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Alfred Lvovich Shmelkin

(12.06.1938 – 22.12.2015)

With deep sorrow and regret we learnt about the death on December 22, 2015 of our colleague, Alfred Lvovich Shmelkin, a prominent algebraist, Honoured Professor of Lomonosov Moscow State University and member of ADM editorial board over the past thirteen years.

Alfred L. Shmelkin was born on June 12, 1938 in Moscow. In 1956 he began his studies at Moscow State University. He graduated in 1961 from the Department of Mathematics and Mechanics. For the next three years he was a PhD student of J.N. Golovin. In 1964 A.L. Shmelkin defended his PhD thesis in Algebra and three years after, in 1967, he got a degree of Doctor of Science for his dissertation entitled “Products of group varieties”. In 1972, by the age of 32, he already became a full Professor at the Department of Higher Algebra of the Lomonosov Moscow State University.

Professor Shmelkin devoted his life to science and teaching. As the founder and leader of the world famous school on group varieties, he made a great contribution to the theory of group varieties and infinite dimensional Lie algebras. His main results, the famous Shmelkin-Neumann Theorem, the construction of verbal wreath products and group varieties of Lie type have found many applications and are widely used by many algebraists. Known for his outstanding teaching abilities, Professor Shmelkin shared his knowledge as a supervisor to 25 PhD students. Many of them became recognizable mathematicians: A. Olshanskii, Yu. Bakhturin, Yu. Razmyslov, A. Krasilnikov, V. Spilrain, R. Stohr, holding the degree of Doctor of Science.

We shall remember Professor Shmelkin as a talented mathematician, a mentor, and a kind, always ready to help, warm-hearted man. He will be greatly missed by all of us.

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Classification of $\mathscr{L}$-cross-sections of the finite symmetric semigroup up to isomorphism

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Communicated by V. Mazorchuk

Abstract. Let $\mathcal{T}_n$ be the symmetric semigroup of full transformations on a finite set with n elements. In the paper we give a counting formula for the number of $\mathscr{L}$-cross-sections of $\mathcal{T}_n$ and classify all $\mathscr{L}$-cross-sections of $\mathcal{T}_n$ up to isomorphism.

Introduction

Let ρ be an equivalence relation on a semigroup S. A subsemigroup S' of S is called a ρ-cross-section of S provided that S' contains exactly one representative from each equivalence class of ρ. Thus, the restriction ρ to the subsemigroup S' is the identity relation. It is natural to investigate the cross-sections with respect to equivalences related somehow to the semigroup operation: Green's relations, conjugacy and various congruences. In general, a semigroup need not to have a ρ-cross-section. It is possible, for example, that a semigroup S has an $\mathscr{R}$-cross-section, while $\mathscr{L}$-cross-sections of S do not exist at all. Thus, the existence of cross-sections of a given semigroup is an essential and non-obvious fact.

The transformation semigroups are classical objects for investigations in semigroup theory (see [1]). For the full finite symmetric semigroup $\mathcal{T}_n$, all $\mathscr{H}$- and $\mathscr{R}$-cross-sections have been described in [3]. It has been proved

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that there exists a unique $\mathscr{R}$-cross-section up to isomorphism. A pair of non-isomorphic $\mathscr{L}$-cross-sections of $\mathscr{T}_4$ has been constructed in [4]. The author has obtained a description of the $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ in [5] (see Theorem 1).

In the present paper we continue to investigate $\mathscr{L}$-cross-sections of $\mathscr{T}_n$. We give necessary information in Section 1. Section 2 is devoted to some additional definitions. In Section 3 we show how to count all different $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ (Theorem 2). In Section 4 we classify all $\mathscr{L}$-cross-sections up to isomorphism (Theorem 3).

1. Preliminaries

For any nonempty set X, the set of all transformations of X into itself, written on the right, constitutes a semigroup under the composition $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in X$. This semigroup is denoted by $\mathscr{T}(X)$ and called the symmetric semigroup. If $|X| = n$, then the symmetric semigroup $\mathscr{T}(X)$ is also denoted by $\mathscr{T}_n$. We write id_X for the identity transformation on X, and c_x for the constant transformation whose image is the singleton $\{x\}$, $x \in X$. For the image of a transformation $\alpha \in \mathscr{T}_n$ we write $\mathrm{im}\,(\alpha)$. The cardinality $|\,\mathrm{im}\,(\alpha)|$ of the image of α is called the rank of this transformation and is denoted by $\mathrm{rk}\,(\alpha)$. The kernel of α is denoted by $\ker\alpha$. Recall that $\ker\alpha = \{(a,b) \in X \times X \mid a\alpha = b\alpha\}$. If X' is a subset of X, then $\alpha|_{X'}$ is the restriction α to X'. We will assume X is finite. As the nature of elements of X is not important for us, suppose further that $X = \{1, 2, \ldots, n\}$.

We recall that two elements in a semigroup S are called $\mathscr{L}$-equivalent provided that they generate the same principal left ideal in S. Transformations $\alpha, \beta \in \mathscr{T}_n$ are $\mathscr{L}$-equivalent if and only if $\mathrm{im}\,(\alpha) = \mathrm{im}\,(\beta)$ (see e.g. [2]). The last means that an $\mathscr{L}$-cross-section of $\mathscr{T}_n$ contains exactly one transformation with the image M for each nonempty $M \subseteq X$. We will use the last fact frequently. Suppose further that L is an $\mathscr{L}$-cross-section in $\mathscr{T}_n$.

First we isolate two trivial cases:

(i) $L = \{c_1 = \mathrm{id}_X\}$, if $n = 1$;

(ii) $L = \{\mathrm{id}_X, c_1, c_2\}$, if $n = 2$.

For the rest of the paper we may and will assume that $n \geqslant 3$.

In order to present our description of $\mathscr{L}$-cross-sections for an arbitrary finite $\mathscr{T}_n$ [5], we need following definitions.

Let X be a nonempty finite set and let $<$ be a strict total order on X. We define a strict order $\prec$ on the family of all nonempty subsets of X by: $A \prec B$ if for all $a \in A$ and all $b \in B$, $a < b$.

Denote by $\{1,2\}^+$ the free semigroup of words over the alphabet $\{1,2\}$, and by $\{1,2\}^*$ the free monoid over $\{1,2\}$, with 0 as the empty word. Recall, that *a subsequence* of $b \in \{1,2\}^*$ is a word a that can be derived from b by deleting some symbols without changing the order of the remaining symbols. If a is a subsequence of b we will write $a \subseteq b$.

Definition 1. Let X be a finite set (possibly empty) and let $<$ be a strict total order on X. An indexed family $\{A_a\}_{a\in\{1,2\}^*}$ of subsets of X is called a Γ-*family* over $(X,<)$ if for every $a \in \{1,2\}^*$:

(a) $A_0 = X$;
(b) if $|A_a| \leqslant 1$, then $A_{a1} = A_{a2} = \varnothing$;
(c) if $|A_a| > 1$, then A_{a1} and A_{a2} are nonempty with $A_{a1} \prec A_{a2}$ and $A_a = A_{a1} \cup A_{a2}$.

We will say that $\{A_a\}_{a\in\{1,2\}^*}$ is a Γ-family over X if $\{A_a\}_{a\in\{1,2\}^*}$ is a Γ-family over $(X,<)$ for some strict total order $<$ on X (necessarily unique). For simplicity, we will write $\Gamma = \{A_a\}$ instead of $\Gamma = \{A_a\}_{a\in\{1,2\}^*}$.

Recall that a tree is a connected graph without cycles. A full binary tree is defined as a tree in which there is exactly one vertex of degree two (referred to as the root) and each of the remaining vertices is of degree one or three. Vertices of degree one are called leaves. Each vertex except the root has a unique parent, that is, the vertex connected to it on the path to the root. A child of a vertex v is a vertex of which v is the parent. Thus, in a full binary tree each vertex v either is a leaf or has exactly two children that we refer to as the left child of v and the right child of v.

