroup of such periodically defined permutations from G, that minimum periods are divisors of a supernatural number u. Then the group $G(u)$ of inverse elements coincides with $S(u)$. Hence, for $u_1 \neq u_2$ semigroups $G(u_1)$ and $G(u_2)$ are non-isomorphic because of groups of their inverse elements are non-isomorphic too. □

The semigroup $G(u)$, $u \in \mathbb{SN}$, can be defined as a limit of the direct spectrum of semigroups with so-called diagonal emmbeding. According to [14] the emmbeding of the transitive transformations group (G, X) into the transformations group (H, Y) is called a diagonal emmbeding if orbits of G onto the set Y either are trivial (consist of one point), or have the same orbit cardinality $|X|$. Note, that the act G on such orbit is isomorphic to the permutation group (G, X). A diagonal emmbeding of a group (G, X) into a permutation group (H, Y) is called a strictly diagonal emmbeding, if there are no trivial orbits of group G onto Y. In this case, with some $k \in \mathbb{N}$ we have $|Y| = k|X|$. Note that strictly diagonal emmbeding can be defined for arbitrary, not necessarily transitive permutations groups. So, its can be defined for transformations semigroups too.

Definition 10. An emmbeding of a transformation semigroup (V, X) into a transformation semigroup (W, Y) will be called (strictly) diagonal emmbeding, if there is a partition of Y onto subsets of the capacity $|X|$, which are invariant under the action of the image of V. Moreover, the action of the image of V onto each of these subsets is isomorphic as a semigroup of transformations to semigroup (V, X).

As before, let $G_n \in \{PT_n, T_n, IS_n, S_n\}$.

Lemma 8. *Let the mapping $\delta_k : G_n \to G_{nk}$ is defined by equality*

$$\delta_k(\pi) = \widehat{\pi}^{(k)},$$

for any permutation $\pi \in G_n$. Then δ_k is an isomorphic emmbeding of the semigroup G_n into the semigroup G_{nk}.

A proof is done by a direct check as at lemma 6.

For an arbitrary divisible sequence $\chi = \langle n_1, n_2, \ldots \rangle \in \mathbb{SN}, n_{i+1}/n_i = k_i$ $(i = 1, 2, \ldots)$ we define a direct spectrum of semigroups

$$G_{n_i} \longrightarrow^{\delta_{k_1}} G_{n_2} \longrightarrow^{\delta_{k_2}} G_{n_3} \longrightarrow \cdots .$$

So, we also have to consider the limit semigroup of the spectrum

$$G[\chi] = \lim_{\to i}(G_{n_i}, \delta_{k_i}).$$

Theorem 4. *For any supernatural number $\chi \in \mathbb{SN}$ semigroups $G[\chi]$ and $G(\chi)$ are isomorphic as semigroups of transformations.*

Proof. An isomorphism of semigroups is constructed in the standard way, and the set of an action of $G[\chi]$ is naturally identified with $\mathbb{N}$. Finally, we note that our semigroups act equally on the set $\mathbb{N}$. □

6. Steinitz's lattice of subspaces in Besicovitch's space

A normalized Hamming metric on the set H_n of $(0, 1)$-sequences of a length n is called a metric d_H, defined by the following equality

$$d_H(x, y) = \frac{1}{n}\sum_{i=1}^{n} |x_i - y_i|, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in H_n. \quad (8)$$

So, let now $\{0,1\}^{\mathbb{N}}$ be a set of infinite $(0,1)$-sequences. We will represent the distance function $\widehat{d}_B$ by the rule

$$\widehat{d}_B(x,y) = \lim_{n\to\infty} \sup d_H((x_1,\dots,x_n),(y_1,\dots,y_n)), \tag{9}$$

$$x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \{0,1\}^n.$$

Since (8) is a metric, it is easy to verify that the function $\widehat{d}_B$, defined by (9), is a pseudometric, i.e., it differs from a metric so that there are infinite sequences with the distance between them equals to 0. The binary relation

$$x \sim_B y \Leftrightarrow \widehat{d}_B(x,y) = 0$$

is an equivalence on $\{0,1\}^{\mathbb{N}}$. Define

$$\mathcal{X}_B = \{0,1\}^{\mathbb{N}}/_{\sim_B}.$$

The function $\widehat{d}_B$ is consistent with the equivalence $\sim_B$. Hence, it determines a function d_B on $\mathcal{X}_B$ as follows:

$$d_B([x],[y]) = \widehat{d}_B(x,y), \tag{10}$$

where $[x]$, $[y]$ are equivalence classes of $\sim_B$, and $x \in [x]$, $y \in [y]$ are arbitrary representatives of these classes. Defined by the equality (10) the function d_B is a metric. So, a metric space $(\mathcal{X}_B, d_B)$ is called Besicovitch's space (see [15]).

For any natural number n normalized Hamming space H_n is isometric embedded into Besicovitch's space $\mathcal{X}_B$.

Lemma 9. *The mapping $h_n : H_n \to \mathcal{X}_B$, defined by setting*

$$h_n((x_1,\dots,x_n)) = [(x_1,\dots,x_n,x_1,\dots,x_n,\dots)],\ (x_1,\dots,x_n) \in H_n,$$

is an isometric emmbeding.

Proof. By the definition the distance (10) between classes of the equivalence $\sim_B$, which defined by periodic sequences

$$\overline{x} = x_1,\dots,x_n,x_1,\dots,x_n,\dots \quad \text{and} \quad \overline{y} = y_1,\dots,y_n,y_1,\dots,y_n,\dots,$$

is equal to

$$\frac{1}{n} d_H((x_1,\dots,x_n),(y_1,\dots,y_n)).$$

□

Denote by $\widetilde{H}_n$ the image of Hamming space H_n for the mapping h_n.

Lemma 10. *The inclusion $\widetilde{H}_n \subseteq \widetilde{H}_m$ $(n, m \in \mathbb{N})$ holds if and only if then $n|m$.*

Proof. It is obvious. □

So that, Besicovitch's space $\mathcal{X}_B$ contains the family of subspaces $\widetilde{H}_n$, $n \in \mathbb{N}$. It is easy to see, that for this family conditions $(A)-(D)$ of the lattice completion hold. It follows from Theorem 1, that the space $\mathcal{X}_B$ contains a family of subspaces $\widetilde{H}_u$, $u \in \mathbb{SN}$, indexing by supernatural numbers. Note, that the subspace $\widetilde{H}_u$ consists of all possible periodic $(0,1)$-sequences, such that lengthes of their minimum periods are divisors of a supernatural number u. By properties of the general construction we obtain the following

Theorem 5. *A family of subspaces $\widetilde{H}_u$, $u \in \mathbb{SN}$, of the space $\mathcal{X}_B$ forms a lattice over the inclusion which is isomorphic to the lattice of supernatural numbers. If $u_1 \neq u_2$, then subspaces $\widetilde{H}_{u_1}$ and $\widetilde{H}_{u_2}$ are not isometric.*

Proof. The first part of the assertion follows from Theorem 1. The proof of the third part given in the article [16]. □

By the Theorem 1 each of spaces $\widetilde{H}_u$, $u \in \mathbb{SN} \setminus \mathbb{N}$, is isometric to inductive limit of the sequence of finite Hamming spaces H_{m_1}, H_{m_2}, $\ldots$, where $\chi = \langle m_1, m_2, \ldots \rangle$ is such divisible sequence that $\operatorname{char} \chi = u$. Monomorphisms are the diagonal emmbedings $f_i : H_{m_i} \to H_{m_{i+1}}$, defined by following equalities

$$f_i((x_1, \ldots, x_{m_i})) = (\underbrace{x_1, \ldots, x_{m_i}, \ldots, x_1, \ldots, x_{m_i}}_{k_i m_i}),\ (x_1, \ldots, x_{m_i}) \in H_{m_i}, \tag{11}$$

where $k_i = m_{i+1}/m_i$, $i = 1, 2, \ldots$. Note, that a construction of limit cube, that is built as inductive limit of the sequence of Hamming spaces H_{2^i} of the dimension 2^i with emmbedings of following doubling coordinates:

$$\delta_i((x_1, \ldots, x_{2^i})) = (x_1, x_1, x_2, x_2, \ldots, x_{2^i}, x_{2^i}), \quad i = 1, 2, \ldots,$$

is considered in [17]. So that, it is easy to understand that the limit space of such direct spectrum is isometric to the space $\widetilde{H}_{2^\infty}$ with emmbedings of the type (11). Note, that in [18] was considered continuum family of subspaces of the Besicovitch space on some alphabet B, naturally parametrized by supernatural numbers. Every subspace is defined as a diagonal limit of finite Hamming spaces on the alphabet B. So, our construction is more general.

Other generalizations of this construction are proposed in [19], and a generalization of construction of lattice completion for linear groups is considered in the article [14].

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RESEARCH ARTICLE

Generalization of primal superideals

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Abstract. Let R be a commutative super-ring with unity $1 \neq 0$. A proper superideal of R is a superideal I of R such that $I \neq R$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function, where $\mathfrak{I}(R)$ denotes the set of all proper superideals of R. A homogeneous element $a \in R$ is *ϕ-prime* to I if $ra \in I - \phi(I)$ where r is a homogeneous element in R, then $r \in I$. We denote by $\nu_\phi(I)$ the set of all homogeneous elements in R that are not ϕ-prime to I. We define I to be *ϕ-primal* if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

forms a superideal of R. For example if we take $\phi_\varnothing(I) = \varnothing$ (resp. $\phi_0(I) = 0$), a *ϕ-primal* superideal is a primal superideal (resp., a weakly primal superideal). In this paper we study several generalizations of primal superideals of R and their properties.

1. Introduction

A supercase on a ring is a $\mathbb{Z}_2$-grading on that ring. In general the grading on a ring, or a module, usually leads computation by allowing one to focus on the homogeneous elements, which are simpler and easier than random elements. However, to do this work you need to know that the constructions being studied are graded. One approach to this issue is to

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redefine the constructions entirely in terms of graded modules and avoid any consideration of non-graded modules or non-homogeneous elements. Unfortunately, while such an approach helps to understand the graded modules, it will only help to understand the original construction, where the graded version of the concept coincide with original one. Therefore, notably, the studying of the graded rings (or modules) is very important.

Because of the importance of the grading, the author made many researches in different subjects in mathematics in super-rings and graded rings few years ago. For example in [1,2,4], the author studied existence of superinvolutions and pseudo superinvolutions of kinds one and two, also in [3, 5] he studied Division $\mathbb{Z}_3$-Algebra, and primitive $\mathbb{Z}_3$-algebra with $\mathbb{Z}_3$-involution. Moreover, in [7] he studied Δ-supergraded submodules and in [6] he studied product of graded submodules. Finally, in [8] the author studied weakly primal graded superideals.

A few years ago Y. A. Bahturin and A. Giambruno in [12] studied Group Gradings on associative algebras with involution.

Let R be any ring with unity, then R is called a super-ring if R is a $\mathbb{Z}_2$-graded ring such that if $a, b \in \mathbb{Z}_2$ then $R_a R_b \subseteq R_{a+b}$ where the subscripts are taken modulo 2. Let $h(R) = R_0 \cup R_1$. Then $h(R)$ is the set of homogeneous elements in R and $1 \in R_0$.

Throughout, R will be a commutative super-ring with unity. By a proper superideal of R we mean a superideal I of R such that $I \neq R$. We will denote the set of all proper superideals of R by $\mathfrak{I}(R)$. If I and J are in $\mathfrak{I}(R)$, then the superideal $\{r \in R \,:\, rJ \subseteq I\}$ is denoted by $(I : J)$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function and let $I \in \mathfrak{I}(R)$, we say that I is a *ϕ-prime* if whenever $x, y \in h(R)$ with $xy \in I - \phi(I)$, then $x \in I$ or $y \in I$. Since $I - \phi(I) = I - (\phi(I) \cap I)$, there is no loss of generality to assume that $\phi(I) \subseteq I$ for every proper superideal I of R.

Given two functions $\psi_1, \psi_2 : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$, we define $\psi_1 \leqslant \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathfrak{I}(R)$.

Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function, then an element $a \in h(R)$ is *ϕ-prime* to I, if whenever $ra \in I - \phi(I)$, where $r \in h(R)$, then $r \in I$. That is $a \in h(R)$ is ϕ-prime to I, if

$$h((I : a)) - h((\phi(I) : a)) \subseteq h(I).$$

Let $\nu_\phi(I)$ be the set of all homogeneous elements in R that are not ϕ-prime to I. We define I to be *ϕ-primal* if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

forms a superideal in R. In this case we say that I is a ϕ-P-primal superideal of R, and P is the adjoint superideal of I.

In the next example we give some famous functions $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ and their corresponding ϕ-primal superideals.

Example 1.1.
- $\phi_\varnothing$, $\phi_\varnothing(I) = \varnothing \forall I \in \mathfrak{I}(R)$ — primal superideal.
- ϕ_0, $\phi_0(I) = \{0\} \forall I \in \mathfrak{I}(R)$ — weakly primal superideal.
- ϕ_2, $\phi_2(I) = I^2 \forall I \in \mathfrak{I}(R)$ — almost primal superideal.
- ϕ_n, $\phi_n(I) = I^n \forall I \in \mathfrak{I}(R)$ — n-almost primal superideal.
- ϕ_ω, $\phi_\omega(I) = \cap_{n=1}^{\infty} I^n \forall I \in \mathfrak{I}(R)$ — ω-primal superideal.

Observe that $\phi_\varnothing \leqslant \phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2$.

For the nongraded case one can easily check that if I is a ϕ-P-primal ideal of R, with $\phi \neq \phi_\varnothing$, then $P = (\nu_\phi(I) \cup \{0\}) + \phi(I)$ if and only if $P = \nu_\phi(I) \cup \phi(I)$. But if $\phi = \phi_\varnothing$ then $P = \nu_\phi(I)$.

Y. Darani in [13] defined that for a commutative ring R with unity and for a function $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ a proper ideal I of R is a ϕ-P-primal ideal of R if $P = \phi(I) \cup \nu_\phi(I)$ is an ideal in R, where $\nu_\phi(I)$ is the set of all elements in R that are not ϕ-prime to I.

By comparing the two definitions (in the trivial case and in the supercase), we can see that the definition of ϕ-primal superideals is a generalization of the definition of the ϕ-primal ideals to the supercase.

In section 2, we give some examples and properties of ϕ-primal superideals of R. Also, we prove that if R is ϕ-torsion free super-ring, then every ϕ-primary superideal of R is ϕ-primal and hence if R is torsion free super-ring then every weakly primary (i.e., ϕ_0-primary) superideal of R is weakly primal.

In section 3, we introduce some conditions under which ϕ-primal superideals are primal.

2. ϕ-Primal superideals

Let R be a commutative super-ring with unity $1 \neq 0 \in R_0$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function and let I be a proper superideal of R. Suppose that $\nu_\phi(I)$ is the set of all homogeneous elements in R that are not ϕ-prime to I, we recall that I is a ϕ-primal superideal of R if the set

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

forms a superideal in R. In this case P is called the adjoint superideal of I.

In the next examples we show that the concepts "primal superideals" and "ϕ-primal superideals" are different.

Example 2.1. Let $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$, where $u^2 = 0$, be a commutative super-ring and assume that $\phi = \phi_0$. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$.

(1) Since $0 \neq \bar{2} \cdot \bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get that $\bar{2}$ and $\bar{4}$ are not ϕ-prime to I. Easy computations imply that $\bar{2} + \bar{4} = \bar{6}$ is ϕ-prime to I. Thus we obtain that I is not a ϕ-primal superideal of R.

(2) Set $P = 2\mathbb{Z}_{24} + u\mathbb{Z}_{24}$. We show that I is a primal superideal of R. It is easy to check that every element of $h(P)$ is not prime to I. Conversely, assume that $\bar{a} \in h(R) - h(P)$, then $\bar{a} \in \mathbb{Z}_{24}$ with $gcd(a, 8) = 1$. If $\bar{a} \cdot \bar{n} \in I$ for some $\bar{n} \in \mathbb{Z}_{24}$, then 8 divides n; hence $\bar{n} \in I$. Therefore, $h(P)$ is exactly the set of elements in $h(R)$ which are not prime to I. Thus I is a primal superideal of R.

Example 2.2. Let $\phi = \phi_0$, and let $T(R)$ be the collection of all homogeneous zero divisors of R. If R is not a superdomain such that $Z(R) = T_0(R) + T_1(R)$ is not a superideal of R, then the trivial superideal of R is a ϕ-primal superideal which is not primal.

According to Examples 2.1 and 2.2 a primal superideal of R need not to be ϕ-primal and a ϕ-primal superideal of R need not to be primal.

In the next lemma we show that if I is a ϕ-primal superideal in R, then $I \subseteq P$. The same result for the non graded case has been proved in [13].

Lemma 2.3. *Let I be a superideal of R, and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. Suppose that I is ϕ-primal superideal of R with the adjoint superideal P. Then*

(1) $I \subseteq P$.

(2) $h(P) = h(\phi(I)) \cup \nu_\phi(I)$.

Proof. (1) Let r be any homogeneous element in I, if $r \in \phi(I)$, then $r \in P$. If $r \in h(I) - h(\phi(I))$, then $1.r \in I - \phi(I)$ with $1 \notin I$, hence $r \in P$. Thus, $I \subseteq P$.

(2) It is trivial that $\nu_\phi(I) \subseteq h(P) - h(\phi(I))$. For the reverse inclusion, let $x \in h(P) - h(\phi(I))$ then $x = x_\alpha + y_\alpha$, where $x_\alpha \neq 0 \in \nu_\phi(I)$ and $y_\alpha \in (\phi(I))_\alpha$, for some α in $\mathbb{Z}_2$. Since $x_\alpha \neq 0 \in \nu_\phi(I)$, there exists $r \in h(R) - h(I)$ with $rx_\alpha \in I - \phi(I)$. Thus, $rx = rx_\alpha + ry_\alpha \in I - \phi(I)$ since $ry_\alpha \in \phi(I)$. Hence $x \in \nu_\phi(I)$. □

Proposition 2.4. *Let I, P be proper superideals of R. Then the following statements are equivalent.*

(1) I is a ϕ-primal superideal of R with the adjoint superideal P.

(2) For $x \in h(R)$ with $x \notin h(P) - h(\phi(I))$ we have $h((I : x)) = h(I) \cup h((\phi(I) : x))$. If $x \in h(P) - h(\phi(I))$ then $h((I : x)) \supsetneqq h(I) \cup h((\phi(I) : x))$.

Proof. (1) $\Rightarrow$ (2) If $x \in h(P) - h(\phi(I))$, then $x \in \nu_\phi(I)$, so there exists $r \in h(R) - h(I)$ with $rx \in I - \phi(I)$. Thus $r \in h((I : x))$ and $r \notin h(I) \cup h((\phi(I) : x))$. Since it is easy to see that $h((I : x)) \supseteq h(I) \cup h((\phi(I) : x))$, we have that $h((I : x)) \supsetneqq h(I) \cup h((\phi(I) : x))$.

Now let $x \notin h(P) - h(\phi(I))$, where $x \in h(R)$, then $x \notin \nu_\phi(I)$ hence x is ϕ-prime to I. Let $r \in h((I : x))$, if $rx \notin \phi(I)$ then $r \in h(I)$. If $rx \in \phi(I)$ then $r \in h((\phi(I) : x))$. Hence

$$h((I : x)) \subseteq h(I) \cup h((\phi(I) : x)) \subseteq h((I : x)).$$

(2) $\Rightarrow$ (1) From part (2) we have $h(P) - h(\phi(I)) = \nu_\phi(I)$. Thus I is a ϕ-primal superideal of R. $\square$

Theorem 2.5. *If I is a ϕ-primal superideal of R, then*

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

is a ϕ-prime superideal of R.

Proof. Suppose that $a, b \in h(R) - h(P)$ we show that $ab \in \phi(P)$ or $ab \notin P$. Assume that $ab \notin \phi(P)$, then $ab \notin \phi(I)$, since $\phi(I) \subseteq \phi(P)$. Let $rab \in I - \phi(I)$ for some $r \in h(R)$. Then by Proposition 2.4, we have $ra \in h((I : b)) = h(I) \cup h((\phi(I) : b))$, but $ra \notin (\phi(I) : b)$; hence $ra \in h(I)$. Moreover $ra \notin h(\phi(I))$, for if $ra \in h(\phi(I))$, then $rab \in h(\phi(I))$, which is a contradiction. Therefore, $ra \in h(I) - h(\phi(I))$ and again by Proposition 2.4, $r \in h((I : a)) = h(I) \cup h((\phi(I) : a))$. Since $ra \notin \phi(I)$, we have $r \notin h((\phi(I) : a))$, so $r \in h(I)$. Hence ab is ϕ-prime to I which implies that $ab \notin P$. $\square$

Remark 2.6. Let I is a ϕ-primal superideal of R then by Theorem 2.5,

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

is a ϕ-prime superideal of R. In this case P is called the ϕ-prime adjoint superideal (simply adjoint superideal) of I, and I is called a ϕ-P-primal superideal of R.

The next result shows that every ϕ-prime superideal of R is ϕ-primal.

Theorem 2.7. *Every ϕ-prime superideal of R is ϕ-primal.*

Proof. Let P be a ϕ-prime superideal of R, we show that P is a ϕ-P-primal superideal of R. Thus we must prove that

$$P = \begin{cases} [(\nu_\phi(P))_0 + (\nu_\phi(P))_1 \cup \{0\}] + \phi(P) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(P))_0 + (\nu_\phi(P))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

Case 1. Suppose that $P \neq \phi(P)$. We show that $h(P) - h(\phi(P)) = \nu_\phi(P)$. Let $a \in h(P) - h(\phi(P))$. Then $a.1 \in P - \phi(P)$ with $1 \notin P$, so $a \in \nu_\phi(P)$. On the other hand let $a \notin h(P) - h(\phi(P))$. If $a \in h(\phi(P))$, then $ra \in \phi(P)$ for all $r \in h(R)$, so a is ϕ-prime to P and hence $a \notin \nu_\phi(P)$. If $a \notin h(\phi(P))$, then $a \notin P$, so for any $r_\alpha \in R_\alpha$ with $r_\alpha a \in P - \phi(P)$ we have $r_\alpha \in P_\alpha$, since P is ϕ-prime. Thus a is ϕ-prime to P, hence $a \notin \nu_\phi(P)$. Therefore, $h(P) - h(\phi(P)) = \nu_\phi(P)$ which implies that P is a ϕ-P-primal superideal of R.

Case 2. Suppose that $P = \phi(P)$ then it is easy to check that $\nu_\phi(P) = \varnothing$, hence P is a ϕ-P-primal superideal of R. □

In the next example we introduce a ϕ-P-primal superideal I of R such that I itself is not ϕ-prime.

Example 2.8. Let $\phi = \phi_0$ and let $R = \mathbb{Z}_8 + u\mathbb{Z}_8$ where $u^2 = 0$. Then R is a commutative super-ring with unity. If $I = 4\mathbb{Z}_8 + u\mathbb{Z}_8$, then I is not a ϕ-prime superideal of R, since $\bar{2} \cdot \bar{2} \neq 0 \in I$, but $\bar{2} \notin I$. Let $P = 2\mathbb{Z}_8 + u\mathbb{Z}_8$, we show that I is a ϕ-P-primal superideal of R. It is enough to show that $\nu(I) = h(P) - \{0\}$. Let $0 \neq \bar{a} \in h(P)$, if $\bar{a} \in 2\mathbb{Z}_8$ then $\bar{a} = 2k \in \mathbb{Z}_8$. If k is an odd number, then $0 \neq \bar{2}\bar{a} \in I$, but $\bar{2} \notin I$, and if k is an even number $0 \neq \bar{1}\bar{a} \in I$ with $\bar{1} \notin I$; hence $\bar{a} \in \nu(I)$. If $\bar{a} \in u\mathbb{Z}_8$ then $\bar{a} \in I \subseteq \nu(I)$. On the other hand, if $\bar{a} \in h(R) - h(P)$, then $\bar{a}$ is an odd number in $\mathbb{Z}_8$. If $0 \neq \bar{a}\bar{m} \in I$ for some $\bar{m} \in \mathbb{Z}_8$ then 4 divides am and so, 4 divides m since $(4, a) = 1$; hence $\bar{m} \in I$. Thus I is a ϕ-P-primal superideal of R.

Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. We assume that for any $I, J \in \mathfrak{I}(R)$, $\phi(J) \subseteq \phi(I)$ if $J \subseteq I$. We produced in Example 2.2 a ψ_2-primal which is not ψ_1-primal, where $\psi_1 \leqslant \psi_2$. In the next theorem we give the condition on ψ_2-P-primal superideal to be ψ_1-P-primal.

Theorem 2.9. *Suppose that $\psi_1 \leqslant \psi_2$, where ψ_1 and ψ_2 are maps from $\mathfrak{I}(R)$ into $\mathfrak{I}(R) \cup \{\varnothing\}$, and let I be a ψ_2-P-primal superideal of R, with $I_0 I_\alpha \neq \psi_2(I)_\alpha$ for all $\alpha \in \mathbb{Z}_2$. If P is a prime superideal of R, then I is ψ_1-P-primal.*

Proof. Since I is a ψ_2-P-primal superideal of R, then

$$P=\begin{cases}[(\nu_{\psi_2}(I))_0+(\nu_{\psi_2}(I))_1\cup\{0\}]+\psi_2(I) & : \quad \text{if } \psi_2\neq\phi_\varnothing\\ (\nu_{\psi_2}(I))_0+(\nu_{\psi_2}(I))_1 & : \quad \text{if } \psi_2=\phi_\varnothing\end{cases}$$

To show that I is a ψ_1-P-primal superideal of R we must prove that

$$P=\begin{cases}[(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1\cup\{0\}]+\psi_1(I) & : \quad \text{if } \psi_1\neq\phi_\varnothing\\ (\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1=\phi_\varnothing\end{cases}$$

If $\psi_2=\phi_\varnothing$, then $\psi_1=\psi_2$ and hence we have that $P=\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1$ which implies that I is a ψ_1-P-primal superideal of R. Now we may assume that $\psi_2\neq\phi_\varnothing$, so we need to prove that

$$P=\begin{cases}[(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1\cup\{0\}]+\psi_1(I) & : \quad \text{if } \psi_1\neq\phi_\varnothing\\ (\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1=\phi_\varnothing\end{cases}.$$

Let $a\in\nu_{\psi_2}(I)$, then there exists $r\in h(R)-h(I)$ with $rs\in I-\psi_2(I)\subseteq I-\psi_1(I)$, so $a\in\nu_{\psi_1}(I)$ which implies that

$$(\nu_{\psi_2}(I))_0+(\nu_{\psi_2}(I))_1\subseteq(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 \tag{1}$$

Now, let $a\in h(\psi_2(I))$, if $a\notin\psi_1(I)$ then $a\in I-\psi_1(I)$, so $1.a\in I-\psi_1(I)$ with $a\notin I$, hence $a\in\nu_{\psi_1}(I)$. Therefore,

$$\psi_2(I)\subseteq\begin{cases}[(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1\cup\{0\}]+\psi_1(I) & : \quad \text{if } \psi_1\neq\phi_\varnothing\\ (\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1=\phi_\varnothing\end{cases}. \tag{2}$$

From (1) and (2) we have that

$$P\subseteq\begin{cases}[(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1\cup\{0\}]+\psi_1(I) & : \quad \text{if } \psi_1\neq\phi_\varnothing\\ (\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1=\phi_\varnothing\end{cases}. \tag{3}$$

Since $\psi_1(I)\subseteq\psi_2(I)\subseteq P$, by (3)

$$P=\begin{cases}[(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1\cup\{0\}]+\psi_1(I) & : \quad \text{if } \psi_1\neq\phi_\varnothing\\ (\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1=\phi_\varnothing\end{cases}$$

if $\nu_{\psi_1}(I)\subseteq P$.

Let $a\in(\nu_{\psi_1}(I))_\alpha$. Then there exists $r_\beta\in R_\beta-I_\beta$ with $ar_\beta\in I-\psi_1(I)$. If $ar_\beta\in I-\psi_2(I)$, then $a\in\nu_{\psi_2}(I)\subseteq P$. So we may assume that $ar_\beta\notin$

$I-\psi_2(I)$, hence $ar_\beta \in \psi_2(I)$. First suppose that $aI_\beta \nsubseteq (\psi_2(I))_{\alpha\beta}$, say $as_\beta \in I_{\alpha\beta}-(\psi_2(I))_{\alpha\beta}$ with $s_\beta \in I_\beta$. Then $a(r_\beta+s_\beta)=ar_\beta+as_\beta \notin \psi_2(I)$ with $r_\beta+s_\beta \in R_\beta - I_\beta$, hence $a \in \nu_{\psi_2}(I) \subseteq P$. Therefore, we may assume that $aI_\beta \subseteq (\psi_2(I))_{\alpha\beta}$.

Now suppose that $r_\beta I_0 \nsubseteq (\psi_2(I))_\beta$, then there exists $c \in I_0$ with $r_\beta c \in I_\beta-(\psi_2(I))_\beta$. Since $a^2 \in R_0$, we have that $(a^2+c)r_\beta \in I_\beta-(\psi_2(I))_\beta$ with $r_\beta \notin I_\beta$, so $a^2+c \in P_0$, but $c \in I_0 \subseteq P_0$, therefore $a^2 \in P$ and hence $a \in P$, since P is a prime superideal. So we may assume that $r_\beta I_0 \subseteq (\psi_2(I))_\beta$. Since $(I_0 I_\beta) \neq (\psi_2(I))_\beta$ there exists $\overline{a} \in I_0$ and $\overline{b} \in I_\beta$ with $\overline{a}\overline{b} \notin (\psi_2(I))_\beta$. Thus, $(a^2+\overline{a})(r_\beta+\overline{b}) = a^2 r_\beta + a^2\overline{b}+\overline{a}r_\beta+\overline{a}\overline{b} \notin \psi_2(I)$, so $(a^2+\overline{a})(r_\beta+\overline{b}) \in I-\psi_2(I)$ with $r_\beta+\overline{b} \in R_\beta - I_\beta$ which implies that $a^2+\overline{a} \in (\nu_{\psi_2}(I))_0 \subseteq P_0$, hence $a^2 \in P_0 \subseteq P$ and then $a \in P$, since P is a prime superideal of R. Therefore, $\nu_{\psi_1}(I) \subseteq P$, so

$$P=\begin{cases}[(\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 \cup \{0\}]+\psi_1(I) & : \quad \text{if } \psi_1 \neq \phi_\varnothing \\ (\nu_{\psi_1}(I))_0+(\nu_{\psi_1}(I))_1 & : \quad \text{if } \psi_1 = \phi_\varnothing\end{cases}$$

and hence I is a ψ_1-P-primal superideal of R. □

We end the section by proving the following results about the relationship between ϕ-primary and ϕ-primal superideals. For more properties about primary and primal superideals see [8, section 4].

Definition 2.10. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function such that $\phi \neq \phi_\varnothing$, then R is a ϕ-torsion free if $ab \in \phi(P)$ where $P \in \mathfrak{I}(R)$, then $a \in \phi(P)$ or $b \in \phi(P)$.

For example if $\phi=\phi_0$, then a ϕ-torsion free super-ring is just a torsion free super-ring.

Theorem 2.11. *Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function, where $\phi \neq \phi_\varnothing$, and let R be a ϕ-torsion free. Then every ϕ-primary superideal of R is ϕ-primal.*

Proof. Let I be a ϕ-primary superideal of R. We show that

$$\sqrt{I}=[(\nu_\phi(I))_0+(\nu_\phi(I))_1 \cup \{0\}]+\phi(I).$$

($\supseteq$) Let $r \in \nu_\phi(I)$, then there exists $a \in h(R)-h(I)$ with $ra \in I-\phi(I)$ which implies that $r \in \sqrt{I}$, since I is ϕ-primary. Moreover, $\phi(I) \subseteq I \subseteq \sqrt{I}$.

($\subseteq$) Let $b \in h(\sqrt{I})$. If $b \in \phi(I)$, then done. So, we may assume that $b \notin \phi(I)$. Let n be the smallest positive integer such that $b^n \in I$. Suppose

$n=1$. If $b \in \phi(I)$, then done. If $b \notin \phi(I)$, then $1.b \in I-\phi(I)$ and $1 \notin I$ so $b \in \nu_\phi(I)$. Therefore we may assume that $n>1$. If $b^n \in \phi(I)$, then $b^n = b^{n-1}b \in \phi(I)$ and $b^{n-1} \notin \phi(I)$, since $b^{n-1} \notin I$ and $\phi(I) \subseteq I$, which is a contradiction since R is ϕ-torsion free. So, $b^n = b^{n-1}b \in I - \phi(I)$ and $b^{n-1} \notin I$, hence $b \in \nu_\phi(I)$. □

Corollary 2.12. *If R is a torsion free, then every weakly primary superideal of R is weakly primal.* □

3. Conditions on ϕ-primal superideals

In this section, we introduce some conditions under which ϕ-primal superideals are primal.

Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. We have to remind that if I is a ϕ-P-primal superideal of R, then

$$P = \begin{cases} [(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \quad \text{if } \phi \neq \phi_\varnothing \\ (\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \quad \text{if } \phi = \phi_\varnothing \end{cases}$$

is a ϕ-prime superideal of R.

Definition 3.1. Let r be a homogeneous element in R, then $|r| = \alpha$ if $r \in R_\alpha$ for some $\alpha \in \mathbb{Z}_2$.

In the next theorem we provide some conditions under which a ϕ-primal superideal is primal.

Theorem 3.2. *Let R be a commutative super-ring with unity and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be any function. Suppose that I is a ϕ-P-primal superideal of R with $I_\gamma I_\delta \nsubseteq \phi(I)$ for each $\gamma, \delta \in \mathbb{Z}_2$. If P is a prime superideal of R, then I is P-primal.*

Proof. Assume that a is a homogeneous element in P. Then $a \in \phi(I)$ or $a \in (\nu_\phi(I))_\alpha$ for some $\alpha \in \mathbb{Z}_2$ or $a = b_\beta + c_\beta$ where $b_\beta \in (\nu_\phi(I))_\beta$ and $c_\beta \in \phi(I)$ for some $\beta \in \mathbb{Z}_2$. If the first two cases hold, then a is not prime to I, since it is not ϕ-prime to I. In the last case, let d be a homogeneous element in R such that $d \notin I$ with $b_\beta d \in I - \phi(I)$. Then $ad = b_\beta d + c_\beta d \in I - \phi(I)$, because $ad \in \phi(I)$ implies that $b_\beta d \in \phi(I)$, since $c_\beta d \in \phi(I)$ which is a contradiction. Thus a is not ϕ-prime to I and hence a is not prime to I. Now assume that $b \in h(R)$ is not prime to I, so $rb \in I$ for some homogeneous element $r \in R - I$. If $rb \notin \phi(I)$, then b is not ϕ-prime to I, so $b \in P$. Thus assume that $rb \in \phi(I)$. Suppose that

$|r| = \alpha$. First suppose that $bI_\alpha \nsubseteq \phi(I)$. Then, there exists $r' \in I_\alpha$ such that $br' \notin \phi(I)$. So $b(r + r') \in I - \phi(I)$, where $r + r'$ is a homogeneous element in $R - I$, implies that b is not ϕ-prime to I, that is $b \in P$. Therefore, we may assume that $bI_\alpha \subseteq \phi(I)$. Let $|b| = \beta$. If $rI_\beta \nsubseteq \phi(I)$, then $rc \notin \phi(I)$ for some $c \in I_\beta$. In this case $r(b + c) \in I - \phi(I)$ with $r \in R - I$, that is $b + c \in P$ and hence $b \in P$, since $c \in I \subseteq P$. So we may assume that $rI_\beta \subseteq \phi(I)$. Since $I_\alpha I_\beta \nsubseteq \phi(I)$, there are $b' \in I_\alpha$ and $a' \in I_\beta$ with $b'a' \notin \phi(I)$. Then $(b + a')(r + b') \in I - \phi(I)$, where $r + b'$ is a homogeneous element in $R - I$, implies that $b + a'$ is a homogeneous element in P. On the other hand $a' \in I \subseteq P$, so that $b \in P$. We have already shown that P is exactly the set of all elements of R that are not prime to I. Hence I is P-primal. □

Let R and S be commutative super-rings. It is easy to prove that the prime superideals of $R \times S$ have the forms $P \times S$ or $R \times Q$ where P is a prime superideal of R and Q is a prime superideal of S. Also we have the following two propositions about primal superideals of $R \times S$. We leave the easy proof for the next two results to the reader. For the trivial case (i.e., $R_1 = \{0\}$) they have proved in [10, Lemma 13] and [9, Theorem 16].

Proposition 3.3. *Let R and S be commutative super-rings. If P is a primal superideal of R and Q is a primal superideal of S, then $P \times S$ and $R \times Q$ are primal superideals of $R \times S$.* □

Proposition 3.4. *Let R_1 and R_2 be commutative super-rings with unities and let $\psi_i : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be functions. Let $\phi = \psi_1 \times \psi_2$. Then ϕ-primes of $R_1 \times R_2$ have exactly one of the following three types:*

(1) *$I_1 \times I_2$ where I_i is a proper superideal of R_i with $\psi_i(I_i) = I_i$;*
(2) *$I_1 \times R_2$ where I_1 is a ψ_1-prime of R_1 which must be prime if $\psi_2(R_2) \neq R_2$;*
(3) *$R_1 \times I_2$ where I_2 is a ψ_2-prime of R_2 which must be prime if $\psi_1(R_1) \neq R_1$.* □

Now let R_1, R_2 be commutative super-rings with unities and let $R = R_1 \times R_2$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be a function. In the next theorem, we provide some conditions under which a ϕ-primal superideal of R is primal, but first we start with the following remark.

Remark 3.5. Let I be a proper superideal of a commutative super-ring R and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be a function. If a homogeneous element a is not ϕ-prime to I, then there is a homogeneous element r in $R - I$ such that $ar \in I - \phi(I) \subseteq I$ so a is not prime to I.

Theorem 3.6. *Let R_1, R_2 be commutative super-rings with unities and let $R = R_1 \times R_2$. Let $\psi_i : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\varnothing\}$ be functions with $\psi_i(R_i) \neq R_i$ for $i = 1, 2$. Let $\phi = \psi_1 \times \psi_2$. Assume that P is a superideal of R with $\phi(P) \neq P$. If I is a ϕ-P-primal superideal of R, then either $I = \phi(I)$ or I is primal.*

Proof. Suppose $\phi(I) \neq I$. By Theorem 2.5, P is a ϕ-prime superideal of R. Therefore, by Proposition 3.4, P has one of the following three cases.

Case 1. $P = P_1 \times P_2$ where P_i is a proper superideal of R_i with $\psi_i(P_i) = P_i$ for $i = 1, 2$. In this case $\phi(P) = P$, a contradiction.

Case 2. $P = P_1 \times R_2$ where P_1 is a ψ_1-prime superideal of R_1. Since $\psi_2(R_2) \neq R_2$, by Proposition 3.4(2), P_1 is a prime superideal of R_1 and so P is a prime superideal of R.

We will show that $I_2 = R_2$. Since $I \neq \phi(I)$, there exists a homogeneous element (a, b) in $I - \phi(I)$. So $(a, 1)(1, b) = (a, b) \in I - \phi(I)$. If $(a, 1) \notin I$, then $(1, b)$ is not ϕ-prime to I, hence $(1, b) \in P = P_1 \times P_2$, so $1 \in P_1$ a contradiction. Thus $(a, 1) \in I = I_1 \times I_2$ i.e., $1 \in I_2$ that is $I_2 = R_2$.

Now we prove that I_1 is a P_1-primal superideal of R_1. Let a_1 be a homogeneous element in P_1. Then $(a_1, 0) \in P_1 \times R_2 = P$. If $(a_1, 0) \in \phi(I) = \psi_1(I_1) \times \psi_2(R_2)$, then $a_1 \in \psi_1(I_1) \subseteq I_1$ so a_1 is not prime to I_1. Therefore, we may assume that $(a_1, 0) \in \nu_\phi(I)$. In this case there exists a homogeneous element $(r_1, r_2) \in R - I$ such that $(a_1, 0)(r_1, r_2) \in I - \phi(I)$ so $a_1 r_1 \in I_1 - \psi_1(I_1)$ with $r_1 \in R_1 - I_1$, since $R - I = (R_1 - I_1) \times R_2$, implies that a_1 is not ψ_1-prime to I_1, hence by Remark 3.5, a_1 is not prime to I_1. Conversely, let b_1 be a homogeneous element in R_1 such that b_1 is not prime to I_1. Then there exists a homogeneous element c_1 in $R_1 - I_1$ with $b_1 c_1 \in I_1$. Since $\psi_2(R_2) \neq R_2$, $(b_1, 1)(c_1, 1) = (b_1 c_1, 1) \in I_1 \times R_2 - (I_1 \times \psi_2(R_2)) \subseteq I - \phi(I)$ with $(c_1, 1) \in R - I$. Hence $(b_1, 1)$ is not ϕ-prime to I which implies that $(b_1, 1) \in P = P_1 \times R_2$ and so $b_1 \in P_1$.

We have already shown that the set of homogeneous elements in P_1 consists exactly of the homogeneous elements of R_1 that are not prime to I_1. Hence I_1 is P_1-primal superideal of R_1 so by Proposition 3.3, I is a P-primal superideal of R.

Case 3. $P = R_1 \times P_2$ where P_2 is a ψ_2-primal superideal of R_2. The proof of case(3) is similar to that of case(2). □

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RESEARCH ARTICLE

Generalizations of semicoprime preradicals

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Abstract. This article introduces the notions quasi-co-n-absorbing preradicals and semi-co-n-absorbing preradicals, generalizing the concept of semicoprime preradicals. We study the concepts quasi-co-n-absorbing submodules and semi-co-n-absorbing submodules and their relations with quasi-co-n-absorbing preradicals and semi-co-n-absorbing preradicals using the lattice structure of preradicals.

1. Introduction

The notion of 2-absorbing ideals of commutative rings was introduced by Badawi in [2], where a proper ideal I of a commutative ring R is called a *2-absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [1] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. In [24], the concept of 2-absorbing ideals was generalized to submodules of a module over a commutative ring. A proper submodule N of an R-module M is said to be a *2-absorbing submodule of* M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. For more studies concerning

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2-absorbing (submodules) ideals we refer to [3],[9],[24],[25]. In [13], Raggi et al. introduced the concepts of prime preradicals and prime submodules over noncommutative rings, and Raggi, Ríos and Wisbauer [18], studied the dual notions of these, coprime preradicals and coprime submodules. A generalization of prime preradicals and submodules, "2-absorbing preradicals and submodules" was investigated by Yousefian and Mostafanasab in [23]. In [14], Raggi et al. defined and investigated semiprime preradicals, and Mostafanasab and Yousefian [10], studied the concepts of quasi-n-absorbing and semi-n-absorbing preradicals. Raggi et al. [11] defined the notions of semicoprime preradicals and submodules. In this paper, we introduce the concepts of "quasi-co-n-absorbing preradicals" and "semi-co-n-absorbing preradicals". As well we investigate"quasi-co-n-absorbing submodules" and "semi-co-n-absorbing submodules" in this paper.