It is easy to see that every Γ-family $\Gamma = \{A_a\}$ over a nonempty set can be represented by a rooted full binary tree $T(\Gamma)$ whose vertices are the nonempty sets from $\{A_a\}$ and a pair $\{A_a, A_b\}$, for $a, b \in \{1,2\}^*$, is an edge if and only if $a = bi$ or $b = ai$, where $i \in \{1,2\}$ (see Fig. 1). For the full binary tree that represents a Γ-family Γ, we will write Γ instead of $T(\Gamma)$, and refer to the tree as a Γ-tree.

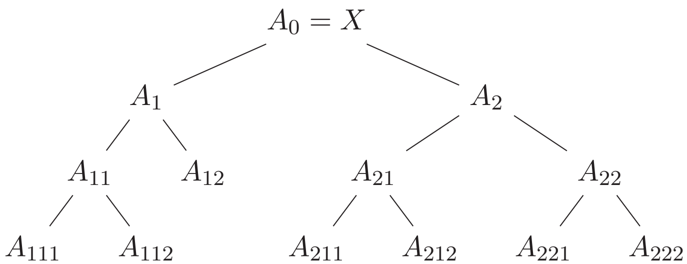

FIGURE 1. A Γ-tree.

Definition 2. A Γ-family $\Gamma = \{A_a\}$ over $(X, <)$ is called an *L-family* over $(X, <)$ if for all $a, b \in \{1, 2\}^*$ and all $i, j \in \{1, 2\}$ with $i \neq j$,

$$|A_{aijb}| \leqslant |A_{ajb}|. \tag{1}$$

We will say that $\{A_a\}_{a\in\{1,2\}^*}$ is an *L-family* over X if $\{A_a\}_{a\in\{1,2\}^*}$ is an L-family over $(X, <)$ for some strict total order $<$ on X.

Example 1. Let $\{1, 2, 3, 4, 5\}$ be naturally ordered. Consider the following Γ-family $\{A_a\}$ (see Fig. 2).

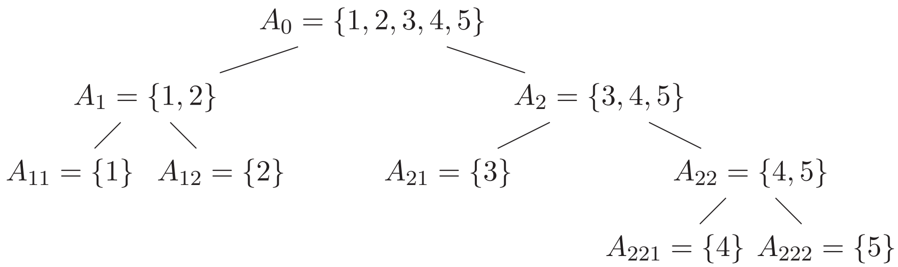

FIGURE 2. Γ-family $\{A_a\}$.

This Γ-family satisfies condition (2) for all $a, b \in \{1, 2\}^*$ and all $i, j \in \{1, 2\}$ with $i \neq j$, hence $\{A_a\}$ is an L-family by definition.

Figure 3 shows a Γ-family $\{B_a\}$ that does not satisfy condition (2) since $|B_{21}| \geqslant |B_1|$.

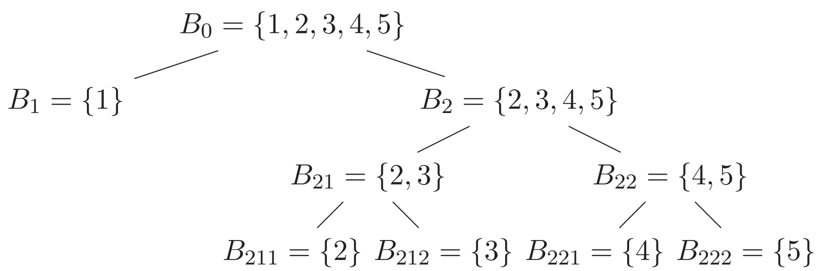

FIGURE 3. Γ-family $\{B_a\}$.

Let Γ be an L-family of subsets of X, $M \subseteq X$ and $M \neq \varnothing$. Our aim now is to construct a map $\alpha_M^{A_a} : A_a \to M$ with $\mathrm{im}\,(\alpha_M^{A_a}) = M$. We construct this map inductively using partial transformations, whose domains go through vertices of a Γ-tree bottom up. For the domain of a partial transformation f we write $\mathrm{dom}\,(f)$.

For functions f and g with disjoint domains, we denote by $f \cup g$ the union of f and g (viewed as sets of pairs). In other words, if $h = f \cup g$, then $\operatorname{dom}(h) = \operatorname{dom}(f) \cup \operatorname{dom}(g)$ and for all $x \in \operatorname{dom}(h)$, $xh = xf$ if $x \in \operatorname{dom}(f)$, and $xh = xg$ if $x \in \operatorname{dom}(g)$.

Definition 3. Let $\Gamma = \{A_a\}$ be an L-family over X and let $M \subseteq X$ with $M \neq \varnothing$. Denote by $\langle M \rangle$ the intersection of all $A_c \in \Gamma$ such that $M \subseteq A_c$, and note that $\langle M \rangle = A_b$ for some $b \in \{1,2\}^*$. For every $a \in \{1,2\}^*$, we define the *mapping* $\alpha_M^{A_a}$ inductively as follows:

(a) if $A_a = \varnothing$ then $\alpha_M^{A_a} = \varnothing$ (empty mapping);
(b) if $M = \{m\}$ and $A_a \neq \varnothing$, then $\operatorname{dom}(x\alpha_M^{A_a}) = A_a$ and $x\alpha_M^{A_a} = m$ for every $x \in A_a$;
(c) if $|M| > 1$ and $A_a \neq \varnothing$, then $\alpha_M^{A_a} = \alpha_{M \cap A_{b1}}^{A_{a1}} \cup \alpha_{M \cap A_{b2}}^{A_{a2}}$.

Lemma 1. *Let $\Gamma = \{A_a\}$ be an L-family over X. If $M \subseteq A_a$ or $A_a \neq \varnothing$ and $M \cap A_a = \varnothing$ then* $\operatorname{dom}(x\alpha_M^{A_a}) = A_a$ *and* $\operatorname{im}(x\alpha_M^{A_a}) = M$.

Proof. The proof is by induction on $|M|$. If $M = \{m\}$, then the statement is true by (b) of Definition 3. Let $|M| > 1$ and suppose the statement is true for every M' with $1 \leqslant |M'| < |M|$. Assume $M \subseteq A_a$ or $A_a \neq \varnothing$ and $M \cap A_a = \varnothing$. By (c) of Definition 3, $\alpha_M^{A_a} = \alpha_{M \cap A_{b1}}^{A_{a1}} \cup \alpha_{M \cap A_{b2}}^{A_{a2}}$. Consider two possible cases.

Case 1. $M \subseteq A_a$. Then $A_b \subseteq A_a$ since A_b is the intersection of all A_c such that $M \subseteq A_c$. If $A_b = A_a$ then

$$M \cap A_{b1} = M \cap A_{a1} \subseteq A_{a1},$$
$$M \cap A_{b2} = M \cap A_{a2} \subseteq A_{a2},$$

and $|M \cap A_{b1}|$, $|M \cap A_{b2}| < |M|$ (since $\langle M \rangle = A_b$). Thus, by the inductive hypothesis, the statement is true for $\alpha_{M \cap A_{b1}}^{A_{a1}}$ and for $\alpha_{M \cap A_{b2}}^{A_{a2}}$. Hence it is true for $\alpha_M^{A_a}$.