2. Preliminaries

Throughout this paper R is an associative ring with nonzero identity, and R-Mod denotes the category of all the unitary left R-modules. We denote by R-simp a complete set of representatives of isomorphism classes of simple left R-modules. For $M \in R$-Mod, we denote by $\mathrm{E}(M)$ the injective hull of M. Let $U, N \in R$-Mod, we say that N is *generated by* U (or N is U*-generated*) if there exists an epimorphism $U^{(\Lambda)} \to N$ for some index set Λ. Dually, we say that N is *cogenerated by* U (or N is U*-cogenerated*) if there exists a monomorphism $N \to U^{\Lambda}$ for some index set Λ. Also, we say that an R-module X is *subgenerated by* M (or X is M*-subgenerated*) if X is a submodule of an M-generated module. The category of M-subgenerated modules (the Wisbauer category) is denoted $\sigma[M]$ (see [21]). A *preradical* over the ring R is a subfunctor of the identity functor on R-Mod. Denote by R-pr the class of all preradicals over R. There is a natural partial ordering in R-pr given by $\sigma \preceq \tau$ if $\sigma(M) \leqslant \tau(M)$ for every $M \in R$-Mod. It is proved in [15] that with this partial ordering, R-pr is an atomic and co-atomic big lattice. The smallest and the largest elements of R-pr are denoted, respectively, 0 and 1.

Let $M \in R$-Mod. Recall ([5] or [15]) that a submodule N of M is called *fully invariant* if $f(N) \leqslant N$ for each R-homomorphism $f : M \to M$. In this paper, the notation $N \leqslant_{fi} M$ means that "N is a fully invariant submodule of M". Obviously the submodule K of M is fully invariant if and only if there exists a preradical τ of R-Mod such that $K = \tau(M)$. If $N \leqslant M$, then the preradicals α_N^M and ω_N^M are defined as follows: For $K \in R$-Mod,

1) $\alpha_N^M(K) = \sum\{f(N) | f \in \mathrm{Hom}_R(M, K)\}$.
2) $\omega_N^M(K) = \bigcap\{f^{-1}(N) | f \in \mathrm{Hom}_R(K, M)\}$.

Notice that for $\sigma \in R$-pr and $M,\ N \in R$-Mod we have that $\sigma(M) = N$ if and only if $N \leqslant_{fi} M$ and $\alpha_N^M \preceq \sigma \preceq \omega_N^M$. We have also that if $K \leqslant N \leqslant M$ with $K,\ N \leqslant_{fi} M$, then $\alpha_K^M \preceq \alpha_N^M$ and $\omega_K^M \preceq \omega_N^M$.

The atoms and coatoms of R-pr are, respectively, $\{\alpha_S^{E(S)} \mid S \in R\text{-simp}\}$ and $\{\omega_I^R \mid I$ is a maximal ideal of $R\}$ (See [15, Theorem 7]).

There are four classical operations in R-pr, namely, $\wedge, \vee, \cdot$ and $:$ which are defined as follows. For $\sigma,\ \tau \in R$-pr and $M \in R$-Mod:

1) $(\sigma \wedge \tau)(M) = \sigma M \cap \tau M$,
2) $(\sigma \vee \tau)(M) = \sigma M + \tau M$,
3) $(\sigma\tau)(M) = \sigma(\tau M)$ and
4) $(\sigma : \tau)(M)$ is determined by $(\sigma : \tau)(M)/\sigma M = \tau(M/\sigma M)$.

The meet $\wedge$ and join $\vee$ can be defined for arbitrary families of preradicals as in [15]. The operation defined in (3) is called *product*, and the operation defined in (4) is called *coproduct*. It is easy to show that for $\sigma,\ \tau \in R$-pr, $\sigma\tau \preceq \sigma \wedge \tau \preceq \sigma \vee \tau \preceq (\sigma : \tau)$. It is clear that in R-pr the operations (1)-(3) are associative, and in [22] it was shown that the coproduct " :″ is associative. Notice the fact that coproduct of preradicals preserves order on both sides, see [8, Remark 2.1]. We denote $\sigma\sigma\cdots\sigma$ (n times) by σ^n and $(\sigma : \sigma : \cdots : \sigma)$ (n times) by $\sigma_{[n]}$. Recall that $\sigma \in R$-pr is an *idempotent* if $\sigma^2 = \sigma$, while σ is a *radical* if $\sigma_{[2]} = \sigma$. Note that σ is a radical if and only if, $\sigma(M/\sigma(M)) = 0$ for each $M \in R$-Mod. We say that σ is *nilpotent* if $\sigma^n = 0$ for some $n \geqslant 1$, while σ is *unipotent* if $\sigma_{[n]} = 1$ for some $n \geqslant 1$.

Using the preradical ω_N^M, in the papers [6], [7] and [18], the following operation was introduced and studied:

$$\omega\text{-coproduct of submodules } K,\ N \leqslant M : (K :_M N) = (\omega_K^M : \omega_N^M)(M).$$

Henceforward, for brevity, $(K : N)$ is written instead of $(K :_M N)$. For any $\sigma \in R$-pr, we will use the following class of R-modules:

$$\mathbb{T}_\sigma = \{M \in R - \mathrm{Mod} \mid \sigma(M) = M\}.$$

Let $\sigma \in R$-pr. By [18, Theorem 8.2], the following classes of modules are closed under taking arbitrary meets and arbitrary joins:

$$\mathcal{A}_e = \{\tau \in R\text{-pr} \mid \tau\sigma = \sigma\} \quad \text{and} \quad \mathcal{A}_t = \{\tau \in R\text{-pr} \mid (\sigma : \tau) = 1\}.$$

As in [16], we define, for $\sigma \in R$-pr, the following preradicals:

- $e(\sigma) = \bigwedge\{\tau \in \mathcal{A}_e\}$ the equalizer of σ;

- $t(\sigma) = \bigwedge\{\tau \in \mathcal{A}_t\}$ the totalizer of σ.

Clearly $e(\sigma)\sigma = \sigma$ and $(\sigma : t(\sigma)) = 1$. For undefined notions we refer the reader to [13, 15–17].

In [18], Raggi et al. defined the notions of coprime preradicals and coprime submodules as follows:

Let $\sigma \in R$-pr. σ is called *coprime in* R-pr if $\sigma \neq 0$ and for any $\tau,\ \eta \in R$-pr, $\sigma \preceq (\tau : \eta)$ implies that $\sigma \preceq \tau$ or $\sigma \preceq \eta$. Let $M \in R$-Mod and let $N \leqslant M$ be a nonzero fully invariant submodule of M. The submodule N is said to be *coprime in* M if whenever $K,\ L$ are fully invariant submodules of M with $N \leqslant (K : L)$, then $N \leqslant K$ or $N \leqslant L$. Also, Raggi et al. [11] defined a preradical σ *semicoprime in* R-pr if $\sigma \neq 0$ and for any $\tau \in R$-pr, $\sigma \preceq (\tau : \tau)$ implies that $\sigma \preceq \tau$. They said that a nonzero fully invariant submodule N of M is *semicoprime in* M if whenever K is a fully invariant submodule of M with $N \leqslant (K : K)$, then $N \leqslant K$. In special case, M is called a *coprime (resp. semicoprime) module* if M is a coprime (resp. semicoprime) submodule of itself.

Yousefian and Mostafanasab in [22] defined the notions of co-2-absorbing preradicals and co-2-absorbing submodules. The preradical $\sigma \in R$-pr is called *co-2-absorbing* if $\sigma \neq 0$ and, for each $\eta, \mu, \nu \in R$-pr, $\sigma \preceq (\eta : \mu : \nu)$ implies that $\sigma \preceq (\eta : \mu)$ or $\sigma \preceq (\eta : \nu)$ or $\sigma \preceq (\mu : \nu)$. More generally, a preradical $0 \neq \sigma$ in R-pr is said to be a *co-n-absorbing preradical* if whenever $\sigma \preceq (\eta_1 : \eta_2 : \cdots : \eta_{n+1})$ for $\eta_1, \eta_2, \ldots, \eta_{n+1} \in R$-pr, there are $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n+1\}$ such that $i_1 < i_2 < \cdots < i_n$ and $\sigma \preceq (\eta_{i_1} : \eta_{i_2} : \cdots : \eta_{i_n})$. They denoted by R-co-ass the class of all R-modules M that the operation ω-coproduct is associative over fully invariant submodules of M, i.e., for any fully invariant submodules K, N, L of M, $((K : N) : L) = (K : (N : L))$. Let $M \in R$-co-ass and K be a fully invariant submodule of M. Then $(K : K : \cdots : K)$ (n times) is simply denoted by $K_{[n]}$. By Proposition 5.4 of [7], we can see that if an R-module M is injective and artinian, then $M \in R$-co-ass. Let $M \in R$-co-ass and N a nonzero fully invariant submodule of M. The submodule N is said to be *co-2-absorbing in* M if whenever J, K, L are fully invariant submodules of M with $N \leqslant (J : K : L)$, then $N \leqslant (J : K)$ or $N \leqslant (J : L)$ or $N \leqslant (K : L)$. The generalization of co-2-absorbing submodules is that, the submodule N is said *co-n-absorbing in* M if whenever $N \leqslant (K_1 : K_2 : \cdots : K_{n+1})$ for fully invariant submodules $K_1, K_2, \ldots, K_{n+1}$ of M, there are $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n+1\}$ such that $i_1 < i_2 < \cdots < i_n$ and $N \leqslant (K_{i_1} : K_{i_2} : \cdots : K_{i_n})$. An R-module M is called a *co-n-absorbing module* if M is a co-n-absorbing submodule of itself.

We say that a preradical $0 \neq \sigma \in R$-pr is called a *quasi-co-n-absorbing preradical* if whenever $\sigma \preceq (\mu_{[n]} : \nu)$ for $\mu, \nu \in R$-pr, then $\sigma \preceq \mu_{[n]}$ or $\sigma \preceq (\mu_{[n-1]} : \nu)$. A preradical $0 \neq \sigma \in R$-pr is called a *semi-co-n-absorbing preradical* if whenever $\sigma \preceq \mu_{[n+1]}$ for $\mu \in R$-pr, then $\sigma \preceq \mu_{[n]}$. Let $M \in R$-co-ass. We say that a nonzero fully invariant submodule N of M is *quasi-co-n-absorbing in M* if for every fully invariant submodules $K,\ L$ of M, $N \leqslant (K_{[n]} : L)$ implies that $N \leqslant K_{[n]}$ or $N \leqslant (K_{[n-1]} : L)$. A nonzero fully invariant submodule N of M is called *semi-co-n-absorbing in M* if for every fully invariant submodule K of M, $N \leqslant K_{[n+1]}$ implies that $N \leqslant K_{[n]}$. An R-module M satisfies *the ω-property* if $(\tau(M) :_M \eta(M)) = (\tau : \eta)(M)$ for every $\tau,\ \eta \in$ *R-pr*, see [22].

We recall the definition of relative epi-projectivity (see [20]). Let M and N be modules. N is said to be *epi-M-projective* if, for any submodule K of M, any epimorphism $f : N \to \frac{M}{K}$ can be lifted to a homomorphism $g : N \to M$

Proposition 1 ([22, Proposition 2.9 (1)]). *Let $M \in$ R-Mod. If for any fully invariant submodule K of M, $\frac{M}{K}$ is epi-M-projective, then M has the ω-property.*

In the next sections we frequently use the following proposition.

Proposition 2 ([12, Proposition 1.2]). *Let $\{M_\gamma\}_{\gamma\in I}$ and $\{N_\gamma\}_{\gamma\in I}$ be families of modules in R-Mod such that for each $\gamma \in I$, $N_\gamma \leqslant M_\gamma$. Let $N = \bigoplus_{\gamma\in I} N_\gamma$, $M = \bigoplus_{\gamma\in I} M_\gamma$, $N' = \prod_{\gamma\in I} N_\gamma$ and $M' = \prod_{\gamma\in I} M_\gamma$.*
(1) If $N \leqslant_{fi} M$, then for each $\gamma \in I$, $N_\gamma \leqslant_{fi} M_\gamma$ and $\alpha^M_N = \bigvee_{\gamma\in I} \alpha^{M_\gamma}_{N_\gamma}$.
(2) If $N' \leqslant_{fi} M'$, then for each $\gamma \in I$, $N_\gamma \leqslant_{fi} M_\gamma$ and $\omega^{M'}_{N'} = \bigwedge_{\gamma\in I} \omega^{M_\gamma}_{N_\gamma}$.

3. Quasi-co-n-absorbing preradicals

Suppose that $m,\ n$ are positive integers with $n > m$. A preradical $\sigma \neq 0$ is called a *quasi-co-(n,m)-absorbing preradical* if whenever $\sigma \preceq (\mu_{[n-1]} : \nu)$ for $\mu, \nu \in R$-pr, then $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : \nu)$.

Proposition 3. *Let $\sigma \in$ R-pr and let $m > 0$. The following conditions are equivalent:*

1. *σ is quasi-co-(n,m)-absorbing for every $n > m$;*
2. *σ is quasi-co-(n,m)-absorbing for some $n > m$;*
3. *σ is quasi-co-m-absorbing.*

Proof. (1)⇒(2) Is trivial.

(2)⇒(3) Assume that σ is quasi-co-(n,m)-absorbing for some $n>m$. Let $\sigma \preceq (\mu_{[m]} : \nu)$ for some $\mu,\nu \in R$-pr. Since $m \leqslant n-1$, then $(\mu_{[m]} : \nu) \preceq (\mu_{[n-1]} : \nu)$ and so $\sigma \preceq (\mu_{[n-1]} : \nu)$. Therefore $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : \nu)$. Consequently σ is quasi-co-m-absorbing.

(3)⇒(1) Suppose that σ is quasi-co-m-absorbing and get $n>m$. Let $\sigma \preceq (\mu_{[n-1]} : \nu)$ for some $\mu,\nu \in R$-pr. Therefore $\sigma \preceq (\mu_{[m]} : (\mu_{[n-1-m]} : \nu))$. Hence $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : (\mu_{[n-1-m]} : \nu)) = (\mu_{[n-2]} : \nu)$. Repeating this method implies that $\sigma \preceq \mu_{[m]}$ or $\sigma \preceq (\mu_{[m-1]} : \nu)$. Thus σ is quasi-co-(n,m)-absorbing. □

Remark 1. Let $\sigma \in R$-pr.

(1) σ is coprime if and only if σ is quasi-co-1-absorbing if and only if σ is co-1-absorbing.
(2) If σ is quasi-co-n-absorbing, then it is quasi-co-i-absorbing for all $i \geqslant n$.
(3) If σ is coprime, then it is quasi-co-n-absorbing for all $n \geqslant 1$.
(4) If σ is quasi-co-n-absorbing for some $n \geqslant 1$, then there exists the least $n_0 \geqslant 1$ such that σ is quasi-co-n_0-absorbing. In this case, σ is quasi-co-n-absorbing for all $n \geqslant n_0$ and it is not quasi-co-i-absorbing for $n_0 > i > 0$.

Proposition 4. *Let $\mathcal{C}$ be a family of coprime preradicals. Then $\bigvee_{\sigma\in\mathcal{C}} \sigma$ is a quasi-co-i-absorbing preradical for every $i \geqslant 2$.*

Proof. Let $\tau = \bigvee_{\sigma\in\mathcal{C}} \sigma$. By Remark 1(2), it is sufficient to show that τ is a quasi-co-2-absorbing preradical. Suppose that $\tau \preceq (\mu_{[2]} : \nu)$ for some $\mu,\nu \in R$-pr. Since every $\sigma \in \mathcal{C}$ is coprime and $\sigma \preceq (\mu_{[2]} : \nu)$, then $\sigma \preceq \mu$ or $\sigma \preceq \nu$. Hence $\tau \preceq (\mu : \nu)$, and so we conclude that τ is a quasi-co-2-absorbing preradical. □

Let $\zeta = \bigvee\{\alpha_S^S \mid S \in R\text{-simp}\}$. Note that for every R-module M, $\zeta(M) = \mathrm{Soc}(M)$. As in [14], ζ is called the socle preradical. Also, let $\kappa = \{\alpha_{R/I}^{R/I} \mid I \text{ a maximal ideal of } R\}$. We call κ the ultrasocle preradical, see [11].

As a direct consequence of Proposition 4 we have the following result.

Proposition 5. *ζ, κ are quasi-co–i-absorbing preradicals for every $i \geqslant 2$.*

Proof. By [18, p. 57], for each simple R-module S, α_S^S is coprime. Also, for every maximal ideal I of R, $\alpha_{R/I}^{R/I}$ is a coprime preradical, [11, Remark 6]. Then by Proposition 4, the claim holds. □

Proposition 6. *If R is a semisimple Artinian ring, then every nonzero preradical $\sigma \in R$-pr is a quasi-co-i-absorbing preradical for every $i \geqslant 2$.*

Proof. Suppose that R is a semisimple Artinian ring. According to [18, Proposition 3.2], every atom $\alpha_S^{E(S)}$ is a coprime preradical. On the other hand [15, Theorem 11] implies that $\sigma = \bigvee\{\alpha_S^{E(S)} \mid S \in R\text{-simp}, \alpha_S^{E(S)} \preceq \sigma\}$. Therefore every nonzero preradical σ in R-pr is quasi-co-i-absorbing for every $i \geqslant 2$, by Proposition 4. $\square$

Remark 2. Let $S_1, S_2, \ldots, S_{n+1} \in R$-simp be distinct. Then by Proposition 4, $\alpha_{S_1}^{S_1} \vee \alpha_{S_2}^{S_2} \vee \cdots \vee \alpha_{S_{n+1}}^{S_{n+1}}$ is a quasi-co-i-absorbing preradical in R-pr for every $i \geqslant 2$. But, [22, Proposition 3.6] implies that $\alpha_{S_1}^{S_1} \vee \alpha_{S_2}^{S_2} \vee \cdots \vee \alpha_{S_{n+1}}^{S_{n+1}}$ is not a co-n-absorbing preradical. This remark shows that the two concepts of quasi-co-n-absorbing preradicals and of co-n-absorbing preradicals are different in general.

Corollary 1. *If R is a ring such that every quasi-co-n-absorbing preradical in R-pr is co-n-absorbing, then $|R\text{-simp}| \leqslant n$.*

Notice the fact that coproduct of preradicals preserves order on both sides.

Proposition 7. *Let R be a ring. The following statements are equivalent:*

(1) *for every $\mu, \nu \in R$-pr, $(\mu_{[n]} : \nu) = \mu_{[n]}$ or $(\mu_{[n]} : \nu) = (\mu_{[n-1]} : \nu)$;*

(2) *for every $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr,*

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]}$$

or

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n-1]} : \sigma_{n+1});$$

(3) *every preredical $0 \neq \sigma \in R$-pr is quasi-co-n-absorbing.*

Proof. (1)$\Rightarrow$(2) If $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, then by part (1) we have that,

$$\begin{aligned}(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) &\preceq ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]} : \sigma_{n+1}) \\ &= (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]},\end{aligned}$$

or

$$\begin{aligned}(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) &\preceq ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n]} : \sigma_{n+1}) \\ &= ((\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n)_{[n-1]} : \sigma_{n+1}).\end{aligned}$$

(2)⇒(1) For preradicals $\mu, \nu \in R$-pr, we have from (2),

$$(\mu_{[n]} : \nu) \preceq (\overbrace{\mu \vee \cdots \vee \mu}^{n \text{ times}})_{[n]} = \mu_{[n]}$$

or

$$(\mu_{[n]} : \nu) \preceq ((\overbrace{\mu \vee \cdots \vee \mu}^{n \text{ times}})_{[n-1]} : \nu) = (\mu_{[n-1]} : \nu).$$

Thus we have that $(\mu_{[n]} : \nu) = \mu_{[n]}$ or $(\mu_{[n]} : \nu) = (\mu_{[n-1]} : \nu)$.

(1)⇔(3) Is evident. □

In the next proposition we use $(\mu_1 : \cdots : \widehat{\mu_i} : \cdots : \mu_{n+1})$ when the *i-th* term is excluded from $(\mu_1 : \cdots : \mu_{n+1})$.

Proposition 8. *Let $0 \neq \sigma \in R$-pr be an idempotent radical.*

(1) *If σ is such that for any $\mu, \nu \in R$-pr, we have*

$$\mu \vee \nu \preceq \sigma \preceq (\mu_{[n]} : \nu) \Rightarrow [\sigma \preceq \mu_{[n]} \text{ or } \sigma \preceq (\mu_{[n-1]} : \nu)],$$

then σ is quasi-co-n-absorbing.

(2) *If σ is such that for any $\mu_1, \mu_2, \ldots, \mu_{n+1} \in R$-pr, we have*

$$\mu_1 \vee \mu_2 \vee \cdots \vee \mu_{n+1} \preceq \sigma \preceq (\mu_1 : \mu_2 : \cdots : \mu_{n+1}) \Rightarrow$$

$$[\sigma \preceq (\mu_1 : \cdots : \widehat{\mu_i} : \cdots : \mu_{n+1}), \quad \text{for some } 1 \leqslant i \leqslant n+1],$$

then σ is a co-n-absorbing preradical.

Proof. (1) Let $\sigma \neq 0$ be an idempotent radical that satisfies the hypothesis in part (1). Let $\sigma \preceq (\tau_{[n]} : \lambda)$ for some $\tau, \lambda \in R$-pr. Then, by [15, Theorem 8(3)] we have

$$\tau\sigma \vee \lambda\sigma \preceq \sigma = \sigma^2 \preceq (\tau_{[n]} : \lambda)\sigma = (\tau_{[n]}\sigma : \lambda\sigma) = ((\tau\sigma)_{[n]} : \lambda\sigma).$$

So, by hypothesis we have $\sigma \preceq (\tau\sigma)_{[n]} = \tau_{[n]}\sigma \preceq \tau_{[n]}$ or $\sigma \preceq ((\tau\sigma)_{[n-1]} : \lambda\sigma) = (\tau_{[n-1]} : \lambda)\sigma \preceq (\tau_{[n-1]} : \lambda)$. Therefore σ is quasi-co-n-absorbing.

(2) The proof is similar to that of (1). □

Proposition 9. *Let $\mathcal{C}$ be a chain of quasi-co-n-absorbing preradicals, that is, a subclass of quasi-co-n-absorbing preradicals which is linearly ordered. Then $\bigvee_{\sigma \in \mathcal{C}} \sigma$ is a quasi-co-n-absorbing preradical.*

Proof. Let $\tau = \bigvee_{\sigma \in \mathcal{C}} \sigma$ and assume that $\tau \preceq (\mu_{[n]} : \nu)$ for some $\mu, \nu \in R$-pr. If $\sigma \preceq \mu_{[n]}$ for each $\sigma \in \mathcal{C}$, then $\tau \preceq \mu_{[n]}$. If there exists $\sigma_0 \in \mathcal{C}$ such that $\sigma_0 \npreceq \mu_{[n]}$, then $\sigma \npreceq \mu_{[n]}$ for each $\sigma_0 \preceq \sigma$. Since all preradicals in $\mathcal{C}$ are quasi-co-n-absorbing, it follows that $\sigma \preceq (\mu_{[n-1]} : \nu)$ for each $\sigma_0 \preceq \sigma$. Thus $\sigma \preceq (\mu_{[n-1]} : \nu)$ for each $\sigma \in \mathcal{C}$, so that $\tau \preceq (\mu_{[n-1]} : \nu)$. Consequently, we deduce that τ is a quasi-co-n-absorbing preradical. $\square$

Proposition 10. *If σ_i is a quasi-co-n_i-absorbing preradical in R-pr for every $1 \leqslant i \leqslant k$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a quasi-co-n-absorbing preradical for $n = n_1 + \cdots + n_k$.*

Proof. For $k = 1$ there is nothing to prove. Then, suppose that $k > 1$. Assume that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq (\mu_{[n]} : \nu)$ for some $\mu, \nu \in R$-pr. Notice that for every $1 \leqslant i \leqslant k$, $\sigma_i \preceq (\mu_{[n]} : \nu) = (\mu_{[n_i]} : \mu_{[n-n_i]} : \nu)$. Then, for every $1 \leqslant i \leqslant k$, either $\sigma_i \preceq \mu_{[n_i]}$ or $\sigma_i \preceq (\mu_{[n_i-1]} : \mu_{[n-n_i]} : \nu) = (\mu_{[n-1]} : \nu)$, because σ_i is quasi-co-n_i-absorbing. On the other hand, for every $1 \leqslant i \leqslant k$, $\mu_{[n_i]} \preceq \mu_{[n-1]}$ and so $\mu_{[n_i]} \preceq (\mu_{[n-1]} : \nu)$. Hence $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq (\mu_{[n-1]} : \nu)$ which shows that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a quasi-co-n-absorbing preradical. $\square$

Proposition 11. *Let $\sigma_1, \sigma_2, \ldots, \sigma_t \in R$-pr.*

(1) *If σ_1 is a quasi-co-n-absorbing preradical and σ_2 is a quasi-co-m-absorbing preradical for $m \leqslant n$, then $\sigma_1 \vee \sigma_2$ is a quasi-co-$(n+1)$-absorbing preradical.*

(2) *If $\sigma_1, \sigma_2, \ldots, \sigma_t$ are quasi-co-n-absorbing preradicals, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is a quasi-co-$(n+t-1)$-absorbing preradical.*

(3) *If σ_i is a quasi-co-n_i-absorbing preradical for every $1 \leqslant i \leqslant t$ with $n_1 < n_2 < \cdots < n_t$ and $t > 2$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_t$ is a quasi-co-$(n_t + 1)$-absorbing preradical.*

Proof. (1) Let $\mu, \nu \in R$-pr be such that $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n+1]} : \nu)$. Since σ_1 is quasi-co-n-absorbing and $\sigma_1 \preceq (\mu_{[n]} : \mu : \nu)$, then either $\sigma_1 \preceq \mu_{[n]}$ or $\sigma_1 \preceq (\mu_{[n-1]} : \mu : \nu) = (\mu_{[n]} : \nu)$. Also, σ_2 is quasi-co-m-absorbing and $\sigma_2 \preceq (\mu_{[m]} : \mu_{[n+1-m]} : \nu)$, so either $\sigma_2 \preceq \mu_{[m]}$ or $\sigma_2 \preceq (\mu_{[m-1]} : \mu_{[n+1-m]} : \nu) = (\mu_{[n]} : \nu)$. There are four cases.

Case 1. Assume that $\sigma_1 \preceq \mu_{[n]}$ and $\sigma_2 \preceq \mu_{[m]}$. Then $\sigma_1 \vee \sigma_2 \preceq \mu_{[n]}$.

Case 2. Assume that $\sigma_1 \preceq \mu_{[n]}$ and $\sigma_2 \preceq (\mu_{[n]} : \nu)$. Then $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n]} : \nu)$.

Case 3. Assume that $\sigma_1 \preceq (\mu_{[n]} : \nu)$ and $\sigma_2 \preceq \mu_{[m]}$. Then $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n]} : \nu)$.

Case 4. Assume that $\sigma_1 \preceq (\mu_{[n]} : \nu)$ and $\sigma_2 \preceq (\mu_{[n]} : \nu)$. Then $\sigma_1 \vee \sigma_2 \preceq (\mu_{[n]} : \nu)$. Hence $\sigma_1 \vee \sigma_2$ is quasi-co-$(n+1)$-absorbing.

(2) We use induction on t. For $t=1$ there is nothing to prove. Let $t>1$ and assume that for $t-1$ the claim holds. Then $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_{t-1}$ is quasi-co-$(n+t-2)$-absorbing. Since σ_t is quasi-co-n-absorbing, then it is quasi-co-$(n+t-2)$-absorbing, by Remark 1(2). Therefore $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_t$ is quasi-co-$(n+t-1)$-absorbing, by part (1).

(3) Induction on t: For $t=3$ apply parts (1) and (2). Let $t>3$ and suppose that for $t-1$ the claim holds. Hence $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_{t-1}$ is quasi-co-$(n_{t-1}+1)$-absorbing. We consider the following cases:

Case 1. Let $n_{t-1}+1<n_t$. In this case $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_t$ is quasi-co-(n_t+1)-absorbing, by part (1).

Case 2. Let $n_{t-1}+1=n_t$. Thus $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_t$ is quasi-co-(n_t+1)-absorbing, by part (2).

Case 3. Let $n_{t-1}+1>n_t$. Then $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_t$ is quasi-co-$(n_{t-1}+2)$-absorbing, by part (1). Since $n_{t-1}+2\leqslant n_t+1$, then $\sigma_1\vee\sigma_2\vee\cdots\vee\sigma_t$ is quasi-co-(n_t+1)-absorbing. $\square$

Proposition 12. *Let $\sigma\in R$-pr be a radical. If σ is quasi-co-n-absorbing, then $e(\sigma)$ is quasi-co-n-absorbing.*

Proof. Assume that σ is quasi-co-n-absorbing, and let $e(\sigma)\preceq(\mu_{[n]}:\nu)$ for some $\mu,\nu\in R$-pr. Then $\sigma=e(\sigma)\sigma\preceq(\mu_{[n]}:\nu)\sigma\preceq((\mu\sigma)_{[n]}:\nu\sigma)$. Since σ is quasi-co-n-absorbing and radical, [15, Theorem 8(3)] implies that either $\sigma\preceq(\mu\sigma)_{[n]}=\mu_{[n]}\sigma\preceq\mu_{[n]}$ or $\sigma\preceq((\mu\sigma)_{[n-1]}:\nu\sigma)=(\mu_{[n-1]}:\nu)\sigma\preceq(\mu_{[n-1]}:\nu)$. Consequently $e(\sigma)$ is quasi-co-n-absorbing. $\square$

Definition 1. For $\tau,\ \rho\in R$-pr define the totalizer of ρ relative to τ as $t_\tau(\rho)=\bigwedge\{\eta\in R\text{-pr}|\ (\rho:\eta)\succeq\tau\}$. Note that $t_1(\rho)=t(\rho)$.

Proposition 13. *Let $\tau\in R$-pr. If τ is quasi-co-2-absorbing, then for each $\lambda\in R$-pr, either $\tau\preceq\lambda_{[n]}$ or $t_\tau(\lambda_{[n]})=t_\tau(\lambda_{[n-1]})$. In particular, if 1 is a quasi-co-2-absorbing preradical, then for each $\lambda\in R$-pr, either $\lambda_{[n]}=1$ or $t(\lambda_{[n]})=t(\lambda_{[n-1]})$.*

Proof. Suppose that τ is quasi-co-2-absorbing and let $\lambda\in R$-pr such that $\tau\npreceq\lambda_{[n]}$. If $\nu\in R$-pr is such that $\tau\preceq(\lambda_{[n]}:\nu)$, then $\tau\preceq(\lambda_{[n-1]}:\nu)$, since σ is quasi-co-2-absorbing. Therefore $t_\tau(\lambda_{[n-1]})\preceq t_\tau(\lambda_{[n]})$. On the other hand $\lambda_{[n-1]}\preceq\lambda_{[n]}$ and so $t_\tau(\lambda_{[n]})\preceq t_\tau(\lambda_{[n-1]})$. Consequently $t_\tau(\lambda_{[n]})=t_\tau(\lambda_{[n-1]})$. $\square$

4. Semi-co-n-absorbing preradicals

Suppose that $m,\ n$ are positive integers with $n > m$. A more general concept than semi-co-n-absorbing preradicals is the concept of semi-co-(n, m)-absorbing preradicals. A preradical $\sigma \neq 0$ is called a *semi-co-(n, m)-absorbing preradical* if whenever $\sigma \preceq \mu_{[n]}$ for $\mu \in R$-pr, then $\sigma \preceq \mu_{[m]}$.

Note that a semicoprime preradical is just a semi-co-1-absorbing preradical.

Theorem 1. *Let $\sigma \in R$-pr and $m,\ n$ be positive integers with $n > m$.*

(1) *If σ is quasi-co-m-absorbing, then it is semi-co-(k, m)-absorbing for every $k > m$.*

(2) *If σ is semi-co-(n, m)-absorbing, then it is semi-co-(i, m)-absorbing for every $m < i < n$, in particular it is semi-co-m-absorbing.*

(3) *σ is semi-co-(n, m)-absorbing if and only if σ is semi-co-(n, k)-absorbing for each $n > k \geqslant m$ if and only if σ is semi-co-(i, j)-absorbing for each $n \geqslant i > j \geqslant m$.*

(4) *If σ is semi-co-(n, m)-absorbing, then it is semi-co-(nk, mk)-absorbing for every positive integer k.*

(5) *If σ is semi-co-(n, m)-absorbing and semi-co-(r, s)-absorbing for some positive integers $r > s$, then it is semi-co-(nr, ms)-absorbing.*

Proof. (1) Is trivial.

(2) Is easy.

(3) Straightforward.

(4) Suppose that σ is semi-co-(n, m)-absorbing. Let $\mu \in R$-pr and let k be a positive integer such that $\sigma \preceq \mu_{[nk]}$. Then $\sigma \preceq \left(\mu_{[k]}\right)_{[n]}$. Since σ is semi-co-(n, m)-absorbing, $\sigma \preceq \left(\mu_{[k]}\right)_{[m]} = \mu_{[mk]}$, and so σ is semi-co-(nk, mk)-absorbing.

(5) Assume that σ is semi-co-(n, m)-absorbing and semi-co-(r, s)-absorbing for some positive integers $r > s$. Let $\sigma \preceq \mu_{[nr]}$. Since σ is semi-co-(n, m)-absorbing, then $\sigma \preceq \mu_{[mr]}$; and since σ is semi-co-(r, s)-absorbing, $\sigma \preceq \mu_{[ms]}$. Hence σ is semi-co-(nr, ms)-absorbing. $\square$

Corollary 2. *Let $\sigma \in R$-pr and n be a positive integer.*

(1) *If σ is quasi-co-n-absorbing, then it is semi-co-n-absorbing.*

(2) *Let $t \leqslant n$ be an integer. If σ is semi-co-$(n+1, t)$-absorbing, then it is semi-co-$(nk+i, tk)$-absorbing for all $k \geqslant i \geqslant 1$.*

(3) *If σ is semi-co-n-absorbing, then it is semi-co-$(nk+i, nk)$-absorbing for all $k \geqslant i \geqslant 1$.*

(4) *If σ is semi-co-n-absorbing, then it is semi-co-$(nk+j)$-absorbing for all $k > j \geqslant 0$.*
(5) *If σ is semi-co-n-absorbing, then it is semi-co-(nk)-absorbing for every positive integer k.*
(6) *If σ is semicoprime, then it is semi-co-k-absorbing for every positive integer k.*
(7) *If σ is semicoprime, then for every $k \geqslant 1$ and every $\mu \in R$-pr, $\sigma \preceq \mu_{[k]}$ implies that $\sigma \preceq \mu$.*
(8) *If σ is semi-co-n-absorbing, then it is semi-co-$((n+1)^t, n^t)$-absorb -ing for all $t \geqslant 1$.*
(9) *If σ is semicoprime, then it is quasi-co-k-absorbing for every $k > 1$.*

Proof. (1) By parts (1), (2) of Theorem 1.

(2) Let σ be semi-co-$(n+1,t)$-absorbing. Then by Theorem 1(4), σ is semi-co-$(nk+k,tk)$-absorbing, for every positive integer k. Hence by Theorem 1(2), σ is semi-co-$(nk+i,tk)$-absorbing for every $k \geqslant i \geqslant 1$.

(3) In part (2) get $t = n$.

(4) By part (3).

(5) Is a special case of (4).

(6) Is a direct consequence of (5).

(7) By part (6).

(8) By Theorem 1(5).

(9) Assume that σ is semicoprime. Let $\sigma \preceq (\mu_{[k]} : \nu)$ for some $\mu, \nu \in R$-pr and some $k > 1$. Then $\sigma \preceq (\mu_{[k]} : \nu) \preceq (\mu : \nu)_{[k]}$. Therefore $\sigma \preceq (\mu : \nu)$, by part (7). So σ is quasi-co-k-absorbing. □

In the following remark we prove Proposition 4 in another way.

Remark 3. Clearly, an arbitrary join of a family of semicoprime (coprime) preradicals is semicoprime, and so it is quasi-co-k-absorbing for every $k > 1$, by Corollary 2(9).

Proposition 14. *Let $\sigma_1, \sigma_2, \ldots, \sigma_n \in R$-pr. If for every $1 \leqslant i \leqslant n$, σ_i is a semicoprime preradical, then $(\sigma_1 : \sigma_2 : \cdots : \sigma_n)$ is a semi-co-n-absorbing preradical. In particular, if σ is a semicoprime preradical, then $\sigma_{[n]}$ is a semi-co-n-absorbing preradical.*

Proof. Apply Corollary 2(7). □

Lemma 1. *Let $\sigma \in R$-pr. If $\sigma_{[n+1]}$ is a semi-co-n-absorbing preradical, then $\sigma_{[n+1]} = \sigma_{[n]}$. In particular, if $\sigma_{[2]}$ is a semicoprime preradical, then σ is radical.*

Proposition 15. *Let $\sigma \in R$-pr, $\sigma \neq 0$ be an idempotent radical. If σ is such that for any $\mu \in R$-pr, we have $\mu \preceq \sigma \preceq \mu_{[n+1]} \Rightarrow \sigma \preceq \mu_{[n]}$, then σ is semi-co-n-absorbing.*

Proof. The proof is similar to that of Proposition 8(1). □

Proposition 16. *Let $\sigma_1, \sigma_2, \ldots, \sigma_n \in R$-pr be semi-co-2-absorbing preradicals. Then $(\sigma_1 : \sigma_2 : \cdots : \sigma_n)$ is a semi-co-$(3^n - 1)$-absorbing preradical.*

Proof. Suppose that $(\sigma_1 : \sigma_2 : \cdots : \sigma_n) \preceq \mu_{[3^n]}$ for some $\mu \in R$-pr. For every $1 \leqslant i \leqslant n$, $\sigma_i \preceq \mu_{[3^n]} = \left(\mu_{[3^{n-1}]}\right)_{[3]}$ and σ_i is semi-co-2-absorbing, then $\sigma_i \preceq \left(\mu_{[3^{n-1}]}\right)_{[2]} = \mu_{[2\cdot 3^{n-1}]} = \left(\mu_{[2\cdot 3^{n-2}]}\right)_{[3]}$. Again, since σ_i is semi-co-2-absorbing, we conclude that $\sigma_i \preceq \mu_{[2^2 . 3^{n-2}]}$. Repeating this method implies that $\sigma_i \preceq \mu_{[2^n]}$. So $(\sigma_1 : \sigma_2 : \cdots : \sigma_n) \preceq \mu_{[n2^n]}$. On the other hand $n2^n \leqslant 3^n - 1$. So $(\sigma_1 : \sigma_2 : \cdots : \sigma_n) \preceq \mu_{[3^n - 1]}$ which shows that $(\sigma_1 : \sigma_2 : \cdots : \sigma_n)$ is semi-co-$(3^n - 1)$-absorbing. □

Proposition 17. *If σ_i is a semi-co-n_i-absorbing preradical in R-pr for every $1 \leqslant i \leqslant k$, then $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a semi-co-$(n-1)$-absorbing preradical for $n = \prod\limits_{i=1}^{k} (n_i + 1)$.*

Proof. Let $\mu \in R$-pr be such that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq \mu_{[n]}$. Thus for every $1 \leqslant i \leqslant k$, $\sigma_i \preceq \left(\mu_{[m]}\right)_{[n_i+1]}$, where $m = \prod\limits_{j=1,\ j\neq i}^{k} (n_j + 1)$. Since σ_i's are semi-co-n_i-absorbing, then, for each $1 \leqslant i \leqslant k$, $\sigma_i \preceq \mu_{[n_i m]}$. Note that for every $1 \leqslant i \leqslant k$,

$$n_i m \leqslant \prod_{i=1}^{k} (n_i + 1) - 1 = n - 1.$$

So we have $\sigma_i \preceq \mu_{[n-1]}$ for every $1 \leqslant i \leqslant k$. Hence $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k \preceq \mu_{[n-1]}$ which implies that $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_k$ is a semi-co-$(n-1)$-absorbing preradical. □

Proposition 18. *Let $\sigma_1, \sigma_2 \in R$-pr and $m,\ n$ be positive integers.*

(1) *If σ_1 is quasi-co-m-absorbing and σ_2 is semi-co-n-absorbing, then $(\sigma_1 : \sigma_2)$ is semi-co-$(n(m+1)+m)$-absorbing.*
(2) *If σ_1 is quasi-co-$(2m)$-absorbing and σ_2 is semi-co-m-absorbing, then $(\sigma_1 : \sigma_2)$ is semi-co-$(m^2 + 2m)$-absorbing.*

Proof. (1) Suppose that $(\sigma_1 : \sigma_2) \preceq \mu_{[(n+1)(m+1)]}$ for some $\mu \in R$-pr. Since σ_1 is quasi-co-m-absorbing and $\sigma_1 \preceq \mu_{[(n+1)(m+1)]}$, then $\sigma_1 \preceq \mu_{[m]}$.

On the other hand σ_2 is semi-co-n-absorbing and $\sigma_2 \preceq \mu_{[(n+1)(m+1)]}$, then $\sigma_2 \preceq \mu_{[n(m+1)]}$. Consequently $(\sigma_1 : \sigma_2) \preceq \mu_{[n(m+1)+m]}$, and so $(\sigma_1 : \sigma_2)$ is semi-co-$(n(m+1)+m)$-absorbing.

(2) Suppose that $(\sigma_1 : \sigma_2) \preceq \mu_{[(m+1)^2]}$ for some $\mu \in R$-pr. Since σ_1 is quasi-co-$(2m)$-absorbing and $\sigma_1 \preceq \mu_{[(m+1)^2]}$, then $\sigma_1 \preceq \mu_{[2m]}$. Since σ_2 is semi-co-m-absorbing and $\sigma_2 \preceq \mu_{[(m+1)^2]}$, then $\sigma_2 \preceq \mu_{[m^2]}$. Hence $(\sigma_1 : \sigma_2) \preceq \mu_{[m^2+2m]}$ which shows that $(\sigma_1 : \sigma_2)$ is semi-co-(m^2+2m)-absorbing. □

Proposition 19. *Let R be a ring. The following statements are equivalent:*

(1) *for every preradical $\sigma \in R$-pr, $\sigma_{[n+1]} = \sigma_{[n]}$;*

(2) *for all preradicals $\sigma_1\sigma_2, \ldots, \sigma_{n+1} \in R$-pr we have*

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n+1})_{[n]};$$

(3) *every preredical $0 \neq \sigma \in R$-pr is semi-co-n-absorbing.*

Proof. (1)⇒(2) If $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, then we get from (1),

$$(\sigma_1 : \sigma_2 : \cdots : \sigma_{n+1}) \preceq (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n+1})_{[n+1]} = (\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n+1})_{[n]}.$$

(2)⇒(1) For a preradical $\sigma \in R$-pr, we have from (2),

$$\sigma_{[n+1]} \preceq (\overbrace{\sigma \vee \cdots \vee \sigma}^{n+1 \text{ times}})_{[n]} = \sigma_{[n]}.$$

So we have that $\sigma_{[n+1]} = \sigma_{[n]}$.

(1)⇔(3) Is clear. □

Remark 4. Let $\{\sigma_\alpha\}_{\alpha \in I} \subseteq R$-pr. If σ_α is semi-co-n-absorbing for every $\alpha \in I$, then $\bigvee_{\alpha \in I} \sigma_\alpha$ is semi-co-n-absorbing.

Proposition 20. *Let $\sigma \in R$-pr be radical. If σ is semi-co-n-absorbing, then $e(\sigma)$ is semi-co-n-absorbing.*

Proof. Is similar to the proof of Proposition 12. □

In Proposition 23 of [11], it was shown that $\sigma^0 := \bigvee\{\sigma \in R\text{-pr} \mid \sigma$ is semicoprime$\}$ is the unique greatest semicoprime preradical.

Proposition 21. *There exists in R-pr a unique greatest semi-co-n-absorbing preradical.*

Proof. Set $\sigma^0_{(n)} = \bigvee\{\sigma \in R\text{-pr} \mid \sigma \text{ is semi-co-}n\text{-absorbing}\}$. By Remark 4, $\sigma^0_{(n)}$ is the greatest semi-co-n-absorbing preradical. □

By notation in the the proof of the previous proposition we have that $\sigma^0_{(1)} = \sigma^0$.

Remark 5. As $\zeta \preceq \kappa \preceq \sigma^0$ are semicoprime preradicals, then $\zeta_{[n]}, \kappa_{[n]}$, $\sigma^0_{[n]}$ are semi-co-n-absorbing preradicals, by Proposition 14. Therefore $\zeta_{[n]} \preceq \kappa_{[n]} \preceq \sigma^0_{[n]} \preceq \sigma^0_{(n)}$.

Proposition 22. *The following statements hold:*
(1) $\sigma^0 = \bigwedge_{n\geqslant 1} \sigma^0_{(n)}$.
(2) $\sigma^0_{(n)} \preceq \sigma^0_{[nk]}$ *for every positive integer* k.
(3) $\sigma_{[n]} \preceq \sigma^0_{(n)}$ *for every semicoprime preradical* σ.

Proof. (1) By Corollary 2(6) every semicoprime preradical is semi-co-n-absorbing for every $n \geqslant 1$. Then $\sigma^0 \preceq \sigma^0_{(n)}$ for every $n \geqslant 1$.

(2) By Corollary 2(5).

(3) By Proposition 14. □

In Proposition 26 of [11] it was shown that $\sigma^0 \preceq \nu_0$, where $\nu_0 = \bigwedge\{\tau \mid \tau \in R\text{-pr},\ \tau \text{ is unipotent}\}$.

The following proposition is straightforward.

Proposition 23. *Suppose that* $\nu_0^{(n)} := \bigwedge\{\tau_{[n]} \mid \tau \in R\text{-}pr,\ \tau_{[n+1]} = 1\}$. *Then:*
(1) $\sigma^0_{(n)} \preceq \nu_0^{(n)}$;
(2) $\nu_0 \preceq \nu_0^{(1)}$.

Corollary 3. *The following statements hold:*
(1) *If* $\zeta_{[n+1]} = 1$, *then* $\zeta_{[n]} = \kappa_{[n]} = \sigma^0_{[n]} = \sigma^0_{(n)} = \nu_0^{(n)}$;
(2) *If* $\zeta_{[2]} = 1$, *then* $\zeta = \kappa = \sigma^0 = \nu_0 = \nu_0^{(1)}$.

Proof. (1) By Remark 5 and Proposition 23 we have that $\zeta_{[n]} \preceq \kappa_{[n]} \preceq \sigma^0_{[n]} \preceq \sigma^0_{(n)} \preceq \nu_0^{(n)}$. If $\zeta_{[n+1]} = 1$, then $\nu_0^{(n)} \preceq \zeta_{[n]}$, and so $\zeta_{[n]} = \kappa_{[n]} = \sigma^0_{[n]} = \sigma^0_{(n)} = \nu_0^{(n)}$.

(2) By part (1) and [11, Corollary 27]. □

Proposition 24. *For a ring* R *the following statements are equivalent:*
(1) *For every* $\mu \in R\text{-}pr$, $\mu_{[n+1]} = 1$ *implies that* $\mu_{[n]} = 1$;

(2) 1 *is a semi-co-n-absorbing preradical;*
(3) $\sigma^0_{(n)} = 1$;
(4) $\nu^{(n)}_0 = 1$.

Proof. Is easy. □

For $\tau \in R$-pr define

$$C^{(n)}(\tau) = \bigvee\{\sigma \in R\text{-pr} \mid \sigma \preceq \tau,\ \sigma \text{ semi-co-}n\text{-absorbing}\},$$

which is the unique greatest semi-co-n-absorbing preradical less than or equal to τ. Notice that in [11], $C^{(1)}$ is denoted by C.

Proposition 25. *Let R be a ring.*
(1) $\sigma^0_{(n)} = C^{(n)}(1) = \bigvee_{\tau\in R\text{-}pr} C^{(n)}(\tau)$.
(2) *For each $\tau \in R$-pr, $C^{(n)}(\tau) \preceq \tau$.*
(3) *For each $\tau, \sigma \in R$-pr we have $\tau \preceq \sigma \Rightarrow C^{(n)}(\tau) \preceq C^{(n)}(\sigma)$.*
(4) *For each $\tau \in R$-pr, $C^{(n)}(\tau_{[n+1]}) = C^{(n)}(\tau_{[n]})$.*
(5) *For each $\tau \in R$-pr, τ is semi-co-n-absorbing if and only if $\tau = C^{(n)}(\tau)$.*
(6) $\{\tau \in R\text{-}pr \mid \tau$ *is semi-co-n-absorbing*$\} = Im\ C^{(n)} = \{C^{(n)}(\sigma) \mid \sigma \in R\text{-}pr\}$.
(7) $\left[C^{(n)}\right]^2 = C^{(n)}$. *Thus, $C^{(n)}$ is a closure operator on R-pr.*
(8) *For each family $\{\tau_\alpha\}_{\alpha\in I} \subseteq R$-pr, we have*

$$C^{(n)}(\bigwedge_{\alpha\in I}\tau_\alpha) = C^{(n)}(\bigwedge_{\alpha\in I} C^{(n)}(\tau_\alpha)).$$

(9) $C^{(n)} = \bigwedge_{k\geqslant 1} C^{(nk)}$, *in particular* $C = \bigwedge_{k\geqslant 1} C^{(k)}$.
(10) $C^{(n)}(\sigma_{[n+1]}) = C^{(n)}(\sigma_{[n]}) = \sigma_{[n]}$ *for any semicoprime preradical σ.*

Proof. The proofs of (1), (2), (3), (5) and (6) is easy.

(4) For any $\tau \in R$-pr, part (3) implies that $C^{(n)}(\tau_{[n]}) \preceq C^{(n)}(\tau_{[n+1]})$. Since $C^{(n)}(\tau_{[n+1]})$ is semi-co-n-absorbing (by Remark 4) and $C^{(n)}(\tau_{[n+1]}) \preceq \tau_{[n+1]}$, then $C^{(n)}(\tau_{[n+1]}) \preceq \tau_{[n]}$. Hence $C^{(n)}(\tau_{[n+1]}) \preceq C^{(n)}(\tau_{[n]})$. So the equality holds.

(7) Is a direct consequence of part (5).

(8) The proof is similar to that of [11, Proposition 31](5).

(9) By Corollary 2(5).

(10) Apply Proposition 14 and parts (4), (5). □

Now consider the operator $\overline{(\_)}$ in R-pr that assigns to each preradical σ the least radical over σ (see [19, p. 137]).

Lemma 2. *Let $\sigma, \tau \in R$-pr be such that σ is radical and τ is semi-co-n-absorbing. Then:*

(1) $C^{(n)}(\sigma) \preceq \overline{C^{(n)}(\sigma)} \preceq \sigma$.
(2) $C^{(n)}(\sigma) = C^{(n)}(\overline{C^{(n)}(\sigma)})$.
(3) $\tau \preceq C^{(n)}(\overline{\tau}) \preceq \overline{\tau}$.
(4) $\overline{\tau} = \overline{C^{(n)}(\overline{\tau})}$.

Proof. Similar to the proof of [11, Lemma 32]. □

Proposition 26. *Let R be a ring.*

(1) *The operator $\overline{C^{(n)}(\_)}$ defines an interior operator on the ordered class of radicals.*
(2) *The operator $C^{(n)}(\overline{(\_)})$ defines a closure operator on the ordered class of semi-co-n-absorbing preradicals.*

Notice that the "open" radicals associated with the interior operator $\overline{C^{(n)}(\_)}$ are

$$\mathcal{O}_{rad}^{(n)} = \{\sigma \text{ radical} \mid \sigma = \overline{\tau} \text{ for some semi-co-}n\text{-absorbing } \tau\}.$$

The "closed" semi-co-n-absorbing preradicals associated with the closure operator $C^{(n)}(\overline{(\_)})$ are

$$\mathcal{C}_{sca}^{(n)} = \{\tau \text{ semi-co-}n\text{-absorbing} \mid \tau = C^{(n)}(\sigma) \text{ for some radical } \sigma\}.$$

The following result is immediate.

Corollary 4. *For a ring R the operators $C^{(n)}(\_)$ and $\overline{(\_)}$ restrict to mutually inverse maps between $\mathcal{O}_{rad}^{(n)}$ and $\mathcal{C}_{sca}^{(n)}$.*

Definition 2. Let $\tau \in R$-pr. Define

$$C_1^{(n)}(\tau) = \bigwedge\{\sigma_{[n]} \mid \sigma \in R\text{-pr}, \tau \preceq \sigma_{[n+1]}\}.$$

Proposition 27. *For a ring R the following conditions hold:*

(1) *For each $\tau \in R$-pr, $C_1^{(n)}(\tau) \preceq \tau_{[n]}$.*
(2) *For each $\tau \in R$-pr, τ is semi-co-n-absorbing if and only if $\tau \preceq C_1^{(n)}(\tau)$.*
(3) *1 is a semi-co-n-absorbing preradical if and only if $C_1^{(n)}(1) = 1$.*

(4) *Let* $\tau,\ \sigma \in R$*-pr. If* $\tau \preceq \sigma$*, then* $C_1^{(n)}(\tau) \preceq C_1^{(n)}(\sigma)$.

(5) *For each family* $\{\tau_\alpha\}_{\alpha\in I} \subseteq R$*-pr,* $C_1^{(n)}(\bigwedge_{\alpha\in I} \tau_\alpha) \preceq \bigwedge_{\alpha\in I} C_1^{(n)}(\tau_\alpha)$ *and* $\bigvee_{\alpha\in I} C_1^{(n)}(\tau_\alpha) \preceq C_1^{(n)}(\bigvee_{\alpha\in I} \tau_\alpha)$.

Proof. The assertions have straightforward verifications. □

We apply an "Amitsur construction" to $C_1^{(n)}$ as follows:

Definition 3. Let $\tau \in R$-pr. We define recursively the preradical $C_\lambda^{(n)}(\tau)$ for each ordinal λ as follows:

(1) $C_0^{(n)}(\tau) = \tau$.

(2) $C_{\lambda+1}^{(n)}(\tau) = C_1^{(n)}(C_\lambda^{(n)}(\tau))$.

(3) If λ is a limit ordinal, then $C_\lambda^{(n)}(\tau) = \bigwedge_{\beta<\lambda} C_\beta^{(n)}(\tau)$.

(4) $C_\Omega^{(n)}(\tau) = \bigwedge_{\lambda \text{ ordinal}} C_\lambda^{(n)}(\tau)$.

Proposition 28. *Let* $\tau \in R$*-pr. Then the following statements are equivalent:*

(1) τ *is semi-co-*n*-absorbing;*

(2) *For each ordinal* λ, $\tau \preceq C_\lambda^{(n)}(\tau)$*;*

(3) $C_\Omega^{(n)}(\tau) = \tau$.

Proof. By Proposition 27 and using transfinite induction we have the claim. □

As is the case with $C_1^{(n)}$, all of the operators $C_\lambda^{(n)}$ preserve order between preradicals.

Proposition 29. *Let* $\tau,\ \sigma \in R$*-pr be such that* $\tau \preceq \sigma$*. Then:*

(1) *For each ordinal* λ, $C_\lambda^{(n)}(\tau) \preceq C_\lambda^{(n)}(\sigma)$.

(2) $C_\Omega^{(n)}(\tau) \preceq C_\Omega^{(n)}(\sigma)$.

Proposition 30. *For each* $\tau \in R$*-pr,* $C^{(n)}(\tau) \preceq C_\Omega^{(n)}(\tau)$.

Proof. Let $\tau \in R$-pr. We use transfinite induction. First, note that $C^{(n)}(\tau) \preceq \tau = C_0^{(n)}(\tau)$. Assume that λ is an ordinal such that $C^{(n)}(\tau) \preceq C_\lambda^{(n)}(\tau)$. Since $C^{(n)}(\tau)$ is semi-co-n-absorbing, $C^{(n)}(\tau) \preceq C_1^{(n)}(C^{(n)}(\tau)) \preceq C_1^{(n)}(C_\lambda^{(n)}(\tau)) = C_{\lambda+1}^{(n)}(\tau)$, by parts (2) and (4) of Proposition 27. If λ is a limit ordinal and $C^{(n)}(\tau) \preceq C_\beta^{(n)}(\tau)$ for each $\beta < \lambda$, then $C^{(n)}(\tau) \preceq \bigwedge_{\beta<\lambda} C_\beta^{(n)}(\tau) = C_\lambda^{(n)}(\tau)$. □

In the following result we give equivalent conditions for the equality $C_\Omega^{(n)}(\tau) = C^{(n)}(\tau)$.

Proposition 31. *For each $\tau \in R$-pr the following statements are equivalent:*

(1) $C_\Omega^{(n)}(\tau)$ *is semi-co-n-absorbing;*
(2) $C_\Omega^{(n)}(\tau) \preceq C_1^{(n)}(C_\Omega^{(n)}(\tau))$*;*
(3) *For each ordinal λ we have* $C_\Omega^{(n)}(\tau) \preceq C_\lambda^{(n)}(C_\Omega^{(n)}(\tau))$*;*
(4) $C_\Omega^{(n)}(C_\Omega^{(n)}(\tau)) = C_\Omega^{(n)}(\tau)$*;*
(5) $C_\Omega^{(n)}(\tau) = C^{(n)}(\tau)$.

Proof. (1)⇒(2) By Proposition 27(2).

(2)⇒(3) It follows by using transfinite induction on λ.

(3)⇒(4) Is easy.

(4)⇒(1) By Proposition 28.

(1)⇒(5) Assume that $C_\Omega^{(n)}(\tau)$ is semi-co-n-absorbing. Since $C_\Omega^{(n)}(\tau) \preceq \tau$, the definition of $C^{(n)}(\tau)$ implies that $C_\Omega^{(n)}(\tau) \preceq C^{(n)}(\tau)$. On the other hand $C^{(n)}(\tau) \preceq C_\Omega^{(n)}(\tau)$, by Proposition 30. So the equality holds.

(5)⇒(1) Is straightforward. □

5. Quasi-co-n-absorbing and semi-co-n-absorbing submodules

Remark 6. Let $M \in R$-co-ass and N be a nonzero fully invariant submodule of M. Then we have:

(1) N is co-n-absorbing in $M \Rightarrow N$ is quasi-co-n-absorbing in $M \Rightarrow N$ is semi-co-n-absorbing in M.
(2) N is a quasi-co-1-absorbing submodule of M if and only if N is a coprime submodule of M.
(3) N is a semi-co-1-absorbing submodule of M if and only if N is a semicoprime submodule of M.

Proposition 32. *Let $\sigma \in R$-pr. If for every $M \in R$-Mod, $\sigma(M)$ is a semicoprime submodule of M, then σ is a semicoprime preradical.*

Proof. By hypothesis, [11, Proposition 19] implies that $\alpha^M_{\sigma(M)}$ is a semicoprime preradical. So $\sigma = \bigvee\{\alpha^M_{\sigma(M)} \mid M \in R\text{-Mod}\}$ (see [17, Remark 1]) is a semicoprime preradical. □

Corollary 5. *Let R be a ring. If every nonzero R-module is semicoprime, then 1 is a semicoprime preradical in R-pr.*

Lemma 3 ([7, Lemma 2.5]). *Let $M \in R$-Mod. Then for any submodules $N,\ K$ of M, $\alpha^M_{N+K} = \alpha^M_N \vee \alpha^M_K$.*

Proposition 33. *Let $M \in R$-Mod. Suppose that $\{N_i\}_{i\in I}$ is a family of semicoprime submodules of M. Then $N = \sum\limits_{i\in I} N_i$ is a semicoprime submodule of M.*

Proof. Let $\{N_i\}_{i\in I}$ be a family of semicoprime submodules of M. Then, by [11, Proposition 19], $\alpha^M_{N_j}$'s are semicoprime preradicals. Thus $\alpha^M_N = \bigvee_{i\in I} \alpha^M_{N_i}$ is a semicoprime preradical. Again by [11, Proposition 19], $N = \sum\limits_{i\in I} N_i$ is a semicoprime submodule of M. □

Proposition 34. *Let R be a ring and $\{M_i\}_{i\in I}$ be a family of semicoprime R-modules. Then $M = \bigoplus\limits_{i\in I} M_i$ is a semicoprime R-module.*

Proof. Since for every $i \in I$, M_i is a semicoprime R-module, then for every $i \in I$, $\alpha^{M_i}_{M_i}$ is a semicoprime preradical, by [11, Proposition 19]. Therefore $\bigvee\limits_{i\in I} \alpha^{M_i}_{M_i} = \alpha^M_M$ is a semicoprime preradical, and so again by [11, Proposition 19], $M = \bigoplus\limits_{i\in I} M_i$ is a semicoprime R-module. □

Proposition 35. *For a ring R the following statements are equivalent:*
1. *R is a finite product of simple rings;*
2. *$\kappa = 1$;*
3. *1 is a semicoprime preradical;*
4. *${}_RR$ is a semicoprime R-module;*
5. *There exists a semicoprime R-module that is a generator in R-Mod.*

Proof. (1)⇔(2) By [11, Theorem 10].

(1)⇔(3) By [11, Theorem 29].

(3)⇔(4) Notice the fact that an R-module G is a generator in R-Mod if and only if $\alpha^G_G = 1$. Since R is a generator in R-Mod, then $\alpha^R_R = 1$. Now, use [11, Proposition 19].

(4)⇒(5) Is trivial.

(5)⇒(3) See the proof of (3)⇔(4). □

Theorem 2. *Let $M \in R$-co-ass and N a fully invariant submodule of M. Consider the following statements.*

(a) N is co-n-absorbing in M.

(b) α^M_N is a co-n-absorbing preradical.

Then $(b) \Rightarrow (a)$, and if M satisfies the ω-property, then $(a) \Rightarrow (b)$.

Proof. The proof is similar to that of [22, Theorem 4.2]. □

Theorem 3. *Let $M \in R$-co-ass and N a fully invariant submodule of M. Consider the following statements:*

(1) *N is quasi-co-n-absorbing (resp. semi-co-n-absorbing) in M.*

(2) *α_N^M is a quasi-co-n-absorbing (resp. semi-co-n-absorbing) preradical.*

Then (2) $\Rightarrow$ (1), and if M satisfies the ω-property, then (1) $\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2) Assume that N is quasi-co-n-absorbing in M and that $(\eta(M) : \mu(M)) = (\eta : \mu)(M)$ for every $\eta,\ \mu \in R$-pr. Since $N \neq 0$ we have $\alpha_N^M \neq 0$. Now let $\eta, \mu \in R$-pr be such that $\alpha_N^M \preceq (\eta_{[n]} : \mu)$. In this case we have

$$N = \alpha_N^M(M) \leqslant (\eta_{[n]} : \mu)(M) = (\eta(M)_{[n]} : \mu(M)).$$

Since N is quasi-co-n-absorbing in M, by hypothesis we have that $N \leqslant \eta(M)_{[n]} = \eta_{[n]}(M)$ or $N \leqslant (\eta(M)_{[n-1]} : \mu(M)) = (\eta_{[n-1]} : \mu)(M)$. It follows from [15, Proposition 5] that $\alpha_N^M \preceq \alpha_{\eta_{[n]}(M)}^M \preceq \eta_{[n]}$ or $\alpha_N^M \preceq \alpha_{(\eta_{[n-1]}:\mu)(M)}^M \preceq (\eta_{[n-1]} : \mu)$, and so α_N^M is quasi-co-n-absorbing.

(2) $\Rightarrow$ (1) Assume that α_N^M is a quasi-co-n-absorbing preradical. Since $\alpha_N^M \neq 0$, we have $N \neq 0$. Suppose that $J,\ K$ are fully invariant submodules of M such that $N \leqslant (J_{[n]} : K)$. Then we have $N \leqslant \left((\omega_J^M)_{[n]} : \omega_K^M\right)(M)$. By [15, Proposition 5], we get

$$\alpha_N^M \preceq \alpha^M_{\left((\omega_J^M)_{[n]}:\omega_K^M\right)(M)} \preceq \left((\omega_J^M)_{[n]} : \omega_K^M\right).$$

Since α_N^M is quasi-co-n-absorbing, we have $\alpha_N^M \preceq (\omega_J^M)_{[n]}$ or $\alpha_N^M \preceq \left((\omega_J^M)_{[n]} : \omega_K^M\right)$. Therefore $N = \alpha_N^M(M) \preceq (\omega_J^M)_{[n]}(M) = J_{[n]}$ or $N = \alpha_N^M(M) \preceq \left((\omega_J^M)_{[n]} : \omega_K^M\right)(M) = (J_{[n-1]} : K)$. Hence N is a quasi-co-n-absorbing submodule. A similar proof can be stated for semi-co-n-absorbing preradicals. □

Remark 7. Note that in Theorem 3, for the case $n = 2$ we can omit the condition $M \in R$-co-ass, by the definition of quasi-co-2-absorbing (semi-co-2-absorbing) submodules.

Definition 4. Let $M \in R$-co-ass. We say that M is a quasi-co-n-absorbing (resp. semi-co-n-absorbing) module if M is a quasi-co-n-absorbing (resp. semi-co-n-absorbing) submodule of itself.

Corollary 6. *Let $M_1, M_2, \dots, M_t$ be injective Artinian R-modules. Suppose that M_i's are quasi-co-n-absorbing modules that satisfy the ω-property. Then $M = \bigoplus_{i=1}^t M_i$ is a quasi-co-$(n+t-1)$-absorbing R-module.*

Proof. Let $M_1, M_2, \dots, M_t$ be quasi-co-n-absorbing R-modules. Then, by Theorem 3, $\alpha_{M_1}^{M_1}, \alpha_{M_2}^{M_2}, \dots, \alpha_{M_t}^{M_t}$ are quasi-co-n-absorbing preradicals, and so $\alpha_M^M = \alpha_{M_1}^{M_1} \vee \alpha_{M_2}^{M_2} \vee \cdots \vee \alpha_{M_t}^{M_t}$ is a quasi-co-$(n+t-1)$-absorbing preradical, by Proposition 11(2). Again by Theorem 3, $M = \bigoplus_{i=1}^t M_i$ is a quasi-co-$(n+t-1)$-absorbing R-module. □

Corollary 7. *Let R be a ring. The following statements hold:*

(1) *If the preradical 1 is quasi-co-2-absorbing (resp. semi-co-2-absorbing), then every generator R-module is a quasi-co-2-absorbing (resp. semi-co-2-absorbing) R-module.*

(2) *If R is a semisimple Artinian ring, then every R-module is quasi-co-i-absorbing for every $i \geqslant 2$.*

Proof. (1) Suppose that 1 is a quasi-co-2-absorbing (resp. semi-co-2-absorbing) preradical and G is a generator R-module. Since $\alpha_G^G = 1$, the result follows from Theorem 3.

(2) By Proposition 6 and Theorem 3. □

Example 1. Let R be a semisimple Artinian ring and $S_1, S_2, \dots, S_{n+1} \in R$-simp be distinct. Then the injective Artinian R-module $\bigoplus_{i=1}^{n+1} S_i$ is quasi-co-n-absorbing, by Corollary 7(2). But note that, by [22, Proposition 3.6] and Theorem 2, $\bigoplus_{i=1}^{n+1} S_i$ is not co-n-absorbing. This example shows that the two concepts of quasi-co-n-absorbing modules and of co-n-absorbing modules are different in general.

The following two propositions can be proved similar to [22, Proposition 4.10] and [22, Theorem 4.11], respectively.

Proposition 36. *Let N, $H \in R$-co-ass such that H be a fully invariant submodule of N and N be self-injective. For a fully invariant submodule K of H,*

(1) *If K is quasi-co-n-absorbing in N, then K is quasi-co-n-absorbing in H.*

(2) *If K is quasi-co-n-absorbing in N and $K \in R$-co-ass, then K is a quasi-co-n-absorbing module.*

(3) *If α_K^N is a quasi-co-n-absorbing preradical and N satisfies the ω-property, then α_K^H is a quasi-co-n-absorbing preradical.*

Proposition 37. *Let N, $Q \in R$-co-ass such that N be a fully invariant submodule of Q and Q be self-injective. Then N is a quasi-co-n-absorbing module if and only if N is quasi-co-n-absorbing in Q.*

Theorem 4. *Let $M \in R$-co-ass that satisfies the ω-property. The following statements are equivalent:*
(1) *M is quasi-co-n-absorbing;*
(2) *α_M^M is quasi-co-n-absorbing;*
(3) *For each $\tau, \eta \in R$-pr, $M \in \mathbb{T}_{(\tau_{[n]}:\eta)} \Rightarrow M \in \mathbb{T}_{\tau_{[n]}}$ or $M \in \mathbb{T}_{(\tau_{[n-1]}:\eta)}$.*

Proof. (1) $\Leftrightarrow$ (2) Is clear by Theorem 3.

(2) $\Rightarrow$ (3) Suppose that α_M^M is quasi-co-n-absorbing. Let $\tau, \eta \in R$-pr such that $M \in \mathbb{T}_{(\tau_{[n]}:\eta)}$. Then $(\tau_{[n]} : \eta)(M) = M$, and so $\alpha_M^M \preceq (\tau_{[n]} : \eta)$. Therefore $\alpha_M^M \preceq \tau_{[n]}$ or $\alpha_M^M \preceq (\tau_{[n-1]} : \eta)$. Hence $\tau_{[n]}(M) = M$ or $(\tau_{[n-1]} : \eta)(M) = M$. Consequently $M \in \mathbb{T}_{\tau_{[n]}}$ or $M \in \mathbb{T}_{(\tau_{[n-1]}:\eta)}$.

(3) $\Rightarrow$ (2) has a routine verification. □

Similarly we can prove the following theorem.

Theorem 5. *Let $M \in R$-co-ass that satisfies the ω-property. The following statements are equivalent:*
(1) *M is semi-co-n-absorbing;*
(2) *α_M^M is semi-co-n-absorbing;*
(3) *For each $\tau \in R$-pr, $M \in \mathbb{T}_{\tau_{[n+1]}} \Rightarrow M \in \mathbb{T}_{\tau_{[n]}}$.*

Theorem 6. *Let $M \in R$-Mod be such that, for each pair K, L of fully invariant submodules of M, we have $\left(\omega_K^M : \omega_L^M\right) = \omega_{(K:L)}^M$. Then, for each quasi-co-n-absorbing (resp. semi-co-n-absorbing) preradical σ such that $\sigma(M) \neq 0$, we have that $\sigma(M)$ is quasi-co-n-absorbing (resp. semi-co-n-absorbing) in M.*

Proof. By hypothesis $M \in R$-co-ass, [22, Lemma 4.12]. Let σ be a quasi-co-n-absorbing preradical such that $\sigma(M) \neq 0$. If K, L are fully invariant submodules of M such that $\sigma(M) \leqslant (K_{[n]} : L)$, then

$$\sigma \preceq \omega_{\sigma(M)}^M \preceq \omega_{(K_{[n]}:L)}^M = \left((\omega_K^M)_{[n]} : \omega_L^M\right).$$

Since σ is quasi-co-n-absorbing, then

$$\sigma \preceq (\omega_K^M)_{[n]} \text{ or } \sigma \preceq \left((\omega_K^M)_{[n-1]} : \omega_L^M\right).$$

In the first case we have $\sigma(M) \leqslant (\omega_K^M)_{[n]}(M) = K_{[n]}$; in the second case we have $\sigma(M) \leqslant \left((\omega_K^M)_{[n-1]} : \omega_L^M\right)(M) = (K_{[n-1]} : L)$. Consequently $\sigma(M)$ is quasi-co-n-absorbing. □

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Extended star graphs

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Abstract. Chordal graphs, which are intersection graph of subtrees of a tree, can be represented on trees. Some representation of a chordal graph often reduces the size of the data structure needed to store the graph, permitting the use of extremely efficient algorithms that take advantage of the compactness of the representation. An *extended star* graph is the intersection graph of a family of subtrees of a tree that has exactly one vertex of degree at least three. An *asteroidal triple* in a graph is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. Several subclasses of chordal graphs (interval graphs, directed path graphs) have been characterized by forbidden asteroids. In this paper, we define, a subclass of chordal graphs, called extended star graphs, prove a characterization of this class by forbidden asteroids and show open problems.

Introduction

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [6] states that a graph is chordal if and only if it is the (vertex) intersection graph of a family of subtrees of a tree. Families of subtrees of a tree together with the tree are called representation of a graph.

Some representation of a chordal graph often reduces the size of the data structure needed to store the graph, permitting the use of

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extremely efficient algorithms that take advantage of the compactness of the representation. Since some chordal graphs have many distinct representations, it is interesting to consider which one is most desirable under various circumstances, for example minimum diameter [1], minimum number of leaves [11], [4], and imposing conditions on trees, subtrees and intersection sizes [15].

The *leafage* of a chordal graph is the minimum integer ℓ such that the graph admits a representation whose tree has exactly ℓ leaves [14]. This number is related with the existence of asteroidal sets [14].

An *asteroidal set* A in a graph G is a set of non-adjacent vertices such that for any $v \in A$ the vertices of $A \setminus \{v\}$ appears in the same connected component of $G \setminus N[v]$. Note that this definition is compatible with the definition of asteroidal triple already given. The *asteroidal number* of a graph G is the maximum integer a such that G admits an asteroidal set of cardinality a. If G is a chordal graph containing an asteroidal set A of size k, then in any representation of G, its tree has at least k leaves. Thus the asteroidal number of a chordal graph is less or equal to its leafage, and this inequality can be strict [14].

Habib and Stacho [11] found a polynomial algorithm to compute the leafage of a chordal graph and built a representation of it.

Natural subclass of chordal graphs are path graphs, directed path graphs, rooted directed path graphs and interval graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. A graph is a *rooted directed path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. A graph is an *interval graph* if it is the intersection graph of a family of subpaths of a path.

By definition we have the following inclusions between the different considered classes (and these inclusion are strict):

$$\text{interval} \subset \text{rooted directed path} \subset \text{directed path} \subset \text{path} \subset \text{chordal}.$$

Chaplick and Stacho [4] proved that for path graphs there is a representation, where the subtrees are paths, that reaches the leafage, and then it is also true for directed path graphs [5]. However, it is not true for rooted directed path graphs [9].

Lekkerkerler and Boland [12] proved that a chordal graph is an interval graph if and only if it contains no asteroidal triple. As byproduct, they found a characterization of interval graphs by forbidden induced subgraphs.

Panda [16] found the characterization of directed path graph by forbidden induced subgraphs and then Cameron, Hoáng and Lévêque [3] gave a characterization of this class in terms of forbidden asteroidal triples.

Lévêque, Maffray and Preissman [13], found the characterization of path graphs by forbidden induced subgraphs but there is still no nice characterization in terms of forbidden asteroids for this class.

Characterizing rooted directed path graph by forbidden induced subgraphs or forbidden asteroids are open problems. It is certainly too difficult to characterizing rooted directed path graphs by forbidden induced subgraphs as there are too many (families of) graphs to exclude but Cameron, Hoáng and Lévêque [2] suggest that directed path graphs could be characterized by forbidding some particular type of asteroidal quadruples (a set of four non-adjacent vertices such that any three of them is an asteroidal triple). Thus, several subclasses of rooted directed path graphs [10], [8] have been characterized by forbidden asteroids, and as byproduct it was found the characterization of them by forbidden induced subgraphs.

Other subclass of chordal graphs is extended star graphs. A graph G is *an extended star* if it is the intersection graph of families of subtrees of a tree which has exactly one vertex of degree at least tree. Clearly this class is a natural generalization of interval graphs.

By definition we have the following inclusions between the different considered classes (and these inclusion are strict):

$$\text{interval} \subset \text{extended star} \subset \text{chordal}$$

On the other hand, this class is hereditary, i.e is closed under vertex-induced subgraphs. It is known that hereditary classes admit a characterization by forbidden induced subgraphs. Characterize extended star graphs by forbidden induced subgraphs or by forbidden asteroids are open problems. Also it is an open problem answer if for extended star graph there is a representation that reaches the leafage.

In this paper we study properties of extended star graphs, and give a characterization of this class by forbidden asteroids.

The paper is organized as follows: in Section 2, we give some definitions and background. In Section 3, we prove a characterization of this class by forbidden asteroids. Finally, in Section 4, we show conclusions and open problems.

1. Definitions and background

A *clique* in a graph G is a set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all maximal cliques* of G. We denote by C_x the set of the maximal cliques that contain x.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$. A vertex s is *simplicial* if its closed neighborhood is a maximal clique.

A *clique tree* T of a graph G is a tree whose vertices are the elements of $\boldsymbol{C}(G)$ and such that for each vertex x of G, C_x induces a subtree of T, which we will denote by T_x.

Note that G is the intersection graph the vertex sets of subtrees $(T_x)_{x \in V(G)}$. Gavril [6] proved that a graph is chordal if and only if it has a clique tree. Clique trees are called *models* of the graph.

It is clear that a graph is an *interval graph* if it admits a clique tree T that is a path such that T_x is a subpath of T for every $x \in V(G)$. A natural generalization of interval graphs are extended star graphs. A graph G is an *extended star* if there is a model of G that has at most exactly one vertex of degree at least three, such models are called *extended star models*. Clearly, interval graphs is a subclass of extended star graphs. Split graphs, minimal forbidden induced subgraphs for interval graphs, and path graphs minimal forbidden induced subgraphs for directed path graphs are examples of extended star graphs.

Let T be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to maximal cliques of G. In order to simplify the notation, we often write $Q \in T$ instead of $Q \in V(T)$, and $e \in T$ instead of $e \in E(T)$. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\cup_{Q \in V(T')} Q$.

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G + G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and the edges are $E(G) \cup E(G')$. Note that if T, T' are two trees such that $|V(T) \cap V(T')| = 0$, then $T + T'$ is a forest.

Let T be a tree. For $V' \subseteq V(T)$, let $T[V']$ be the minimal subtree of T containing V'. Then for $X, Y \in V(T)$, $T[X, Y]$ is the subpath of T between X and Y. Let $T[X, Y) = T[X, Y] \setminus Y$, $T(X, Y] = T[X, Y] \setminus X$ and $T(X, Y) = T[X, Y] \setminus \{X, Y\}$. Note that some of these paths may be empty or reduced to a single vertex when X and Y are equal or adjacent.

We say that $T[X,Y]$ is a *branch* of T if X is a leaf of T and Y is its most next vertex of degree at least three of T.

For $X,Y,Z \in V(T)$ that are not on the same path in T, $T[X,Y,Z]$ is the subtree of T that has X,Y,Z, as its leaves. Let $T[X,Y,Z) = T[X,Y,Z]\backslash Z$ and $T(X,Y,Z) = T[X,Y,Z]\backslash\{X,Z\}$.

In a clique tree T, the *label* of an edge QQ' of T is defined as $lab(QQ') = Q \cap Q'$. Observe that the label of an edge of T is a minimal separator of G.

Let T be a tree, we denote by $ln(T)$ the number of leaves of T. The *leafage* of a chordal graph G is a minimum integer ℓ such that G admits a model T with $ln(T) = \ell$ [14].

In some cases the leafage of a graph decides if a graph is an extended star as shows the following Lemma.

Lemma 1. *Let G be a chordal graph. If $l(G) \leqslant 3$ then G is an extended star graph.*

Proof. Let T be a model of G that reaches the leafage, i.e $ln(T) = l(G)$. Clearly, $ln(T) \leqslant 3$. Thus T has at most exactly one vertex of degree three. Therefore, G is an extended star graph. □

An *asteroidal triple* in a graph G is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. An *asteroidal n-tupla* in a graph G is a set of n non-adjacent vertices such that for any $(n-1)$ of them is an asteroidal $(n-1)$-tupla.

If G is a chordal graph containing an asteroidal n-tupla, then in any model T of G, T has at least n leaves. Thus the leafage of G is greater or equal to n.

In [7] has been proved that for any clique tree that reaches the leafage, every vertex of degree at least three, and every choice of three branches incident to it there is an asteroidal triple on these branches. Thus for extended star graphs we have the same result.

Lemma 2. *Let G be an extended star graph and T be an extended star model of G with minimum number of leaves equal $n > 2$. Then G has $\frac{n(n-1)(n-2)}{6}$ asteroidal triples.*

Proof. Let $H_1, H_2, \ldots, H_n$ be the leaves of T and Q be the vertex of degree at least three in T. Suppose that $G_{T[H_1,H_2,H_3]}$ does not have an asteroidal triple. Then there is an interval model T' of $G_{T[H_1,H_2,H_3]}$. Clearly $T - (T[H_1,Q) + T[H_2,Q) + T[H_3,Q)) + T'$ is an extended star model of

G which has less leaves than T, a contradiction. Hence $G_{T[H_i,H_j,H_k]}$ has an asteroidal triple for any three different $i, j, k \in \{1, 2 \ldots, n\}$. Therefore G has $\frac{n(n-1)(n-2)}{6}$ asteroidal triples. $\square$

Lemma 3. *Let G be an extended star chordal graph and T be an extended star model of G with minimum number of leaves equal $n > 2$. If T has exactly k leaves whose distance to the vertex of degree at least three is greater than one then G has at least an asteroidal $(n-k)-$tuple.*

Proof. Let Q be the vertex of degree n of T, $H_1, \ldots, H_k$ be the leaves of T at distance greater than one to Q in T, and $H_{k+1}, \ldots, H_n$ be the other that are incident to the vertex Q. Let $a_{k+1}, \ldots, a_n$ be simplicial vertices of $H_{k+1}, \ldots, H_n$ respectively. Since a_i is a simplicial vertex of G, $N[a_i] = H_i$ for $i \in \{k+1, \ldots, n\}$. Let $T' = T[H_{k+1}, \ldots, H_n]$. Suppose that $G_{T'} \setminus N[a_n]$ is not a connected graph. So there is at least an edge H_iQ in T' for some $i \in \{k+1, \ldots n-1\}$ such that $lab(H_iQ) \subset H_n$. Then $T_1 = T - H_iQ + H_iH_n$ is an extended star model of G that has less leaves than T, a contradiction. Hence $G_{T'} \setminus N[a_i]$ is a connected graph for all $i \in \{k+1, \ldots, n\}$. Therefore $a_{k+1}, \ldots, a_n$ is an asteroidal $(n-k)$-tupla. $\square$

Lemma 4. *Let s be a simplicial vertex of G, a minimally non extended star graph. Then*

1) *s is a vertex of some asteroidal triple;*
2) *there is a model T of G which has exactly two vertices of degree at least three Q and Q'. Moreover, there is at least two branches $T[Q', H_i']$ for $i = 1, 2$ such that $G_{T[H_1',H_2',Q]}$ is not an interval graph;*
3) *there is a model T of G which has exactly two vertices Q, Q' of degree at least three, it has at least two branches $T[Q', H_i']$ for $i = 1, 2$ such that $G_{T[H_1',H_2',Q]}$ is not an interval graph, and if $T[H_i, Q]$ are the branches of T for $i \in \{1, \ldots, n\}$ then $G_{T[H_i,H_j,Q']}$ are not interval graphs for $i, j \in \{1, \ldots, n\}$, $i \neq j$.*

Proof. 1), 2) Since G is a minimal non extended star graph each simplical vertex of G verifies that if we remove this vertex, the graph obtained has lower number of maximal cliques than G. Let s be a simplicial vertex of G. Clearly, there is a maximal clique $Q' \neq N[s]$ such that $N(s) \subset Q'$. Since G is a minimal non extended star graph, $G \setminus s$ is an extended star graph. By Lemma 1 $l(G) \geqslant 4$, and since s is a simplicial vertex it follows that $l(G \setminus s) \geqslant 3$. Let T' be an extended star model of $G \setminus s$, and Q be the vertex of degree at least three of T'. Clearly $T = T' + N[s]Q'$ is a model of G, and since G is not an extended star graph so $Q' \neq Q$ and Q' is not a leaf of T'. Observe that T has only two vertices of degree at least three Q and Q'. Let $H \neq N[s]$ be the leaf of T such that $Q' \in T[Q, H]$. In case that $G_{T[Q,N[s],H]}$ is an interval graph, there is an interval model T_1' of $G_{T[Q,N[s],H]}$. Let $T_1 = T - T(Q, N[s], H] + T_1'$. Clearly T_1 is an extended star model of G, a contradiction. Hence $G_{T[Q,N[s],H]}$ is not an interval graph, so there is an asteroidal triple, and clearly s must be a vertex of it.

3) Among all the trees in the condition 2), choice that has minimum leafage, and maximum degree in Q' (recall that Q' is a vertex of degree at least three such that there is at least two branches $T[H_i', Q']$ for $i = 1, 2$ such that $G_{T[H_1',H_2',Q]}$ is not an interval graph). If for some $i, j \in \{1, \dots n\}$, $i \neq j$, $G_{T[H_i,H_j,Q']}$ is an interval graph then there is an interval model T_1 of $G_{T[H_i,H_j,Q']}$. Let $T' = T - T[H_i, H_j, Q') + T_1$. Clearly T' is a model of G which has exactly two vertices of degree at least three, a leaf is incident to Q' and $G_{T[Q,N[s],H]}$ is not an interval graph. Moreover, if Q' is a leaf of T_1 then in T' the degree of Q' is the same that in T but $ln(T') < ln(T)$, a contradiction. If Q' is not a leaf of T_1 then $ln(T') = ln(T)$ but the degree of Q' in T' is greater than the degree of Q' in T, a contradiction. Hence for $i, j \in \{1, \dots n\}$, $i \neq j$, $G_{T[H_i,H_j,Q']}$ is not an interval graph. □

The following algorithm is a technical tool necessary in the proof of characterization of extended star graph by forbidden asteroids.

Algorithm

Input: A model T that has minimum number of leaves, exactly two vertices Q, Q' of degree at least three at distance greater than one, $Q^* \in T(Q, Q')$ and $T[H_i, Q]$ the branches incident to Q for $i \in \{1, \dots, n\}$.

Output: A model T' that has exactly two vertices Q^*, Q' of degree at least three whose distance in T' is the same that its distance in T, and Q, Q^*, Q' appear in this order in T'; or it has at least two vertices Q, Q' of degree at least three and at most three vertices Q, Q', Q^* of degree at least three, Q, Q^*, Q' appear in this order in T', and there are two branches $T'[\overline{H}_l, Q]$ and $T'[H_{l+2}, Q]$ for $l \in \{1, \dots, n-2\}$ such that $G_{T'[\overline{H}_l,H_{l+2},Q^*]}$ is not an interval graph.