If $A_b \neq A_a$ then, since $A_a = A_{a1} \cup A_{a2}$ and $A_{a1} \cap A_{a2} = \varnothing$, we get either $A_b \subseteq A_{a1}$ or $A_b \subseteq A_{a2}$. We may assume that $A_b \subseteq A_{a1}$. Then

$$M \cap A_{b1} \subseteq A_{a1},$$
$$(M \cap A_{b2}) \cap A_{a2} = \varnothing.$$

Note that $A_{a2} \neq \varnothing$ (since $M \subseteq A_a$ and $|M| > 1$) and $|M \cap A_{b1}|$, $|M \cap A_{b2}| < |M|$ (since $\langle M \rangle = A_b$). Again, the statement follows by the inductive hypothesis from $\alpha_M^{A_a} = \alpha_{M \cap A_{b1}}^{A_{a1}} \cup \alpha_{M \cap A_{b2}}^{A_{a2}}$.

Case 2. $A_a \neq \varnothing$ and $M \cap A_a = \varnothing$. Then $(M \cap A_{b1}) \cap A_{a1} = \varnothing$ and $(M \cap A_{b2}) \cap A_{a2} = \varnothing$. As before, we get the statement by the inductive hypothesis from $\alpha_M^{A_a} = \alpha_{M \cap A_{b1}}^{A_{a1}} \cup \alpha_{M \cap A_{b2}}^{A_{a2}}$. □

Denote by L_X^Γ the set of all transformations of the form α_M^X, where $M \subseteq X$, $M \neq \varnothing$. We will denote the elements α_M^X also by α_M.

Example 2. Let $\{1,2,3,4,5\}$ be naturally ordered. We will construct the transformation $\alpha = \alpha_M$ with $M = \{1,2,4,5\}$ for the L-family $\{A_a\}$ from Example 1. Clearly, $\langle M \rangle = A_0$, so by definition of α_M

$$\alpha = \alpha_M^{A_0} = \alpha_{M\cap A_1}^{A_1} \cup \alpha_{M\cap A_2}^{A_2} = \alpha_{\{1,2\}}^{A_1} \cup \alpha_{\{4,5\}}^{A_2}.$$

Since $\langle\{1,2\}\rangle = A_1$, $\langle\{4,5\}\rangle = A_{22}$, thus

$$\alpha_{\{1,2\}}^{A_1} = \alpha_{\{1,2\}\cap A_{11}}^{A_{11}} \cup \alpha_{\{1,2\}\cap A_{12}}^{A_{12}} = \alpha_{\{1\}}^{A_{11}} \cup \alpha_{\{2\}}^{A_{12}},$$

$$\alpha_{\{4,5\}}^{A_2} = \alpha_{\{4,5\}\cap A_{221}}^{A_{21}} \cup \alpha_{\{4,5\}\cap A_{222}}^{A_{22}} = \alpha_{\{4\}}^{A_{21}} \cup \alpha_{\{5\}}^{A_{22}}.$$

Thus, since $A_{11} = \{1\}$, $A_{12} = \{2\}$, $A_{21} = \{3\}$, and $A_{22} = \{4,5\}$, we have

$$\alpha = \alpha_{\{1\}}^{A_{11}} \cup \alpha_{\{2\}}^{A_{12}} \cup \alpha_{\{4\}}^{A_{21}} \cup \alpha_{\{5\}}^{A_{22}} = \begin{pmatrix} 12345 \\ 12455 \end{pmatrix}.$$

The other transformations from L_X^Γ can be obtained in the same way (see [5, Example 3]).

The following theorem describes the $\mathscr{L}$-cross-sections of $\mathscr{T}_n$:

Theorem 1 ([5, Theorem 1]). *For each L-family Γ of X, the set L_X^Γ is an $\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$. Conversely, every $\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$ is given by L_X^Γ for a suitable L-family Γ on X.*

2. Alternative definition of L-family

Since the definition of an L-family may seem difficult to use and understand, we try to find a way to make it easy and more visual. We state a new definition in Proposition 1. But first we need some preparation.

Definition 4. Let Γ_1, Γ_2 be the full binary trees that represent Γ-families $\{A_a\}$ over X_1 and $\{B_a\}$ over X_2 respectively. We say that Γ_1 *is less than or equal to* Γ_2, written $\Gamma_1 \leqslant \Gamma_2$, if $|A_a| \leqslant |B_a|$ for all $a \in \{1,2\}^*$.

Let $\Gamma = \{A_a\}_{a\in\{1,2\}^*}$ be a Γ-family over X. For every $a \in \{1,2\}^*$, denote by $\Gamma(a)$ the family $\{B_b\}_{b\in\{1,2\}^*}$ of subsets of A_a such that $B_b = A_{ab}$ for each $b \in \{1,2\}^*$. It is clear that $\Gamma(a)$ is a Γ-family over the set A_a and that, if $A_a \neq \varnothing$, then $\Gamma(a)$ is represented by the subtree $\Gamma(a)$ of the full binary tree Γ with the root A_a.

Definition 5. Let Γ be a Γ-tree. For all $a \in \{1,2\}^*$ and $i \in \{1,2\}$, we call the tree $\Gamma(a)$ the *parent tree* of the subtree $\Gamma(ai)$. We will say that Γ is *monotone* if for all $a \in \{1,2\}^*$ and $i \in \{1,2\}$, $\Gamma(ai) \leqslant \Gamma(a)$.

Proposition 1. *A Γ-tree Γ with root X represents an L-family over X if and only if Γ is monotone.*

Proof. Necessity. Suppose that a Γ-tree Γ with root X represents an L-family over X and let $a \in \{1,2\}^*$. We aim to prove that $\Gamma(a1) \leqslant \Gamma(a)$. If $|A_{a1}| = 1$, then it is clear that $\Gamma(a1) \leqslant \Gamma(a)$. Let $|A_{a1}| > 1$. To prove $\Gamma(a1) \leqslant \Gamma(a)$ we show first $\Gamma(a12) \leqslant \Gamma(a2)$ and then $\Gamma(a11) \leqslant \Gamma(a1)$.

Let $\Gamma(a12)$ represent $\{B_b\}$ and let $\Gamma(a2)$ represent $\{C_b\}$. Then, for every $b \in \{1,2\}^*$,

$$|B_b| = |A_{a12b}| \leqslant |A_{a2b}| = |C_b|,$$

where $\leqslant$ follows by (2). Thus, $\Gamma(a12) \leqslant \Gamma(a2)$.

To prove that $\Gamma(a11) \leqslant \Gamma(a1)$, denote by $\{B_b\}$ and $\{C_b\}$ the L-families that are represented by $\Gamma(a11)$ and $\Gamma(a1)$, respectively. Denote by $\overline{k}$, $k \geqslant 0$, the empty word 0 if $k = 0$ and $\underbrace{11\ldots1}_{k} \in \{1,2\}^*$ if $k \geqslant 1$. Then, for every $b \in \{1,2\}^*$, if $b = \overline{k}$, $k \geqslant 0$, then

$$|B_b| = |A_{a11\overline{k}}| \leqslant |A_{a1\overline{k}}| = |C_b|,$$

since $A_{a11\overline{k}} \subset A_{a1\overline{k}}$; and if $b = \overline{k}2c$ ($k \geqslant 0,\ c \in \{1,2\}^*$), then

$$|B_b| = |A_{a11\overline{k}2c}| \leqslant |A_{a1\overline{k}2c}| = |C_b|,$$

where $\leqslant$ follows by (2).

Now, since $|A_{a1}| < |A_a|$, $\Gamma(a11) \leqslant \Gamma(a1)$ and $\Gamma(a12) \leqslant \Gamma(a2)$, we get $\Gamma(a1) \leqslant \Gamma(a)$. In dual way, one can show that $\Gamma(a2) \leqslant \Gamma(a)$. So any subtree of Γ is less than or equal to the parent tree of this subtree, thus Γ is monotone.