If $G_{T[H_1,H_2,Q^*]}$ is not an interval graph **Then**

RETURN: $T' = T$

Else

Take T_1 an interval model of $G_{T[H_1,H_2,Q^*]}$ and build a model $\overline{T}_1 = T - T[H_1, H_2, Q^*) + T_1$.

If $n = 2$ **Then**

RETURN: $T' = \overline{T}_1$

Else

Let $\overline{T}_1[\overline{H}_1, Q]$ and $\overline{T}_1[H_i, Q]$ be the branches incident to Q for $i \in \{3, \ldots, n\}$.

If $G_{\overline{T}_1[\overline{H}_1, H_3, Q^*]}$ is not an interval graph **Then**

RETURN: $T' = \overline{T}_1$

Else

$i = 2$

$*$ Take T_i an interval model of $G_{\overline{T}_{i-1}[\overline{H}_{i-1}, H_{i+1}, Q^*]}$ and build a model $\overline{T}_i = \overline{T}_{i-1} - \overline{T}_{i-1}[\overline{H}_{i-1}, H_{i+1}, Q^*) + T_i$.

If $n > i + 1$ **Then**

Let $\overline{T}_i[\overline{H}_i, Q]$ and $\overline{T}_i[H_j, Q]$ be the branches incident to Q for $j \in \{i + 2, \ldots, n\}$

If $G_{\overline{T}_i[\overline{H}_i, H_{i+2}, Q^*]}$ is not an interval graph **Then**

RETURN: $T' = \overline{T}_i$

Else

$i = i + 1$ go to $*$

Else

RETURN: $T' = \overline{T}_i$

Observe that T_i is an interval model that does not have Q^* as a leaf, otherwise $ln(\overline{T}_i) < ln(T)$ a contradiction since T is a model of G that has minimum number of leaves.

Note that the way $\overline{T}_i$ was built assure that has at most three vertices of degree at least three Q, Q^*, Q' that appear in this order in $\overline{T}_i$, and $\overline{T}_i[\overline{H}_i, Q]$, $\overline{T}_i[H_j, Q]$ are the branches of $\overline{T}_i$ for $j \in \{i + 2, \ldots, n\}$. Also the degree in $\overline{T}_i$ of Q is $n + 1 - i$ and the degree of Q^* is $i + 2$.

We will see that the algorithm works.

Suppose that the algorithm stopped since $G_{T[H_1, H_2, Q^*]}$ is not an interval graph then $T' = T$ has exactly two vertices Q, Q' of degree at least three whose distance in T' is the same that its distance in T.

Suppose that the algorithm stopped when $i = 1$ and $n = 2$. Since T has minimum number of leaves then Q^* is not a leaf of T_1 then $ln(\overline{T}_1) = ln(T)$. Also $T' = \overline{T}_1$ has exactly two vertices Q^*, Q' of degree at least three whose distance in T' is the same that its distance in T, and Q, Q^*, Q' appear in this order in T'. If $n > 2$ and $G_{\overline{T}_1[\overline{H}_1, H_3, Q^*]}$ is not an interval graph then $T' = \overline{T}_1$ has three vertices Q, Q', Q^* of degree at least three, Q, Q^*, Q' appear in this order in T', and there are two branches $T'[\overline{H}_l, Q]$ and $T'[H_{l+2}, Q]$ for $l \in \{1, \ldots, n - 2\}$ such that $G_{T'[\overline{H}_l, H_{l+2}, Q^*]}$ is not an interval graph.

Suppose that the algorithm stopped when $2 \leqslant i < n-1$. Thus T' has three vertices Q, Q', Q^* of degree at least three; Q, Q^*, Q' appear in this order in T', and there are two branches $T'[\overline{H}_l, Q]$ and $T'[H_{l+2}, Q]$ for $l \in \{1, \ldots, n-2\}$ such that $G_{T'[\overline{H}_l, H_{l+2}, Q^*]}$ is not an interval graph.

Suppose that the algorithm stopped when $i = n-1$. Thus T' has exactly two vertices Q^*, Q' of degree at least three whose distance in T' is the same that its distance in T, and Q, Q^*, Q' appear in this order in T'

2. Forbidden asteroids characterization for extended star graphs

A pair of asteroidal triples in a graph G is *strongly linked* if it contains from two asteroidal triples a_1, a_2, a_3; b_1, b_2, b_3 satisfying the following conditions:

1) $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$.
2) Every path between a_i and b_j has vertices in $N[a_3]$ and in $N[b_3]$ for $i, j \in \{1, 2\}$.
3) Let S, M be minimal separators of G with $S \subset N[b_3]$ and $M \subset N[a_3]$. If a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$ then there is no $Q \in C(G)$ such that $M \cup S \subset Q$.

Observe that if T is a model of a graph G that has a pair of strongly linked asteroidal triples a_1, a_2, a_3; b_1, b_2, b_3 and $Q_i, Q'_i \in C(G)$ such that $a_i \in Q_i$ and $b_i \in Q'_i$ for $i = 1, 2$ then by 2, there are at least two edges $e, e' \in T[Q_i, Q'_i]$ such that $lab(e) \subset N[a_3]$ and $lab(e') \subset N[b_3]$. Also $T_{a_i} \cap T_{b_j} = \varnothing$ for $i, j \in \{1, 2\}$.

Notice that if G has a pair of strongly linked asteroidal triples by item 2 of the definition: a_i, b_j are in different connected component of $G \setminus N[a_3]$ and $G \setminus N[b_3]$ or $a_i \in N[b_3]$ or $b_j \in N[a_3]$ for $i, j \in \{1, 2\}$.

Theorem 1. *Let G be a chordal graph. G is an extended star graph if and only if G does not have a pair of strongly linked asteroidal triples.*

Proof. $\Rightarrow$ Suppose that G has a pair of strongly linked asteroidal triples a_1, a_2, a_3 ; b_1, b_2, b_3, and it is an extended star graph. Then there is an extended star model T of G. Since G has an asteroidal triple then $l(G) \geqslant 3$. Let Q be the vertex of degree at least three in T. Since T is an extended star model, T_{a_i} and T_{b_i} induce paths in T for $i \in \{1, 2, 3\}$. Let H_1, H_2, H_3 be leaves of T such that T_{a_i} induces a path in $T(Q, H_i]$ for $i \in \{1, 2, 3\}$.

In the follows, we prove that T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$.

Suppose that T_{b_1} induces a path in $T(Q, H_1]$.

Let $T_{a_1} = T[Q_1, Q_2]$ and $T_{b_1} = T[Q_3, Q_4]$ be such that $Q_1 \in T[Q, Q_2]$ and $Q_3 \in T[Q, Q_4]$. Since $a_1, a_2, a_3; b_1, b_2, b_3$ is a pair of strongly linked asteroidal triples it follows that $T_{a_1} \cap T_{b_1} = \varnothing$. Thus $Q, Q_3, Q_4, Q_1, Q_2, H_1$ or $Q, Q_1, Q_2, Q_3, Q_4, H_1$ appear in this order in $T[Q, H_1]$.

In case that $Q, Q_3, Q_4, Q_1, Q_2, H_1$ appear in this order in $T[Q, H_1]$, by the item 2) of the definition of a pair of strongly linked asteroidal triples, there is an edge $e \in T[Q_4, Q_1]$ such that $lab(e) \subset N[a_3]$ so each path between a_1 and a_2 in G has neighbors of a_3 contradicting that a_1, a_2, a_3 is an asteroidal triple.

In case that $Q, Q_1, Q_2, Q_3, Q_4, H_1$ appear in this order in $T[Q, H_1]$, by the item 2) of the definition of a pair of strongly linked asteroidal triples, there is an edge $e' \in T[Q_3, Q_2]$ such that $lab(e') \subset N[b_3]$. Then each path between b_1 and b_2 in G has neighbors of b_3 contradicting that b_1, b_2, b_3 is an asteroidal triple.

Following the earlier argument, we can conclude that T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$.

Finally, we prove that T_{b_3} does not induce a path in $T(Q, H_3]$.

Suppose that T_{b_3} induces a path in $T(Q, H_3]$. Let $T_{a_3} = T[Q_5, Q_6]$ and $T_{b_3} = T[Q_7, Q_8]$ be such that $Q_5 \in T[Q, Q_6]$ and $Q_7 \in T[Q, Q_8]$. Observe that $T_{a_3} \cap T_{b_3}$ may be different from $\varnothing$. Clearly Q, Q_5, Q_7, H_3 or Q, Q_7, Q_5, H_3 appear in this order in $T[Q, H_3]$. As T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$, and T_{b_3} induces a path in $T(Q, H_3]$ then there exist H_4, H_5 leaves of T such that T_{b_1} and T_{b_2} induce paths in $T(Q, H_4]$ and $T(Q, H_5]$ respectively.

In case that Q, Q_5, Q_7, H_3 appear in this order in $T[Q, H_3]$, there is an edge $e' \in T[Q_1, Q]$ such that $lab(e') \subset N[b_3]$. By the position in T of Q_5, $lab(e') \subset N[a_3]$ so each path between a_1 and a_2 in G has neighbors of a_3 contradicting that a_1, a_2, a_3 is an asteroidal triple.

In case that Q, Q_7, Q_5, H_3 appear in this order in $T[Q, H_3]$, there is an edge $e \in T[Q_3, Q]$ such that $lab(e) \subset N[a_3]$, following the earlier argument each path between b_1 and b_2 in G has neighbors of b_3 contradicting that b_1, b_2, b_3 is an asteroidal triple.

Hence T_{b_3} does not induce a path in $T(Q, H_3]$.

By before exposed, T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$ and T_{b_3} does not induce a path in $T(Q, H_3]$.

Suppose that T_{b_1} does not induce a path in $T(Q, H_j]$ for $j \in \{1, 2, 3\}$.

Let H_4 be a leaf different from H_1, H_2, H_3 such that T_{b_1} induces a path in $T(Q, H_4]$. We can assume that T_{b_3} does not induce a path in $T[H_1, Q]$. By the item 2) of the definition of a pair of strongly linked asteroidal triples, there are edges e, e', $e \in T[H_1, Q]$ and $e' \in T[Q, H_4]$ such that $lab(e) \subset N[b_3]$ and $lab(e') \subset N[a_3]$. Let $S = lab(e)$ and $M = lab(e')$. Clearly S and M are minimal separators of G such that a_1, a_2 are in different connected components of $G \setminus S$, and b_1, b_2 are in different connected components of $G \setminus M$. By the position in T of the maximal cliques $N[b_3]$ and $N[a_3]$ it follows that $S \cup M \subset Q$, contradicting the item 3) of the definition of a pair of strongly linked asteroidal triples.

Thus the pair of strongly linked asteroidal triples do not have way of being located on an extended star model. Therefore, G is not an extended star graph.

$\Leftarrow$ Suppose that G is a minimally non extended star graph. By Lemma 1, $l(G) \geqslant 4$ and by Lemma 4. 3), there is a model T of G that has exactly two vertices Q, Q' of degree at least three. Let $H_1, \ldots, H_n$ be the leaves of T such that $T[H_i, Q]$ are branches of T for $i = 1, \ldots, n$, and let $H'_1, \ldots, H'_m$ be the leaves of T such that $T[H'_j, Q']$ are branches of T for $j = 1, \ldots, m$. Moreover, by Lemma 4. 3), Q' has maximum degree and there are at least two leaves H'_k, H'_l for $k \neq l$, $k, l \in \{1, \ldots, m\}$ such that $G_{T[H'_k, H'_l, Q]}$ is not an interval graph. Also for all $i \neq j$, $i, j \in \{1, \ldots, n\}$ $G_{T[H_i, H_j, Q']}$ are not interval graphs. Recall that T has minimum leafage. Among all the trees in these conditions choice one that minimizing the distance in T between Q and Q'.

- In case that the distance in T between Q and Q' is greater than one we analyze two situations:

Case 1. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T; or Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T.

Case 2. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$, and applying the Algorithm to Tconsidering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$, in both cases it does not output T.

Observe that applying the Algorithm to T considering $Q^* \in T(Q,\, Q')$, the branches $T[H_i, Q]$ for $i = 1, \ldots, n$, and by our election of T, which minimizing the distance in T between Q and Q', if the Algorithm outputs a tree with exactly two vertices of degree at least three then it must be T. More clearly, if it outputs a tree T' with exactly two vertices of degree at least three, which are not Q and Q', then they must be Q^* and Q'. Also by the way T' was built $l(T') = ln(T)$, and the distance between Q^* and Q' in T' is the same that its distance in T, and it is lower that the distance in T between Q and Q', contradicting this way the election of T that has exactly two vertices of degree at least three to minimum distance.

Case 1. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T; or applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T.

Suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$, it outputs T. In this case we can assume that $G_{T[H_1,H_2,Q^*]}$ is not an interval graph. We will analyze two situations: applying the Algorithm considering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T or not.

Case 1.1. Suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T. Also suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$ and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T. In this case we can assume that $G_{T[H'_1,H'_2,Q^*]}$ is not an interval graph.

Since $G_{T[H_1,H_2,Q^*]}$ is not an interval graph then there is an asteroidal triple a_1, a_2, a_3. Analogously, there is an asteroidal triple b_1, b_2, b_3 in $G_{T[H'_1,H'_2,Q^*]}$.

Suppose that $a_3 \in Q_3$ with $Q_3 \in T(Q, Q^*]$, and $b_3 \in Q'_3$ with $Q'_3 \in T[Q^*, Q')$. Thus $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$. Then the item 1) of the definition of a pair of strongly linked asteroidal triples was checked.

Given that $Q_3, Q'_3 \in T(Q, Q')$ each path between a_i and b_j must have vertices in Q_3 and Q'_3 for $i, j \in \{1, 2\}$. So each path between a_i and b_j has neighbors of a_3 and b_3 for $i, j \in \{1, 2\}$. Then the item 2) of the definition of a pair of strongly linked asteroidal triples was checked.

Finally, by our choice of a_1, a_2, a_3; b_1, b_2, b_3, there are not minimal separators $S \subset N[b_3]$, $M \subset N[a_3]$ satisfying a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 are a pair of strongly linked asteroidal triples.

Case 1.2. Suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T. Let T_0 be the connected component of $T - T(Q^*, Q')$ that contains Q and Q^*.

Also, assume that applying the Algorithm to T considering $Q^* \in T(Q, Q')$ and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it does not output T. Let $\overline{T'}$ be the tree outputs by the Algorithm, and $\overline{T_0}$ be the connected component of $\overline{T'} - \overline{T'}(Q, Q^*)$ that contains Q' and Q^*.

Let $T'' = T_0 + \overline{T_0}$. Clearly T'' is a model of G.

By the way T'' was built Q, Q^*, Q' appear in this order in T'', T'' has three vertices Q, Q^*, Q' of degree at least three. Also there are four branches in T'', $T''[H_1, Q] = T_0[H_1, Q] = T[H_1, Q]$, $T''[\overline{H_2}, Q] = T_0[H_2, Q] = T[H_2, Q]$, $T''[H'_j, Q'] = \overline{T_0}[H'_j, Q'] = \overline{T'}[H'_j, Q']$, $T''[\overline{H'_l}, Q'] = \overline{T_0}[\overline{H'_l}, Q'] = \overline{T'}[\overline{H'_l}, Q']$ for $j \neq l$, $j, l \in \{1, \ldots, m\}$ such that $G_{T''[H_1, H_2, Q^*]}$ and $G_{T''[H'_j, \overline{H'_l}, Q^*]}$ are not interval graphs. Suppose that $j = 1$ and $l = 2$.

In each situations describing before, we can assume that there is an asteroidal triple a_1, a_2, a_3 in $G_{T''[H_1, H_2, Q^*]}$ and there is an asteroidal triple b_1, b_2, b_3 in $G_{T''[H'_1, \overline{H'_2}, Q^*]}$. Suppose that $a_3 \in Q_3$ with $Q_3 \in T''(Q, Q^*]$, and $b_3 \in Q'_3$ with $Q'_3 \in T''[Q^*, Q')$. Thus $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$. Then the item 1) of the definition of a pair of strongly linked asteroidal triples was checked.

Given that $Q_3, Q'_3 \in T''(Q, Q')$ each path between a_i and b_j must have vertices in Q_3 and Q'_3 for $i, j \in \{1, 2\}$. So each path between a_i and b_j has neighbors of a_3 and b_3 for $i, j \in \{1, 2\}$. Then the item 2) of the definition of a pair of strongly linked asteroidal triples was checked.

Finally, by our choice of a_1, a_2, a_3; b_1, b_2, b_3, there are not minimal separators $S \subset N[b_3]$, $M \subset N[a_3]$ satisfying a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 are a pair of strongly linked asteroidal triples.

Case 2. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ and applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_j, Q']$ for $i = j, \ldots, m$, in both cases it does not output T. Let T' and $\overline{T'}$ be the subtrees obtained respectively. By our assumption $T' \neq T$ and $\overline{T'} \neq T$.

Let T_0 be the connected component of $T' - T'(Q^*, Q')$ that contains Q and Q^*, and $\overline{T_0}$ be the connected component of $\overline{T'} - \overline{T'}(Q, Q^*)$ that contains Q' and Q^*. Let $T'' = T_0 + \overline{T_0}$.Clearly T'' is a model of G.

By the way T'' was built Q, Q^*, Q' appear in this order in T'', T'' has at least two vertices Q, Q' of degree at least three and at most three vertices Q, Q^*, Q' of degree at least three. Also there are four branches in T'', $T''[H_i, Q] = T_0[H_i, Q] = T'[H_i, Q]$, $T''[\overline{H_k}, Q] = T_0[\overline{H_k}, Q] = T'[\overline{H_k}, Q]$, $T''[H'_j, Q'] = \overline{T_0}[H'_j, Q'] = \overline{T'}[H'_j, Q']$, $T''[\overline{H'_l}, Q'] = \overline{T_0}[\overline{H'_l}, Q'] = \overline{T'}[\overline{H'_l}, Q']$ for $i \neq k$, $j \neq l$, $i, k \in \{1, \ldots, n\}$ and $j, l \in \{1, \ldots, m\}$ such that $G_{T''[H_i, \overline{H_k}, Q^*]}$ and $G_{T''[H'_j, \overline{H'_l}, Q^*]}$ are not interval graphs. Suppose that $i = 1$, $k = 2$, $j = 1$ and $l = 2$.

We can assume that there is an asteroidal triple a_1, a_2, a_3 of $G_{T''[H_1,\overline{H_2},Q^*]}$ and there is an asteroidal triple b_1, b_2, b_3 of $G_{T''[H'_1,\overline{H'_2},Q^*]}$. Suppose that $a_3 \in Q_3$ with $Q_3 \in T''(Q, Q^*]$, and $b_3 \in Q'_3$ with $Q'_3 \in T''[Q^*, Q')$. Thus $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$. Then the item 1) of the definition of a pair of strongly linked asteroidal triples was checked.

Given that $Q_3, Q'_3 \in T''(Q, Q')$ each path between a_i and b_j must have vertices in Q_3 and Q'_3 for $i, j \in \{1, 2\}$. So each path between a_i and b_j has neighbors of a_3 and b_3 for $i, j \in \{1, 2\}$. Then the item 2) of the definition of a pair of strongly linked asteroidal triples was checked.

Finally, by our choice of a_1, a_2, a_3; b_1, b_2, b_3, there are not minimal separators $S \subset N[b_3]$, $M \subset N[a_3]$ satisfying a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 are a pair of strongly linked asteroidal triples.

- In case that the distance in T between Q and Q' is one.

By our election of T, we can assume that there is an asteroidal triple a_1, a_2, a_3 of $G_{T[H_1,H_2,Q']}$ and there is an asteroidal triple b_1, b_2, b_3 of $G_{T[H'_1,H'_2,Q]}$. Clearly $a_3 \in Q'$ and $b_3 \in Q$. It is easy to verify that a_1, a_2, a_3; b_1, b_2, b_3 satisfy the items 1), 2) of the definition of a pair of strongly linked asteroidal triples.

Finally, we check the item 3) of the definition of a pair of strongly asteroidal triples. Let $Q_1, Q_2 \in T[H_1, H_2]$ be such that minimizing the distance to Q and $a_i \in Q_i$ for $i = 1, 2$. Observe that each minimal separator $S \subset N[b_3]$, which satisfies a_1, a_2 are in different connected components of $G \setminus S$, is the label of an edge in $T[H_1, H_2]$. Moreover it is in $T[Q_1, Q_2]$. Analogously, each minimal separator $M \subset N[a_3]$, which satisfies b_1, b_2 are in different connected components of $G \setminus M$, is the label of an edge in $T[H_3, H_4]$, and it is in $T[Q_3, Q_4]$ with $Q_3, Q_4 \in T[H'_1, H'_2]$ minimizing the distance to Q and $b_i \in Q_{i+2}$ for $i \in \{1, 2\}$. Suppose that there is Q^* such that $S \cup M \subset Q^*$. Let T_1, T_2 be subtrees of T such that $T_1 + T_2 + T[Q, Q'] = T$, $T_1 \cap T_2 = \varnothing$, $T_1 \cap T[Q, Q'] = \{Q\}$, $T_2 \cap T[Q, Q'] = \{Q'\}$. Suppose that $Q^* \in T_1$.It is clear that Q^*, Q, Q' appear in this order in T. Since $M \subset N[a_3]$, there is an edge $e' \in T_2$ such that $lab(e') = M \subset Q^*$. Given that $e' \in T[Q_3, Q_4]$ and by the order in that appear Q^*, Q in T it follows that $lab(e') \subset Q$. As $b_3 \in Q$, $Q \subset N[b_3]$. It follows that $lab(e') \subset N[b_3]$. Thus each path between b_1 and b_2 in G has vertices in $N[b_3]$ contradicting that b_1, b_2, b_3 is an asteroidal triple of G. Hence $Q^* \notin T_1$. Suppose that $Q^* \in T_2$. Following an argument similar to the previous one, we arrive to a contradiction since a_1, a_2, a_3 is an asteroidal triple of G.

Hence there is no $Q^* \supset S \cup M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 is a pair of strongly linked asteroidal triples. $\square$

Corollary 1. *Let G be a minimal non extended star graph. Then $l(G) = 4$*

Proof. Suppose that $l(G) > 4$. Thus each model of G has at least five leaves. As a consequence of the proof of Theorem 1, there are a model T of G and H_1, H_2, H_3, H_4 four leaves of T such that $G_{T[H_1,H_2,H_3,H_4]} \neq G$ has a pair of strongly linked asteroidal triples contradicting that G is a minimal non extended star graph. $\square$

Conclusions

The characterization of interval graphs given by Lekkerkerker-Boland, related chordal non interval graphs with asteroidal triples. This kind of characterization is given by Cameron, Hoáng and Lévêque for chordal non directed path graphs. In this paper we have defined a subclass of chordal graphs, extended star graphs, and we related chordal non extended star graphs with asteroids. For this purpose we defined a particular type of asteroidal triple to obtain a characterization of this class by forbidden asteroids. On the other hand, this class is hereditary so it admits a characterization by forbidden induced subgraphs. Our result is useful to build forbidden induced subgraphs, it may be choice two forbidden induced subgraphs for interval graphs whose asteroidal triples are a_1, a_2, a_3 and b_1, b_2, b_3 and add a path between a_3 and b_3 or identify a_3 and b_3.

On the other hand, it is known that for path graphs and directed path graphs there is a model that reaches the leafage. But it is not true for rooted directed path graphs. An interesting questions is if for extended star graphs there is a model that reaches the leafage or if it is possible to build a model with minimum number of leaves.

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RESEARCH ARTICLE

Involution rings with unique minimal *-biideal

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Abstract. The structure of certain involution rings which have exactly one minimal *-biideal is determined. In addition, involution rings with identity having a unique maximal biideal are characterized.

1. Introduction

In the category of involution rings, it is not plausible to use the concept of left (right) ideal, since a left (right) ideal which is closed under involution is an ideal. An appropriate generalization which has been efficient in playing the role of these in the case of involution rings is that of *-biideal, first used by Loi [9] for proving structure theorems for involution rings. For semiprime involution rings, Loi also investigated the interrelation between the existence of minimal *-biideals and minimal biideals and Aburawash [3] characterized minimal *-biideals by means of idempotent elements. In [12], the author described minimal *-biideals of an arbitrary involution ring. The structure and properties of certain classes of right subdirectly irreducible rings (that is, rings in which the intersection of all nonzero right ideals is nonzero) were determined by

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Desphande ([6] , [7]). It seems, therefore, pertinent to consider involution rings in which the intersection of all nonzero *-biideals is nonzero. In a broader setting, we shall determine the structure of involution rings, belonging to certain classes, having exactly one minimal *-biideal.

All rings considered are associative and do not necessarily have identity. Let us recall that an involution ring A is a ring with an additional unary operation $*$, called involution, such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in A$. An element of an involution ring A, which is either symmetric or skew-symmetric, shall be called a *-element. A biideal B of a ring A is a subring of A satisfying the inclusion $BAB \subseteq B$. An ideal (biideal) B of an involution ring A is called a *-ideal (*-biideal) of A if B is closed under involution; that is, $B^* = \{a^* \in A : a \in B\} \subseteq B$. An involution ring A is semiprime if and only if, for any *-ideal I of A, $I^2 = 0$ implies $I = 0$. An involution ring A is called *-subdirectly irreducible if the intersection of all nonzero *-ideals of A (called the *-heart of A) is nonzero.

2. Involution rings with unique minimal *-biideal

We begin by considering involution rings in which the intersection of nonzero *-biideals is nonzero, which are obviously *-subdirectly irreducible. These will be called *-bi-subdirectly irreducible rings. If p is a prime, then $Z(p)$ denotes the zero ring on the cyclic additive group of order p.

Proposition 1. *Let A be a *-bi-subdirectly irreducible (with unique minimal *-biideal B). Then one of the following holds:*

(i) *A is a division ring with involution;*
(ii) *$A \cong D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution;*
(iii) *A is *-subdirectly irreducible involution ring with *-heart $B \cong Z(p)$ for some prime p;*
(iv) *A is a *-subdirectly irreducible involution ring with *-heart $H = K \oplus K^*$, where $K \cong Z(2) \cong K^*$ and $B = \{a + a^* : a \in K\} \cong Z(2)$.*

Proof. Since the intersection of the nonzero *-biideals of A is nonzero, B generates the *-heart H of A.

Case 1. $(H^2 \neq 0)$. Either H is a simple prime ring or $H = K \oplus K^*$, where the ideals K and K^* of A are simple prime rings [5].

The *-biideal B is contained in every nonzero *-biideal B_1 of H. Indeed, $0 \neq B_1HB_1$ is a *-biideal of A so that $B \subseteq B_1HB_1 \subseteq B_1$. Therefore,

H is a *-simple involution ring having a minimal *-biideal, namely B. If H is simple prime, then H has a minimal left ideal L and $L = He$ for some idempotent element e in H [1]. Then $0 \neq L^*L = e^*He$ is a minimal *-biideal of H. So $B = L^*L \subseteq L$. The *-ideal H does not contain other minimal left ideals besides L, for if L_1 is a minimal left ideal of H, then $B = L_1^*L_1 \subseteq L_1$. Now, $0 \neq B \subseteq L \cap L_1 \subseteq L_1$ and since L and L_1 are minimal left ideals, it follows that $L_1 = L$. Thus $H = L$ and H is a division ring. Since the *-essential *-ideal H has identity, we have, by ([11] , Lemma 8) that $A = H$. Thus A is a division ring. If $H = K \oplus K^*$, then it is clear, from [1] , that K and K^* have minimal left ideals. Moreover, it can be deduced that K and K^* have unique minimal left ideals and this implies that K and K^* are division rings. Consequently, $H = B$ and we have $A = H = K \oplus K^* \cong K \oplus K^{op}$ endowed with the exchange involution.

Case 2. $(H^2 = 0)$. In this case, the *-biideal $B \cong Z(p)$ for some prime p, according to ([12], Corollary 4(iii)). Moreover, every subgroup of H is a biideal of A. By ([8], Proposition 6.2), H^+, the additive group of H, is an elementary abelian p-group and hence is a direct sum of cyclic groups of order p. By our assumption on A, either $H \cong Z(p)$ or $H = K \oplus K^*$, where $K \cong Z(p) \cong K^*$. If $p \neq 2$, then the case $H = K \oplus K^*$ cannot occur, for then $\{a + a^* : a \in K\}$ and $\{a - a^* : a \in K\}$ would be two distinct minimal *-biideals of A. □

The following corollary is immediate:

Corollary 2. *An involution ring A is semiprime *-bi-subdirectly irreducible if and only if it is one of the following types:*

(i) *a division ring;*

(ii) *$D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution.*

Next, we study certain classes of involution rings having exactly one atom in their lattice of *-biideals. In the sequel, $[a]$ and $\langle a \rangle$ denote, respectively, the subring of A and the biideal of A generated by $a \in A$. Furthermore, if B is a biideal of A with p elements (p prime), we let $A_B = \{a \in A : pa = 0 = a^2 \text{ and } a \notin B\}$.

Lemma 3. *Let A be a nilpotent involution p-ring (p prime). Then A has a unique minimal *-biideal if and only if A is *-bi-subdirectly irreducible.*

Proof. Let A have a unique minimal *-biideal B. Then $B^2 = 0$, B contains a minimal *-subring S of order p and $B = S + SAS$, the *-biideal generated by S. But SAS is a *-biideal of A and $SAS = sAs$ for some *-element $s \in S$. Hence, either $sAs = 0$ or $sAs = B$. The latter case cannot occur, because then we would have $0 \neq s = sas$ for some $a \in A$; a contradiction with the fact that A is nilpotent. Therefore $B = S \cong Z(p)$. Now we will show that S is contained in every nonzero *-biideal of A. Let B_1 be any nonzero *-biideal of A. There exists a nonzero *-element s_1 in B_1, of order p and such that $s_1^2 = 0$. If $s_1As_1 \neq 0$, then there exists a nonzero *-element s_2 in s_1As_1. Now $s_2As_2 \subseteq s_1As_1 \subseteq B_1$. Continuing in this way, we obtain a chain $\ldots \subseteq s_iAs_i \ldots \subseteq s_2As_2 \subseteq s_1As_1 \subseteq B_1$. Since A is nilpotent, eventually we must obtain $s_iAs_i = 0$ for some nonzero *-element $s_i \in B_1$. Hence $\langle s_i \rangle = [s_i] = S$ and so $S \subseteq B_1$.

The converse is clear. □

Proposition 4. *If A is a nilpotent involution p-ring ($p \neq 2$ and p prime), then the following conditions are equivalent:*

(i) *A has a unique minimal *-biideal B;*

(ii) *A is subdirectly irreducible with heart $B \cong Z(p)$ and, for each $a \in A_B$, at least one of the following holds: $aAa \neq 0$, $aAa^* \neq 0$, $a^*Aa \neq 0$, $a^*a \neq 0$, $aa^* \neq 0$.*

Proof. Suppose that (i) holds. From the Lemma 3, we know that B is contained in every nonzero *-biideal of A. By Proposition 1, A is *-subdirectly irreducible with *-heart $B \cong Z(p)$. Next, we show that A is, in fact, subdirectly irreducible. Let I be any nonzero ideal of A such that $I \neq I^*$. We claim that $I \cap I^* \neq 0$. Suppose, on the contrary, that $I \cap I^* = 0$. Since A is nilpotent, there exists a least positive integer $n \geqslant 2$ such that $I^n = 0$. If n is even, let $J = I^{\frac{n}{2}}$ and if n is odd, let $J = I^{\frac{n+1}{2}}$. Hence $J^2 = JJ^* = J^*J = 0$. Then, for $0 \neq j \in J$ such that $pj = 0$ and $K = [j]$, it is easy to see that $\{k + k^* : k \in K\}$ and $\{k - k^* : k \in K\}$ are two distinct *-biideals of A of order p, which is a contradiction with our assumption. Therefore $I \cap I^* \neq 0$ and $B \subseteq I \cap I^* \subseteq I$. Hence A is a subdirectly irreducible ring with heart B. Suppose that there exists $a \in A_B$ such that $aAa = aAa^* = a^*Aa = 0$ and $a^*a = aa^* = 0$. If a is a *-element, then $[a]$ is a minimal *-biideal of A, which is a contradiction with our assumption. If a is not a *-element, and $T = [a]$, then $\{a + a^* : a \in T\}$ and $\{a - a^* : a \in T\}$ are distinct minimal *-biideals of A, which is again a contradiction.

Suppose that (ii) holds and let C be a minimal *-biideal of A and $C \neq B$. Clearly there exists a *-element $a \in C \cap A_B$ and $CAC = 0$, whence $aAa = aAa^* = a^*Aa = 0$ and $a^*a = aa^* = 0$, contradicting (ii). □

Corollary 5. *If A is an involution p-ring ($p \neq 2$ and p prime) and $A^2 = 0$, then the following conditions are equivalent:*

(i) *A has a unique minimal *-biideal B;*
(ii) *A has a unique minimal *-subring B;*
(iii) *A has a unique minimal subring B;*
(iv) *A is subdirectly irreducible with heart $B \cong Z(p)$ and $A_B = \varnothing$.*

The following example illustrates that Corollary 5 is not true, in general, when $p = 2$.

Example 6. *The 2-ring $A = Z(2) \oplus Z(2)$, with the exchange involution, is such that $A^2 = 0$ and has a unique minimal *-biideal, $B = \{(0,0),(1,1)\}$. However, A is not subdirectly irreducible.*

As usual, a ring A with identity 1 is called a local ring if $A/\mathcal{J}(A)$ is a division ring, where $\mathcal{J}(A)$ denotes the Jacobson radical of A.

Proposition 7. *Let A be a local involution ring of characteristic p^n ($p \neq 2$, p prime and $n \geqslant 1$) and with nonzero nilpotent Jacobson radical $\mathcal{J}(A)$. Then*

(i) *if $\mathcal{J}(A)$ has a unique minimal *-biideal B, then B is the unique minimal *-biideal of A;*
(ii) *$B = \{a \in A : a\mathcal{J}(A) = a^*\mathcal{J}(A) = 0\}$;*
(iii) *for a fixed nonzero $b \in B$, $\mathcal{J}(A) = \{a \in A : ba = ba^* = 0\} = \{a \in A : aB = a^*B = 0\}$;*
(iv) *for any $b \in B$, $a \in \mathcal{J}(A) \backslash B$, there exist $a_1, a_2 \in \mathcal{J}(A) \backslash B$ such that either $b = aa_1 = a_2a$ (if a is a *-element) or $b = (a + a^*)a_1 = a_2(a + a^*)$ (if a is not a *-element).*

Proof. (i) Taking into account Proposition 1 and the fact that a local ring contains only the trivial idempotents, it is clear that any minimal *-biideal of A must be contained in the Jacobson radical $\mathcal{J}(A)$ of A. If $\mathcal{J}(A)$ has a unique minimal *-biideal B, then we know that $B \cong Z(p)$ (Proposition 4). Clearly, $BAB \subseteq \mathcal{J}(A)$ and so, if $BAB \neq 0$, then $B \subseteq BAB$. However, this is impossible since $\mathcal{J}(A)$ is nilpotent. Thus $BAB = 0$ and so B is a biideal of A. Since any minimal *-biideal C of A is contained in $\mathcal{J}(A)$, we must have $C = B$.

(ii) From Proposition 1, $B \cong Z(p)$ and B is a *-ideal of A. Hence, for any nonzero $b \in B$, $b\mathcal{J}(A) \subseteq B$ implies that $b\mathcal{J}(A) = 0$ or $b\mathcal{J}(A) = B$. However, the latter case cannot occur since $\mathcal{J}(A)$ is nilpotent. Similarly, $b^*\mathcal{J}(A) = 0$. Thus $B \subseteq \{a \in A : a\mathcal{J}(A) = a^*\mathcal{J}(A) = 0\}$. Now to prove the other inclusion, let $a \in A$ such that $a\mathcal{J}(A) = a^*\mathcal{J}(A) = 0$. Then

$a \in \mathcal{J}(A)$ and $a^2 = 0$. Moreover, we claim that $pa = 0$. Indeed, since $(p1)^n = p^n 1 = 0$, $p1$ is not invertible and hence $p1 \in \mathcal{J}(A)$ and $pa = a(p1) = 0$. Taking into account Proposition 4, it follows that $a \in B$.

(iii) Let b be a fixed nonzero element in B. If $x \in \mathcal{J}(A)$, then also $x^* \in \mathcal{J}(A)$ and it follows from (ii) that $bx = bx^* = 0$ and so $x \in \{a \in A : ba = ba^* = 0\}$. On the other hand, if $x \in A$ such that $bx = bx^* = 0$, then $x \in \mathcal{J}(A)$, since $\mathcal{J}(A)$ contains all the zero divisors of A. Since $Ab = B$, it is now clear that $\mathcal{J}(A) = \{a \in A : ba = ba^* = 0\} = \{a \in A : Ba = Ba^* = 0\} = \{a \in A : aB = a^*B = 0\}$.

(iv) Let $b \in B$ and $a \in \mathcal{J}(A) \backslash B$. If a is a *-element, then $b \in Aa \cap aA$. If, on the other hand, a is not a *-element, then $b \in A(a + a^*) \cap (a + a^*) A$. □

Lemma 8. *Let A be a direct sum of rings, $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$, and let B be a biideal of A. There exist biideals B_k of A_k, $k = 1, 2, \ldots, n$, such that $B \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_n$. In particular, if B is a minimal biideal of A, then there exist minimal biideals B_k of A_k, $k = 1, 2, \ldots, n$, such that $B \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_n$.*

Proof. For each $k = 1, 2, \ldots, n$, consider the epimorphism $\pi_k : A_1 \oplus A_2 \oplus \ldots \oplus A_n \to A_k$ given by $\pi_k((a_1, a_2, \ldots, a_n)) = a_k$ and let $\pi_k(B) = B_k$. Then B_k is a biideal of A_k. For $b = (b_1, b_2, \ldots, b_n) \in B$, $\pi_k(b) = b_k$ and hence $b \in B_1 \oplus B_2 \oplus \ldots \oplus B_n$. Therefore $B \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_n$. Clearly, if B is a minimal biideal of A, then $\pi_k(B) = B_k$ is a minimal biideal of A_k, $k = 1, 2, \ldots, n$. □

For any prime p, let A_p denote, as usual, the p-component of an involution ring A. In addition, an involution ring A is said to be a *CI*-involution ring if every idempotent in A is central. Now we are in a position to give the following classification theorem.

Theorem 9. *Let A be a CI-involution ring with descending chain condition on *-biideals. Then A is *-bi-subdirectly irreducible if and only if A is one of the following rings:*

(i) *A is a division ring with involution;*

(ii) *$A \cong D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution;*

(iii) *A is a local involution ring of characteristic p^n (p prime and $n \geqslant 1$) with nonzero nilpotent Jacobson radical, having a unique minimal *-biideal;*

(iv) *$A \cong L \oplus L^{op}$ where each of the rings L and L^{op} is a local ring of characteristic 2^n $(n \geqslant 1)$ with nonzero nilpotent Jacobson radical having a unique minimal biideal and $L \oplus L^{op}$ is endowed with the exchange involution;*

(v) *A is a nilpotent involution p-ring (p prime) having a unique minimal *-biideal.*

Proof. First we prove the direct implication. It is well-known that an involution ring A has d.c.c. on *-biideals if and only if it is an artinian ring with artinian Jacobson radical $\mathcal{J}(A)$ and $\mathcal{J}(A)$ is nilpotent. Moreover, $A = F \oplus T$, where the *-ideal T is the maximal torsion ideal of A and F is a torsion-free *-ideal with identity and $\mathcal{J}(A) \subseteq T$ ([2], [4], [10]). Our assumption on A implies that the intersection of all nonzero *-biideals of A is a nonzero *-biideal and either $A = T = A_p$, for some prime p, or $A = F$. Suppose that $A = A_p$. Since A is artinian, either A_p has a nonzero idempotent or A_p is nilpotent. First, we consider the case when A_p has a nonzero idempotent. Then A_p has a nonzero idempotent e which is a *-element. Then e must be the identity of A_p. Indeed, if e is not the identity of A_p, then eA_p and $(1-e)A_p = \{a - ea : a \in A_p\}$ are nonzero *-biideals with zero intersection, contradicting our assumption. If e is the only nonzero idempotent in A_p, then, A_p, being artinian without nontrivial idempotents, is a local ring of characteristic p^n, for some integer $n \geqslant 1$, having a unique minimal *-biideal, and so (i) or (iii) holds.

If there is another nonzero idempotent element $f \neq e$ in A_p, then f is not a *-element and $ff^* = 0$. Indeed, if $ff^* \neq 0$, then $ff^* = 1$ and so $f = ff^*$, which is a contradiction with the fact that f is not a *-element. Likewise, $f^*f = 0$. Hence $f + f^*$ is the identity element of A_p. Furthermore, $A_p = fA_p \oplus f^*A_p$, where f and f^* are the only nonzero idempotents in fA_p and f^*A_p, respectively. Hence each of the ideals fA_p and f^*A_p is a local ring of characteristic p^n $(n \geqslant 1)$ with nilpotent Jacobson radical, having a unique minimal biideal. Thus (ii) or (iv) holds.

Notice that if $p \neq 2$ and S is the unique minimal biideal of fA_p, then $\{a + a^* : a \in S\}$ and $\{a - a^* : a \in S\}$ are two distinct minimal *-biideals of A_p. If A_p is nilpotent, then (v) holds. Suppose now that $A = F$. From Proposition 1 and the fact that A is torsion-free, it follows that A is either a division ring of characteristic zero or $A \cong D \oplus D^{op}$, where D is a division ring of characteristic zero and $D \oplus D^{op}$ is endowed with the exchange involution.

Conversely, it is clear that the involution rings in (i) and (ii) are *-bi-subdirectily irreducible (see [12]), and so are the involution rings in (iii) and (v). Taking into consideration Lemma 8, the involution rings in (iv) have a unique minimal *-biideal, so the descending chain condition on *biideals implies that these are *-bi-subdirectly irreducible. □

3. Involution rings with unique maximal biideal

The next proposition states that an involution ring with identity which has a unique maximal biideal B is a local involution ring with Jacobson radical B. The proof is an easy adaptation of the well-known result that if a ring A with identity has a unique maximal right ideal R, then R is in fact an ideal of A and $R = \mathcal{J}(A)$.

Proposition 10. *Let A be an involution ring with identity. If A has a unique maximal biideal B, then B is a *-ideal of A and $B = \mathcal{J}(A)$.*

Proof. Let $a \in A$. Then Ba is a biideal of A. If $Ba \neq A$, then Ba is contained in a maximal biideal of A. Indeed, it is easily deduced, using Zorn's Lemma, that every biideal is contained in a maximal biideal. Since B is the unique maximal biideal of A, $Ba \subseteq B$. On the other hand, if $Ba = A$, then $ba = 1$ and $b'a = a$ for certain $b, b' \in B$. Now $0 \neq ab = b'ab \in B$; hence $ab \neq 1$ and $1 - b'$ is not invertible and so $A(1 - b') \neq A$. But then $A(1 - b')$ is contained in a maximal biideal; that is, $1 - b' \in A(1 - b') \subseteq B$, whence $1 \in B$, which is a contradiction. Thus $Ba = A$ is impossible and so B is a right ideal of A. Since every right ideal is a biideal, we have that B is the unique maximal right ideal of A. As is well-known, B is therefore an ideal of A, it is also the unique maximal left ideal of A and $B = \mathcal{J}(A)$ is a *-ideal of A. □

Corollary 11. *A ring A with identity has a unique maximal biideal B if and only if it has a unique maximal right (left) ideal.*

Proof. The direct implication was proved in the previous proposition. Conversely, let A have a unique maximal right ideal R and let B_1 be a maximal biideal of A. Then $B_1 \subseteq B_1A \subseteq R$ and, since a right ideal is also a biideal, the maximality of B_1 implies that $B_1 = R$. □

We now terminate with a result which permits us to conclude that an involution ring with identity having a unique maximal *-biideal may not be a local ring.