Sufficiency. Let $a \in \{1,2\}^*$ and $i, j \in \{1,2\}$ with $i \neq j$. Let the subtrees $\Gamma(ai)$ and $\Gamma(a)$ of Γ represent $\{B_b\}_{b\in\{1,2\}^*}$ and $\{C_b\}_{b\in\{1,2\}^*}$, respectively. Since $\Gamma(ai) \leqslant \Gamma(a)$,

$$|A_{aijb}| = |B_{jb}| \leqslant |C_{jb}| = |A_{ajb}|.$$

Hence (2) holds, that is, Γ is an L-family. □

Definition 6. For $n \in \mathbb{N}$ we will write Γ^n to mean an L-family over a set with n elements. Let $\Gamma^n = \{A_a\}$ be an L-family with $n \geqslant 2$. Let $s, t \in \{1, 2, \ldots, n\}$ with $s + t \leqslant n$. We denote by $Q_{s,t}$ the set of all pairs (Γ^s, Γ^t) of L-families Γ^s and Γ^t such that:

(a) $\Gamma^s = \Gamma^n(a)$ and $\Gamma^t = \Gamma^n(b)$ for $a, b \in \{1, 2\}^*$ such that $A_a \cap A_b = \varnothing$;

(b) if $s > 1$ then $\Gamma^s(2) \leqslant \Gamma^t$, and if $t > 1$ then $\Gamma^t(1) \leqslant \Gamma^s$.

Example 3. Figure 4 shows a pair of L-families (Γ^4, Γ^5) that does not belong to $Q_{4,5}$. To simplify the picture we denote the nodes of the trees by their cardinalities.

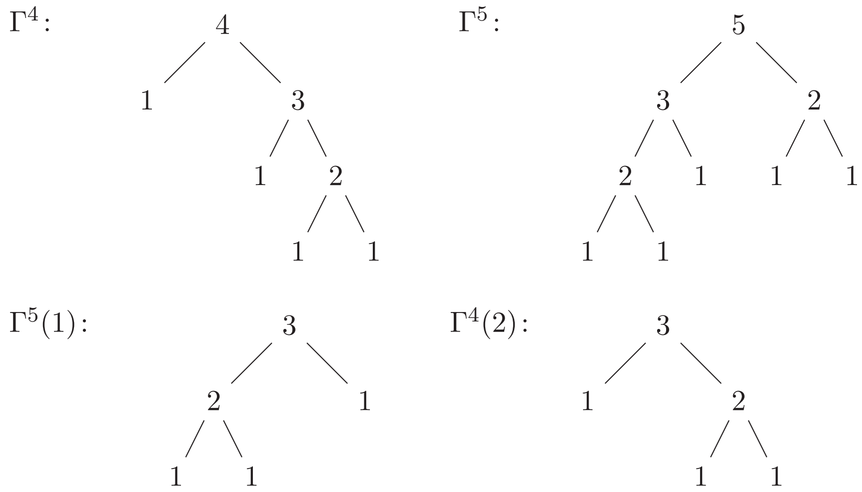

FIGURE 4. Γ^4 and Γ^5 such that $(\Gamma^4, \Gamma^5) \notin Q_{4,5}$.

As the picture shows, Γ^4 and $\Gamma^5(1)$ do not satisfy the condition $\Gamma^5(1) \leqslant \Gamma^4$ ($2 > 1$ in the first position). However, Γ^5 and $\Gamma^4(2)$ satisfy the condition $\Gamma^4(2) \leqslant \Gamma^5$.

Fix a total order $<$ on an n-element set X and denote by Q_n the number of L-families over X.

Proposition 2. *The number Q_n of all distinct L-families Γ on the totally ordered set $(X, <)$, with $|X| = n$, is given by the formula:*

$$Q_1 = 1, \quad Q_n = \sum_{\substack{s,t \\ s+t=n}} |Q_{s,t}| \quad \text{if} \quad n \geqslant 2.$$

Proof. Obviously, $Q_1 = 1$. Let $n \geqslant 2$. Let Γ^n be an L-family over $(X, <)$ and let $\Gamma^s = \Gamma^n(1)$ and $\Gamma^t = \Gamma^n(2)$. It is clear that $s + t = n$. Using

Proposition 1, we get $\Gamma^s \leqslant \Gamma^n$, $\Gamma^t \leqslant \Gamma^n$, whence $\Gamma^s(2) \leqslant \Gamma^t$, $\Gamma^t(1) \leqslant \Gamma^s$ and thus $(\Gamma^s, \Gamma^t) \in Q_{s,t}$. It is then clear that the mapping $\Gamma^n \to (\Gamma^{n(1)}, \Gamma^{n(2)})$ is a bijection from the set of L-families over $(X, <)$ onto the union of the sets $Q_{s,t}$ with $s + t = n$.

Second variant of proof, using the first definition of an L-family (condition (2)): Since Γ^n is an L-family, if $|A_1| > 1$ then $|A_{12b}| \leqslant |A_{2b}|$ for all $b \in \{1, 2\}^*$, therefore $\Gamma^s(2) \leqslant \Gamma^t$. Analogously we obtain $\Gamma^t(1) \leqslant \Gamma^s$.

Thus, $\Gamma^s, \Gamma^t \in Q_{s,t}$ and $Q_n = \sum_{s+t=n} |Q_{s,t}|$, for $n \geqslant 2$. □

We give the initial values of Q_n, $n \in \mathbb{N}$ below. To calculate them we have used a computer programm.

n	1	2	3	4	5	6	7	8	9	10
Q_n	1	1	2	3	6	10	18	32	58	101

3. The number of $\mathscr{L}$-cross-sections of $\mathscr{T}_n$

Suppose $a \in \{1, 2\}^*$ is an arbitrary word. The word obtained from a by replacing each 1 by 2 and each 2 by 1, is denoted by $\bar{a}$.

Definition 7. Let $\Gamma_1 = \{A_a\}$, $\Gamma_2 = \{B_a\}$ be L-families over X_1 and X_2, respectively. We say that Γ_1 and Γ_2 are *similar* if

$$\forall a \in \{1, 2\}^* \ |A_a| = |B_a| \ \text{ or } \ \forall a \in \{1, 2\}^* \ |A_a| = |B_{\bar{a}}|.$$

The similarity of L-families Γ_1 and Γ_2 is denoted by $\Gamma_1 \sim \Gamma_2$.

The relation of similarity is clearly an equivalence and partitions the set of all L-families over the n-element set into disjoint equivalence classes.

Lemma 2. *Let $<_1$, $<_2$ be strict total orders on X, $\Gamma_1 = \{A_a\}$, $\Gamma_2 = \{B_a\}$ be arbitrary L-families over $(X, <_1)$ and $(X, <_2)$, respectively. If $L_1 = L^{\Gamma_1}_{<_1}$, $L_2 = L^{\Gamma_2}_{<_2}$ are corresponding $\mathscr{L}$-cross-sections of $\mathscr{T}_n$, then $L_1 = L_2$ if and only if one of the following conditions is satisfied:*

(i) $\Gamma_1 = \Gamma_2$ *(i. e. $\Gamma_1 \sim \Gamma_2$ and $<_1 = <_2$);*

(ii) $\Gamma_1 \sim \Gamma_2$ *and* $<_2 = <_1^{-1}$.

Proof. Sufficiency. Obviously (i) implies $L_1 = L_2$. Suppose (ii) holds. Then $A_a = B_{\overline{a}}$ for all $a \in \{1, 2\}^*$. To prove that $L_1 = L_2$, it suffices to show that $\alpha_M^{A_a} = \alpha_M^{B_{\overline{a}}}$ for all $a \in \{1, 2\}^*$ and $M \subseteq X$ with $M \neq \varnothing$. We proceed by induction on $|M|$. Let $M = \{m\}$. If $A_a = \varnothing$, then $B_{\overline{a}} = A_a = \varnothing$, and so $\alpha_M^{A_a} = \varnothing = \alpha_M^{B_{\overline{a}}}$. If $A_a \neq \varnothing$ then $\mathrm{dom}\,(\alpha_M^{A_a}) = A_a = B_{\overline{a}} = \mathrm{dom}\,(\alpha_M^{B_{\overline{a}}})$ and for all x in the common domain, $x\alpha_M^{A_a} = m = x\alpha_M^{B_{\overline{a}}}$, which implies $\alpha_M^{A_a} = \alpha_M^{B_{\overline{a}}}$.

Let $|M| > 1$ and suppose that the statement is true for all $M_1 \subseteq X$ with $|M_1| < M$. Again, if $A_a = \varnothing$, then $B_{\overline{a}} = A_a = \varnothing$, and so $\alpha_M^{A_a} = \varnothing = \alpha_M^{B_{\overline{a}}}$. Suppose that $A_a \neq \varnothing$ and let $\langle M\rangle = A_b$, $b \in \{1,2\}^*$. Then $B_{\overline{b}} = A_b = \langle M\rangle$, and so

$$\begin{aligned}\alpha_M^{A_a} &= \alpha_{M\cap A_{b1}}^{A_{a1}} \cup \alpha_{M\cap A_{b2}}^{A_{a2}},\\ \alpha_M^{B_{\bar a}} &= \alpha_{M\cap A_{\bar b1}}^{B_{\bar a1}} \cup \alpha_{M\cap A_{\bar b2}}^{B_{\bar a2}} = \alpha_{M\cap A_{\overline{b2}}}^{B_{\overline{a2}}} \cup \alpha_{M\cap A_{\overline{b1}}}^{B_{\overline{a1}}}.\end{aligned}$$

By the inductive hypothesis, $\alpha_{M\cap A_{b1}}^{A_{a1}} = \alpha_{M\cap A_{\overline{b1}}}^{B_{\overline{a1}}}$ and $\alpha_{M\cap A_{b2}}^{A_{a2}} = \alpha_{M\cap A_{\overline{b2}}}^{B_{\overline{a2}}}$. Thus $\alpha_M^{A_a} = \alpha_M^{B_{\bar a}}$.

Necessity. Let $L_1 = L_2$. According to [5, Corollary 4], Γ_1 and $\bigcup_{\alpha\in L_1} X/\ker\alpha$ coincide as unindexed families of sets. The same result is true for Γ_2 and L_2. Since $L_1 = L_2$, it follows that Γ_1 and Γ_2 are the same as unindexed families of sets.

If $<_1 = <_2$, then Γ_1 and Γ_2 coincide as L-families, so (i) holds. Suppose $<_2 = <_1^{-1}$. Then $A_a = B_{\overline{a}}$ for all $a \in \{1,2\}^*$, which implies $\Gamma_1 \sim \Gamma_2$, so (ii) holds.

To complete the proof we show that in all other cases one gets a contradiction. Let $<_1 \neq <_2 \neq <_1^{-1}$. Since Γ_1 and Γ_2 are the same as unindexed families of sets, we have either $A_1 = B_1$ and $A_2 = B_2$ or $A_1 = B_2$ and $A_2 = B_1$. First suppose that $A_i = B_i$, $i \in \{1,2\}$. Let $x, y \in X$ such that $x <_1 y$, $y <_2 x$. Then $\begin{pmatrix} A_1 & A_2 \\ x & y\end{pmatrix} \in L_1$, $\begin{pmatrix} B_1 & B_2 \\ y & x\end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ y & x\end{pmatrix} \in L_2$ and we get a contradiction with $L_1 = L_2$.

Suppose now that $A_1 = B_2$, $A_2 = B_1$. Let $x, y \in X$ such that $x <_1 y$ and $x <_2 y$. In this case we have $\begin{pmatrix} A_1 & A_2 \\ x & y\end{pmatrix} \in L_1$, $\begin{pmatrix} B_1 & B_2 \\ x & y\end{pmatrix} = \begin{pmatrix} A_2 & A_1 \\ x & y\end{pmatrix} \in L_2$. The last is impossible since $L_1 = L_2$. □

Theorem 2. *The number of different $\mathscr{L}$–cross-sections in the semigroup $\mathscr{T}_n$, $n \geqslant 2$, equals $Q_n \cdot \frac{n!}{2}$.*

Proof. Let A_Γ and A_L be the sets of L-families over X and $\mathscr{L}$-cross-sections in $\mathscr{T}_n$, respectively. Since there are $n!$ strict orders on X and Q_n L-families for each strict order $<$, $|A_\Gamma| = Q_n \cdot n!$. Define a mapping $\omega : A_\Gamma \to A_L$ by $\Gamma\omega = L^\Gamma$. By Theorem 1, ω is onto. Suppose that $\Gamma_1\omega = \Gamma_2\omega$ with $\Gamma_1 \neq \Gamma_2$. Let $\Gamma_1 = \{A_a\}$ and $\Gamma_2 = \{B_a\}$. By Lemma 2, we have $B_a = A_{\bar{a}}$ for every $a \in \{1,2\}$. Thus ω is two-to-one, and so $|A_L| = \frac{|A_\Gamma|}{2} = Q_n \cdot \frac{n!}{2}$. □

4. The classification of $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ up to isomorphism

It is well known that not all $\mathscr{L}$-cross-sections in semigroup $\mathscr{T}(X)$ are isomorphic to each other (see [4]). We now investigate when two L-families correspond to isomorphic $\mathscr{L}$-cross-sections. Throughout this section let L_1 and L_2 be two $\mathscr{L}$-cross-sections of $\mathscr{T}(X_n)$; $\Gamma_1 = \{A_a\}$, $\Gamma_2 = \{B_a\}$ be the L-families associated with L_1 and L_2, i.e. $L_1 = L_X^{\Gamma_1}$ and $L_2 = L_X^{\Gamma_2}$.

Note that if $|X| \leqslant 3$ all the possible $\mathscr{L}$-cross-sections are isomorphic and all the possible L-families are similar. The following is true for an arbitrary finite set X.

Lemma 3. *If* $\Gamma_1 \sim \Gamma_2$, *then* $L_1 \cong L_2$.

Proof. If $|A_a| = |B_a|$, for all $a \in \{1,2\}^*$, then set

$$\theta : \Gamma_1 \to \Gamma_2 : A_a \mapsto B_a$$

and if $|A_a| = |B_{\bar{a}}|$ for all $a \in \{1,2\}^*$, then set

$$\theta : \Gamma_1 \to \Gamma_2 : A_a \mapsto B_{\bar{a}}.$$

Without loss of generality we can assume that $|A_a| = |B_a|$, for all $a \in \{1,2\}^*$. Let $x, y \in X$ be arbitrary elements and $A_a = \{x\}$, $A_a \in \Gamma_1$, $a \in \{1,2\}^*$. Set

$$\psi : X \to X : x \mapsto y \Leftrightarrow A_a\theta = \{y\}.$$

It is clear that this mapping is a bijection, and for all $a \in \{1,2\}^*$, we have $A_a\psi = A_a\theta$, where $A_a\psi = \{x\psi \mid x \in A_a\}$. Let

$$\tau : L_1 \to L_2 : \varphi \mapsto \varphi' = \psi^{-1}\varphi\psi.$$

Now we verify that $\varphi' \in L_X^{\Gamma_2}$. To be more precise, we show that $\varphi' = \alpha_{(\operatorname{im}\varphi)\psi}$ for $\varphi \in L_1$ with $\varphi\tau = \varphi'$. Let $a \in \{1,2\}^*$ be an arbitrary element such that $A_a \neq \varnothing$. Consider the image of B_a under the map φ'. Since ψ is a bijection, we have

$$\langle B_a\varphi' \rangle = \langle (A_a\psi)(\psi^{-1}\varphi\psi) \rangle = \langle A_a(\varphi\psi) \rangle = \langle (A_a\varphi)\psi \rangle. \tag{2}$$

We denote by M the image of φ, so $\varphi=\alpha_M$. Let $M=\{m_1, m_2, \ldots, m_k\}$ for $m_1, m_2, \ldots, m_k \in X$. By definition of α_M we have

$$\alpha_M = \alpha_{\{m_1\}}^{A_{b_1}} \cup \alpha_{\{m_2\}}^{A_{b_2}} \cup \ldots \cup \alpha_{\{m_k\}}^{A_{b_k}}$$

for suitable $b_1, b_2, \ldots, b_k \in \{1,2\}^*$. In virtue of arbitrariness of $a \in \{1,2\}^*$ in (2) we obtain $\langle B_{b_i}\varphi'\rangle = \langle (A_{b_i}\varphi)\psi\rangle = \langle \{m_i\psi\}\rangle, 1 \leqslant i \leqslant k$. Since $B_{bi}, 1 \leqslant i \leqslant k$, are pairwise disjoint and $|B_{b_1} \cup B_{b_2} \cup \ldots B_{b_k}| = |A_{b_1} \cup A_{b_2} \cup \ldots A_{b_k}| = |X|$, we get $B_{b_1} \cup B_{b_2} \cup \ldots B_{b_k} = X$, consequently $\operatorname{im}(\varphi') = (\operatorname{im}\varphi)\psi$.