Proposition 12. *If B is a maximal *-biideal of an involution ring A with identity, then one of the following conditions holds:*

(i) *B is a maximal biideal of A;*
(ii) *there exist maximal biideals K and K^* of A such that $B = K \cap K^*$.*

Proof. Let B be a maximal *-biideal of A. If B is not a maximal biideal of A, then B is contained in a maximal biideal K of A. Since B is closed under involution, B is also contained in K^*. Now $B \subseteq K \cap K^*$, where $K \cap K^*$ is a *-biideal of A. The maximality of B now implies that $B = K \cap K^*$. $\square$

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The action of Sylow 2-subgroups of symmetric groups on the set of bases and the problem of isomorphism of their Cayley graphs

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Abstract. Base (minimal generating set) of the Sylow 2-subgroup of S_{2^n} is called diagonal if every element of this set acts non-trivially only on one coordinate, and different elements act on different coordinates. The Sylow 2-subgroup $P_n(2)$ of S_{2^n} acts by conjugation on the set of all bases. In presented paper the stabilizer of the set of all diagonal bases in $S_n(2)$ is characterized and the orbits of the action are determined. It is shown that every orbit contains exactly 2^{n-1} diagonal bases and 2^{2^n-2n} bases at all. Recursive construction of Cayley graphs of $P_n(2)$ on diagonal bases ($n \geqslant 2$) is proposed.

Introduction

Let n be a positive integer greater then 1 and let p be a prime. By $P_n(p)$ we denote the Sylow p-subgroup of the symmetric group S_{p^n}. In this paper by *base of a group* we mean a minimal set of generators of this group (whitch further is simply called *a base*).

It is known that

$$P_n(p) \cong \underbrace{C_p \wr C_p \wr \ldots \wr C_p}_{n},$$

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where C_p is a cyclic permutation group of order p. For every finite p-group G the following equality holds:

$$\Phi(G) = G' \cdot G^p,$$

where $\Phi(G)$ is a Frattini subgroup of G (see e.g. [2]). If $G = P_n(p)$ then $G' = G^p$, thus

$$\Phi(P_n(p)) = (P_n(p))'.$$

So

$$P_n(p)/(P_n(p))' \cong \mathbb{Z}_p^n,$$

but $\mathbb{Z}_p^n$ is a vector space over $\mathbb{Z}_p$ and every basis of $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$ induces a base of $P_n(p)$. Thus every base of $P_n(p)$ has exactly n elements. The group $P_n(p)$ acts on the set of bases of $P_n(p)$ by inner automorphisms. The purpose of this article is to investigate orbits of this action and the respective Cayley graphs of $P_n(p)$. We will consider the case $p = 2$, because group $P_n(2)$ is of particular interest. Namely group $P_n(2)$ is the full group of automorphisms of 2-adic rooted tree of height n (see eg. [3]) and the inverse limit of such groups is a group of automorphisms of 2-adic rooted tree, which is widely investigated because of its properties (for the survey, see e.g. [1]). On the other hand, $p = 2$ is also the only case for which considered diagonal bases generate undirected Cayley graphs.

In Section 2 we recall basic facts about Sylow p-subgroups of symmetric groups and the polynomial (Kaluzhnin) representation of such subgroups. Section 3 shows a special type of bases of Sylow 2-subgroups of S_{2^n} called diagonal bases and some of their properties (an exemplary construction of a diagonal base is presented in [5]). Also in this section we present some further investigations of these bases, which lead us to the definition of primal diagonal bases and characterize the orbits of the action of $P_n(2)$ by inner automorphisms on the set of all diagonal bases. In Section 4 we present a recursive algorithm for construction of Cayley graphs of $P_n(2)$ on diagonal bases. In Section 5 we give some examples of Cayley graphs constructed with the proposed algorithm and present two non-isomorphic Cayley graphs of $P_3(n)$.

1. Preliminaries

Let X_i be the vector of variables $x_1, x_2, \ldots, x_i$. Polynomial representation of group $P_n(p)$ (see e.g. [4], [6]) states that every element $f \in P_n(p)$ can be written in form

$$f = [f_1, f_2(X_1), f_3(X_2), \ldots, f_n(X_{n-1})], \tag{1}$$

where $f_1 \in \mathbb{Z}_p$ and $f_i : \mathbb{Z}_p^{i-1} \to \mathbb{Z}_p$ for $i = 2, \ldots, n$ are reduced polynomials from the quotient ring $\mathbb{Z}_p[X_i]/\langle x_1^p - x_1, \ldots, x_i^p - x_i\rangle$. Following the original paper of L. Kaluzhnin ([4]) we call such element f a *tableau*. By $[f]_i$ we denote the i-th coordinate of tableau f and by $f_{(i)}$ we denote the table

$$f_{(i)} = [f_1, f_2(X_1), \ldots, f_i(X_{i-1})] \in P_i(p),$$

where $i \leqslant n$.

For tableaux $f, g \in P_n(p)$, where f has the form (1) and

$$g = [g_1, g_2(X_1), g_3(X_2), \ldots, g_n(X_{n-1})]$$

the product fg has the form

$$\begin{aligned} fg = [&f_1 + g_1,\ f_2(X_1) + g_2(x_1 + f_1),\ \ldots, \\ &f_n(X_{n-1}) + g_n(x_1 + f_1, x_2 + f_2(X_1), \ldots, x_{n-1} + f_{n-1}(X_{n-2}))], \end{aligned}$$

and the inverse

$$\begin{aligned} f^{-1} = \Big[&- f_1, -f_2(x_1 - f_1), \ldots, \\ &- f_n(x_1 - f_1, x_2 - f_2(x_1 - f_1), \ldots, x_{n-1} - f_{n-1}(x_1 - f_1, \ldots))\Big]. \end{aligned}$$

Let $\mathfrak{B}$ be the set of all bases of $P_n(p)$. $P_n(p)$ acts on the set $\mathfrak{B}$ by conjugation:

$$B^u = \langle u^{-1}B_1u,\ u^{-1}B_2u,\ \ldots,\ u^{-1}B_nu\rangle \tag{2}$$

for all $B = \{B_1, \ldots, B_n\} \in \mathfrak{B}$.

Lemma 1. *The center of group $P_n(p)$ has the form*

$$Z(P_n(p)) = \{[0, \ldots, 0, \alpha] :\ \alpha \in \mathbb{Z}_p\}.$$

Proof. See [4]. □

Proposition 1. *The action (2) of $P_n(p)$ on the set $\mathfrak{B}$ is semi-regular. The length of every orbit of this action is equal to $p^{\frac{p^n-1}{p-1}-1}$.*

Proof. An action of a group G on a set X is semi-regular, iff every orbit of G on X has the same length. Let $B = \{B_1, B_2, \ldots, B_n\}$ be a base of $P_n(p)$. For any $u \in P_n(p)$ we have $B^u = B$ if and only if $u^{-1}B_iu = B_i$ for every $i = 1, \ldots, n$. Since $\langle B_1, \ldots, B_n\rangle = P_n(p)$, it follows that for every

$g \in P_n(2)$, equality $u^{-1}gu = g$ holds if and only if $u \in Z(P_n(2))$. But following Lemma 1:

$$|Z(P_n(p))| = p,$$

hence the length of orbit containing B is equal to $\frac{|P_n(p)|}{p}$. Thus the length of every orbit is the same regardless of the choice of base B. Hence the action (2) is semi-regular. The length of every orbit is equal to

$$\frac{|P_n(p)|}{p} = p^{\frac{p^n-1}{p-1}-1}. \qquad \square$$

2. Diagonal bases of $P_n(2)$

From now on we assume that $p = 2$.

2.1. Definitions and basic facts

Let $\overline{x_n}$ be the monomial $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ and let $\overline{x_n}/x_i$ be the monomial $x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_n$ for $i = 1, \ldots, n$.

In [6] the authors defined so-called triangular bases of group $P_n(p)$. In the following article we consider a special type of triangular bases, which we call diagonal. However, the notion of diagonal bases can be formulated independently of triangularity.

Definition 1. Base $B = \{B_1, \ldots, B_n\} \in \mathfrak{B}$ is called diagonal if for any i, $1 \leqslant i \leqslant n$, the table B_i is i-th coordinative, i.e. $[B_i]_j = 0$ for $j \neq i$.

It is well known that in every base B of $P_n(2)$ for every i there exists a tableaux $B' \in B$ which contains a monomial $\overline{x_{i-1}}$ on i-th coordinate. Thus, the nonzero coordinates of elements of diagonal base $B = \{B_1, \ldots, B_n\}$ have form $[B_1]_1 = 1$ and $[B_i]_i = b_i(X_{i-1})$, where b_i contains monomial $\overline{x_{i-1}}$ for every $i = 2, \ldots, n$.

Diagonal bases $B = \{B_1, \ldots, B_n\}$ and $C = \{C_1, \ldots, C_n\}$ of $P_n(2)$ are conjugate if there exists element $u \in P_n(2)$ such that $u^{-1}Bu = C$, i.e.

$$u^{-1} B_i u = C_i \tag{3}$$

for every $i = 1, \ldots, n$.

Definition 2. The length $l(m)$ of a nonzero monomial $m = x_{i_1} \ldots x_{i_k}$ is the number of variables of this monomial. We assume that $l(0) = -1$ and $l(1) = 0$. The length of the reduced polynomial is equal to the maximal length of its monomials.

For every polynomials f and g the following inequality holds:

$$l(f+g) \leqslant \max\{l(f), l(g)\}.$$

Definition 3. Reduced polynomial $f_n : \mathbb{Z}_2^{n-1} \to \mathbb{Z}_2$ is called primal if

$$f_n = \overline{x_{n-1}} + \beta_n(X_{n-1}),$$

where $l(\beta_n) \leqslant n-3$.

Diagonal base $B = \{B_1, \ldots, B_n\}$ is called primal if $[B_n]_n$ is primal polynomial.

Let $\delta(P_n(2))$ and $\delta'(P_n(2))$ be the numbers of different diagonal bases and different primal diagonal bases of $P_n(2)$, respectively.

Theorem 1. *The following equalities holds:*

$$\delta(P_n(2)) = 2^{2^n-(n+1)} \quad \textit{and} \quad \delta'(P_n(2)) = 2^{2^n-2n}.$$

Proof. Let $B = \{B_1, \ldots, B_n\}$ be a diagonal base of $P_n(2)$, i.e. every tableau B_i has on i-th coordinate a polynomial of length $i-1$ for $1 \leqslant i \leqslant n$. Every polynomial $[B_i]_i$ contains monomial $\overline{x_{i-1}}$. There are 2^{i-1} monomials on variables $x_1, \ldots, x_{i-1}$. Thus there are $2^{2^{i-1}-1}$ polynomials on $(i-1)$ variables, which length equal to $i-1$. So the number of diagonal bases of $P_n(2)$ is equal to

$$\prod_{i=0}^{n-1} 2^{2^i-1} = 2^{\gamma},$$

where $\gamma = \sum_{i=0}^{n-1}(2^i-1) = 2^n-(n+1)$.

Let B be a primal diagonal base, i.e. $[B_n]_n$ be the primal polynomial. There are $2^{2^{n-1}-n}$ primal polynomials on $(n-1)$ variables. So the number of different primal diagonal bases of $P_n(2)$ is equal to

$$\left(\prod_{i=0}^{n-2} 2^{2^i-1}\right) \cdot 2^{2^{n-1}-n} = 2^{\gamma'},$$

where $\gamma' = \left(\sum_{i=0}^{n-2}(2^i-1)\right) + 2^{n-1} - n = 2^{n-1} - n + 2^{n-1} - n = 2^n - 2n$. $\square$

2.2. Properties of diagonal bases

Let
$$\Lambda = \{[\lambda_1, \ldots, \lambda_n] : \lambda_i \in \mathbb{Z}_2,\, 1 \leqslant i \leqslant n\}$$
be an maximal elementary abelian 2-subgroup of group $P_n(2)$. For any $\lambda = [\lambda_1, \ldots, \lambda_n] \in \Lambda$ and vector X_{n-1} we denote
$$X_{n-1} + \lambda = (x_1 + \lambda_1, \ldots, x_{n-1} + \lambda_{n-1}).$$

We can define the left and right actions of group Λ on the set of reduced polynomial on $(n-1)$ variables in the following way. For a reduced polynomial $f : \mathbb{Z}_2^{n-1} \to \mathbb{Z}_2$ and $\lambda = [\lambda_1, \ldots, \lambda_n] \in \Lambda$ let
$$\lambda \star f(X_{n-1}) = f(X_{n-1} + \lambda) + \lambda_n \quad \text{and} \quad f(X_{n-1}) \star \lambda = f(X_{n-1}) + \lambda_n.$$
As we can can see, this actions resemble the multiplication of tables in $P_n(p)$.

Lemma 2. *Let $\lambda = [\lambda_1, \ldots, \lambda_n] \in \Lambda$ and let $f(X_{n-1}) = \overline{x_{n-1}}$. Then*
$$\lambda^{-1} \star f(X_{n-1}) \star \lambda = \overline{x_{n-1}} + \sum_{i=1}^{n-1} \lambda_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}),$$
where h is some reduced polynomial such that $l(h) \leqslant n-3$.

Proof. We have
$$\begin{aligned}\lambda^{-1} \star f(X_{n-1}) &= (x_1 + \lambda_1)(x_2 + \lambda_2) \ldots (x_{n-1} + \lambda_{n-1}) + \lambda_n \\ &= x_1 x_2 \ldots x_{n-1} + (\lambda_1 x_2 \ldots x_{n-1} + \lambda_2 x_1 x_3 \ldots x_{n-1} + \ldots + \lambda_{n-1} x_1 \ldots x_{n-2}) \\ &\quad + \ldots + \lambda_1 \lambda_2 \ldots \lambda_{n-1} + \lambda_n \\ &= \overline{x_{n-1}} + \sum_{i=1}^{n-1} \lambda_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}) + \lambda_n,\end{aligned}$$
where h is some reduced polynomial such that $l(h) \leqslant n-3$. Thus
$$\begin{aligned}\lambda^{-1} \star f(X_{n-1}) \star \lambda &= \overline{x_{n-1}} + \sum_{i=1}^{n-1} \lambda_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}) + \lambda_n + \lambda_n \\ &= \overline{x_{n-1}} + \sum_{i=1}^{n-1} \lambda_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}). \qquad \square\end{aligned}$$

There is also an important relation between polynomials of maximal length and the primal polynomials.

Lemma 3. *For every reduced polynomial $f : \mathbb{Z}_2^{n-1} \to \mathbb{Z}_2$ such that $l(f) = n-1$, there exists a tableau $\lambda \in \Lambda$ such that $\lambda^{-1} \star f \star \lambda$ is the primal polynomial.*

Proof. Every polynomial $f(X_{n-1})$ such that $l(f) = n-1$ can be written in the form

$$f(X_{n-1}) = \overline{x_{n-1}} + \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}),$$

where $\alpha_i \in \mathbb{Z}_2$ for $i = 1, \ldots, n-1$ and $l(h) \leqslant n-3$.

Let $f_1(X_{n-1}) = \overline{x_{n-1}}$ and $f_2^{(i)}(X_{n-1}) = \alpha_i(\overline{x_{n-1}}/x_i)$ for every $i = 1, \ldots, n-1$. Then

$$f = f_1 + \sum_{i=1}^{n-1} f_2^{(i)} + h$$

and

$$\lambda^{-1} \star f \star \lambda = \lambda^{-1} \star f_1 \star \lambda + \sum_{i=1}^{n-1} (\lambda^{-1} \star f_2^{(i)} \star \lambda) + \lambda^{-1} \star h \star \lambda. \quad (4)$$

We construct the tableau λ using coefficients α_i from the polynomial f in form $\lambda = [\alpha_1, \ldots, \alpha_{n-1}, u_n]$, where $u_n \in \mathbb{Z}_2$ is fixed. Let us investigate the form of sum (4). From Lemma 2 we have

$$\lambda^{-1} \star f_1(X_{n-1}) \star \lambda = \overline{x_{n-1}} + \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + h'(X_{n-1})$$

where h' is some reduced polynomial such that $l(h') \leqslant n-3$, and

$$\begin{aligned}\lambda^{-1} \star f_2^{(i)}(X_{n-1}) \star \lambda =& \alpha_i(\overline{x_{n-1}}/x_i) \\ &+ \alpha_i \sum_{j=1, j\neq i}^{n-1} \beta_j\big((\overline{x_{n-1}}/x_i)/x_j\big) + \alpha_i k^{(i)}(X_{n-1}),\end{aligned}$$

where $\beta_j \in \mathbb{Z}_2$ and $k^{(i)}$ is some reduced polynomial such that $l(k^{(i)}) \leqslant n-4$. Thus

$$\begin{aligned}&\sum_{i=1}^{n-1} \Big(\lambda^{-1} \star f_2^{(i)}(X_{n-1}) \star \lambda\Big) \\ &\quad = \sum_{i=1}^{n-1} \alpha_i \left(\overline{x_{n-1}}/x_i + \sum_{j=1, j\neq i}^{n-1} \beta_j\big((\overline{x_{n-1}}/x_i)/x_j\big) + k^{(i)}(X_{n-1})\right)\end{aligned}$$

$$= \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + h''(X_{n-1}),$$

where h'' is some reduced polynomial such that $l(h'') \leqslant n-3$.

The last element in sum (4) has the form

$$\lambda^{-1} \star h(X_{n-1}) \star \lambda = h'''_n(X_{n-1}),$$

where h''' is some reduced polynomial such that $l(h''') \leqslant n-3$. Thus finally

$$\begin{aligned}\lambda^{-1} \star f(X_{n-1}) \star \lambda = \overline{x_{n-1}} + \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + h'(X_{n-1}) \\ + \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + h''(X_{n-1}) + h'''(X_{n-1}) \\ = \overline{x_{n-1}} + h'(X_{n-1}) + h''(X_{n-1}) + h'''(X_{n-1}) \\ = \overline{x_{n-1}} + b(X_{n-1}),\end{aligned}$$

where $b = h' + h'' + h'''$ and $l(b) \leqslant n-3$. So $\lambda^{-1} \star f \star \lambda$ is a primal polynomial. □

Theorem 2. *Every*

$$f = [0, 0, \ldots, 0, f_n(X_{n-1})] \in P_n(2)$$

where $l(f_n) = n-1$, *is conjugate to a tableau*

$$b = [0, 0, \ldots, 0, b_n(X_{n-1})],$$

where b_n *is the primal polynomial.*

Proof. Similarly like in the proof of Lemma 3, tableau f can be written in form

$$f = \left[0,\ \ldots,\ 0,\ \overline{x_{n-1}} + \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + h_n(X_{n-1})\right],$$

where $\alpha_i \in \mathbb{Z}_2$ for $i = 1, \ldots, n-1$ and $l(h_n) \leqslant n-3$.

Let us construct the tableau u using coefficients α_i from tableau f. Let $u = [\alpha_1,\ \ldots,\ \alpha_{n-1}, u_n]$, where $u_n \in \mathbb{Z}_2$ is fixed. Notice that $u \in \Lambda$. Of course the equality

$$[u^{-1} f u]_j = 0$$

holds for every $j = 1, \ldots, n-1$. From Lemma 3 we get that $[u^{-1} f u]_n$ is the primal polynomial. □

Let us denote the set of all diagonal bases of $P_n(2)$ by $\mathfrak{D}$. Now we describe stabilizer of the set $\mathfrak{D}$ in the group $P_n(2)$ with respect to the action (2).

Theorem 3. *The stabilizer of the subset $\mathfrak{D} \subset \mathfrak{B}$ in the group $P_n(2)$ acting on the set $\mathfrak{B}$ according to (2) is equal to Λ. The kernel of this action coincide with the center of $P_n(2)$.*

Proof. To show that Λ is the stabilizer of $\mathfrak{D}$ we have to prove the following.

1) If $B = \{B_1, \dots, B_n\}$ is a diagonal base of $P_n(2)$ and $\lambda \in \Lambda$, then $\lambda^{-1} B \lambda$ is a diagonal base of $P_n(2)$.
2) For every diagonal bases $B = \{B_1, \dots, B_n\}$ and $C = \{C_1, \dots, C_n\}$ of $P_n(2)$ if there exists $u \in P_n(2)$ such that $u^{-1} B u = C$, then $u \in \Lambda$.

A set conjugate to a base is always a base. Let $1 \leqslant s \leqslant n$ and let $B_s \in P_n(2)$ be a tableau with the only nonzero element on its s-th coordinate. Let $j \neq s$. Then

$$[\lambda^{-1} B_s \lambda]_j = 0.$$

Thus the first condition is proved.

We now prove the second condition. Let $[B_1]_1 = 1$ and $[B_i]_i = b_i(X_{i-1})$ for $i = 2, \dots, n$. Base B is diagonal, so $b_i(X_{i-1}) \neq 0$ for every $i = 2, \dots, n$. Let

$$u = [\alpha_1, u_2(X_1), \dots, u_n(X_n)].$$

We will show that for every $s = 1, \dots, n-1$, the reduced polynomial u_i for $i = 2, \dots, n$ does not contain variable x_s. Variable x_s can be contained only in polynomials u_i for which $i > s$. Every such polynomial can be described as

$$u_i(X_{i-1}) = u_i'(X_{i-1}) \cdot x_s + u_i''(X_{i-1}),$$

where polynomials u_i' and u_i'' do not contain variable x_s. Equality $u^{-1} B_s u = C_s$ can be written in form $B_s u = u C_s$. Thus

$$[B_s u]_k = [u C_s]_k \tag{5}$$

for every $k = 1, \dots, n$. For $k > s$ we have $[B_s]_k = [C_s]_k = 0$, so in this case

$$\begin{aligned}[B_s u]_k &= 0 + u_i'(X_{i-1}) \cdot (x_s + b_i(X_{i-1})) + u_i''(X_{i-1}) \\ &= u_i'(X_{i-1}) \cdot x_s + u_i'(X_{i-1}) \cdot b_i(X_{i-1}) + u_i''(X_{i-1})\end{aligned}$$

and

$$[u C_s]_k = u_i'(X_{i-1}) \cdot x_s + u_i''(X_{i-1}) + 0 = u_i'(X_{i-1}) \cdot x_s + u_i''(X_{i-1}).$$

Thus

$$[B_s u]_k = [uC_s]_k,$$
$$u_i'(X_{i-1})\, x_s + u_i'(X_{i-1})\, b_i(X_{i-1}) + u_i''(X_{i-1}) = u_i'(X_{i-1})\, x_s + u_i''(X_{i-1}),$$
$$u_i'(X_{i-1})\, b_i(X_{i-1}) = 0.$$

We know that $b_i(X_{i-1}) \neq 0$, so $u_i'(X_{i-1}) = 0$ and hence

$$u_i = 0 \cdot x_s + u_i''(X_{i-1}) = u_i''(X_{i-1}),$$

where u_i'' does not contain variable x_s.

We have shown that any variable x_s for $1 \leqslant s \leqslant n$ is not contained in polynomials u_i for $i = 2, \ldots, n$, so $u_i(X_{i-1}) = \alpha_i$, where α_i is constant and hence $u = [\alpha_1, \alpha_2, \ldots, \alpha_n] \in \Lambda$. Thus indeed Λ is the stabilizer of σ on $\mathfrak{D}$. Lemma 1 implies that the center of $P_n(2)$ contains only the tableaux $[0, \ldots, 0, 0]$ and $[0, \ldots, 0, 1]$.

Let

$$b_n(X_{n-1}) = \overline{x_{n-1}} + \sum_{i=1}^{n-1} \alpha_i(\overline{x_{n-1}}/x_i) + \beta_n(X_{n-1}),$$

where β_n is some reduced polynomial such that $l(\beta_n) \leqslant n-3$. Thus

$$b_n(x_1+\lambda_1, \ldots, x_{n-1}+\lambda_{n-1}) = \overline{x_{n-1}} + \sum_{i=1}^{n-1} (\alpha_i+\lambda_i)(\overline{x_{n-1}}/x_i) + \overline{\beta_n}(X_{n-1}),$$

where $\overline{\beta_n}$ is a reduced polynomial such that $l(\overline{\beta_n}) \leqslant n-3$. So the necessary condition for the equality $\lambda^{-1} B_n \lambda = B_n$ to hold is

$$\alpha_i = \alpha_i + \lambda_i$$

for all $i = 1, \ldots, n-1$. So $\lambda_i = 0$ for all such i. It follows that $\overline{\beta_n} = \beta_n$. Hence

$$\lambda^{-1} B_n \lambda = B_n$$

if and only if $\lambda_1 = \ldots = \lambda_{n-1} = 0$. $\square$

Corollary 1. *If B and C are two conjugated diagonal bases of $P_n(2)$ such that for tableaux $u, v \in \Lambda$ the following equalities hold:*

$$u^{-1} B u = C \quad \textit{and} \quad v^{-1} B v = C,$$

then

$$u = v + [0, \ldots, 0, \alpha],$$

where $\alpha \in \mathbb{Z}_2$.

2.3. Properties of primal diagonal bases

Let $B = \{B_1, \dots, B_n\}$ be a diagonal base of $P_n(2)$. Theorem 2 implies that tableau B_n is conjugate with some tableau $C_n = [0, \dots, 0, c_n(X_{n-1})]$, where c_n is the primal polynomial. As we could see in the proof of Theorem 2, the tableau u which conjugate tableaux B_n and C_n belongs to the subgroup Λ. Thus, by Theorem 3 we can formulate

Corollary 2. *Every diagonal base of $P_n(2)$ is conjugate to some primal diagonal base.*

Primal diagonal bases have another important property.

Theorem 4. *If B and C are different primal diagonal bases of $P_n(2)$, then B and C are not conjugated.*

Proof. Let us assume that bases

$$B = \{B_1, \dots, B_n\} \quad \text{and} \quad C = \{C_1, \dots, C_n\}$$

are conjugated. Then according to Theorem 3 there exists tableau $u \in \Lambda$ such that

$$u^{-1} B u = C. \tag{6}$$

Let

$$B_n = [0, \dots, 0, \overline{x_{n-1}} + \beta_n(X_{n-1})], \quad \text{where } l(\beta_n) \leqslant n-3,$$

and

$$C_n = [0, \dots, 0, \overline{x_{n-1}} + \gamma_n(X_{n-1})], \quad \text{where } l(\gamma_n) \leqslant n-3.$$

From (6) we get the equality

$$[u^{-1} B_n u]_n = [C_n]_n. \tag{7}$$

By Lemma 2, we have

$$[u^{-1} B_n u]_n = \overline{x_{n-1}} + \sum_{i=1}^{n-1} u_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}),$$

where $l(h) \leqslant n-2$. So equation (7) implies that

$$\overline{x_{n-1}} + \sum_{i=1}^{n-1} u_i(\overline{x_{n-1}}/x_i) + h(X_{n-1}) = \overline{x_{n-1}} + \gamma_n(X_{n-1}).$$

Thus $h(X_{n-1}) = \gamma_n(X_{n-1})$ and $u_i(\overline{x_{n-1}}/x_i) = 0$ for every $i = 1, \ldots, n-1$, so $u_i = 0$ for every $i = 1, \ldots, n-1$, that is, $u = [0, \ldots, 0, u_n]$. But if $u = [0, \ldots, 0, u_n]$ then $u^{-1}Bu = B$ and from (6) we get that $B = C$, which contradicts the assumption that B and C are different primal diagonal bases. □

The orbit of $P_n(2)$ on $\mathfrak{B}$ by action (2) which contains a diagonal base is called $\mathfrak{D}$-orbit. Summing up previous results we can formulate following

Theorem 5. *The following statement holds:*
1) *every $\mathfrak{D}$-orbit contains exactly one primal diagonal base;*
2) *every $\mathfrak{D}$-orbit contains exactly 2^{n-1} diagonal bases and 2^{2^n-2} bases at all;*
3) *the number of different $\mathfrak{D}$-orbits is equal to 2^{2^n-2n}.*

Proof. 1) Corollary 2 states that every diagonal base is conjugate with some primal diagonal base. Thus every $\mathfrak{D}$-orbit contains a primal diagonal base. From Theorem 4 we get that this primal diagonal base is unique in every $\mathfrak{D}$-orbit.

2) From Theorem 3 we know that the elements which conjugate diagonal bases are of form $u = [u_1, \ldots, u_{n-1}, u_n]$, where $u_i \in \mathbb{Z}_2$ for $i = 1, \ldots, n$. Theorem 3 also states that conjugation does not depend on u_n, so the number of conjugated diagonal bases is equal to the number of different tableaux of the form $[u_1, \ldots, u_{n-1}, 0]$. There are 2^{n-1} such tableaux. The number of all bases in single $\mathfrak{D}$-orbit is determined by Theorem 1.

3) Every $\mathfrak{D}$-orbit contains exactly one primal diagonal base, so the number of $\mathfrak{D}$-orbits is equal to the number of different primal diagonal bases, which is equal to $2^{2^n} - 2n$ by Theorem 1. □

3. Cayley graphs of $P_n(2)$ on diagonal bases

We recall the definition of Cayley graphs.

Definition 4. Let G be a group and S be a set of generators of G. The Cayley graph of group G on set S is a graph $\mathrm{Cay}(G, S)$ in which vertex set is equal to G and two vertices u, v are connected by an edge iff there exists $s \in S$ such that $u = v \cdot s$. Such edge will be denoted as uv.

If $S = S^{-1}$, then $\mathrm{Cay}(G, S)$ is undirected. Thus Cayley graphs of $P_n(2)$ on diagonal bases are undirected.

From now on in this section we assume that $n > 2$.

Let $B = \{B_1, \dots, B_n\}$ be a diagonal base of $P_n(2)$. By Theorem 5 base B is in the same orbit with some primal diagonal base $D = \{D_1, \dots, D_n\}$, so

$$\mathrm{Cay}(P_n(2), B) \cong \mathrm{Cay}(P_n(2), D).$$

Thus investigation of Cayley graphs of $P_2(n)$ on diagonal bases is equivalent with investigation of Cayley graphs only on primal diagonal bases.

Let $B' = \{(B_1)_{(n-1)}, \dots, (B_{n-1})_{(n-1)}\}$. Set B' is a diagonal base of group $P_{n-1}(2)$.

Theorem 6. *Let $D = \{D_1, \dots, D_{n-1}, D_n\}$ be a diagonal base of $P_n(2)$ and let $D' = \{(D_1)_{(n-1)}, \dots, (D_{n-1})_{(n-1)}\}$ be a diagonal base of $P_{n-1}(2)$. Let Γ be a graph obtained from $\mathrm{Cay}(P_n(2), D)$ by removing edges of form uD_n for every $u \in P_n(2)$. Then*

1) *Γ is not connected;*
2) *Γ contains $2^{2^{n-1}}$ connected components;*
3) *every connected component of Γ is isomorphic to the Cayley graph $\mathrm{Cay}(P_{n-1}(2), D')$.*

Proof. Let $(D_{j_1}, D_{j_2}, \dots, D_{j_l})$ be a tuple of (not necessarily different) elements of $D\backslash\{D_n\}$, i.e. $D_{j_k} \in \{D_1, \dots, D_{n-1}\}$ for every $k = 1, \dots, l$. Thus

$$\left[\prod_{k=1}^{l} D_{i_k}\right]_n = 0. \tag{8}$$

We now prove stated properties.

1) Consider vertices $f_1 = [0, \dots, 0]$ and $f_2 = [0, \dots, 0, 1]$ of graph Γ. Equality (8) implies that

$$\left[f_1 \cdot \prod_{k=1}^{l} D_{i_k}\right]_n = 0.$$

Thus in Γ there is no path from vertex f_1 to vertex f_2, which implies that Γ is not connected.

2) Let $f = [0, \dots, 0, f_n(X_{n-1})]$. Equality (8) implies that

$$\left[f \cdot \prod_{k=1}^{l} D_{i_k}\right]_n = f_n(X_{n-1}).$$

Thus if $g = [0, \dots, 0, g_n(X_{n-1})]$ and $g_n \neq f_n$, then vertices f and g are contained in different connected components of Γ.

Let f' be a tableau for which $[f']_n = [f]_n$. Set D' is a base of $P_{n-1}(2)$, and there exists a set $\{D_{j_1}, D_{j_2}, \ldots, D_{j_l}\}$ of elements of $D\backslash\{D_n\}$ such that

$$f' \cdot \prod_{k=1}^{l} D_{i_k} = f.$$

Thus every vertex

$$f' = [f_1, \ldots, f_n(X_{n-1})]$$

of Γ is contained in the same connected component of Γ as vertices of the form

$$[0, \ldots, 0, f_n(X_{n-1})], \tag{9}$$

and different vertices of form (9) lays in different connected components of Γ, so the number of connected component of Γ is equal to the number of different reduced polynomials $f_n : \mathbb{Z}_2^{n-1} \to \mathbb{Z}_2$, which is equal to $2^{2^{n-1}}$.

3) We have shown that every connected component of Γ contains a vertex made of tableaux with fixed last coordinate. Let V_{f_n} be the subgroup of $P_n(2)$ such that if $g \in V_{f_n}$ iff $[g_n] = f_n$. Thus $V_{f_n} \cong P_{n-1}(2)$, hence

$$\mathrm{Cay}(V_{f_n}, D') \cong \mathrm{Cay}(P_{n-1}(2), D'). \qquad \square$$

Theorem 6 implies the recurrent construction of Cayley graphs of $P_n(2)$ on primal diagonal bases. Let $D = \{D_1, \ldots, D_n\}$ be a primal diagonal base of $P_n(2)$. Graph $\mathrm{Cay}(P_n(2), D)$ can be constructed in following way.

1) We construct $2^{2^{n-1}}$ Cayley graphs $\mathrm{Cay}(P_{n-1}(2), D')$, where

$$D' = \{(D_1)_{(n-1)}, \ldots, (D_{n-1})_{(n-1)}\}.$$

Every such Cayley graph may be labeled with a different reduced polynomial $f_n : \mathbb{Z}_2^{n-1} \to \mathbb{Z}_2$. Denote the Cayley graph corresponding to polynomial f_n by Cay_{f_n}.

2) In every graph Cay_{f_n} we replace the set of vertices $V(\mathrm{Cay}_{f_n}) = P_{n-1}(2)$ by the set of vertices $V' \subset P_n(2)$ in following way: we replace $u = [u_1, \ldots, u_{n-1}(X_{n-2})]$ by

$$u' = [u_1, \ldots, u_{n-1}(X_{n-2}), f_n(X_{n-1})]$$

for every $u \in V(\mathrm{Cay}_{f_n})$.

3) For every pair of vertices u', v' of obtained graph, if $u'B_n = v'$, then we add an edge $u'v'$.

So in the construction we need to start with the case $n = 2$, which is presented in the next section.

Above construction suggests the dependence between Cayley graphs and Schreier coset graphs on diagonal bases of $P_n(2)$.

Let us recall the definition of the latter graphs.

Definition 5. Let G be a group, S be a set of generators of G and H be a subgroup of finite index in G. The Schreier coset graph $\mathrm{Sch}(G,S,H)$ is a graph whose vertices are the right cosets of H in G and two vertices Hu and Hv are connected by an edge iff there exists $s \in S$ such that $Hu = Hv \cdot s$.

Let us notice that every Cayley graph of group G is a Schreier coset graph of G in which H is a trivial subgroup.

We consider a subgroup $\overline{P}_n(2)$ of group $P_n(2)$ in which in every tableuax the last coordinate is equal to 0, i.e. if $f \in \overline{P}_n(2)$,then

$$f = [f_1, f_2(X_1), \dots, f_{n-1}(X_{n-2}), 0].$$

Of course $\overline{P}_n(2) \cong P_{n-1}(2)$.

Theorem 7. *Let $D = \{D_1, \dots, D_n\}$ be a diagonal base of $P_n(2)$. Then the following conditions hold.*

1) *Two vertices $\overline{P}_n(2)u$ and $\overline{P}_n(2)v$ of graph $\mathrm{Sch}(P_n(2), D, \overline{P}_n(2))$ are connected by an edge, iff*

$$\overline{P}_n(2)u = \overline{P}_n(2)v \cdot D_n.$$

2) *Graph $\mathrm{Sch}(P_n(2), D, \overline{P}_n(2))$ is bipartite.*

Proof. If $i = 1, \dots, n-1$, then $[D_i]_n = 0$. Thus in this case

$$\overline{P}_n(2)u \cdot D_i = \overline{P}_n(2)u,$$

so elements $D_1, \dots, D_{n-1}$ do not generate edges of $\mathrm{Sch}(P_n(2), D, \overline{P}_n(2))$.

We now prove the second statement.

Vertex set $V(\mathrm{Sch})$ can be described as a sum of sets V_1 and V_2, where V_1 is made of cosets in which the last coordinate in all tableaux in this coset is a polynomial which contains a monomial $\overline{x_{n-1}}$ and V_2 is made of cosets in which the last coordinate in all tableaux are polynomials which do not contain such a monomial. $[D_n]_n$ contains a monomial $\overline{x_{n-1}}$, thus for every $\overline{P}_n(2)v_1 \in V_1$ and $\overline{P}_n(2)v_2 \in V_2$:

$$\overline{P}_n(2)v_1 \cdot D_n \in V_2 \text{ and } \overline{P}_n(2)v_2 \cdot D_n \in V_1. \qquad \square$$

Hence for diagonal base $D = \{D_1, \ldots, D_n\}$ we can obtain a Cayley graph $\mathrm{Cay}(P_n(2))$ from a graph $\mathrm{Sch}(P_n(2), D, \overline{P}_n(2))$ by replacing every vertex of $\mathrm{Sch}(P_n(2), D, \overline{P}_n(2))$ by a graph $\mathrm{Cay}(P_{n-1}(2), D')$ and replacing every edge of $\mathrm{Sch}(P_n(2), D, \overline{P}_n(2))$ by a set of corresponding edges between elements $P_n(2)$ due to generator D_n (see point 3 of above construction).

4. Cayley graphs of $P_n(2)$ for small n

4.1. The case $n = 2$

Group $P_2(2)$ is isomorphic with the dihedral group D_4. It has two different diagonal bases and 12 different bases at all. The list of bases is as follows:

$$
\begin{aligned}
B_1 &= D_1 = \{[1,0],[0,x_1]\}, & B_2 &= D_2 = \{[1,0],[0,x_1+1]\},\\
B_3 &= \{[1,1],[0,x_1]\}, & B_4 &= \{[1,1],[0,x_1+1]\},\\
B_5 &= \{[1,0],[1,x_1]\}, & B_6 &= \{[1,0],[1,x_1+1]\},\\
B_7 &= \{[1,1],[1,x_1]\}, & B_8 &= \{[1,1],[1,x_1+1]\},\\
B_9 &= \{[0,x_1],[1,x_1]\}, & B_{10} &= \{[0,x_1],[1,x_1+1]\},\\
B_{11} &= \{[0,x_1+1],[1,x_1]\}, & B_{12} &= \{[0,x_1+1],[1,x_1+1]\}.
\end{aligned}
$$

The only primal diagonal base in $P_n(2)$ is B_1. The action on the set of all bases has 3 different orbits of length 4:

$$
O_1 = \{D_1,\, D_2,\, B_3,\, B_4\}, \qquad O_2 = \{B_5,\, B_6,\, B_7,\, B_8\},
$$
$$
O_3 = \{B_9,\, B_{10},\, B_{11},\, B_{12}\}.
$$

The orbit O_1 is the only $\mathfrak{D}$-orbit. Cayley graphs of $P_2(2)$ on bases from O_2 and O_3 are isomorphic (Fig. 1).

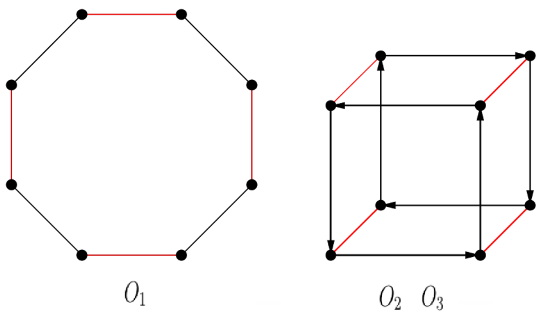

FIGURE 1. Cayley graphs of $P_2(2)$ in bases from respective orbits.

4.2. The case $n = 3$

There are four different primal diagonal bases of $P_3(2)$:

$$\begin{aligned}
D_1 &= \{[1,0,0],\ [0,x_1,0],\ [0,0,x_1x_2]\},\\
D_2 &= \{[1,0,0],\ [0,x_1,0],\ [0,0,x_1x_2+1]\},\\
D_3 &= \{[1,0,0],\ [0,x_1+1,0],\ [0,0,x_1x_2]\},\\
D_4 &= \{[1,0,0],\ [0,x_1+1,0],\ [0,0,x_1x_2+1]\},
\end{aligned}$$

Thus there are four different $\mathfrak{D}$-orbits and every such orbit contains exactly four diagonal bases and exactly 60 bases, which are not diagonal. Schreier coset graph $\mathrm{Sch}(P_3(2), D, \overline{P}_3(2))$ on bases from orbits $\mathfrak{D}$-orbits have form presented in Figure 2.

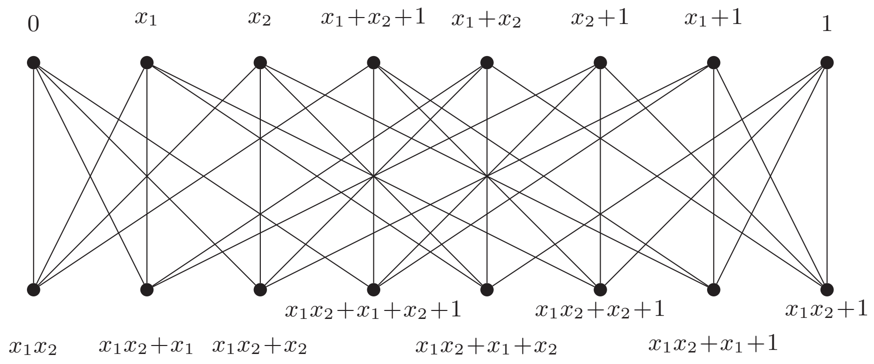

FIGURE 2. $\mathrm{Sch}(P_3(2), D, \overline{P}_3(2))$, where D is a diagonal base (vertex indexed by polynomials on last coordinate).

As we can see, $\mathrm{Sch}(P_3(2), D, \overline{P}_3(2))$ is a 4-regular bipartite graph. Every edge of this graph corresponds to connections with subgraphs isomorphic to $\mathrm{Cay}(P_2(2), D')$ (i.e. undirected cycle on 8 vertices, see 5.1). Every such connected cycles in $\mathrm{Cay}(P_3(2), D)$ are connected by two edges and form of connection depends of bases (Fig. 3)

Thus the length of the shortest cycle in graphs on bases D_1 and D_2 is equal to 8, and length of the shortest cycle in graphs on bases D_3 and D_4 is equal to 4. This means that these Cayley graphs of $P_3(2)$ on diagonal bases are not isomorphic.