Now to prove $\varphi' = \alpha_{(\operatorname{im}\varphi)\psi}$ it suffices to show $\varphi'|_{B_a} = \alpha_{(A_a\varphi)\psi}^{B_a}$ for all $a \in \{1,2\}^*$. We proceed by induction on $|(A_a\varphi)\psi|$. If $B_a = \varnothing$, then $\varphi'|_{B_a} = \varnothing = \alpha_{(A_a\varphi)\psi}^{B_a}$. If $|B_a| = |A_a| \neq 0$ and $(A_a\varphi)\psi = \{m\}$ then $\operatorname{dom}(\alpha_{(A_a\varphi)\psi}^{B_a}) = B_a = \operatorname{dom}(\varphi'|_{B_a})$ and by (2)

$$\langle \operatorname{im}(\varphi'|_{B_a})\rangle = \langle (A_a\varphi)\psi\rangle = \langle\{m\}\rangle,$$

thus, for all x in the common domain, $x\varphi'|_{B_a} = m = x\alpha_{(A_a\varphi)\psi}^{B_a}$, which implies $\varphi'|_{B_a} = \alpha_{(A_a\varphi)\psi}^{B_a}$.

Let $|(A_a\varphi)\psi| > 1$ and suppose the statement is true for all $M_1 \subseteq X$ with $M_1 \neq \varnothing$ and $|M_1| < |(A_a\varphi)\psi|$. Again, if $B_a = \varnothing$, then $\varphi'|_{B_a} = \varnothing = \alpha_{(A_a\varphi)\psi}^{B_a}$. Suppose $B_a \neq \varnothing$, then, clearly, $\varphi'|_{B_a} = \varphi'|_{B_{a1}} \cup \varphi'|_{B_{a2}}$. By the inductive hypothesis $\varphi'|_{B_{a1}} = \alpha_{(A_{a1}\varphi)\psi}^{B_{a1}}$ and $\varphi'|_{B_{a2}} = \alpha_{(A_{a2}\varphi)\psi}^{B_{a2}}$. Thus

$$\varphi'|_{B_a} = \alpha_{(A_{a1}\varphi)\psi}^{B_{a1}} \cup \alpha_{(A_{a2}\varphi)\psi}^{B_{a2}} = \alpha_{(A_a\varphi)\psi}^{B_a} \quad \text{for all } a \in \{1,2\}^*.$$

Hence,

$$\varphi' = \alpha_{(A_1\varphi)\psi}^{B_1} \cup \alpha_{(A_2\varphi)\psi}^{B_2} = \alpha_{(A_1\varphi \cup A_2\varphi)\psi}^{B_1 \cup B_2} = \alpha_{(\operatorname{im}\varphi)\psi} \in L_X^{\Gamma_2}.$$

Since $\alpha_M\tau = \alpha_{M\psi}$, $M \subseteq X$ and ψ is bijective, we get τ is bijective too.

Finally, for all $\beta, \gamma \in L_1$, we have

$$(\beta)\tau(\gamma)\tau = \psi^{-1}(\beta\gamma)\psi = (\beta\gamma)\tau. \qquad \square$$

To prove the converse we first need some preparations.

Let $\tau : L_1 \to L_2$ be an isomorphism. In both L_1 and L_2, the set $\{c_x \mid x \in X\}$ of constant transformations is the minimum ideal. Thus, τ maps $\{c_x \mid x \in X\}$ onto $\{c_x \mid x \in X\}$. For $x \in X$, denote by x' the element of X such that $c_x\tau = c_{x'}$.

If $A_a \in \Gamma_1$ and $x \in A_a$ is an arbitrary fixed element, then denote by $\varphi(A_a, x)$ the transformation in L_1 with the image $(X \setminus A_a) \cup \{x\}$.

Whenever we say that $L_1 \cong L_2$, we will assume that τ is an isomorphism from L_1 to L_2.

It is clear, that if $\Gamma_1 \sim \Gamma_2$, then $|A_a\theta| = |B_a|$ ($|A_a\bar{\theta}| = |B_a|$). Obviously, if $L_1 \cong L_2$, then $|\Gamma_1| = |\Gamma_2|$. We show in following Lemma, that if $L_1 \cong L_2$, then for every set in Γ_1 there exists a unique set in Γ_2 with the same cardinality.

Lemma 4. *Let $L_1 \cong L_2$. For every $A_a \in \Gamma_1$, $x \in A_a$, the following statements hold true:*

(i) $\varphi(A_a, x)|_{X\setminus A_a} = \mathrm{id}_{X\setminus A_a}$, $\varphi(A_a, x)|_{A_a} = c_x$.

(ii) *there exists $B_{a'} \in \Gamma_2$ such that $|A_a| = |B_{a'}|$ and $\varphi(A_a, x)\tau = \varphi(B_{a'}, x')$, where $c_x\tau = c_{x'}$.*

Proof. (i) For every $A_a \in \Gamma_1$, $x \in A_a$, consider the elements $\varphi(A_a, x) \in L_1$ such that

$$\mathrm{im}\,(\varphi(A_a, x)) = (X \setminus A_a) \cup \{x\}.$$

If $A_a = X$ we get $\varphi(A_a, x) = c_x$, and $\varphi(A_a, x) = \mathrm{id}_X$ if $|A_a| = 1$.

Suppose $A_a \neq X$, $|A_a| > 1$. In this case denote subsets of Γ_1 as follows: put $X = X_1 \uplus X_1'$, if $A_a \subseteq X_1'$; $X_1' = X_2 \uplus X_2'$ if $A_a \subseteq X_2'$; …, etc., until we get, for a natural p, that $X_{p-1}' = X_p \uplus X_p'$ and $A_a = X_p'$, where $C = D \uplus E$ means that $C = D \cup E$ and $D \cap E = \varnothing$.

In the proof of [5, Lemma 4, (ii)] it was shown that

$$\sigma_p = \begin{pmatrix} X_1 & X_2 & \dots & X_p & X_p' \\ x_1 & x_2 & \dots & x_p & x_p' \end{pmatrix} \in L_1, \tag{3}$$

where $x_p' \in X_p'$, $x_j \in X_j$, $1 \leqslant j \leqslant p$. Since $X \setminus A_a = X_1 \cup X_2 \cup \ldots \cup X_p$, and $x \in A_a = X_p'$ with $X_p' \cap X_i = \varnothing$ for all $1 \leqslant i \leqslant p$, we get

$$\mathrm{im}\,(\varphi(A_a, x)\sigma_p) = \mathrm{im}\,(\sigma_p).$$

From $\varphi(A_a, x)\sigma_p$, $\sigma_p \in L_1$, we obtain $\varphi(A_a, x)\sigma_p = \sigma_p$. The last equality is true for every σ_p as in (3), which is only possible if $\varphi(A_a, x)|_{X\setminus A_a} = \mathrm{id}_{X\setminus A_a}$, $\varphi(A_a, x)|_{A_a} = c_x$.

(ii) Let $x, t \in A_a$, $z \in X \setminus A_a$. On the one hand,

$$(c_z\varphi(A_a, x))\tau = c_z\tau = c_{z'} \quad \text{and} \quad (c_z\tau)(\varphi(A_a, x)\tau) = c_{z'}(\varphi(A_a, x)\tau), \tag{4}$$

so $c_{z'}(\varphi(A_a, x)\tau) = c_{z'}$. On the other hand,

$$(c_t\varphi(A_a, x))\tau = c_x\tau = c_{x'} \quad \text{and} \quad (c_t\tau)(\varphi(A_a, x)\tau) = c_{t'}(\varphi(A_a, x)\tau), \tag{5}$$

so $c_{t'}(\varphi(A_a,x)\tau)=c_{x'}$. Consider $x'(\varphi(A_a,x)\tau)^{-1}\in\ker\varphi(A_a,x)\tau$. For all $t\in A_a$, $z\in X\setminus A_a$ we have $c_{z'}\neq c_{x'}$ (since $x\neq z$), $c_{z'}(\varphi(A_a,x)\tau)=c_{z'}$, $c_{t'}(\varphi(A_a,x)\tau)=c_{x'}$. It follows that

$$x'(\varphi(A_a,x)\tau)^{-1}=\{t'\mid t\in A_a\},$$

and so $\{t'\mid t\in A_a\}\in X/\ker\varphi(A_a,x)\tau$. By [5, Corollary 4], Γ_2 and $\cup_{\alpha\in L_2}X/\ker\alpha$ are the same as unindexed families of sets, thus there exists $B_{a'}\in\Gamma_2$, for some $a'\in\{1,2\}^*$, with $B_{a'}=\{t'\mid t\in A_a\}$. Due to bijectivity of τ, we have $|A_a|=|B_{a'}|$. Furthermore, by (4) and (5), $(\varphi(A_a,x)\tau)|_{X\setminus B_{a'}}=\mathrm{id}_{X\setminus B_{a'}}$ and $(\varphi(A_a,x)\tau)|_{B_{a'}}=c_{x'}$, $x'\in B_{a'}$. Hence, $\varphi(A_a,x)\tau=\varphi(B_{a'},x')$ and $c_{x'}=c_x\tau$. □

Denote the set of all nonempty subsets of X by $U(X)$.

Lemma 5. *Let $L_1\cong L_2$, $\psi:U(X)\to U(X):M\mapsto M'\Leftrightarrow\ \alpha_M\tau=\alpha_{M'}$. The following statements hold true:*

(i) *for all $A_a\in\Gamma_1$, $A_a\psi\in\Gamma_2$;*

(ii) *for all $A_a\in\Gamma_1$ and $\beta\in L_1$, $(A_a\beta)\psi=A_a\psi(\beta\tau)$.*

Proof. (i) It is clear that if $|A_a|=1$, then $A_a\psi\in\Gamma_2$. Let $|A_a|>1$ and $\alpha=\alpha_{A_a}\in L_1$. Let $x\in A_a$ be an arbitrary fixed element, and $\varphi(A_a,x)\tau=\varphi(B_{a'},x')$, $B_{a'}\in\Gamma_2$, $x'\in B_{a'}$, $|A_a|=|B_{a'}|$. Since $\alpha\varphi(A_a,x)=c_x$, we have $(\alpha\tau)\varphi(B_{a'},x')=c_{x'}$, therefore $\mathrm{im}\,(\alpha\tau)\subseteq B_{a'}$. Suppose that $\mathrm{rk}\,(\alpha)>\mathrm{rk}\,(\alpha\tau)$ and denote by β' the transformation from L_2 with $\mathrm{im}\,(\beta')=B_{a'}$.

Let $\delta\in L_1$ such that $\mathrm{im}\,(\delta)\subseteq A_a$. Just as in [5, Lemma 4,(iv)] it can be shown, that there exists $\gamma\in L_1$ with $\mathrm{im}\,(\gamma|_{A_a})=\mathrm{im}\,(\delta)$. We denote this transformation by γ^δ. Thus, for all $\delta\in L_1$ with $\mathrm{im}\,(\delta)\subseteq A_a$, there exists $\gamma^\delta\in L_1$ such that $\delta=\alpha\gamma^\delta$.

Let $\beta=\beta'\tau^{-1}$. Since $\beta'\varphi(B_{a'},x')=c'_x$, it follows that $(\beta'\varphi(B_{a'},x'))\tau^{-1}=\beta\varphi(A_a,x)=c_x$, hence $\mathrm{im}\,(\beta)\subseteq A_a$. Thus, $\beta=\alpha\delta^\beta$, whence $\beta'=(\alpha\tau)(\delta^\beta\tau)$. But

$$\mathrm{rk}\,(\beta')=\mathrm{rk}\,((\alpha\tau)(\delta^\beta\tau))\leqslant\mathrm{rk}\,(\alpha\tau)<\mathrm{rk}\,(\alpha).$$

The latter contradiction proves that $\mathrm{rk}\,(\alpha)=\mathrm{rk}\,(\alpha\tau)$. Hence $|\,\mathrm{im}\,(\alpha\tau)|=|A_a|=|B_{a'}|$ and $\mathrm{im}\,(\alpha\tau)\subseteq B_{a'}$, which implies $\mathrm{im}\,(\alpha\tau)=B_{a'}$. Thus, $\alpha_{A_a}\tau=\alpha_{B_{a'}}$, hence $A_a\psi=B_{a'}\in\Gamma_2$.

(ii) Suppose that $A_a\in\Gamma_1$ and $\beta\in L_1$. Let $\alpha_{A_a}\tau=\alpha_{B_{a'}}$, $B_{a'}\in\Gamma_2$. Denote $\beta\tau$ by β'. Then

$$(\alpha_{A_a\beta})\tau=(\alpha_{A_a}\beta)\tau=(\alpha_{B_{a'}})\beta'=\alpha_{A_a\psi}\beta'=\alpha_{(A_a\psi)\beta'},$$

which implies $(A_a\beta)\psi=A_a\psi(\beta\tau)$ by the definition of ψ. □

Corollary 1. *Let* $L_1 \cong L_2$, ψ *be the function from* Lemma 5. *Then, for all* $A_a, A_b \in \Gamma_1$*:*

(i) $|A_a| = |A_a\psi|$;

(ii) *if* $A_a \subseteq A_b$, *then* $A_a\psi \subseteq A_b\psi$;

(iii) *if* $A_a \cap A_b = \varnothing$, *then* $A_a\psi \cap A_b\psi = \varnothing$.

Proof. (i) Let $A_a \in \Gamma_1$, $x \in A_a$, and $(\varphi(A_a, x))\tau = \varphi(B_{a'}, x')$ for $B_{a'} \in \Gamma_2$, $x' \in B_{a'}$, with $|A_a| = |B_{a'}|$ (see Lemma 4, (ii)). The proof of Lemma 5, (i) implies that $A_a\psi = B_{a'}$. So $|A_a| = |A_a\psi|$.

(ii) Let $A_a \subseteq A_b$ and $z \in A_b$ be an arbitrary fixed element. Suppose $(\varphi(A_b, z))\tau = \varphi(B_{b'}, z')$ with $B_{b'} \in \Gamma_2$, $z' \in B_{b'}$. By Lemma 4, (ii), $c_z\tau = c_{z'}$, consequently $\{z\}\psi = \{z'\}$. On the one hand,

$$(A_a\varphi(A_b, z))\psi = \{z\}\psi = \{z'\}.$$

On the other hand, by Lemma 5, (ii),

$$(A_a\varphi(A_b, z))\psi = (A_a\psi)(\varphi(A_b, z)\tau) = (A_a\psi)\varphi(B_{b'}, z').$$

So $(A_a\psi)\varphi(B_{b'}, z') = \{z'\}$, which is implies $A_a\psi \subseteq B_{b'} = A_b\psi$.