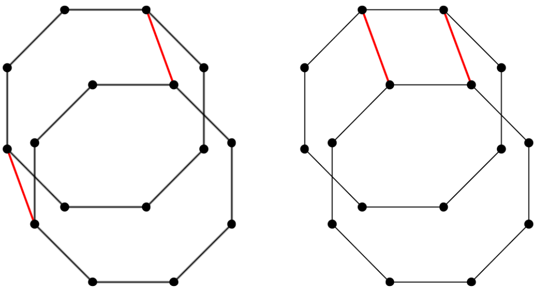

Figure 3. Connections between subgraphs of $\mathrm{Cay}(P_3(2), D)$ isomorphic with $\mathrm{Cay}(P_2(2), D')$ for different diagonal bases.

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The comb-like representations of cellular ordinal balleans

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Abstract. Given two ordinal λ and γ, let $f:[0,\lambda)\to[0,\gamma)$ be a function such that, for each $\alpha<\gamma$, $\sup\{f(t):t\in[0,\alpha]\}<\gamma$. We define a mapping $d_f:[0,\lambda)\times[0,\lambda)\longrightarrow[0,\gamma)$ by the rule: if $x<y$ then $d_f(x,y)=d_f(y,x)=\sup\{f(t):t\in(x,y]\}$, $d(x,x)=0$. The pair $([0,\lambda),d_f)$ is called a $\gamma-$comb defined by f. We show that each cellular ordinal ballean can be represented as a $\gamma-$comb. In *General Asymptology*, cellular ordinal balleans play a part of ultrametric spaces.

Introduction

In [3], a function $f:[0,1]\to[0,\infty)$ is called a *comb* if, for every $\varepsilon>0$, the set $\{t\in[0,1]:f(t)\geqslant\varepsilon\}$ is finite. Each comb f defines a pseudo-metric d_f on the set $I_f=\{t\in[0,1]:f(t)=0\}$ by the rule: if $x<y$ then

$$d_f(x,y)=\max\{f(t):t\in(x,y)\},$$

$$d_f(y,x)=d_f(x,y),\quad d(x,x)=0.$$

After some reduced completion of (I_f,d_f), the authors get a compact ultrametric space and show that each compact ultrametric space with no isolated points can be obtained in this way.

In this note, we modify the basic construction from [3] to get the comb-like representations of cellular ordinal balleans which, in *General Asymptology* [7], play a part of ultrametric spaces.

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1. Balleans

Following [5], [7], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii*.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},$$

$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha) \quad \text{and} \quad B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha') \quad \text{and} \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

- for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

We note that a ballean can be considered as an asymptotic counterpart of a uniform space, and could be defined [8] in terms of the entourages of the diagonal $\Delta_X = \{(x, x) : x \in X\}$ in $X \times X$. In this case a ballean is called a *coarse structure*.

For categorical look at the balleans and coarse structures as "two faces of the same coin" see [2].

Let $\mathcal{B} = (X, P, B), \mathcal{B}' = (X', P', B')$ be balleans. A mapping $f \colon X \to X'$ is called a $\prec$-*mapping* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that, for every $x \in X$, $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$.

A bijection $f \colon X \to X'$ is called an *asymorphism* between $\mathcal{B}$ and $\mathcal{B}'$ if f and f^{-1} are $\prec$-mappings. In this case $\mathcal{B}$ and $\mathcal{B}'$ are called *asymorphic*.

Given a ballean $\mathcal{B} = (X, P, B)$, we define a preodering $<$ on P by the rule: $\alpha < \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for each $x \in X$. A subset P' of P is called *cofinal* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha < \alpha'$. A ballean $\mathcal{B}$ is called *ordinal* if there exists a cofinal well-ordered (by $<$) subset P' of P.

For a ballean $\mathcal{B} = (X, P, B)$, $x, y \in X$ and $\alpha \in P$, we say that x and y are α-*path connected* if there exists a finite sequence $x_0, \dots, x_n$, $x_0 = x$,

$x_n = y$ such that $x_{i+1} \in B(x_i, \alpha)$ for each $i \in \{0, \dots, n-1\}$. For any $x \in X$ and $\alpha \in P$, we set

$$B^\diamond(x, \alpha) = \{y \in X : x, y \text{ are } \alpha\text{-path connected}\},$$

and say that the ballean $\mathcal{B}^\diamond = (X, P, B^\diamond)$ is a *cellularization* of $\mathcal{B}$. A ballean $\mathcal{B}$ is called *cellular* if the identity $id\colon X \to X$ is an asymorphism between $\mathcal{B}$ and $\mathcal{B}^\diamond$.

Each metric space (X, d) defines a metric ballean

$$\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_\alpha),$$

where $B_d(x, r) = \{y \in X : d(x, y) \leqslant r\}$. Clearly, $\mathcal{B}(X, d)$ is ordinal and, if d is an ultrametric then $\mathcal{B}(X, d)$ is cellular.

For examples, decompositions and classification of cellular ordinal balleans see [1], [2], [4], [6].

2. Representations

For ordinals α, β, $\alpha < \beta$, we use the standard notations

$$[\alpha, \beta] = \{t : \alpha \leqslant t \leqslant \beta\}, \qquad [\alpha, \beta) = \{t : \alpha \leqslant t < \beta\},$$
$$(\alpha, \beta] = \{t : \alpha < t \leqslant \beta\}.$$

Let X be a set and γ be an ordinal. We say that a mapping $d\colon X \times X \to [0, \gamma)$ is a γ-*ultrametric* if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}.$$

Clearly, each ultrametric space with integer valued metric is an ω-ultrametric space. By [7, Theorem 3.1.1], every cellular metrizable ballean is asymorphic to some ω-ultrametric space.

Given two γ-ultrametric spaces (X, d), (X', d'), a bijection $h\colon X \to X'$ is called an *isometry* if, for any $x, y \in X$, we have

$$d(x, y) = d'(h(x), h(y)).$$

Now let λ, γ be ordinal and $f\colon [0, \lambda) \to [0, \gamma)$ be a function such that, for each $\alpha < \lambda$, $\sup\{f(t) : t \in [0, \alpha]\} < \gamma$. We define a mapping $d_f\colon [0, \lambda) \times [0, \lambda) \to [0, \gamma)$ by the rule: if $x < y$ then

$$d_f(x, y) = d_f(y, x) = \sup\{f(t) : t \in (x, y]\}, d(x, x) = 0,$$

and say that $([0, \lambda), d_f)$ is a γ-*comb* determined by f. Evidently, each γ-comb is a γ-ultrametric space.

Theorem. *Every γ-ultrametric space (X, d) is isometric to some γ-comb $([0, \lambda), d_f)$.*

Proof. We proceed on induction by γ. For $\gamma = 1$, we just enumerate X as $[0, \lambda)$ and take $f \equiv 0$. Assume that we have proved the statement for all ordinals less than γ and consider two cases.

Case 1. Let γ is not a limit ordinal, so $\gamma = \gamma' + 1$. We partition $X = \bigcup\{X_\delta : \delta \in [0, \nu)\}$ into classes of the equivalence $\sim$ defined by $x \sim y$ if and only if $d(x, y) < \gamma'$. If $\delta < \delta' < \nu$ and $x \in X_\delta$, $y \in X_{\delta'}$ then $d(x, y) = \gamma'$.

By the inductive hypothesis, each X_δ is isometric to some γ'-comb $([0, \lambda_\delta), d_{f_\delta})$. We replace inductively each $\delta \in [0, \nu)$ with consecutive intervals $\{[l_\delta, l_\delta + \lambda_\delta) : \delta \in [0, \nu)\}$, $l_0 = 0$ and define a function $f\colon [0, \lambda) \to [0, \gamma)$, $[0, \lambda) = \bigcup\{[l_\delta,\ l_\delta + \lambda_\delta) : \delta \in [0, \nu)\}$ as follows. We put $f = f_0$ on $[0, \lambda_0)$. If $\delta > 0$ then we put $f(l_\delta) = \gamma'$ and $f(l_\delta + x) = f_\delta(x)$ for $x \in (0, \lambda_\delta)$.

After $|\nu|$ steps, we get the desired γ-comb $([0, \lambda), d_f)$.

Case 2. γ is a limit ordinal. We fix some $x_0 \in X$ and, for each $\delta < \gamma$, denote $X_\delta = \{x \in X : d(x_0, x) < \delta\}$. By the inductive hypothesis, there is an isometry $h_\delta\colon X_\delta \to ([0, \lambda_\delta), d_{f_\delta})$. Moreover, in view of Case 1, $f_{\delta+1}$ and $h_{\delta+1}$ can be chosen as the extensions of f_δ and h_δ. Hence, we can use induction by δ to get the desired γ-comb and isometry. □

Every γ-ultrametric space (X, d) can be considered as the ballean $(X, [0, \gamma), B_d)$, where $B_d(x, \alpha) = \{y \in X : d(x, y) \leqslant \alpha\}$.

On the other hand, let (X, P, B) be a cellular ordinal ballean. We may suppose that $P = [0, \gamma)$ and $B(x, \alpha) = B^\diamond(x, \alpha)$ for all $x \in X$, $\alpha \in [0, \gamma)$. We define a γ-ultrametric d on X by $d(x, y) = \min\{\alpha \in [0, \gamma) : y \in B(x, \alpha)\}$. Then (X, P, B) is asymorphic to (X, d).

Corollary. *Every cellular ordinal ballean is asymorphic to some γ-comb.*

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RESEARCH ARTICLE

Weak Frobenius monads and Frobenius bimodules

Robert Wisbauer*

Abstract. As observed by Eilenberg and Moore (1965), for a monad F with right adjoint comonad G on any category $\mathbb{A}$, the category of unital F-modules $\mathbb{A}_F$ is isomorphic to the category of counital G-comodules $\mathbb{A}^G$. The monad F is Frobenius provided we have $F = G$ and then $\mathbb{A}_F \simeq \mathbb{A}^F$. Here we investigate which kind of isomorphisms can be obtained for non-unital monads and non-counital comonads. For this we observe that the mentioned isomorphism is in fact an isomorphisms between $\mathbb{A}_F$ and the category of bimodules $\mathbb{A}_F^F$ subject to certain compatibility conditions (Frobenius bimodules). Eventually we obtain that for a weak monad (F, m, η) and a weak comonad (F, δ, ε) satisfying $Fm \cdot \delta F = \delta \cdot m = mF \cdot F\delta$ and $m \cdot F\eta = F\varepsilon \cdot \delta$, the category of compatible F-modules is isomorphic to the category of compatible Frobenius bimodules and the category of compatible F-comodules.

Introduction

A monad (F, m, η) on a category $\mathbb{A}$ is called a *Frobenius monad* provided the functor F is (right) adjoint to itself (e.g. Street [6]). Then F also allows for a comonad structure (F, δ, ε) and the (Eilenberg-Moore) category $\mathbb{A}_F$ of F-modules is isomorphic to the category $\mathbb{A}^F$ of F-comodules. As shown in [5, Theorem 3.13], this isomorphism characterises a functor with monad and comonad structure as Frobenius monad. It is not difficult to see that the categories $\mathbb{A}_F$ and $\mathbb{A}^F$ are in fact isomorphic to the

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category $\mathbb{A}_F^F$ of (unital and counital) *Frobenius bimodules.* In this setting units and counits play a crucial role.

Here we are concerned with the question what is left from these correspondences when the conditions on units and counits are weakened. An elementary approach to this setting is offered in [7] and [8] where adjunctions between functors are replaced by *regular pairings* (L, R) of functors $L : \mathbb{A} \to \mathbb{B}$, $R : \mathbb{B} \to \mathbb{A}$ (see 1.4). The composition LR (resp. RL) yields endofunctors on $\mathbb{A}$ (resp. $\mathbb{B}$) and these are closely related to *weak (co)monads* as considered by Böhm et al. in [1,3] (see Remark 1.12). In Section 1 we recall the definitions and collect basic results needed for our investigations.

Given a non-unital monad (F, m) on any category $\mathbb{A}$, a non-unital module $\varrho : F(A) \to A$ is called *firm* (see [2]) if the defining fork

$$FF(A) \overset{m_A}{\underset{F(\varrho)}{\rightrightarrows}} F(A) \xrightarrow{\varrho} A$$

is a coequaliser in the category of non-unital F-modules. This notion is generalised in Section 2 by restricting the coequaliser requirement to certain classes $\mathbb{K}$ of morphisms of F-modules. It turns out that compatible modules of a weak monad (F, m, η) satisfy the resulting conditions for a suitable class $\mathbb{K}$ (Proposition 2.10). Similar results hold for weak comonads.

In Section 3, we return to parings of the functors L and R. Given natural transformations $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{B}}$, one obtains a non-unital monad $(LR, L\widetilde{\varepsilon}R)$ and a non-counital comonad $(LR, L\eta R)$ on $\mathbb{B}$ for which the Frobenius condition is satisfied and this motivates the definition of Frobenius bimodules (see 3.1). Given a non-counital LR-comodule $\omega : B \to LR(B)$, the question arises when it can be extended to a Frobenius bimodule by some $\varrho : LR(B) \to B$. As sufficient condition it turns out that the defining cofork for ϱ is a coequaliser in the category of non-counital comodules (see Proposition 3.6). Further situations are investigated, in particular for regular pairings (Theorems 3.9, 3.10).

In Section 4, the results about the pairings (L, R) from Section 3 are reformulated for the (co)monad LR, that is, we consider an endofunctor F on $\mathbb{B}$ endowed with a weak monad structure (F, m, η), a weak comonad structure (F, δ, ε), and the compatibility between m and δ is postulated as the Frobenius property (see 4.1). (For $L\eta R$ and $L\widetilde{\varepsilon}R$ the latter follows by naturality, see (3.1)). The constructions lead to various functors between (compatible) module, comodule and bimodule categories (see 4.2, 4.3, 4.6). For proper (co)monads we get some results obtained by Böhm and Gómez-Torrecillas in [2] as Corollaries 4.7, 4.8.

1. Regular pairings

Throughout $\mathbb{A}$ and $\mathbb{B}$ will denote any categories. The symbols I_A, A, or just I will stand for the identity morphism on an object A, I_F or F denote the identity transformation on the functor F, and $I_{\mathbb{A}}$ means the identity functor on $\mathbb{A}$.

Given an endofunctor T on $\mathbb{A}$, an idempotent natural transformation $e : T \to T$ is said to *split* if there are an endofunctor $\underline{T}$ on $\mathbb{A}$ and natural transformations $p : T \to \underline{T}$ and $i : \underline{T} \to T$ such that $e = i \cdot p$ and $p \cdot i = I_{\underline{T}}$.

We recall some notions from [7], [8], [3].

1.1. Non-counital comodules. Let (G, δ) be a pair with an endofunctor $G : \mathbb{A} \to \mathbb{A}$ and a coassociative natural transformation (coproduct) $\delta : G \to GG$. Then (non-counital) G-comodules are defined as objects $A \in \mathbb{A}$ with a morphism $\upsilon : A \to G(A)$ satisfying $G(\upsilon) \cdot \upsilon = \delta_A \cdot \upsilon$ and the category of these G-comodules is denoted by $\underset{\to}{\mathbb{A}}^G$.

Consider a triple (G, δ, ε), with (G, δ) a pair as above and $\varepsilon : G \to I_{\mathbb{A}}$ any natural transformation (quasi-counit). Then a G-comodule (A, υ) is said to be *compatible* provided $\upsilon = G\varepsilon_A \cdot \delta_A \cdot \upsilon$. The full subcategory of $\underset{\to}{\mathbb{A}}^G$ consisting of compatible comodules is denoted by $\underline{\mathbb{A}}^G$.

(G, δ, ε) is called a *weak comonad* if

$$\varepsilon = \varepsilon \cdot G\varepsilon \cdot \delta, \;\; \delta = G\varepsilon G \cdot G\delta \cdot \delta, \quad \text{and} \quad G\varepsilon \cdot \delta = \varepsilon G \cdot \delta.$$

Then a G-comodule (A, υ) is compatible if $\varepsilon G_A \cdot \delta_A \cdot \upsilon = \upsilon = \upsilon \cdot \varepsilon_A \cdot \upsilon$. Furthermore, $G\varepsilon \cdot \delta : G \to G$ is idempotent and in case this is split by $G \xrightarrow{p} \underline{G} \xrightarrow{i} G$, one obtains a comonad $(\underline{G}, \underline{\delta}, \underline{\varepsilon})$ by putting

$$\underline{\delta} : \underline{G} \xrightarrow{i} G \xrightarrow{\delta} GG \xrightarrow{pp} \underline{GG}, \quad \underline{\varepsilon} : \underline{G} \xrightarrow{i} G \xrightarrow{\varepsilon} I_{\mathbb{A}}.$$

1.2. Non-unital modules. Let (F, m) be a pair with an endofunctor $F : \mathbb{A} \to \mathbb{A}$ and an associative natural transformation (product) $m : FF \to F$. Then (non-unital) F-modules are defined as objects $A \in \mathbb{A}$ with a morphism $\varrho : F(A) \to A$ satisfying $\varrho \cdot F\varrho = \varrho \cdot m_A$ and the category of these F-modules is denoted by $\underset{\to}{\mathbb{A}}_F$.

Consider a triple (F, m, η), with (F, m) a pair as above and any natural transformation $\eta : I_{\mathbb{A}} \to F$ (quasi-unit). An F-module (A, ϱ) is said to be *compatible* provided $\varrho = \varrho \cdot m_A \cdot F\eta_A$ and the full subcategory of $\underset{\to}{\mathbb{A}}_F$ consisting of compatible modules is denoted by $\underline{\mathbb{A}}_F$.

(F, m, η) is called a *weak monad* if

$$\eta = m \cdot F\eta \cdot \eta, \;\; m = m \cdot mF \cdot F\eta F, \quad \text{and} \quad m \cdot F\eta = m \cdot \eta F.$$

Then an F-module (A,ϱ) is compatible if $\varrho\cdot m_A\cdot\eta F_A=\varrho=\varrho\cdot\eta_A\cdot\varrho$. Furthermore, $m\cdot F\eta : F\to F$ is idempotent and in case this is split by $F\xrightarrow{p}\underline{F}\xrightarrow{i}F$, one obtains a monad $(\underline{F},\underline{m},\underline{\eta})$ by putting

$$\underline{m}:\underline{FF}\xrightarrow{ii}FF\xrightarrow{m}F\xrightarrow{p}\underline{F},\quad \underline{\eta}:I_{\mathbb{A}}\xrightarrow{\eta}F\xrightarrow{p}\underline{F}.$$

1.3. Pairings of functors. For functors $L:\mathbb{A}\to\mathbb{B}$ and $R:\mathbb{B}\to\mathbb{A}$, *pairings* are defined as maps, natural in $A\in\mathbb{A}$ and $B\in\mathbb{B}$,

$$\mathrm{Mor}_{\mathbb{B}}(L(A),B)\underset{\beta}{\overset{\alpha}{\rightleftarrows}}\mathrm{Mor}_{\mathbb{A}}(A,R(B)),$$

$$\mathrm{Mor}_{\mathbb{A}}(R(B),A)\underset{\widetilde{\beta}}{\overset{\widetilde{\alpha}}{\rightleftarrows}}\mathrm{Mor}_{\mathbb{A}}(B,L(A))\ .$$

These - and their compositions - are determined by natural transformations obtained as images of the corresponding identity morphisms,

map	natural transformation	map	natural transformation
α	$\eta: I_{\mathbb{A}}\to RL,$	$\widetilde{\alpha}$	$\widetilde{\eta}: I_{\mathbb{B}}\to LR,$
β	$\varepsilon: LR\to I_{\mathbb{B}},$	$\widetilde{\beta}$	$\widetilde{\varepsilon}: RL\to I_{\mathbb{A}},$
$\beta\cdot\alpha$	$\ell: L\xrightarrow{L\eta}LRL\xrightarrow{\varepsilon L}L$	$\widetilde{\beta}\cdot\widetilde{\alpha}$	$\widetilde{r}: R\xrightarrow{R\widetilde{\eta}}RLR\xrightarrow{\widetilde{\varepsilon}R}R$
$\alpha\cdot\beta$	$r: R\xrightarrow{\eta R}RLR\xrightarrow{R\varepsilon}R$	$\widetilde{\alpha}\cdot\widetilde{\beta}$	$\widetilde{\ell}: L\xrightarrow{\widetilde{\eta}L}LRL\xrightarrow{L\widetilde{\varepsilon}}L\ .$

β (resp. α) is said to be *symmetric* if $Lr=\ell R$ (resp. $R\ell=rL$) (see [8, §3]). Under the given conditions (see [8]),

(i) $(LR,L\eta R,\varepsilon)$ is a non-counital comonad on $\mathbb{B}$ with quasi-counit ε;
(ii) $(RL,R\varepsilon L,\eta)$ is a non-unital monad on $\mathbb{A}$ with quasi-unit η;
(iii) $(LR,L\widetilde{\varepsilon}R,\widetilde{\eta})$ is a non-unital monad on $\mathbb{B}$ with quasi-unit $\widetilde{\eta}$;
(iv) $(RL,R\widetilde{\eta}L,\widetilde{\varepsilon})$ is a non-counital comonad on $\mathbb{A}$ with quasi-counit $\widetilde{\varepsilon}$.

Clearly, if α is a bijection, then (L,R) is an adjoint pair, if $\widetilde{\alpha}$ is a bijection, then (R,L) is an adjoint pair, and if α and $\widetilde{\alpha}$ are bijections, then LR and RL are Frobenius functors.

1.4. Regular pairings. A pairing (L,R,α,β) is said to be *regular* if

$$\alpha\cdot\beta\cdot\alpha=\alpha\text{ and }\beta\cdot\alpha\cdot\beta=\beta.$$

In this case, $\ell:L\to L$ and $r:R\to R$ (see 1.3) are idempotents and

$$\varepsilon=\varepsilon\cdot\ell r=\varepsilon\cdot\ell R=\varepsilon\cdot Lr,$$

$$\eta=r\ell\cdot\eta=R\ell\cdot\eta=rL\cdot\eta.$$

If β is symmetric, $\ell r = Lr = \ell R$; if α is symmetric, $r\ell = R\ell = rL$.

Assume the idempotents ℓ, r to be splitting, that is,

$$L \xrightarrow{\ell} L = L \xrightarrow{p} \underline{L} \xrightarrow{i} L, \quad R \xrightarrow{r} R = R \xrightarrow{p'} \underline{R} \xrightarrow{i'} R.$$

Then, for the natural morphisms

$$\underline{\eta}: \ I_{\mathbb{A}} \xrightarrow{\ \eta\ } RL \xrightarrow{\ p'p\ } \underline{RL}, \quad \underline{\varepsilon}: \ \underline{LR} \xrightarrow{\ ii'\ } LR \xrightarrow{\ \varepsilon\ } I_{\mathbb{B}},$$

one gets $\underline{\varepsilon L} \cdot \underline{L\eta} = I_{\underline{L}}$ and $\underline{R\varepsilon} \cdot \underline{\eta R} = I_{\underline{R}}$, hence yielding an adjunction $(\underline{L}, \underline{R}, \underline{\alpha}, \underline{\beta})$.

1.5. Proposition. *For functors* $\mathbb{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{B}$ *, there are equivalent:*

(a) (L,R) *allows for a regular pairing* (L,R,α,β) *with splitting idempotents* ℓ, r*;*

(b) *there are retractions* $\underline{L} \xrightarrow{i} L \xrightarrow{p} \underline{L}$ *and* $\underline{R} \xrightarrow{i'} R \xrightarrow{p'} \underline{R}$ *such that* $(\underline{L}, \underline{R})$ *allows for an adjunction.*

Proof. (a)$\Rightarrow$(b) The data from 1.4 yield an adjunction $(\underline{L}, \underline{R}, \underline{\alpha}, \underline{\beta})$ and the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mor}_{\mathbb{B}}(L(A),B) & \xrightarrow{\ \alpha\ } & \mathrm{Mor}_{\mathbb{A}}(A,R(B)) & \xrightarrow{\ \beta\ } & \mathrm{Mor}_{\mathbb{B}}(L(A),B) \\
\downarrow{\scriptstyle \mathrm{Mor}(i_A,B)} & & \downarrow{\scriptstyle \mathrm{Mor}(A,p'_B)} & & \downarrow{\scriptstyle \mathrm{Mor}(i_A,B)} \\
\mathrm{Mor}_{\mathbb{B}}(\underline{L}(A),B) & \xrightarrow{\ \underline{\alpha}\ } & \mathrm{Mor}_{\mathbb{A}}(A,\underline{R}(B)) & \xrightarrow{\ \underline{\beta}\ } & \mathrm{Mor}_{\mathbb{B}}(\underline{L}(A),B).
\end{array}$$

(b)$\Rightarrow$(a) Given an adjunction $(\underline{L}, \underline{R}, \underline{\alpha}, \underline{\beta})$ and retracts $\underline{L} \xrightarrow{i} L \xrightarrow{p} \underline{L}$ and $\underline{R} \xrightarrow{i'} R \xrightarrow{p'} \underline{R}$, the above diagram tells us how to define (new) α and β to get commutativity. Then it is routine to check that (L,R,α,β) is a regular pairing and the resulting idempotents are split by (p,i) and (p',i'), respectively. □

Now assume that (L,R,α,β) and $(R,L,\widetilde{\alpha},\widetilde{\beta})$ are regular pairings. Then ℓ, $\widetilde{\ell}$ are two natural transformations on L and r, $\widetilde{r}$ are two natural transformations on R. We are interested in the case when they coincide. Applying 1.5 and its dual yields:

1.6. Proposition. *For functors* $\mathbb{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{B}$ *, there are equivalent:*

(a) *(L,R) allows for regular pairings (L,R,α,β) and $(R,L,\widetilde{\alpha},\widetilde{\beta})$ with splitting idempotents $\ell=\widetilde{\ell}$, $r=\widetilde{r}$;*

(b) *there are retractions $\underline{L}\xrightarrow{i}L\xrightarrow{p}\underline{L}$ and $\underline{R}\xrightarrow{i'}R\xrightarrow{p'}\underline{R}$ such that $(\underline{L},\underline{R})$ and $(\underline{R},\underline{L})$ allow for adjunctions, that is, $(\underline{L},\underline{R})$ is a Frobenius pair of functors.*

1.7. Remark. Let (G,δ,ε) be a non-counital comonad on the category $\mathbb{A}$ with quasi-unit ε. For the Eilenberg-Moore category $\underrightarrow{\mathbb{A}}^G$ of non-counital G-comodules there are the free and the forgetful functors

$$\phi^G:\mathbb{A}\to\underrightarrow{\mathbb{A}}^G,\ A\mapsto(G(A),\delta_A),\qquad U^G:\underrightarrow{\mathbb{A}}^G\to\mathbb{A},\ (A,\omega)\mapsto A.$$

There is a pairing $(\phi^G,U^G,\alpha^G,\beta^G)$ with the maps, for $X\in\mathbb{A}$, $(A,\omega)\in\underrightarrow{\mathbb{A}}^G$,

$$\begin{aligned}\alpha^G&:\mathrm{Mor}_{\mathbb{A}}(U^G(A),X)\to\mathrm{Mor}^G(A,\phi^G(X)),\quad f\mapsto G(f)\cdot\omega,\\ \beta^G&:\mathrm{Mor}^G(A,\phi^G(X))\to\mathrm{Mor}_{\mathbb{A}}(U^G(A),X),\quad g\mapsto\varepsilon_X\cdot g.\end{aligned}$$

Compatible G-comodules $\upsilon:A\to G(A)$ are those with $\alpha^G\beta^G(\upsilon)=\upsilon$.

(G,δ,ε) is a weak comonad if and only if $(\phi^G,U^G,\alpha^G,\beta^G)$ is a regular pairing with β^G symmetric (see [8, Proposition 4.4]).

Similar characterisations hold for weak monads ([8, Proposition 3.4]).

1.8. Related comonads. *Let (L,R,α,β) be a regular pairing (see* 1.4*).*

(1) *For the coproduct*

$$\delta:LR\xrightarrow{L\eta R}LRLR\xrightarrow{\ell RLr}LRLR,$$

(LR,δ,ε) is a weak comonad on $\mathbb{B}$. If β is symmetric, $\delta=L\eta R$.

(2) *$\ell r:LR\to LR$ induces morphisms of non-counital comonads respecting the quasi-counits,*

$$(LR,L\eta R,\varepsilon)\to(LR,L\eta R,\varepsilon)\ \textit{and}\ (LR,L\eta R,\varepsilon)\to(LR,\delta,\varepsilon),$$

and an endomorphism of weak comonads $(LR,\delta,\varepsilon)\to(LR,\delta,\varepsilon)$.

Proof. Direct verification shows $\varepsilon LR\cdot\delta=\ell r=LR\varepsilon\cdot\delta$, the conditions for a weak comonad. For the next claims, consider the commutative diagram

$$\begin{tikzcd}
LR \arrow[r, "\ell r"] \arrow[d, "L\eta R"'] & LR \arrow[d, "L\eta R"'] \arrow[dr, "\delta"] & \\
LRLR \arrow[r, "\ell RLr"] \arrow[dr, "\ell r\ell r"'] & LRLR \arrow[r, "\ell RLr"] \arrow[d, "Lr\ell R"] & LRLR \arrow[d, "Lr\ell R"] \\
 & LRLR \arrow[r, "\ell RLr"'] & LRLR;
\end{tikzcd}$$

the left hand part proves the assertion about the first morphism and the outer paths show the properties of the second and third morphisms. □

1.9. Related monads. *Let* (L,R,α,β) *be a regular pairing (see* 1.4*).*

(1) *For the product*

$$m : RLRL \xrightarrow{rLR\ell} RLRL \xrightarrow{R\varepsilon L} RL,$$

(RL,m,η) *is a weak monad on* $\mathbb{A}$. *If* α *is symmetric,* $m = R\varepsilon L$.

(2) $r\ell : RL \to RL$ *yields morphisms of non-unital monads respecting the quasi-units,*

$$(RL,R\varepsilon L,\eta) \to (RL,R\varepsilon L,\eta) \text{ and } (RL,R\varepsilon L,\eta) \to (RL,m,\eta).$$

and an endomorphism of weak monads $(RL,m,\eta) \to (RL,m,\eta)$.

Proof. One easily verifies $m \cdot \eta RL = r\ell = m \cdot RL\eta$, the condition for a weak monad. The other claims are shown similarly to 1.8 □

Combining the preceding observations we have shown:

1.10. Proposition. *Let* (L,R,α,β) *be a regular pairing and assume the idempotents* ℓ *and* r *to split. With the notation from* 1.4, $(\underline{LR},\underline{L}\eta\underline{R},\underline{\varepsilon})$ *is a comonad on* $\mathbb{B}$ *and* $(\underline{RL},\underline{R\varepsilon L},\underline{\eta})$ *is monad on* $\mathbb{A}$. *Then,*

(1) *the natural transformation* $pp' : LR \to \underline{LR}$ *induces morphisms of non-counital comonads* $(LR,L\eta R,\varepsilon) \to (\underline{LR},\underline{L}\eta\underline{R},\underline{\varepsilon})$, *and morphisms of weak comonads* $(LR,\delta,\varepsilon) \to (\underline{LR},\underline{L}\eta\underline{R},\underline{\varepsilon})$;

(2) *the natural transformation* $p'p : RL \to \underline{RL}$ *induces morphisms of non-unital monads* $(RL,R\varepsilon L,\eta) \to (\underline{RL},\underline{R\varepsilon L},\underline{\eta})$ *and morphisms of weak monads* $(RL,m,\eta) \to (\underline{RL},\underline{R\varepsilon L},\underline{\eta})$.

1.11. Regular pairings and comodules. Let (L,R,α,β) be a regular pairing and consider the weak comonad (LR,δ,ε) defined in 1.8. Then a non-counital (LR,δ,ε)-comodule (B,υ) is compatible (see 1.1) if $\upsilon = \varepsilon LR_B \cdot \delta_B \cdot \upsilon = r\ell_B \cdot \upsilon$.

Write $\overline{\mathbb{B}}^{LR,\delta}$ for the full subcategory of $\underrightarrow{\mathbb{B}}^{LR,\delta}$ formed by the compatible (LR,δ,ε)-comodules. For any $B \in \mathbb{B}$, $(LR(B),\delta_B)$ is a compatible (LR,δ,ε)-comodule, and thus we have a functor

$$\phi^{LR,\delta} : \mathbb{B} \to \overline{\mathbb{B}}^{LR,\delta}, \quad B \mapsto (LR(B),\delta_B).$$

The obvious forgetful functor $U^{LR,\delta}:\overline{\mathbb{B}}^{LR,\delta}\to\mathbb{B}$ need not be (left) adjoint to ϕ^{LR} but $(\phi^{LR},U^{LR,\delta})$ allows for a regular pairing (see 1.7).

Denoting by $\underrightarrow{\mathbb{B}}^{LR,\eta}$ the non-counital comodules for $(LR, L\eta R,\varepsilon)$, the natural transformation $(LR, L\eta R,\varepsilon)\to(LR,\delta,\varepsilon)$ induced by ℓr (see 1.8) defines a functor $t_{\ell r}:\underrightarrow{\mathbb{B}}^{LR,\eta}\to\underrightarrow{\mathbb{B}}^{LR,\delta}$. It is easy to see that hereby the image of any comodule in $\underrightarrow{\mathbb{B}}^{LR,\eta}$ is a compatible comodule in $\underrightarrow{\mathbb{B}}^{LR,\delta}$ leading to a commutative diagram

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "\phi^{LR,\eta}"] \arrow[dr, "\phi^{LR,\delta}"'] & \underrightarrow{\mathbb{B}}^{LR,\eta} \arrow[d, "t_{\ell r}"] \\
 & \overline{\mathbb{B}}^{LR,\delta}.
\end{tikzcd}$$

In case the idempotents ℓ and r are splitting, we get the splitting natural transformation $pp':LR\to\underline{LR}$ (from 1.4) which induces functors $\overline{\mathbb{B}}^{LR,\delta}\to\overline{\mathbb{B}}^{\underline{LR}}$ and $\overline{\mathbb{B}}^{LR,\eta}\to\overline{\mathbb{B}}^{\underline{LR}}$, also denoted by pp', with commutative diagram

$$\begin{tikzcd}
\underrightarrow{\mathbb{B}}^{LR,\eta} \arrow[r, "t_{\ell r}"] \arrow[dr, "pp'"] & \overline{\mathbb{B}}^{LR,\delta} \arrow[d, "pp'"] \\
\mathbb{B} \arrow[u, "\phi^{LR,\eta}"] \arrow[r, "\phi^{\underline{LR}}"'] & \overline{\mathbb{B}}^{\underline{LR}}.
\end{tikzcd}$$

Since $\underline{LR}$ is a comonad, every non-counital $\underline{LR}$-comodule is compatible, that is $\overline{\mathbb{B}}^{\underline{LR}}=\underrightarrow{\mathbb{B}}^{\underline{LR}}$, but need not be counital.

1.12. Remark. As pointed out by an anonymous referee, a regular pairing (L,R,α,β) defined in 1.4 is in fact the same as an adjunction in the local idempotemt closure $\overline{\mathrm{Cat}}$ of the 2-category Cat of categories and hence corresponds to a comonad in $\overline{\mathrm{Cat}}$. This lives on the 1-cell $(LR,\ell r)$ with coproduct $\ell RLr\cdot L\eta R$ and counit ε (see [3]). In this approach, similar to Proposition 1.6, the properties of the weak comonad LR are described by properties of a related comonad $\underline{LR}$.

We are also interested in the modules and comodules induced directly by RL and LR, respectively.

2. (Co)firm (co)modules

To develope further constructions for pairings of functors symmetry conditions are needed and so we consider weak (co)monads.

The notion of (co-)equalisers in categories may be modified in the following way.

2.1. Definitions. Let $\mathbb{K}$ be a class of morphisms in a category $\mathbb{A}$ closed under composition. A cofork

$$B \xrightarrow{k} C \overset{g}{\underset{f}{\rightrightarrows}} D$$

is said to be a $\mathbb{K}$-*equaliser* provided $k \in \mathbb{K}$ and, for any $h : Q \to C$ in $\mathbb{K}$ with $f \cdot h = g \cdot h$, there exists a unique $q : Q \to B$ in $\mathbb{K}$ such that $h = k \cdot q$. If this holds, then, for morphisms $r, s : X \to B$ in $\mathbb{K}$, $k \cdot r = k \cdot s$ implies $r = s$.

Similarly, a fork

$$B \overset{g}{\underset{f}{\rightrightarrows}} C \xrightarrow{s} D$$

is said to be a $\mathbb{K}$-*coequaliser* provided $s \in \mathbb{K}$ and, for any $h : C \to Q$ in $\mathbb{K}$ with $h \cdot f = h \cdot g$, there is a unique $q : D \to Q$ in $\mathbb{K}$ such that $h = q \cdot s$. In this case, for morphisms $t, u : D \to Y$ in $\mathbb{K}$, $t \cdot s = u \cdot s$ implies $t = u$.

A class $\mathbb{K}$ of morphisms in $\mathbb{A}$ is called an *ideal class* if for any morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{A}$, f or g in $\mathbb{K}$ implies that $g \cdot f$ is in $\mathbb{K}$.

Taking for $\mathbb{K}$ the class of all morphisms in $\mathbb{A}$, the notions defined above yield the usual equalisers and coequalisers in the category $\mathbb{A}$.

2.2. $\mathbb{K}$-cofirm comodules. Let (G, δ) be a non-counital comonad. Given an ideal class $\mathbb{K}$ of morphisms in the category $\underrightarrow{\mathbb{B}}^{LR}$ of non-counital G-comodules, a comodule (B, ω) is called $\mathbb{K}$-*cofirm* provided the defining cofork

$$B \xrightarrow{\omega} G(B) \overset{\delta_B}{\underset{G(\omega)}{\rightrightarrows}} GG(B)$$

is a $\mathbb{K}$-equaliser. If we choose for $\mathbb{K}$ all morphisms in $\underrightarrow{\mathbb{B}}^{LR}$, a $\mathbb{K}$-cofirm comodule is just called *cofirm*.

2.3. Compatible comodule morphisms. Now let (G, δ, ε) be a weak comonad and $\gamma := G\varepsilon \cdot \delta : G \to G$ the idempotent comonad morphism. We call a morphism h between G-comodules (B, ω) and (B', ω') γ-*compatible*, provided it induces commutativity of the triangles in the diagram

$$\begin{tikzcd}
B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[drr, "h"] & G(B) \arrow[r, "\varepsilon_B"] & B \arrow[d, "h"] \\
B' \arrow[r, "\omega'"'] & G(B') \arrow[r, "\varepsilon_{B'}"'] & B'.
\end{tikzcd}$$

Clearly, since the outer diagram is always commutative for comodule morphisms, it is enough to require commutativity for one of the triangles. Thus one readily obtains:

(1) *The class* $\mathbb{K}_\gamma$ *of all* γ*-compatible morphisms in* $\overline{\mathbb{B}}^G$ *is an ideal class.*

(2) *A morphism* $h : Q \to G(B)$ *of* G*-comodules is in* $\mathbb{K}_\gamma$ *if and only if* $\gamma_B \cdot h = h$.

(3) *A morphism* $h : G(B) \to Q$ *of* G*-comodules is in* $\mathbb{K}_\gamma$ *if and only if* $h \cdot \gamma_B = h$.

Evidently, a G-comodule (B, ω) is compatible (as in 1.1) if and only if $\omega \in \mathbb{K}_\gamma$, that is, $\omega = \gamma_B \cdot \omega$.

Notice that $\gamma = I_G$ implies that every non-counital G-comodule is γ-compatible, that is, $\underrightarrow{\mathbb{B}}^G = \overline{\mathbb{B}}^G$; in this case, however, not every G-comodule morphism need to be γ-compatible and a G-comodule (B, ω) need not be counital but only satisfies $\omega = \omega \cdot \varepsilon_B \cdot \omega$.

2.4. Proposition. *If* (G, δ, ε) *is a weak comonad, then any compatible* G*-comodule* (B, ω) *is* $\mathbb{K}_\gamma$*-cofirm.*

Proof. We have to show that the cofork

$$\begin{tikzcd} B \arrow[r, "\omega"] & G(B) \arrow[r, shift left, "\delta_B"] \arrow[r, shift right, "G(\omega)"'] & GG(B) \end{tikzcd}$$

is a $\mathbb{K}_\gamma$-equaliser. Let (Q, κ) be a G-comodule and $h : Q \to G(B)$ a morphism in $\mathbb{K}_\gamma$ with $G(\omega) \cdot h = \delta_B \cdot h$. In the diagram

$$\begin{tikzcd} Q \arrow[rr, "h"] \arrow[dr, "h"] \arrow[dd, "\kappa"'] & & G(B) \arrow[r, "\varepsilon_B"] \arrow[d, "G(\omega)"] & B \arrow[dd, "\omega"] \\ & G(B) \arrow[r, "\delta_B"] \arrow[d, "\delta_B"] & GG(B) \arrow[dr, "\varepsilon_{G(B)}"] & \\ G(Q) \arrow[r, "G(h)"'] & GG(B) \arrow[rr, "G\varepsilon_B"'] & & G(B), \end{tikzcd} \tag{2.1}$$

all inner diagrams are commutative. This shows that $\widetilde{h} := \varepsilon_B \cdot h : Q \to B$ is a G-comodule morphism with

$$\begin{aligned} \omega \cdot \widetilde{h} &= \omega \cdot \varepsilon_B \cdot h = \varepsilon_{G(B)} \cdot G(\omega) \cdot h \\ &= \varepsilon_{G(B)} \cdot \delta_B \cdot h = \gamma_B \cdot h = h, \\ \varepsilon_B \cdot \omega \cdot \widetilde{h} &= \varepsilon_B \cdot \gamma_B \cdot h = \varepsilon_B \cdot h = \widetilde{h}, \end{aligned}$$

thus $\widetilde{h} \in \mathbb{K}_\gamma$. Moreover, for any $q : Q \to B$ in $\mathbb{K}_\gamma$ with $\omega \cdot q = h$, we have $\varepsilon_B \cdot h = \varepsilon_B \cdot \omega \cdot q = q$, showing uniqueness of q. $\square$

Replacing (Q, h) in diagram (2.1) by (B, ω), we see that $\varepsilon_B \cdot \omega$ is a comodule morphism and this leads to the following observation.

2.5. Proposition. *If (G, δ, ε) is a (proper) comonad, then any non-counital G-comodule (B, ω) is cofirm if and only if it is counital.*

Proof. Since we have a comonad, $\gamma = I_G$, every G-comodule (B, ω) is γ-compatible, and $\omega = \omega \cdot \varepsilon_B \cdot \omega$ (see 2.3).

If (B, ω) is cofirm, then ω is monomorph in $\underrightarrow{\mathbb{B}}^{LR}$; since $\varepsilon_B \cdot \omega$ and I_B are morphisms in $\underrightarrow{\mathbb{B}}^{G}$ we conclude $\varepsilon_B \cdot \omega = I_B$, that is, (B, ω) is counital.

It is folklore that any counital G-comodule is cofirm. $\square$

2.6. $\mathbb{K}$-firm modules. Let (F, m) be a non-unital monad on $\mathbb{B}$. Given an ideal class of morphisms in the category $\underrightarrow{\mathbb{B}}_F$ of non-unital F-modules (see [8]), a module (B, ϱ) is called $\mathbb{K}$-*firm* provided the defining fork

$$\begin{tikzcd} FF(B) \arrow[r, shift left, "m_B"] \arrow[r, shift right, "F(\varrho)"'] & F(B) \arrow[r, "\varrho"] & B \end{tikzcd}$$

is a $\mathbb{K}$-coequaliser (Definitions 2.1).

2.7. Remark. Following [2, 2.3], a non-unital F-module (B, ϱ) is called *firm* provided it is $\mathbb{K}$-firm for the class $\mathbb{K}$ of all morphisms in $\underrightarrow{\mathbb{B}}_F$ and ϱ is an epimorphism in $\mathbb{B}$. The term *firm* was coined by Quillen for non-unital algebras A over a commutative ring k with the property that the map $A \otimes_A A \to A$, $a \otimes b \mapsto ab$, is an isomorphism. Then, an A-module is firm provided it is firm for the monad $A \otimes_k -$ on the category of k-modules. In the category of non-unital A-modules, coequalisers are induced by coequalisers of k-modules and hence are epimorph (in fact surjective) as k-module morphisms (e.g. [2, 6.1]).

2.8. Compatible module morphisms. Let (F, m, η) be a weak monad with idempotent monad morphism $\vartheta := m \cdot \eta F : F \to F$. A morphism h between F-modules (B, ϱ) and (B', ϱ') is called ϑ-*compatible*, provided it induces commutativity of the triangles in the diagram

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "h"'] \arrow[drr, "h"] & F(B) \arrow[r, "\varrho"] & B \arrow[d, "h"] \\ B' \arrow[r, "\eta_{B'}"'] & F(B') \arrow[r, "\varrho'"'] & B'. \end{tikzcd}$$

Similar to 2.3 one obtains:

(1) *The class $\mathbb{K}_\vartheta$ of all ϑ-compatible morphisms in $\mathbb{B}_F$ is an ideal class.*

(2) *A morphism $h : Q \to F(B)$ of F-modules is in $\mathbb{K}_\vartheta$ if and only if $\vartheta_B \cdot h = h$.*

(3) *A morphism $h : LR(B) \to Q$ of F-modules is in $\mathbb{K}_\vartheta$ if and only if $h \cdot \vartheta_B = h$.*

Clearly, an F-module (B, ϱ) is compatible (see 1.2) if and only if $\varrho \in \mathbb{K}_\vartheta$, that is, $\varrho \cdot \vartheta_B = \varrho$.

2.9. Remark. Given the assumptions in 2.8, one may consider the subcategory of $\mathbb{B}_F$ consisting of the same objects and as morphisms the ϑ-compatible morphisms. Then the identity morphism on a ϑ-compatible module (B, ϱ) is $\varrho \cdot \eta_B : B \to B$ and equalisers in this category are essentially the $\mathbb{K}_\vartheta$-equalisers. This situation is also addressed in [3, Remark 2.5] (with different terminology).

Dual to the Propositions 2.4 and 2.5 we now have:

2.10. Proposition. *If (F, m, η) is a weak monad, then any ϑ-compatible F-module (B, ϱ) is $\mathbb{K}_\vartheta$-firm.*

2.11. Proposition. *If (F, m, η) is a (proper) monad, then a non-unital F-module (B, ϱ) is firm if and only if it is unital.*

3. Frobenius property and Frobenius bimodules

In the setting of 1.3, assume α and $\widetilde{\beta}$ to be given, that is, there are natural transformations $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$. Then $(LR, L\eta R)$ is a non-counital comonad, and $(LR, L\widetilde{\varepsilon}R)$ is a non-unital monad on $\mathbb{B}$ (see [8]). This section is for studying the interplay between the corresponding module and comodule structures.

Let $\mathbb{B}^{LR}_{\underrightarrow{LR}}$ denote the category of objects in $\mathbb{B}$ which have an LR-module as well as an LR-comodule structure (LR-bimodules) and with morphisms which are LR-module and LR-comodule morphisms.

By naturality, we have the commutative diagram (Frobenius property)

$$\begin{tikzcd}
 & LRLRLR \arrow[dr, "LRL\widetilde{\varepsilon}R"] & \\
LRLR \arrow[ur, "L\eta RLR"] \arrow[r, "L\widetilde{\varepsilon}R"] \arrow[dr, "LRL\eta R"'] & LR \arrow[r, "L\eta R"] & LRLR \\
 & LRLRLR \arrow[ur, "L\widetilde{\varepsilon}RLR"'] &
\end{tikzcd} \qquad (3.1)$$

.

We are interested in LR-modules and LR-comodules subject to a reasonable compatibility condition.

3.1. Frobenius bimodules. A triple (B, ϱ, ω) with an object $B \in \mathbb{B}$ and two morphisms $\varrho : LR(B) \to B$ and $\omega : B \to LR(B)$ is called a *Frobenius bimodule* provided the data induce commutativity of the diagram

$$\begin{tikzcd}
LRLR(B) \arrow[r, "LR(\varrho)"] \arrow[d, "L\widetilde{\varepsilon}R"'] \arrow[dr, phantom, "\mathrm{(I)}"] & LR(B) \arrow[r, "LR(\omega)"] \arrow[d, "\varrho"] \arrow[dr, phantom, "\mathrm{(II)}"] & LRLR(B) \arrow[d, "L\widetilde{\varepsilon}R"] \\
LR(B) \arrow[r, "\varrho"] \arrow[d, "L\eta R"'] \arrow[dr, phantom, "\mathrm{(III)}"] & B \arrow[r, "\omega"] \arrow[d, "\omega"] \arrow[dr, phantom, "\mathrm{(IV)}"] & LR(B) \arrow[d, "L\eta R"] \\
LRLR(B) \arrow[r, "LR(\varrho)"'] & LR(B) \arrow[r, "LR(\omega)"'] & LRLR(B).
\end{tikzcd}$$

This implies that $\varrho : LR(B) \to B$ defines a (non-unital) LR-module and $\omega : B \to LR(B)$ a (non-counital) LR-comodule; if that is already known, the conditions on Frobenius bimodules reduce to commutativity of the diagrams (II) and (III), that is commutativity of (Frobenius property for modules)

$$\begin{tikzcd}
 & LRLR(B) \arrow[dr, "LR(\varrho)"] & \\
LR(B) \arrow[ur, "L\eta R"] \arrow[r, "\varrho"] \arrow[dr, "LR(\omega)"'] & B \arrow[r, "\omega"] & LR(B) \\
 & LRLR(B) \arrow[ur, "L\widetilde{\varepsilon}R"'] &
\end{tikzcd} \quad . \tag{3.2}$$

Denote by $\mathbb{B}^{LR}_{LR}$ the category with the Frobenius LR-bimodules as objects and morphisms which are LR-module as well as LR-comodule morphisms.

By the commutative diagram (3.1), for any $B \in \mathbb{B}$, $LR(B)$ is a Frobenius bimodule with the canonical structures, that is, there is a functor

$$K^{LR}_{LR} : \mathbb{B} \to \mathbb{B}^{LR}_{LR}, \quad B \mapsto (LR(B), L\widetilde{\varepsilon}_{R(B)}, L\eta_{R(B)}).$$

3.2. Natural mappings. Assume again $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given (see 1.3). Then there are maps, natural in $A, A' \in \mathbb{A}$,

$$\Phi_{A,A'} : \mathrm{Mor}_{\mathbb{B}}(L(A), L(A')) \to \mathrm{Mor}_{\mathbb{A}}(A, A'), \quad g \mapsto \widetilde{\varepsilon}_{A'} \cdot R(g) \cdot \eta_A,$$
$$L_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(L(A), L(A')), \quad f \mapsto L(f),$$

$$\Phi_{A,A'} \cdot L_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{A}}(A, A'), \; f \mapsto f \cdot \widetilde{\varepsilon}_A \cdot \eta_A = \widetilde{\varepsilon}_{A'} \cdot \eta_{A'} \cdot f.$$

- If $\widetilde{\varepsilon} \cdot \eta = I_{\mathbb{A}}$, then $\Phi \cdot L_{-,-}$ is the identity (L is separable).
- If $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$, then $\Phi \cdot L_{-,-} \cdot \Phi = \Phi$ ($\Phi \cdot L_{-,-}$ is idempotent).

The natural transformation

$$\theta : LR \xrightarrow{L\eta R} LRLR \xrightarrow{L\widetilde{\varepsilon}R} LR$$

is an LR-module as well as an LR-comodule morphism. From diagram (3.1) one immediately obtains the equalities

$$\begin{aligned} L\eta R \cdot \theta = LR\,\theta \cdot L\eta R = \theta\, LR \cdot L\eta R, \\ \theta \cdot L\widetilde{\varepsilon}R = L\widetilde{\varepsilon}R \cdot \theta\, LR = L\widetilde{\varepsilon}L \cdot LR\,\theta. \end{aligned}$$

Similar relations are obtained for Frobenius bimodules.

3.3. Proposition. *Given $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$, let (B, ϱ, ω) be a Frobenius LR-bimodule (see 3.1). Then*

$$\varrho \cdot \omega \cdot \varrho = \varrho \cdot \theta_B \quad \textit{and} \quad \omega \cdot \varrho \cdot \omega = \theta_B \cdot \omega.$$

(1) *If $\widetilde{\varepsilon} \cdot \eta = I_{\mathbb{A}}$, then $\varrho \cdot \omega \cdot \varrho = \varrho$ and $\omega \cdot \varrho \cdot \omega = \omega$. Then, if ϱ is an epimorphism in $\underrightarrow{\mathbb{B}}^{LR}$ or ω is a monomorphism in $\underrightarrow{\mathbb{B}}_{LR}$, one gets $\varrho \cdot \omega = I_B$.*

(2) *If $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$ or $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$, then $\omega \cdot \varrho$ is an idempotent morphism.*

Proof. The equalities claimed and (1) can be derived from the commutative diagram

$$\begin{tikzcd}
 & & LRLR(B) \arrow[r, "L\widetilde{\varepsilon}_{R(B)}"] \arrow[dr, "LR(\varrho)"] & LR(B) \arrow[dr, "\varrho"] & \\
B \arrow[r, "\omega"] \arrow[dr, "\omega"'] & LR(B) \arrow[r, "\varrho"] \arrow[ur, "L\eta_{R(B)}"] \arrow[dr, "LR(\omega)"] & B \arrow[r, "\omega"] & LR(B) \arrow[r, "\varrho"'] & B \\
 & LR(B) \arrow[r, "L\eta_{R(B)}"'] & LRLR(B) \arrow[ur, "L\widetilde{\varepsilon}_{R(B)}"'] & &
\end{tikzcd} .$$

(2) To show this, extend the above diagram by ω on the right or by ϱ on the left, respectively. $\square$

3.4. Compatible bimodule morphisms. Assume $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given. A morphism h between Frobenius modules (B, ϱ, ω)

and (B', ϱ', ω') is called *θ-compatible*, provided it induces commutativity of the diagram

$$\begin{tikzcd}
B \arrow[r, "\omega"] \arrow[d, "h"'] \arrow[rrd, "h"] & LR(B) \arrow[r, "\varrho"] & B \arrow[d, "h"] \\
B' \arrow[r, "\omega'"'] & LR(B') \arrow[r, "\varrho'"'] & B'.
\end{tikzcd}$$

One easily obtains the following.

(1) *The class $\mathbb{K}_\theta$ of all θ-compatible bimodule morphisms in $\mathbb{B}^{LR}_{\underrightarrow{LR}}$ is an ideal class.*

(2) *A morphism $h : Q \to LR(B)$ of LR-bimodules is in $\mathbb{K}_\theta$ if and only if $\theta_B \cdot h = h$.*

(3) *A morphism $h : LR(B) \to Q$ of LR-bimodules is in $\mathbb{K}_\theta$ if and only if $h \cdot \theta_B = h$.*

(4) *If $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$, then $L\widetilde{\varepsilon}R = \theta \cdot L\widetilde{\varepsilon}R$, that is, $L\widetilde{\varepsilon}R$ is θ-compatible.*

(5) *If $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$, then $L\eta R = L\eta R \cdot \theta$, that is, $L\eta R$ is θ-compatible.*

(6) *For a Frobenius bimodule (B, ω, ϱ), ω is θ-compatible if and only if $\omega = \omega \cdot \varrho \cdot \omega$, and ϱ is θ-compatible if and only if $\varrho = \varrho \cdot \omega \cdot \varrho$.*

The next result shows how (co)firm (co)modules enter the picture.

3.5. Proposition. *Let $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ be given and consider a Frobenius LR-bimodule (B, ϱ, ω).*

(1) *If ω is θ-compatible, then (B, ω) is $\mathbb{K}_\theta$-cofirm;*
if ϱ is θ-compatible, then (B, ϱ) is $\mathbb{K}_\theta$-firm.

(2) *If $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$, then $(LR(B), L\widetilde{\varepsilon}_{R(B)})$ is a $\mathbb{K}_\theta$-firm module;*
if $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$, $(LR(B), L\eta_{R(B)})$ is a $\mathbb{K}_\theta$-cofirm comodule.

Proof. (compare Proposition 2.4) (1) For a non-counital LR-comodule (Q, κ), let $h : Q \to LR(B)$ be a comodule morphism with $L\eta R \cdot h = LR(\omega) \cdot h$ and $h = \theta \cdot h$. For $\widetilde{h} := \varrho \cdot h$ we get

$$\omega \cdot \widetilde{h} = \omega \cdot \varrho \cdot h = L\widetilde{\varepsilon}R \cdot LR(\omega) \cdot h = L\widetilde{\varepsilon}R \cdot L\eta R \cdot h = h.$$

For any θ-compatilbe comodule morphism $q : Q \to B$ with $\omega \cdot q = h$, we have $g = \varrho \cdot \omega \cdot q = \varrho \cdot h = \widetilde{h}$, showing uniqueness of $\widetilde{h}$.

The second claim is shown similarly.

(2) In view of 3.4, (4) and (5), the assertions follow from (1). □

3.6. Proposition. *Assume $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given. Let $\mathbb{K}$ be an ideal class of LR-comodule morphisms and suppose $L\widetilde{\varepsilon}_{R(B)}$ in $\mathbb{K}$ for any $B \in \mathbb{B}$.*

(1) *If (B, ω) in $\underrightarrow{\mathbb{B}}^{LR}$ is a $\mathbb{K}$-cofirm comodule (see 2.2), there is a unique $\varrho : LR(B) \to B$ in $\mathbb{K}$ making (B, ϱ, ω) a Frobenius bimodule.*

(2) *With this module structure, LR-comodule morphisms between $\mathbb{K}$-cofirm LR-comodules (B, ω) and (B', ω') are morphisms of the Frobenius bimodules (B, ω, ϱ) and (B', ω', ϱ').*

Proof. (1) Consider the diagram (see 3.1)

$$\begin{tikzcd}
LRLR(B) \arrow[r, dotted, "LR(\varrho)"] \arrow[d, "L\widetilde{\varepsilon}R"'] \arrow[dr, phantom, "(\mathrm{I})"] & LR(B) \arrow[r, "LR(\omega)"] \arrow[d, dotted, "\varrho"] \arrow[dr, phantom, "(\mathrm{II})"] & LRLR(B) \arrow[d, "L\widetilde{\varepsilon}R"] \\
LR(B) \arrow[r, dotted, "\varrho"] \arrow[d, "L\eta R"'] \arrow[dr, phantom, "(\mathrm{III})"] & B \arrow[r, "\omega"] \arrow[d, "\omega"] \arrow[dr, phantom, "(\mathrm{IV})"] & LR(B) \arrow[d, "L\eta R"] \\
LRLR(B) \arrow[r, dotted, "LR(\varrho)"'] & LR(B) \arrow[r, "LR(\omega)"'] & LRLR(B),
\end{tikzcd}$$

where (IV) is assumed to be a $\mathbb{K}$-equaliser. Since

$$\begin{aligned}
L\eta_{R(B)} \cdot L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega) &= L\widetilde{\varepsilon}_{RLR(B)} \cdot LRL\eta_{R(B)} \cdot LR(\omega) \\
&= L\widetilde{\varepsilon}_{RLR(B)} \cdot LRLR(\omega) \cdot LR(\omega) \\
&= LR(\omega) \cdot L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega),
\end{aligned}$$

and $L\widetilde{\varepsilon}R_B \cdot LR(\omega)$ is in $\mathbb{K}$, there exists a unique $\varrho : LR(B) \to B$ in $\mathbb{K}$ leading to the commutative diagram (II), and (III) commutes since ϱ is required to be a comodule morphism. Moreover,

$$\begin{aligned}
\omega \cdot \varrho \cdot L\widetilde{\varepsilon}_{R(B)} &= LR(\varrho) \cdot L\eta_{R(B)} \cdot L\widetilde{\varepsilon}_{R(B)} \\
&= LR(\varrho) \cdot L\widetilde{\varepsilon}_{RLR(B)} \cdot LRL\eta_{R(B)} \\
&= L\widetilde{\varepsilon}_{R(B)} \cdot LRLR(\varrho) \cdot LRL\eta_{R(B)} \\
&= L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega) \cdot LR(\varrho) = \omega \cdot \varrho \cdot LR(\varrho),
\end{aligned}$$

and hence $\varrho \cdot L\widetilde{\varepsilon}_{R(B)} = \varrho \cdot LR\varrho$ since ω is a $\mathbb{K}$-equaliser. This means that the diagram (I) is also commutative.

(2) Now let $h : B \to B'$ be an LR-comodule morphism. Then

$$\begin{aligned}\omega' \cdot h \cdot \varrho &= LR(h) \cdot \omega \cdot \varrho \\ &= LR(h) \cdot L\widetilde{\varepsilon}_{R(B)} \cdot LR(\omega) \\ &= L\widetilde{\varepsilon}_{R(B')} \cdot LRLR(h) \cdot LR(\omega) \\ &= L\widetilde{\varepsilon}_{R(B')} \cdot LR(\omega') \cdot LR(h) = \omega' \cdot \varrho' \cdot LR(h)\end{aligned}$$

and, since both $h \cdot \varrho$ and $\varrho' \cdot LR(h)$ are in $\mathbb{K}$, this implies that they are equal (see Definition 2.1), that is, h is also an LR-module morphism. □

Symmetric to Proposition 3.6 we get:

3.7. Proposition. *Assume $\eta : I_{\mathbb{A}} \to RL$ and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ to be given. Let $\mathbb{K}'$ be an ideal class of LR-module morphisms and suppose $L\eta_{R(B)}$ belongs to $\mathbb{K}'$ for any $B \in \mathbb{B}$.*

1. *If (B, ϱ) in $\underrightarrow{\mathbb{B}}_{LR}$ is a $\mathbb{K}'$-firm module (see 2.6), there is a unique $\omega : B \to LR(B)$ in $\mathbb{K}'$ making (B, ϱ, ω) a Frobenius bimodule.*
2. *With this comodule structure, LR-module morphisms between $\mathbb{K}'$-firm LR-modules (B, ϱ) and (B', ϱ') are morphisms of the Frobenius bimodules (B, ω, ϱ) and (B', ω', ϱ').*

So far we have only considered the case when α and $\widetilde{\beta}$ (in 1.3) exist. Now we want to include more mappings in our assumptions.

3.8. Lemma. *Refer to the notation in 1.3 and 3.2.*

1. *Let (L, R, α, β) be any pairing and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ a natural transformation satisfying $\eta \cdot \widetilde{\varepsilon} \cdot \eta = \eta$. Then $\ell r \cdot \theta = \ell r$.*
2. *Let $(R, L, \widetilde{\alpha}, \widetilde{\beta})$ be any pairing and $\eta : I_{\mathbb{A}} \to RL$ a natural transformation satisfying $\widetilde{\varepsilon} \cdot \eta \cdot \widetilde{\varepsilon} = \widetilde{\varepsilon}$. Then $\theta \cdot \widetilde{\ell r} = \widetilde{\ell r}$.*

Proof. The assertions follow immediately from the definitions. □

3.9. Theorem. *Let (L, R, α, β) be a regular pairing with β symmetric and $\widetilde{\varepsilon} : RL \to I_{\mathbb{A}}$ any natural transformation. Then,*

1. *$\widehat{\varepsilon} := \widetilde{\varepsilon} \cdot r\ell : RL \to I_{\mathbb{A}}$ is a natural transformation with $\widehat{\varepsilon} = \widehat{\varepsilon} \cdot r\ell$. Furthermore, $\ell r \cdot L\widehat{\varepsilon}R = L\widehat{\varepsilon}R$, that is, $L\widehat{\varepsilon}R$ is ℓr-compatible as an LR-comodule morphism;*
2. *$(LR, L\eta R, \varepsilon)$ is a weak comonad and if $\omega : B \to LR(B)$ is an ℓr-compatible LR-comodule, there is a unique $\varrho : LR(B) \to B$ in $\mathbb{K}_{\ell r}$ making (B, ϱ, ω) a Frobenius $(LR, \eta, \widehat{\varepsilon})$-module, given by*

$$\varrho : LR(B) \xrightarrow{LR(\omega)} LRLR(B) \xrightarrow{L\widehat{\varepsilon}_{R(B)}} LR(B) \xrightarrow{\varepsilon_B} B;$$

(3) *morphisms between ℓr-compatible LR-comodules (B,ω), (B',ω') are LR-bimodule morphisms between (B,ϱ,ω) and (B',ϱ',ω').*

Proof. (1) By our symmetry assumption, $\ell R = Lr$ and the diagram

$$\begin{tikzcd} LRL \arrow[r, "Lr\ell R", "\ell RLr"'] \arrow[d, "L\widehat{\varepsilon}R"'] \arrow[dr, "L\widehat{\varepsilon}R"] & LRLR \arrow[d, "L\widehat{\varepsilon}R"] \\ LR \arrow[r, "\ell r"'] & LR \end{tikzcd}$$

commutes, showing $L\widehat{\varepsilon}R = \ell r \cdot L\widehat{\varepsilon}R$.

(2) As shown in Proposition 2.4, (B,ω) is $\mathbb{K}_{\ell r}$-cofirm and hence the existence of ϱ follows by Proposition 3.6. For the Frobenius module (B,ϱ,ω), we have the commutative diagram

$$\begin{tikzcd} & LRLR(B) \arrow[r, "\varepsilon LR"] \arrow[dr, "LR(\varrho)"] & LR(B) \arrow[dr, "\varrho"] & \\ LR(B) \arrow[ur, "L\eta R"] \arrow[r, "\varrho"'] \arrow[dr, "LR(\omega)"'] & B \arrow[r, "\omega"'] & LR(B) \arrow[r, "\varepsilon_B"'] & B \\ & LRLR(B) \arrow[ur, "L\widehat{\varepsilon}R"'] & & \end{tikzcd}$$

.

Since ϱ is ℓR-compatible, the upper paths yields $\varrho \cdot \ell R = \varrho$. The lower path is the composite given for ϱ.

(3) Since $\mathbb{K}_{\ell r}$ is an ideal class, the assertion about the bimodule morphisms follows by Proposition 3.6. $\square$

Instead of (L,R,α,β), we may require $(R,L,\widetilde{\alpha},\widetilde{\beta})$ to be a regular pairing (see 1.3) and relate the bimodules for $(LR,\eta,\widehat{\varepsilon})$ with modules for $(LR,L\widetilde{\varepsilon}R)$. By symmetry we obtain:

3.10. Theorem. *Let $(R,L,\widetilde{\alpha},\widetilde{\beta})$ be a regular pairing of functors with $\widetilde{\alpha}$ symmetric and $\eta : I_{\mathbb{A}} \to RL$ any natural transformation. Then,*

(1) *$\widehat{\eta} := \widetilde{r}\widetilde{\ell}\cdot\eta : I_{\mathbb{A}} \to RL$ is a natural transformation with $\widehat{\eta} = \widetilde{r}\widetilde{\ell}\cdot\widehat{\eta}$ and $L\widehat{\eta}R = \widetilde{r}\widetilde{\ell}\cdot L\widehat{\eta}R$, that is, $L\widehat{\eta}R$ is $\widetilde{r}\widetilde{\ell}$-compatible as an LR-module morphism (see 2.8);*

(2) *$(LR, L\widetilde{\varepsilon}R, \widetilde{\eta})$ is a weak monad and if $\varrho : LR(B) \to B$ is an $\widetilde{\ell}\widetilde{r}$-compatible LR-module, there is a unique $\omega : B \to LR(B)$ in $\mathbb{K}_{\widetilde{\ell r}}$ making (B,ϱ,ω) a Frobenius $(LR,\widehat{\eta},\widehat{\varepsilon})$-bimodule given by*

$$\omega : B \xrightarrow{\widetilde{\eta}_B} LR(B) \xrightarrow{L\widehat{\eta}_{R(B)}} LRLR(B) \xrightarrow{LR(\varrho)} LR(B);$$

(3) *morphisms between $\widetilde{\ell r}$-compatible LR-modules (B,ϱ), (B',ϱ') are $(LR,\widehat{\eta},\widetilde{\varepsilon})$-bimodule morphism between (B,ϱ,ω) and (B',ϱ',ω').*

4. Weak Frobenius monads

As we have seen in the previous section, for results on the interplay between (co)module and bimodule structures for Frobenius monads symmetry conditions on our pairings were needed, that is, the intrinsic non-(co)unital (co)monads became weak (co)monads. Hence we will concentrate in this section on this kind of (co)monads and also apply results from Section 2.

4.1. Frobenius property. Let (F,m) be a non-unital monad, (F,δ) a non-counital comonad, $(B,\varrho)\in \underrightarrow{\mathbb{B}}_F$ and $(B,\omega)\in \underrightarrow{\mathbb{B}}^F$. We say that (F,m,δ) satisfies the *Frobenius property* and (B,ϱ,ω) is a *Frobenius bimodule*, provided they induce commutativity of the respective diagrams,

$$\begin{tikzcd} & FFF \arrow[dr, "Fm"] & \\ FF \arrow[ur, "\delta F"] \arrow[r, "m"] \arrow[dr, "F\delta"'] & F \arrow[r, "\delta"] & FF, \\ & FFF \arrow[ur, "mF"'] & \end{tikzcd} \qquad \begin{tikzcd} & FF(B) \arrow[dr, "F(\varrho)"] & \\ F(B) \arrow[ur, "\delta_B"] \arrow[r, "\varrho"] \arrow[dr, "F(\omega)"'] & B \arrow[r, "\omega"] & F(B). \\ & FF(B) \arrow[ur, "m_B"'] & \end{tikzcd}$$

The Frobenius bimodules as objects and the morphisms, which are F-module as well as F-comodule morphisms, form a category which we denote by $\underrightarrow{\mathbb{B}}_F^F$. Transferring the Propositions 3.6 and 3.7 yields:

4.2. Theorem. *Assume (F,m,δ) to satisfy the Frobenius property. Let (F,δ,ε) be a weak comonad, $\gamma := \varepsilon F\cdot\delta$, and assume $m=\gamma\cdot m$. Then,*

(1) *for any γ-compatible F-comodule (B,ω), there is a unique γ-compatible F-comodule morphism*

$$\varrho : F(B)\xrightarrow{F(\omega)} FF(B)\xrightarrow{m_B} F(B)\xrightarrow{\varepsilon_B} B$$

making (B,ϱ,ω) a Frobenius bimodule;

(2) *any F-comodule morphism between γ-compatible comodules (B,ω) and (B',ω') becomes a morphism between the Frobenius bimodules (B,ϱ,ω) and (B',ϱ',ω');*

(3) *there is an isomorphism of categories*

$$\Psi : \overline{\mathbb{B}}^F \to \overline{\mathbb{B}}^F_F, \quad (B,\omega) \mapsto (B,\varrho,\omega),$$

with the forgetful functor $U_F : \overline{\mathbb{B}}^F_F \to \overline{\mathbb{B}}^F$ *as inverse, where* $\overline{\mathbb{B}}^F_F$ *denotes the category of Frobenius bimodules which are* γ*-compatible as* F*-comodules.*

Proof. By our compatibility condition on m, we can apply Proposition 3.9 and the formula for ϱ given there. The assertions about the functors follow directly from the constructions. $\square$

4.3. Theorem. *Assume* (F,m,δ) *to satisfy the Frobenius property. Let* (F,m,η) *be a weak monad,* $\vartheta := m \cdot F\eta$*, and assume* $\delta = \delta \cdot \vartheta$*. Then,*

(1) *for a* ϑ*-compatible* F*-module* (B,ϱ)*, there is a unique* ϑ*-compatible module morphism*

$$\omega : B \xrightarrow{\eta_B} F(B) \xrightarrow{\delta_B} FF(B) \xrightarrow{F(\varrho)} F(B)$$

making (B,ϱ,ω) *a Frobenius bimodule;*

(2) *any* F*-morphism between* ϑ*-compatible modules* (B,ϱ)*,* (B',ϱ') *becomes a morphism between the Frobenius bimodules* (B,ϱ,ω) *and* (B',ϱ',ω')*;*

(3) *there is an isomorphism of categories*

$$\Phi : \underline{\mathbb{B}}_F \to \underline{\mathbb{B}}^F_F, \quad (B,\varrho) \mapsto (B,\varrho,\omega),$$

with the forgetful functor $U^F : \underline{\mathbb{B}}^F_F \to \underline{\mathbb{B}}_F$ *as inverse, where* $\underline{\mathbb{B}}^F_F$ *denotes the category of Frobenius modules which are* ϑ*-compatible as* F*-modules.*

Proof. By Proposition 3.10 and the formula for ω given there. $\square$

4.4. Definition. We call $(F,m,\eta;\delta,\varepsilon)$ a *weak Frobenius monad* provided (F,m,η) is a weak monad, (F,δ,ε) is a weak comonad, (F,m,δ) has the Frobenius property (see (4.1)), and $m \cdot F\eta = F\varepsilon \cdot \delta$ (i.e. $\vartheta = \gamma$).

As a first property we observe:

4.5. Proposition. *Let* $(F,m,\eta;\delta,\varepsilon)$ *be a weak Frobenius monad and assume the idempotent* $m \cdot F\eta = F\varepsilon \cdot \delta$ *to be split by* $F \to \underline{F} \to F$*. Then* $\underline{F}$ *has a canonical monad and comonad structure* $(\underline{F},\underline{m},\underline{\eta};\underline{\delta},\underline{\varepsilon})$ *which makes it a Frobenius monad.*

Proof. The monad and comonad structures on $\underline{F}$ are obtained from 1.1 and 1.2 and a routine diagram chase shows that the Frobenius property (see 4.1) is satisfied. □

Summarising we obtain our main result for these structures.

4.6. Theorem. *Let $(F, m, \eta; \delta, \varepsilon)$ be a weak Frobenius monad. Then the constructions in* 4.2 *and* 4.3 *yield category isomorphisms*

$$\overline{\mathbb{B}}^F \xrightarrow{\Psi} \overline{\underline{\mathbb{B}}}_F^F \xrightarrow{U^F} \underline{\mathbb{B}}_F\ , \qquad \underline{\mathbb{B}}_F \xrightarrow{\Phi} \overline{\underline{\mathbb{B}}}_F^F \xrightarrow{U_F} \overline{\mathbb{B}}^F\ .$$

where $\overline{\underline{\mathbb{B}}}_F^F$ denotes the category of those Frobenius F-bimodules which are (γ-)compatible as F-comodules and (ϑ-)compatible as F-modules.

Proof. For a weak monad (F, m, η), m is ϑ-compatible and hence γ-compatible by our assumption $\gamma = \vartheta$. Similarly, δ is ϑ compatible and hence the conditions in the preceding propositions are satisfied. □

For (proper) monads and comonads the assertions simplify. For Proposition 4.2 this situation is considered in [2, Section 4] and our results for this case correspond essentially to [2, Lemma 2, Corollary 1].

4.7. Corollary. *Let (F, m, δ) satisfy the Frobenius property and assume (F, δ, ε) to be a comonad.*

1. *For any counital F-comodule $\omega : B \to F(B)$, there is some F-module morphism $\varrho : F(B) \to B$ making (B, ϱ, ω) a Frobenius bimodule.*
2. *If (F, m) allows for a unit, then (B, ϱ) is a unital F-module.*
3. *If $m \cdot \delta = I_F$, then, for any Frobenius bimodule (B, ϱ, ω), (B, ϱ) is a firm F-module.*

Proof. (1), (2) hold by Theorem 4.2; (3) follows from Theorem 3.5. □

4.8. Corollary. *Let (F, m, δ) satisfy the Frobenius property and assume (F, m, η) to be a monad.*

1. *For any unital F-module $\varrho : F(B) \to B$, there is some F-comodule morphism $\omega : B \to F(B)$ (given in* 3.10*) making (B, ϱ, ω) a Frobenius bimodule.*
2. *If (F, δ) allows for a counit, then (B, ω) is a counital F-comodule.*
3. *If $m \cdot \delta = I_F$, then, for any Frobenius bimodule (B, ϱ, ω), (B, ω) is a firm F-comodule.*

For proper monads and comonads F, all non-unital F-modules are compatible and all non-counital F-comodules are compatible, that is, $\underrightarrow{\mathbb{B}}_F = \underline{\mathbb{B}}_F$ and $\underrightarrow{\mathbb{B}}^F = \overline{\mathbb{B}}^F$. Thus we have:

4.9. Corollary. *Let $(F, m, \eta; \delta, \varepsilon)$ be a Frobenius monad. There are category isomorphisms*

$$\Psi : \underrightarrow{\mathbb{B}}^F \to \underrightarrow{\mathbb{B}}_F^F, \quad \Phi : \underrightarrow{\mathbb{B}}_F \to \underrightarrow{\mathbb{B}}_F^F,$$

where $\underrightarrow{\mathbb{B}}_F^F$ denotes the category of non-unital and non-counital Frobenius F-bimodules, and

$$\Psi' : \mathbb{B}^F \to \mathbb{B}_F^F, \quad \Phi' : \mathbb{B}_F \to \mathbb{B}_F^F,$$

where $\mathbb{B}_F^F$ is the category of unital and counital Frobenius F-bimodules.

It is easy to see that (by (co)restriction) these isomorphisms induce isomorphisms between the category of unital F-modules, counital F-comodules, and of unital and counital Frobenius bimodules, an observation following from Eilenberg-Moore [4], which may be considered as the starting point for the categorical treatment of Frobenius algebras.

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Automorphisms of the endomorphism semigroup of a free commutative *g*-dimonoid

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Abstract. We determine all isomorphisms between the endomorphism semigroups of free commutative g-dimonoids and prove that all automorphisms of the endomorphism semigroup of a free commutative g-dimonoid are quasi-inner.

1. Introduction

A *dimonoid* is an algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ such that for all $x, y, z \in D$ the following conditions hold:

$$\begin{aligned}
&(D_1) \qquad (x \dashv y) \dashv z = x \dashv (y \vdash z),\\
&(D_2) \qquad (x \vdash y) \dashv z = x \vdash (y \dashv z),\\
&(D_3) \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z).
\end{aligned}$$

This notion was introduced by Jean-Louis Loday in [1] and now it plays a prominent role in problems from the theory of Leibniz algebras. A vector space equipped with the structure of a dimonoid is called a *dialgebra*. Thus, a dialgebra is a linear analog of a dimonoid. It is known that Leibniz algebras are a non-commutative variation of Lie algebras and dialgebras are a variation of associative algebras.

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There exist some generalizations of dimonoids, for example, 0-dialgebras and duplexes (see, e.g., [2], [3]), g-dimonoids etc. Omitting the axiom (D_2) of an inner associativity in the definition of a dimonoid, we obtain the notion of a g-dimonoid. An associative 0-dialgebra, that is, a vector space equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms (D_1) and (D_3), is a linear analog of a g-dimonoid. Free g-dimonoids and free n-nilpotent g-dimonoids were constructed in [4], [5] and [5], respectively. The construction of a free commutative g-dimonoid and the least commutative congruence on a free g-dimonoid were described in [6]. Defining identities of a g-dimonoid appear also in axioms of trialgebras and of trioids [7–9].

Endomorphism semigroups of algebraic systems have been studied by numerous authors. The problem of studying the endomorphism semigroup for free algebras in a certain variety was raised by B.I. Plotkin in his papers on universal algebraic geometry (see, e.g., [10], [11]). In this direction there are many papers devoted to describing automorphisms of endomorphism semigroups of free finitely generated universal algebras of some varieties: groups [12], semigroups [13], associative algebras [14], inverse semigroups [15], modules and semimodules [16], Lie algebras [17] and other algebras (see also [18]). In this paper we solve the similar problem for the variety of commutative g-dimonoids.

The paper is organized in the following way. In Section 2, we give necessary definitions and statements. In Section 3, we define the notion of a crossed isomorphism of g-dimonoids and prove auxiliary lemmas. In Section 4, we describe all isomorphisms between the endomorphism monoids of free commutative g-dimonoids of rank 1. In Section 5, we prove that automorphisms of the endomorphism semigroup of a free commutative g-dimonoid of a non-unity rank are inner or "mirror inner". We show also that the automorphism group of the endomorphism semigroup of a free commutative g-dimonoid is isomorphic to the direct product of a symmetric group and a 2-element group.

2. Preliminaries

Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary g-dimonoids. A mapping $\varphi : D_1 \to D_2$ is called a *homomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$ if

$$(x \dashv_1 y)\varphi = x\varphi \dashv_2 y\varphi, \ \ (x \vdash_1 y)\varphi = x\varphi \vdash_2 y\varphi$$

for all $x, y \in D_1$.

A bijective homomorphism $\varphi : D_1 \to D_2$ is called an *isomorphism* of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. In this case g-dimonoids $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are called *isomorphic.*

A g-dimonoid $(D, \dashv, \vdash)$ is called *commutative* if for all $x, y \in D$,

$$x \dashv y = y \dashv x, \qquad x \vdash y = y \vdash x.$$

Firstly we give an example of a g-dimonoid which is not a dimonoid.

Let A be an arbitrary nonempty set and $\overline{A} = \{\overline{x} \mid x \in A\}$. For every $x \in A$ assume $\widetilde{\overline{x}} = x$ and introduce a mapping $\alpha = \alpha_A : A \cup \overline{A} \to A$ by the following rule:

$$y\alpha = \begin{cases} y, & y \in A, \\ \widetilde{y}, & y \in \overline{A}. \end{cases}$$

Give an arbitrary semigroup S and define operations $\prec$ and $\succ$ on $S \cup \overline{S}$ as follows:

$$a \prec b = (a\alpha_S)(b\alpha_S), \qquad a \succ b = \overline{(a\alpha_S)(b\alpha_S)}$$

for all $a, b \in S \cup \overline{S}$. The algebra $(S \cup \overline{S}, \prec, \succ)$ is denoted by $S^{(\alpha)}$.

Proposition 1 ([6])**.** *$S^{(\alpha)}$ is a g-dimonoid but not a dimonoid.*

We note that if X is a generating set of a semigroup S, then $S^{(\alpha)} \setminus \overline{X}$ is a g-subdimonoid of $S^{(\alpha)}$ generated by X.

For an arbitrary commutative semigroup S, obviously, $S^{(\alpha)}$ is a commutative g-dimonoid.

Recall the construction of a free commutative g-dimonoid. Let $F[A]$ be the free commutative semigroup generated by a set A.

Theorem 1 ([6])**.** *$F[A]^{(\alpha)} \setminus \overline{A}$ is the free commutative g-dimonoid.*

Observe that A is a generating set of $F[A]^{(\alpha)} \setminus \overline{A}$, the cardinality of A is the *rank* of $F[A]^{(\alpha)} \setminus \overline{A}$ and this g-dimonoid is uniquely determined up to an isomorphism by $|A|$.

Further the free commutative g-dimonoid generated by A will be denoted by $\mathfrak{FCD}_A^g$.

In particular, we consider the free commutative g-dimonoid of rank 1. Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}^* = (\mathbb{N} \cup \overline{\mathbb{N}}) \setminus \{\overline{1}\}$. Define operations $\prec$ and $\succ$ on $\mathbb{N}^*$ by

$$m \prec n = m + n, \qquad \overline{q} \prec \overline{r} = q + r,$$
$$m \prec \overline{r} = m + r, \qquad \overline{q} \prec n = q + n,$$
$$a \succ b = \overline{a \prec b},$$

for all $m, n \in \mathbb{N}$, $\overline{q}, \overline{r} \in \overline{\mathbb{N}} \setminus \{\overline{1}\}$ and $a, b \in \mathbb{N}^*$.

Proposition 2 ([6]). *The free commutative g-dimonoid $\mathfrak{FCD}^g_A$ of rank 1 is isomorphic to* $(\mathbb{N}^*, \prec, \succ)$.

Recall that the content of $\omega = x_1x_2 \dots x_n \in F[A]$ is the set $c(\omega) = \{x_1, x_2, \dots, x_n\}$ and the length of ω is the number $l(\omega) = n$.

For every $\omega \in \mathfrak{FCD}^g_A$, the set $c(\omega\alpha)$ and the number $l(\omega\alpha)$ we call the *content* and the *length* of ω, respectively, and denote it by $c(\omega)$ and $l(\omega)$. For example, for $w = \overline{bacda}$ we have $c(w) = \{a, b, c, d\}$ and $l(w) = 5$.

3. Auxiliary statements

We start this section with the following lemma.

Lemma 1. *Let $\mathfrak{FCD}^g_X$ and $\mathfrak{FCD}^g_Y$ be free commutative g-dimonoids generated by X and Y, respectively. Every bijection $\varphi : X \to Y$ induces an isomorphism $\varepsilon_\varphi : \mathfrak{FCD}^g_X \to \mathfrak{FCD}^g_Y$ such that*

$$\omega\varepsilon_\varphi = \begin{cases} x_1\varphi \prec x_2\varphi \prec \dots \prec x_m\varphi, & \omega = x_1x_2 \dots x_m,\ m \geqslant 1, \\ x_1\varphi \succ x_2\varphi \succ \dots \succ x_m\varphi, & \omega = \overline{x_1x_2 \dots x_m},\ m > 1 \end{cases}$$

for all $\omega \in \mathfrak{FCD}^g_X$.

Proof. The proof of this statement is obvious. □

Now we introduce the notion of a crossed isomorphism of g-dimonoids. A mapping $\varphi : D_1 \to D_2$ we call a *crossed homomorphism* of a g-dimonoid $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ into a g-dimonoid $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ if for all $x, y \in D_1$,

$$(x \dashv_1 y)\varphi = x\varphi \vdash_2 y\varphi, \qquad (x \vdash_1 y)\varphi = x\varphi \dashv_2 y\varphi.$$

A bijective crossed homomorphism $\varphi : D_1 \to D_2$ will be called a *crossed isomorphism* of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. In such case g-dimonoids $\mathfrak{D}_1$ and $\mathfrak{D}_2$ we call *crossed isomorphic*.

An example of crossed isomorphic g-dimonoids gives the next lemma.

Lemma 2. *Let $\mathfrak{FCD}^g_X$ and $\mathfrak{FCD}^g_Y$ be free commutative g-dimonoids generated by X and Y, respectively. Every bijection $\varphi : X \to Y$ induces a crossed isomorphism $\varepsilon^*_\varphi : \mathfrak{FCD}^g_X \to \mathfrak{FCD}^g_Y$ such that*

$$\omega\varepsilon^*_\varphi = \begin{cases} x_1\varphi \succ x_2\varphi \succ \dots \succ x_m\varphi, & \omega = x_1x_2 \dots x_m,\ m \geqslant 1, \\ x_1\varphi \prec x_2\varphi \prec \dots \prec x_m\varphi, & \omega = \overline{x_1x_2 \dots x_m},\ m > 1 \end{cases}$$

for all $\omega \in \mathfrak{FCD}^g_X$.

Proof. It is clear that ε^*_φ is a bijection. Take arbitrary $u, v \in \mathfrak{FCD}^g_X$ and consider the following cases.

Case 1. $u = u_1u_2\dots u_m, v = v_1v_2\dots v_n \in F[X]$, then

$$
\begin{aligned}
(u \prec v)\varepsilon^*_\varphi &= (u\alpha v\alpha)\varepsilon^*_\varphi = (uv)\varepsilon^*_\varphi \\
&= u_1\varphi \succ \dots \succ u_m\varphi \succ v_1\varphi \succ \dots \succ v_n\varphi = u\varepsilon^*_\varphi \succ v\varepsilon^*_\varphi,
\end{aligned}
$$

$$
\begin{aligned}
(u \succ v)\varepsilon^*_\varphi &= (\overline{u\alpha v\alpha})\varepsilon^*_\varphi = (\overline{uv})\varepsilon^*_\varphi \\
&= u_1\varphi \prec \dots \prec u_m\varphi \prec v_1\varphi \prec \dots \prec v_n\varphi \\
&= \overline{u_1\varphi\dots u_m\varphi} \prec \overline{v_1\varphi\dots v_n\varphi} = u\varepsilon^*_\varphi \prec v\varepsilon^*_\varphi.
\end{aligned}
$$

Case 2. $u = u_1u_2\dots u_m \in F[X], \overline{v} = \overline{v_1v_2\dots v_n} \in \overline{F[X]} \setminus \overline{X}$, then

$$
\begin{aligned}
(u \prec \overline{v})\varepsilon^*_\varphi &= (uv)\varepsilon^*_\varphi = \overline{u_1\varphi\dots u_m\varphi v_1\varphi\dots v_n\varphi} \\
&= \overline{u_1\varphi\dots u_m\varphi} \succ (v_1\varphi\dots v_n\varphi) = u\varepsilon^*_\varphi \succ \overline{v}\varepsilon^*_\varphi,
\end{aligned}
$$

$$
\begin{aligned}
(u \succ \overline{v})\varepsilon^*_\varphi &= (\overline{uv})\varepsilon^*_\varphi = u_1\varphi\dots u_m\varphi v_1\varphi\dots v_n\varphi \\
&= \overline{u_1\varphi\dots u_m\varphi} \prec (v_1\varphi\dots v_n\varphi) = u\varepsilon^*_\varphi \prec \overline{v}\varepsilon^*_\varphi.
\end{aligned}
$$

Case 3, where $\overline{u} = \overline{u_1u_2\dots u_m} \in \overline{F[X]} \setminus \overline{X}, v = v_1v_2\dots v_n \in F[X]$, can be omited since operations $\prec$ and $\succ$ are commutative.

Case 4, where $\overline{u} = \overline{u_1u_2\dots u_m}, \overline{v} = \overline{v_1v_2\dots v_n} \in \overline{F[X]} \setminus \overline{X}$, is analogous to the case 1.

From cases 1–4 it follows that ε^*_φ is a crossed homomorphism which completes the proof of this statement. □

For an arbitrary algebra A, we denote the endomorphism semigroup and the automorphism group of A by $\mathrm{End}(A)$ and $\mathrm{Aut}(A)$, respectively.

Anywhere the composition of mappings is defined from left to right.

Lemma 3. *Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary g-dimonoids, and φ be any isomorphism or a crossed isomorphism of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. The mapping*

$$
\Phi : f \mapsto f\Phi = \varphi^{-1} f \varphi, \quad f \in \mathrm{End}(\mathfrak{D}_1),
$$

is an isomorphism of $\mathrm{End}(\mathfrak{D}_1)$ onto $\mathrm{End}(\mathfrak{D}_2)$.

Proof. Let φ be a crossed isomorphism of $\mathfrak{D}_1$ onto $\mathfrak{D}_2$. Clearly, φ^{-1} is a crossed isomorphism of $\mathfrak{D}_2$ onto $\mathfrak{D}_1$. For all $u, v \in D_2$ and $f \in \mathrm{End}(\mathfrak{D}_1)$,

$$\begin{aligned}(u \dashv_2 v)\varphi^{-1}f\varphi &= (u\varphi^{-1} \vdash_1 v\varphi^{-1})f\varphi\\ &= (u\varphi^{-1}f \vdash_1 v\varphi^{-1}f)\varphi = u(\varphi^{-1}f\varphi) \dashv_2 v(\varphi^{-1}f\varphi).\end{aligned}$$

In similar way, $\varphi^{-1}f\varphi \in \mathrm{End}(D_2, \vdash_2)$ and so $f\Phi \in \mathrm{End}(\mathfrak{D}_2)$ for all $f \in \mathrm{End}(\mathfrak{D}_1)$. The remaining part of the proof is trivial. $\square$

We call Φ from Lemma 3 as the isomorphism induced by the isomorphism or the crossed isomorphism φ.

For an arbitrary nonempty set X the identity transformation of X is denoted by id_X. By Lemma 2, $\varepsilon^*_{id_X}$ is a crossed automorphism of the free commutative g-dimonoid $\mathfrak{FCD}^g_X$.

By Lemma 3, a transformation Φ_1 of the endomorphism monoid $\mathrm{End}(\mathfrak{FCD}^g_X)$ defined by $\eta\Phi_1 = (\varepsilon^*_{id_X})^{-1}\eta\varepsilon^*_{id_X}$ for all $\eta \in \mathrm{End}(\mathfrak{FCD}^g_X)$, is an automorphism. Obviously, $(\varepsilon^*_{id_X})^{-1} = \varepsilon^*_{id_X}$.

The automorphism Φ_1 we will call the *mirror automorphism* of the endomorphism monoid $\mathrm{End}(\mathfrak{FCD}^g_X)$. By Φ_0 we denote the identity automorphism of $\mathrm{End}(\mathfrak{FCD}^g_X)$. It is clear that $\{\Phi_0, \Phi_1\}$ is a group with respect to the composition of permutations.

Let $\mathfrak{FCD}^g_X$ be the free commutative g-dimonoid generated by X. Each endomorphism ξ of $\mathfrak{FCD}^g_X$ is uniquely determined by a mapping $\varphi : X \to \mathfrak{FCD}^g_X$. Really, to define ξ, it suffices to put

$$\omega\xi = \begin{cases} x_1\varphi \prec x_2\varphi \prec \ldots \prec x_m\varphi, & \omega = x_1x_2\ldots x_m,\ m \geqslant 1,\\ x_1\varphi \succ x_2\varphi \succ \ldots \succ x_m\varphi, & \omega = \overline{x_1x_2\ldots x_m},\ m > 1\end{cases}$$

for all $\omega \in \mathfrak{FCD}^g_X$.

In particular, an endomorphism ξ of $\mathfrak{FCD}^g_X$ is an automorphism if and only if a restriction ξ on X belong to the symmetric group $S(X)$. Therefore, the group $\mathrm{Aut}(\mathfrak{FCD}^g_X)$ is isomorphic to $S(X)$ (see [6]).

Let $u \in \mathfrak{FCD}^g_X$. An endomorphism $\theta_u \in \mathrm{End}(\mathfrak{FCD}^g_X)$ is called *constant* if $x\theta_u = u$ for all $x \in X$.

Lemma 4. (i) *Let $u \in \mathfrak{FCD}^g_X$, $\xi \in \mathrm{End}(\mathfrak{FCD}^g_X)$. Then $\theta_u\xi = \theta_{u\xi}$.*

(ii) *An endomorphism ξ of $\mathfrak{FCD}^g_X$ is constant if and only if $\psi\xi = \xi$ for all $\psi \in \mathrm{Aut}(\mathfrak{FCD}^g_X)$.*

(iii) *An endomorphism ξ of $\mathfrak{FCD}^g_X$ is constant idempotent if and only if $\xi = \theta_x$ for some $x \in X$.*

Proof. (i) It is obvious.

(ii) Take a constant $\theta_u \in \mathrm{End}(\mathfrak{FCD}_X^g)$ for some $u \in \mathfrak{FCD}_X^g$, and let $\psi \in \mathrm{Aut}(\mathfrak{FCD}_X^g)$. Then $x(\psi\theta_u) = (x\psi)\,\theta_u = u = x\theta_u$ for all $x \in X$.

Now let $\psi\xi = \xi$ for all $\psi \in \mathrm{Aut}(\mathfrak{FCD}_X^g)$ and some $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$. Fixing $x \in X$, we obtain $x\xi = x\,(\psi\xi) = (x\psi)\,\xi = y\xi$, where $y = x\psi$. Since $\{x\psi \mid \psi \in \mathrm{Aut}(\mathfrak{FCD}_X^g)\} = X$, then $x\xi = y\xi$ for all $y \in X$. Consequently, $\xi = \theta_u$ for $u = x\xi$.

(iii) Let $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ be a constant idempotent. Then $\xi = \theta_u, u \in \mathfrak{FCD}_X^g$, and by (i) of this lemma, $\theta_u = \theta_u\theta_u = \theta_{u\theta_u}$. This implies $u = u\theta_u$ and, therefore, $l(u) = 1$ and $u \in X$. Converse is obvious. □

4. The automorphism group of $\mathrm{End}(\mathfrak{FCD}_X^g)$, $|X| = 1$

The free commutative g-dimonoid $\mathfrak{FCD}_X^g$ on an n-element set X we denote by $\mathfrak{FCD}_n^g$. Recall that the g-dimonoid $\mathfrak{FCD}_1^g$ is isomorphic to $(\mathbb{N}^*, \prec, \succ)$ (see Proposition 2). Therefore, we will identify elements of $\mathfrak{FCD}_1^g$ with corresponding elements of $(\mathbb{N}^*, \prec, \succ)$.

Define a binary operation $\odot$ on $\mathbb{N}^* = (\mathbb{N} \cup \overline{\mathbb{N}}) \setminus \{\overline{1}\}$ by

$$m \odot n = m \odot \overline{n} = m \cdot n, \qquad \overline{m} \odot n = \overline{m} \odot \overline{n} = \overline{m \cdot n},$$
$$1 \odot x = x \odot 1 = x$$

for all $m, n \in \mathbb{N} \setminus \{1\}$, $\overline{m}, \overline{n} \in \overline{\mathbb{N}} \setminus \{\overline{1}\}$ and $x \in \mathbb{N}^*$.

Lemma 5. (i) *The operation $\odot$ is associative.*

(ii) *The operation $\odot$ is distributive with respect to $\prec$ and $\succ$.*

Proof. It can be verified directly. □

From Lemma 5 (i) it follows that $(\mathbb{N}^*, \odot)$ is a semigroup.

Lemma 6. *The semigroups* $\mathrm{End}(\mathfrak{FCD}_1^g)$ *and* $(\mathbb{N}^*, \odot)$ *are isomorphic.*

Proof. Let φ be an arbitrary endomorphism of $(\mathbb{N}^*, \prec, \succ)$ and $1\varphi = k$ for some $k \in \mathbb{N}^*$. For all $a \in \mathbb{N}$ and $\overline{b} \in \overline{N} \setminus \{\overline{1}\}$ we obtain

$$a\varphi = (\underbrace{1 \prec 1 \prec \ldots \prec 1}_{a})\varphi = a \odot k, \quad \overline{b}\varphi = (\underbrace{1 \succ 1 \succ \ldots \succ 1}_{b})\varphi = \overline{b \odot k}.$$

Converse, any transformation $\varphi_k : \mathbb{N}^* \to \mathbb{N}^*, k \in \mathbb{N}^*$, defined by

$$a\varphi_k = a \odot k$$

is an endomorphism of $(\mathbb{N}^*, \prec, \succ)$. Indeed, using the condition (ii) of Lemma 5, for all $a, b \in \mathbb{N}^*$ and $\star \in \{\prec, \succ\}$ we obtain

$$(a \star b)\varphi_k = (a \star b) \odot k = (a \odot k) \star (b \odot k) = a\varphi_k \star b\varphi_k.$$

Consequently,

$$\mathrm{End}(\mathbb{N}^*, \prec, \succ) = \{\varphi_k \mid k \in \mathbb{N}^*\}.$$

Define a mapping Θ of $\mathrm{End}(\mathbb{N}^*, \prec, \succ)$ into $(\mathbb{N}^*, \odot)$ by $\varphi_k\Theta = k$ for all $\varphi_k \in \mathrm{End}(\mathbb{N}^*, \prec, \succ)$. An immediate verification shows that Θ is an isomorphism. $\square$

Remark 1. Note that all endomorphisms of a g-dimonoid $(\mathbb{N}^*, \prec, \succ)$ are injective but they are not surjective (except an identity automorphism). So that the automorphism group of $(\mathbb{N}^*, \prec, \succ)$ is singleton.

Let $\mathbb{P}$ be the set of all prime numbers, $\overline{\mathbb{P}} = \{\overline{x} \mid x \in \mathbb{P}\}$ and $\mathbb{P}^* = \mathbb{P} \cup \overline{\mathbb{P}}$. For any mapping $f : A \to B$ and a nonempty subset $C \subseteq A$, we denote the restriction of f to C by $f|_C$.

Further let $A, B \subseteq N \setminus \{1\}$, $C \subseteq \overline{N} \setminus \{\overline{1}\}$ be nonempty subsets and $\varphi : A \to B$, $\psi : B \to C$ be arbitrary mappings. Denote by $\overline{\varphi}$ and $\overrightarrow{\psi}$ the mappings $\overline{A} \to \overline{B}$ and, respectively, $\overline{B} \to C\alpha$ (the mapping α was defined in Section 2) such that

$$\overline{a}\,\overline{\varphi} = \overline{b} \ \ if \ \ a\varphi = b \ \ and \ \ \overline{b}\,\overrightarrow{\psi} = c \ \ if \ \ b\psi = \overline{c}.$$

Proposition 3. *Let* $\mathrm{End}(\mathfrak{FCD}_X^g) \cong \mathrm{End}(\mathfrak{FCD}_Y^g)$, *where* X *is a singleton set,* Y *is an arbitrary set. Then* $|Y| = 1$ *and the isomorphisms of* $\mathrm{End}(\mathfrak{FCD}_X^g)$ *onto* $\mathrm{End}(\mathfrak{FCD}_Y^g)$ *are in a natural one-to-one correspondence with permutations* $f : \mathbb{P}^* \to \mathbb{P}^*$ *such that*

$$\mathbb{P}f = \mathbb{P}, f|_{\overline{\mathbb{P}}} = \overline{f|_{\mathbb{P}}} \ \ or \ \ \mathbb{P}f = \overline{\mathbb{P}}, f|_{\overline{\mathbb{P}}} = \overrightarrow{f|_{\mathbb{P}}}.$$

Proof. According to Lemma 6, $\mathrm{End}(\mathfrak{FCD}_1^g) \cong (\mathbb{N}^*, \odot)$. Let $|Y| \geqslant 2$ and $a, b \in Y, a \neq b$. Define a binary relation ρ on $\mathbb{N}^*$ by

$$(a; b) \in \rho \Leftrightarrow a = b = 1 \ \ or \ \ a \neq 1 \neq b, a \odot b = b \odot a.$$

Obviously, ρ is an equivalence and $\mathbb{N}^*/\rho = \{\mathbb{N} \setminus \{1\}, \overline{\mathbb{N}} \setminus \{\overline{1}\}, \{1\}\}$. Since $\mathrm{End}(\mathfrak{FCD}_Y^g) \cong (\mathbb{N}^*, \odot)$, we will use the relation ρ for $\mathrm{End}(\mathfrak{FCD}_Y^g)$ too. For constants $\theta_{\overline{ab}}, \theta_a, \theta_{ab} \in \mathrm{End}(\mathfrak{FCD}_Y^g)$ and some $y \in Y$ we have

$$y(\theta_{\overline{ab}}\theta_a) = \overline{ab}\theta_a = \overline{aa} \neq \overline{ab} = a\theta_{\overline{ab}} = y(\theta_a\theta_{\overline{ab}}),$$
$$y(\theta_{\overline{ab}}\theta_{ab}) = \overline{ab}\theta_{ab} = \overline{abab} \neq abab = ab\theta_{\overline{ab}} = y(\theta_{ab}\theta_{\overline{ab}}),$$

therefore $(\theta_{\overline{ab}}, \theta_a) \notin \rho$ and $(\theta_{\overline{ab}}, \theta_{ab}) \notin \rho$. From here it follows that $(\theta_{ab}, \theta_a) \in \rho$ which contradicts the fact that $\theta_{ab}\theta_a \neq \theta_a\theta_{ab}$. Then $|Y| = 1$.

It is clear that the semigroup $(\mathbb{N}^* \setminus \{1\}, \odot)$ is generated by $\mathbb{P}^*$ and $\mathbb{P}^* f = \mathbb{P}^*$ for all $f \in \mathrm{Aut}(\mathbb{N}^*, \odot)$. Assume that there exist $p, q \in \mathbb{P}$ such that $pf = p' \in \mathbb{P}$ and $qf = \overline{q'} \in \overline{\mathbb{P}}$ for some $f \in \mathrm{Aut}(\mathbb{N}^*, \odot)$. Then

$$p' \cdot q' = p' \odot \overline{q'} = (p \cdot q)f = (q \cdot p)f = \overline{q'} \odot p' = \overline{p' \cdot q'}.$$

It means that $\mathbb{P}f = \mathbb{P}$ and so $\overline{\mathbb{P}}f = \overline{\mathbb{P}}$, or $\mathbb{P}f = \overline{\mathbb{P}}$ and then $\overline{\mathbb{P}}f = \mathbb{P}$.

If $\mathbb{P}f = \mathbb{P}$, then for all $p \in \mathbb{P}$ we have $(pf)^2 = p^2 f = (p \odot \overline{p})f = pf \odot \overline{p}f$, whence $\overline{p}f = \overline{pf}$. Thus, $f|_{\overline{\mathbb{P}}} = \overline{f|_{\mathbb{P}}}$. In a similar way it can be shown that in the case $\mathbb{P}f = \overline{\mathbb{P}}$ we obtain $f|_{\overline{\mathbb{P}}} = \overrightarrow{f|_{\mathbb{P}}}$.

On the other hand, as it is not hard to check, every permutation $f : \mathbb{P}^* \to \mathbb{P}^*$ such that $\mathbb{P}f = \mathbb{P}, f|_{\overline{\mathbb{P}}} = \overline{f|_{\mathbb{P}}}$, or $\mathbb{P}f = \overline{\mathbb{P}}, f|_{\overline{\mathbb{P}}} = \overrightarrow{f|_{\mathbb{P}}}$, uniquely determines an automorphism of $(\mathbb{N}^*, \odot)$. These permutations and hence the isomorphisms $\mathrm{End}(\mathfrak{FCD}^g_X) \to \mathrm{End}(\mathfrak{FCD}^g_Y)$, are in a natural one-to-one correspondence. $\square$

An automorphism $\Phi : \mathrm{End}(\mathfrak{FCD}^g_X) \to \mathrm{End}(\mathfrak{FCD}^g_X)$ is called *quasi-inner* if there exists a permutation α of $\mathfrak{FCD}^g_X$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in \mathrm{End}(\mathfrak{FCD}^g_X)$. If α turns out to be an automorphism of $\mathfrak{FCD}^g_X$, Φ is an inner automorphism of $\mathrm{End}(\mathfrak{FCD}^g_X)$.

We denote the symmetric group on a set X by $S(X)$. A 2-element group with identity e is denoted by $C_2 = \{e, a\}$.

Proposition 4. *Automorphisms of the monoid* $\mathrm{End}(\mathfrak{FCD}^g_1)$ *are quasi-inner. In addition, the automorphism group of* $\mathrm{End}(\mathfrak{FCD}^g_1)$ *is isomorphic to the direct product* $S(\mathbb{P}) \times C_2$.

Proof. Let $\Psi : \mathrm{End}(\mathfrak{FCD}^g_1) \to \mathrm{End}(\mathfrak{FCD}^g_1)$ be an arbitrary automorphism. Define a bijection $\psi : N^* \to N^*$ putting $x\psi = y$ if $\varphi_x\Psi = \varphi_y$. It is clear that $\psi \in \mathrm{Aut}(\mathbb{N}^*, \odot)$, however $\psi \notin \mathrm{Aut}(\mathbb{N}^*, \prec, \succ)$ except the identity permutation (see Remark 1). Then for all $x \in N^*$ and some $\varphi_i \in \mathrm{End}(\mathfrak{FCD}^g_1)$, $i \in N^*$, we have

$$\begin{aligned} x(\psi^{-1}\varphi_i\psi) &= (x\psi^{-1})\varphi_i\psi = ((x\psi^{-1}) \odot i)\psi \\ &= (x\psi^{-1})\psi \odot i\psi = x \odot i\psi = x\varphi_{i\psi}. \end{aligned}$$

Thus, $\psi^{-1}\varphi_i\psi = \varphi_i\Psi$ and Ψ is a quasi-inner automorphism.

The immediate check shows that a mapping Θ of $\mathrm{Aut}(\mathbb{N}^*, \odot)$ onto $S(\mathbb{P}) \times C_2$ defined as follows:

$$\xi\Theta = \begin{cases} (\xi|_P, e), & P\xi = P, \\ (\xi|_P, a), & P\xi = \overline{P} \end{cases}$$

for all $\xi \in \mathrm{Aut}(\mathbb{N}^*, \odot)$, is an isomorphism.

By Lemma 6, $\mathrm{End}(\mathfrak{FCD}_1^g) \cong (\mathbb{N}^*, \odot)$, therefore $\mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}_1^g))$ and $S(\mathbb{P}) \times C_2$ are isomorphic. □

5. The automorphism group of $\mathrm{End}(\mathfrak{FCD}_X^g)$, $|X| \geqslant 2$

An automorphism Ψ of the endomorphism monoid $\mathrm{End}(\mathfrak{FCD}_X^g)$ of the free commutative g-dimonoid $\mathfrak{FCD}_X^g$ is called *stable* if Ψ induces the identity permutation of X, that is, $\theta_x\Psi = \theta_x$ for all $x \in X$.

Lemma 7. *For all $u, v \in F[X] \setminus X$ the following equalities hold:*

$$\theta_u\theta_v = \theta_u\theta_{\overline{v}} \text{ and } \theta_{\overline{u}}\theta_{\overline{v}} = \theta_{\overline{u}}\theta_v.$$

Proof. It is obvious. □

Lemma 8. *Let Ψ be a stable automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$, $u, v \in F[X] \setminus X$, $x \in X$ and $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$. Then*

(i) $\theta_{x\xi}\Psi = \theta_{x(\xi\Psi)}$;

(ii) $\theta_u\Psi = \theta_v$ *implies* $\theta_{\overline{u}}\Psi = \theta_{\overline{v}}$;

(iii) $\theta_u\Psi = \theta_{\overline{v}}$ *implies* $\theta_{\overline{u}}\Psi = \theta_v$.

Proof. (i) By Lemma 4 (i), $\theta_{x\xi}\Psi = (\theta_x\xi)\Psi = \theta_x(\xi\Psi) = \theta_{x(\xi\Psi)}$.

(ii) Let $\theta_u\Psi = \theta_v$. By (i) of this lemma, $\theta_{\overline{u}}\Psi = \theta_w$ for some $w \in \mathfrak{FCD}_X^g$. Using Lemma 7, we obtain

$$\begin{aligned}\theta_{v^{l(v)}} &= \theta_v^2 = (\theta_u\Psi)^2 = (\theta_u^2)\Psi \\ &= (\theta_u\theta_{\overline{u}})\Psi = \theta_u\Psi\theta_{\overline{u}}\Psi = \theta_v\theta_w = \theta_{w^{l(v)}},\end{aligned}$$

where $w^{l(v)} = \underbrace{w \prec w \prec \ldots \prec w}_{l(v)}$. From here $w = v$ or $w = \overline{v}$. In the first case we have $\theta_{\overline{u}}\Psi = \theta_v$ which contradicts to injectivity of Ψ, therefore $\theta_{\overline{u}}\Psi = \theta_{\overline{v}}$.

(iii) This statement is anologous to the case (ii). □

An endomorphism θ of the free commutative g-dimonoid $\mathfrak{FCD}_X^g$ is called *linear* if $x\theta \in X$ for all $x \in X$.

Lemma 9. *Let* Ψ *be a stable automorphism of* $\mathrm{End}(\mathfrak{FCD}_X^g)$, $u, v \in \mathfrak{FCD}_X^g$, $x \in X$ *and* $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$. *The following conditions hold:*

(i) $\xi\Psi = \xi$, *if* ξ *is linear;*

(ii) $c(u) = c(v)$, *if* $\theta_u\Psi = \theta_v$;

(iii) $l(x\xi) = l(x(\xi\Psi))$.

Proof. (i) If ξ is linear, then $x\xi \in X$ for all $x \in X$. Hence by stability of Ψ, $\theta_{x(\xi\Psi)} = \theta_{x\xi}\Psi = \theta_{x\xi}$. From here, $\xi\Psi = \xi$.

(ii) Let $\theta_u\Psi = \theta_v$ and $c(u) \setminus c(v) \neq \varnothing$. We take $z \in c(u) \setminus c(v)$, and $x \in X, x \neq z$, and $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ such that $z\xi = x$ and $y\xi = y$ for all $y \in X, y \neq z$. Then ξ is linear, $v\xi = v$ and

$$\theta_u\Psi = \theta_v = \theta_{v\xi} = \theta_v\xi = (\theta_u\Psi)(\xi\Psi) = (\theta_u\xi)\Psi = \theta_{u\xi}\Psi.$$

From here $\theta_u = \theta_{u\xi}$ and then $u = u\xi$ which contradicts to the definition of ξ, so $c(u) \setminus c(v) = \varnothing$. If $z \in c(v) \setminus c(u) \neq \varnothing,\ z \neq x$ and $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ the same as above, then

$$\theta_v = \theta_u\Psi = \theta_{u\xi}\Psi = (\theta_u\xi)\Psi = (\theta_u\Psi)(\xi\Psi) = \theta_v\xi = \theta_{v\xi},$$

whence $v = v\xi$ which contradicts to the definition of ξ. Thus, $c(v) \setminus c(u) = \varnothing$ and therefore, $c(u) = c(v)$.

(iii) Let $\xi_1, \xi_2 \in \mathrm{End}(\mathfrak{FCD}_X^g)$ such that $l(x\xi_1) = l(x\xi_2) = m$ and $l(x(\xi_1\Psi)) = r,\ \ l(x(\xi_2\Psi)) = s$. For all $t \in X$ we obtain

$$t(\theta_x\xi_1\theta_x) = (x\xi_1)\theta_x = \begin{cases} x^m = t\theta_{x^m}, & x\xi_1 \in F[X], \\ \overline{x^m} = t\theta_{\overline{x^m}}, & x\xi_1 \in \overline{F[X]} \setminus \overline{X}. \end{cases}$$

Analogously it is proved that $\theta_x\xi_2\theta_x = \begin{cases} \theta_{x^m}, & x\xi_2 \in F[X], \\ \theta_{\overline{x^m}}, & x\xi_2 \in \overline{F[X]} \setminus \overline{X}. \end{cases}$

Consider following four cases.

Case 1. $x\xi_1, x\xi_2 \in F[X]$. Using that Ψ is stable, we have

$$\theta_{x^m}\Psi = (\theta_x\xi_1\theta_x)\Psi = \theta_x(\xi_1\Psi)\theta_x = \begin{cases} \theta_{x^r}, & x(\xi_1\Psi) \in F[X], \\ \theta_{\overline{x^r}}, & x(\xi_1\Psi) \in \overline{F[X]} \setminus \overline{X}, \end{cases}$$

$$\theta_{x^m}\Psi = (\theta_x\xi_2\theta_x)\Psi = \theta_x(\xi_2\Psi)\theta_x = \begin{cases} \theta_{x^s}, & x(\xi_2\Psi) \in F[X], \\ \theta_{\overline{x^s}}, & x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}. \end{cases}$$

If $x(\xi_1\Psi) \in F[X], x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}$ or $x(\xi_1\Psi) \in \overline{F[X]} \setminus \overline{X}, x(\xi_2\Psi) \in F[X]$, then we obtain $\theta_{x^r} = \theta_{\overline{x^s}}$ or $\theta_{\overline{x^r}} = \theta_{x^s}$ which is false. Otherwise, we have $r = s$.

Case 2. $x\xi_1, x\xi_2 \in \overline{F[X]} \setminus \overline{X}$. It is similar to the case 1.

Case 3. $x\xi_1 \in F[X]$, $x\xi_2 \in \overline{F[X]} \setminus \overline{X}$. Assume that $\theta_{x^m}\Psi = \theta_{x^r}$, then by (ii) of Lemma 8 we have $\theta_{\overline{x^m}}\Psi = \theta_{\overline{x^r}}$. On the other hand,

$$\theta_{\overline{x^m}}\Psi = (\theta_x \xi_2 \theta_x)\Psi = \theta_x(\xi_2\Psi)\theta_x = \begin{cases} \theta_{x^s}, & x(\xi_2\Psi) \in F[X], \\ \theta_{\overline{x^s}}, & x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}. \end{cases}$$

For $x(\xi_2\Psi) \in F[X]$ we obtain $\overline{x^r} = x^s$ which is false. If $x(\xi_2\Psi) \in \overline{F[X]} \setminus \overline{X}$, then $\theta_{\overline{x^r}} = \theta_{\overline{x^s}}$, whence $r = s$.

In similar way we can show that $r = s$ if $\theta_{x^m}\Psi = \theta_{\overline{x^r}}$.

Case 4. $x\xi_1 \in \overline{F[X]} \setminus \overline{X}$, $x\xi_2 \in F[X]$. It is analogous to the case 3.

Thus, cases 1–4 imply that r and s coincides.

Further, let A be a nonempty finite subset of X and

$$\mathrm{End}_A^m(x) = \{\xi \in \mathrm{End}(\mathfrak{FCD}_X^g) \mid l(x\xi) = m, c(x\xi) = A\}.$$

For $\theta_{x\xi} \in \mathrm{End}_A^m(x)$ by (i) of Lemma 8 we have $\theta_{x\xi}\Psi = \theta_{x(\xi\Psi)}$. By (ii) of given lemma, $c(x\xi) = c(x(\xi\Psi))$. Taking into account the previous arguments, there exists k such that $\mathrm{End}_A^m(x)\Psi \subseteq \mathrm{End}_A^k(x)$. Since Ψ is bijective, $k = m$. Thus, $l(x\xi) = l(x(\xi\Psi))$ for all $\xi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ and $x \in X$. □

Corollary 1. *Let Ψ be a stable automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$ and $x_1, x_2 \in X$ are distinct. Then*

$$\theta_{x_1x_2}\Psi = \theta_{x_1x_2} \ \text{or}\ \theta_{x_1x_2}\Psi = \theta_{\overline{x_1x_2}}.$$

Proof. By Lemma 8 (i), $\theta_{x_1x_2}\Psi = \theta_u$ for some $u \in \mathfrak{FCD}_X^g$, and by (ii) of Lemma 9, $c(u) = \{x_1, x_2\}$. By (iii) of Lemma 9, $l(u) = 2$. Thus, $u = x_1x_2$ or $u = \overline{x_1x_2}$. □

Lemma 10. *Let Ψ be a stable automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$ and $x_1, x_2 \in X$ are distinct. Then*

(i) $\theta_{x_1x_2}\Psi = \theta_{x_1x_2}$ *implies* $\Psi = \Phi_0$*;*

(ii) $\theta_{x_1x_2}\Psi = \theta_{\overline{x_1x_2}}$ *implies* $\Psi = \Phi_1$.

Proof. (i) By induction on the length of u we show that $\theta_u\Psi = \theta_u$ for all $u \in F[X]$. By stability of Ψ, $\theta_x\Psi = \theta_x$ for all $x \in X$. Assume that $\theta_v\Psi = \theta_v$ for all $v \in F[X]$ with $l(v) < n$, and let $u = u_1 \dots u_n \in F[X]$, where $n \geqslant 2$. Let $v_1 = u_1 \dots u_{n-1}$, $v_2 = u_n$ and $f \in \mathrm{End}(\mathfrak{FCD}_X^g)$ is such that $x_1 f = v_1$, $x_2 f = v_2$ and $yf = y$ for all $y \in X \setminus \{x_1, x_2\}$. Then $x(\theta_{x_1x_2} f) = (x_1x_2)f = x_1 f x_2 f = u = x\theta_u$ for all $x \in X$.

By Lemma 8 (i) and the induction hypothesis, we have

$$\theta_{x_i(f\Psi)} = \theta_{x_i f}\Psi = \theta_{v_i}\Psi = \theta_{v_i} = \theta_{x_i f}, \quad i \in \{1,2\},$$
$$\theta_{x(f\Psi)} = \theta_{xf}\Psi = \theta_x\Psi = \theta_x = \theta_{xf}, \quad x \in X \setminus \{x_1, x_2\}.$$

So, $f\Psi = f$ and then for all $u \in F[X]$ with $l(u) \geqslant 2$,

$$\theta_u\Psi = (\theta_{x_1x_2}f)\Psi = (\theta_{x_1x_2}\Psi)(f\Psi) = \theta_{x_1x_2}f = \theta_u.$$

By (ii) of Lemma 8, $\theta_{\overline{u}}\Psi = \theta_{\overline{u}}$ for all $\overline{u} \in \overline{F[X]} \setminus \overline{X}$, so that $\theta_u\Psi = \theta_u$ for all $u \in \mathfrak{FCD}_X^g$. Now for all $x \in X$ and $\varphi \in \mathrm{End}(\mathfrak{FCD}_X^g)$,

$$\theta_{x(\varphi\Psi)} = \theta_{x\varphi}\Psi = \theta_{x\varphi}.$$

This implies $\varphi\Psi = \varphi$ for all $\varphi \in \mathrm{End}(\mathfrak{FCD}_X^g)$, that is, $\Psi = \Phi_0$.

(ii) Take the crossed automorphism $\varepsilon^*_{id_X}$ of $\mathfrak{FCD}_X^g$ (see Lemma 2). For all $u \in \mathfrak{FCD}_X^g$ and $f \in \mathrm{End}(\mathfrak{FCD}_X^g)$ we use denotations $u^* = u\varepsilon^*_{id_X}$ and $f^* = (\varepsilon^*_{id_X})^{-1}f\varepsilon^*_{id_X}$.

By induction on $l(u)$ we show that $\theta_u\Psi = \theta_{u^*}$ for all $u \in F[X]$. The induction base follows from the fact that Ψ is stable.

Let us suppose that $\theta_v\Psi = \theta_{v^*}$ for all $v \in F[X]$ such that $l(v) < n$, and let $u = u_1 \dots u_n \in F[X], n \geqslant 2$. We put $v_1 = u_1, v_2 = u_2 \dots u_n$, and take the endomorphism f of $\mathfrak{FCD}_X^g$ such that $x_1 f = v_1$, $x_2 f = v_2$, and $yf = y$ for all $y \in X \setminus \{x_1, x_2\}$.

Similarly as in (i) of this lemma, we can show that $\theta_{x_1x_2}f = \theta_u$. By Lemma 8 (i) and the induction hypothesis,

$$\theta_{x_i(f\Psi)} = \theta_{x_i f}\Psi = \theta_{v_i}\Psi = \theta_{v_i^*} = \theta_{x_i f^*}, \quad i \in \{1,2\},$$
$$\theta_{x(f\Psi)} = \theta_{xf}\Psi = \theta_x\Psi = \theta_{x^*} = \theta_{xf^*}, \quad x \in X \setminus \{x_1, x_2\}.$$

From here, $f\Psi = f^*$. Then for all $u \in F[X]$ with $l(u) \geqslant 2$,

$$\theta_u\Psi = (\theta_{x_1x_2}f)\Psi = (\theta_{x_1x_2}\Psi)(f\Psi) = \theta_{\overline{x_1x_2}}f^* = \theta_{\overline{u}} = \theta_{u^*}.$$

Taking into account Lemma 8 (iii), $\theta_{\overline{u}}\Psi = \theta_u$ for all $\overline{u} \in \overline{F[X]} \setminus \overline{X}$. It means that $\theta_u\Psi = \theta_{u^*}$ for all $u \in \mathfrak{FCD}_X^g$.

Finally, for all $x \in X$ and $\varphi \in \mathrm{End}(\mathfrak{FCD}_X^g)$ we have

$$\theta_{x(\varphi\Psi)} = \theta_{x\varphi}\Psi = \theta_{(x\varphi)^*} = \theta_{x\varphi^*}.$$

Hence, $\varphi\Psi = \varphi^*$ for all $\varphi \in \mathrm{End}(\mathfrak{FCD}_X^g)$, that is, $\Psi = \Phi_1$. $\square$

Theorem 2. *Let X be an arbitrary set with $|X| \geqslant 2$. Every isomorphism $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_Y^g)$ is induced either by the isomorphism ε_f or by the crossed isomorphism ε_f^* of $\mathfrak{FCD}_X^g$ onto $\mathfrak{FCD}_Y^g$ for a uniquely determined bijection $f : X \to Y$.*

Proof. Let $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_Y^g)$ be an arbitrary isomorphism. In similar way as in the case of free abelian dimonoids (see [19, Theorem 3]), using Lemma 4 can be shown that for every $x \in X$ there exists $y \in Y$ such that $\theta_x\Phi = \theta_y$. Define a bijection $f : X \to Y$ putting $xf = y$ if $\theta_x\Phi = \theta_y$. In this case we say that f is induced by Φ.

By Lemma 1, f induces the isomorphism $\varepsilon_f : \mathfrak{FCD}_X^g \to \mathfrak{FCD}_Y^g$. According to Lemma 3, $E_f : \eta \mapsto \varepsilon_f^{-1}\eta\varepsilon_f$ is an isomorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$ onto $\mathrm{End}(\mathfrak{FCD}_Y^g)$. From this it follows that the composition ΦE_f^{-1} is an automorphism of $\mathrm{End}(\mathfrak{FCD}_X^g)$.

Further for all $x \in X$ we have

$$\theta_x(\Phi E_f^{-1}) = (\theta_x\Phi)E_f^{-1} = \theta_{xf}E_f^{-1} = \theta_{(xf)f^{-1}} = \theta_x,$$

which implies stability of ΦE_f^{-1}.

Using Corollary 1 and Lemma 10, we obtain ΦE_f^{-1} is either the identity automorphism Φ_0 or the mirror automorphism Φ_1. Assume, $\Phi E_f^{-1} = \Phi_0$, then $\Phi = E_f$ which means that Φ is an isomorphism induced by ε_f. If $\Phi E_f^{-1} = \Phi_1$, then $\Phi = \Phi_1 E_f$ which means that Φ is an isomorphism induced by ε_f^*. □

The following statement gives the positive solution of the definability problem of free commutative g-dimonoids by its endomorphism semigroups.

Corollary 2. *Let $\mathfrak{FCD}_X^g$ and $\mathfrak{FCD}_Y^g$ be free commutative g-dimonoids such that $\mathrm{End}(\mathfrak{FCD}_X^g) \cong \mathrm{End}(\mathfrak{FCD}_Y^g)$. Then $\mathfrak{FCD}_X^g$ and $\mathfrak{FCD}_Y^g$ are isomorphic.*

Proof. As shown in the proof of Theorem 2, every isomorphism $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_Y^g)$ induces a bijection $X \to Y$, therefore obviously we obtain $\mathfrak{FCD}_X^g \cong \mathfrak{FCD}_Y^g$. □

We recall that an automorphism $\Phi : \mathrm{End}(\mathfrak{FCD}_X^g) \to \mathrm{End}(\mathfrak{FCD}_X^g)$ is quasi-inner if there exists $\alpha \in S(\mathfrak{FCD}_X^g)$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in \mathrm{End}(\mathfrak{FCD}_X^g)$.

At the end we consider the automorphism group of $\mathrm{End}(\mathfrak{FCD}_X^g)$.

Theorem 3. *Let X be an arbitrary set with $|X| \geqslant 2$. Then*

(i) *all automorphisms of* $\mathrm{End}(\mathfrak{FCD}^g_X)$ *are quasi-inner;*

(ii) *the automorphism group* $\mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}^g_X))$ *is isomorphic to the direct product* $S(X) \times C_2$.

Proof. (i) Let $X = Y$ in Theorem 2, then it will be the part of Theorem 3. It is not hard to see that every automorphism Φ of $\mathrm{End}(\mathfrak{FCD}^g_X)$ is either an inner automorphism or the product of a mirror automorphism and an inner automorphism. Namely, we have $\Phi = E_\varphi$ or $\Phi = \Phi_1 E_\varphi$ for a suitable bijection $\varphi : X \to X$. It means that all automorphisms of $\mathrm{End}(\mathfrak{FCD}^g_X)$ are quasi-inner.

(ii) It is clear that the automorphism group $\{\Phi_0, \Phi_1\}$ of $\mathrm{End}(\mathfrak{FCD}^g_X)$ is isomorphic to C_2. Define a mapping $\zeta : \mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}^g_X)) \to S(X) \times C_2$ as follows:

$$\Phi\zeta = \begin{cases} (\varphi, \Phi_0), & \Phi = E_\varphi, \\ (\varphi, \Phi_1), & \Phi = \Phi_1 E_\varphi \end{cases}$$

for all $\Phi \in \mathrm{Aut}(\mathrm{End}(\mathfrak{FCD}^g_X))$.

It is easy to see that ζ is a bijection. Since for all $\varphi, \psi \in S(X)$ and $f \in \mathrm{End}(\mathfrak{FCD}^g_X)$,

$$\begin{aligned} f(E_\varphi E_\psi) &= (\varepsilon_\varphi^{-1} f \varepsilon_\varphi) E_\psi = (\varepsilon_\varphi \varepsilon_\psi)^{-1} f (\varepsilon_\varphi \varepsilon_\psi) \\ &= \varepsilon_{\varphi\psi}^{-1} f \varepsilon_{\varphi\psi} = f E_{\varphi\psi} \end{aligned}$$

and

$$\begin{aligned} f(E_\varphi \Phi_1) &= (\varepsilon_\varphi^{-1} f \varepsilon_\varphi)\Phi_1 = (\varepsilon^*_{id_X} \varepsilon_\varphi^{-1}) f (\varepsilon_\varphi \varepsilon^*_{id_X}) \\ &= (\varepsilon_\varphi^{-1} \varepsilon^*_{id_X}) f (\varepsilon^*_{id_X} \varepsilon_\varphi) = (\varepsilon^*_{id_X} f \varepsilon^*_{id_X}) E_\varphi = f(\Phi_1 E_\varphi), \end{aligned}$$

we obtain $E_\varphi E_\psi = E_{\varphi\psi}$ and $E_\varphi \Phi_1 = \Phi_1 E_\varphi$.

The immediate check shows that ζ is a homomorphism. □

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