(iii) Let $A_a \cap A_b = \varnothing$. Fix $z \in A_b$ and let $(\varphi(A_b, z))\tau = \varphi(B_{b'}, z')$ with $B_{b'} \in \Gamma_2$, $z' \in B_{b'}$. Suppose $y \in A_a$ is an arbitrary element, and $c_y\tau = c_{y'}$. By definition of ψ then we have $\{y\}\psi = \{y'\}$. On the one hand,

$$(y\varphi(A_b, z))\psi = \{y\}\psi = \{y'\}, \text{ where } y' \neq z'.$$

On the other hand, by Lemma 5, (ii),

$$(\{y\}\varphi(A_b, z))\psi = (\{y\}\psi)(\varphi(A_b, z)\tau) = \{y'\}\varphi(B_{b'}, z').$$

So $y'\varphi(B_{b'}, z') = y'$, $y' \neq z'$ for all $y \in A_a$. Thus

$$\{y' \mid c_{y'} = c_y\tau,\ y \in A_a\} \cap A_b\psi = \{y' \mid c_{y'} = c_y\tau,\ y \in A_a\} \cap B_{b'} = \varnothing.$$

By (ii) of this Corollary $\{y'\} = \{y\}\psi \subseteq A_a\psi$. Since τ is a bijection, we have

$$|A_a| = |\{y' \mid c_{y'} = c_y\tau,\ y \in A_a\}|.$$

By (i), we get $|A_a| = |A_a\psi|$, thus $A_a\psi = \{y' \mid c_{y'} = c_y\tau,\ y \in A_a\}$. Hence $A_a\psi \cap A_b\psi = \varnothing$. □

Now we are ready to prove

Lemma 6. *If* $L_1 \cong L_2$, *then* $\Gamma_1 \sim \Gamma_2$.

Proof. The result is clearly true if $|X| = 1$. Suppose that $|X| \geqslant 2$. Consider the restriction of the function ψ from Lemma 5 to Γ_1 (which we will also call ψ). By (i) of Lemma 5, $\psi : \Gamma_1 \to \Gamma_2$. It easily follows from Corollary 1 that either $A_1\psi = B_1$ and $A_2\psi = B_2$ or $A_1\psi = B_2$ and $A_2\psi = B_1$. Suppose that $A_1\psi = B_1$ and $A_2\psi = B_2$. We will prove by induction on $|a|$ that for all $a \in \{1,2\}^*$, $A_a\psi = B_a$. We already know that this is true if $|a| = 1$. Let $k \geqslant 1$ and suppose that $A_a\psi = B_a$ for every $a \in \{1,2\}^*$ with $|a| \leqslant k$.

Note, that for all $\mathscr{T}_n$, $n \in \mathbb{N}$ if $|A_1| = 1$ or $|A_2| = 1$, $A_1, A_2 \in \Gamma_1$, then the structure of Γ_1 is uniquely determined in virtue of (2). Thus, if $L_1 = L_X^{\Gamma_1} \cong L_2$, then we get immediately that $\Gamma_1 \sim \Gamma_2$.

Assume further that $|A_1|, |A_2| > 1$. We will prove by induction on $|a|$ that for all $a \in \{1,2\}^*$ $A_a\psi = B_a$ or $A_a\psi = B_{\bar{a}}$.

Suppose, that condition $A_a\psi = B_a$ or $A_a\psi = B_{\bar{a}}$ holds for all $A_a \in \Gamma_1$, $|a| \leqslant k$, $k \in \mathbb{N}$. Without loss of generality set $A_a\psi = B_a$, for all $A_a \in \Gamma_1$, $B_a \in \Gamma_2$ if $|a| \leqslant k$, $k \in \mathbb{N}$.

Let $bi \in \{1,2\}^*$, $|bi| = k$ and $A_{bi} \in \Gamma_1$. As has been shown in [5, Lemma 4, (iv)], there exists a transformation $\gamma \in L_1$ such that $A_b\gamma = A_{bi}$, i.e., $\gamma|_{A_b} = \alpha_{A_{bi}}^{A_b}$. According to Definition 3,

$$\alpha_{A_{bi}}^{A_b} = \alpha_{A_{bi} \cap A_{bi1}}^{A_{b1}} \cup \alpha_{A_{bi} \cap A_{bi2}}^{A_{b2}} = \alpha_{A_{bi1}}^{A_{b1}} \cup \alpha_{A_{bi2}}^{A_{b2}},$$

so $A_{bj}\gamma = A_{bij}$, $j \in \{1,2\}$. Moreover, by the induction hypothesis, the following conditions hold:

$$(\alpha_{A_b}\gamma)\tau = (\alpha_{A_{bi}})\tau = \alpha_{A_{bi}\psi} = \alpha_{B_{bi}},$$
$$(\alpha_{A_b}\gamma)\tau = (\alpha_{A_b}\tau)(\gamma\tau) = (\alpha_{A_b\psi})(\gamma\tau) = \alpha_{B_b}(\gamma\tau).$$

Consequently, $\alpha_{B_b}(\gamma\tau) = \alpha_{B_{bi}}$, and so $B_b(\gamma\tau) = B_{bi}$. Since $\gamma\tau \in L_2$, we have $(\gamma\tau)|_{B_b} = \alpha_{B_{bi}}^{B_b}$. According to Definition 3,

$$\alpha_{B_{bi}}^{B_b} = \alpha_{B_{bi} \cap B_{bi1}}^{B_{b1}} \cup \alpha_{B_{bi} \cap B_{bi2}}^{B_{b2}} = \alpha_{B_{bi1}}^{B_{b1}} \cup \alpha_{B_{bi2}}^{B_{b2}},$$

so $B_{bj}(\gamma\tau) = B_{bij}$, $j \in \{1,2\}$.

Now, on the one hand, we have $A_{bj}\psi(\gamma\tau) = (A_{bj}\gamma)\psi = A_{bij}\psi$, $j \in \{1,2\}$, by (ii) of Lemma 5. On the other hand, using the induction hypothesis, we get $A_{bj}\psi(\gamma\tau) = B_{bj}(\gamma\tau) = B_{bij}$, $j \in \{1,2\}$. Thus, $A_{bij}\psi = B_{bij}$, $j \in \{1,2\}$. Since A_{bj} is an arbitrary element with $|bj| = k$, we get $A_c\psi = B_c$ for all $c \in \{1,2\}^*$, $|c| = k+1$, $A_c \in \Gamma_1$. So for all $a \in \{1,2\}^*$ $A_a\psi = B_a$.

In a dual way, we can prove that if $A_1\psi = B_2$ and $A_2\psi = B_1$, then $A_a\psi = B_{\bar{a}}$ for every $a \in \{1,2\}^*$. Since $|A_a| = |A_a\psi|$ for every $a \in \{1,2\}^*$ (by Corollary 1), it follows that $\Gamma_1 \sim \Gamma_2$. □

Now Lemmas 3 and 6 yield

Theorem 3. *Two $\mathscr{L}$-cross-sections of $\mathcal{T}_n$ are isomorphic if and only if the L-families associated with them are similar.*

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On classification of pairs of potent linear operators with the simplest annihilation condition

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Abstract. We study the problem of classifying the pairs of linear operators $\mathcal{A}, \mathcal{B}$ (acting on the same vector space), when the both operators are potent and $\mathcal{A}\mathcal{B} = 0$. We describe the finite, tame and wild cases and classify the indecomposable pairs of operators in the first two of them.

Introduction

Throughout the paper, k is an algebraic closed field of characteristic $\operatorname{char} k = 0$. All k-vector space are finite-dimensional. Under consideration maps, morphisms, etc., we keep the right-side notation.

We call a Krull-Schmidt category (i.e. an additive k-category with local endomorphism algebras for all indecomposable objects) *of tame* (respectively, *wild*) *type* if so is the problem of classifying its objects up to isomorphism (see precise general definitions in [1]). For formal reasons we exclude the categories of finite type (i.e. with finite number of the isomorphism classes of indecomposable objects) from those of tame type.

In this paper we study the problem of classifying the pairs of annihilating potent linear operators (an operator $\mathcal{C}$ is called potent or, more precisely, s-potent if $\mathcal{C}^s = \mathcal{C}$, where $s > 1$).

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Formulate our problem more precisely and in the category language.

Let $\mathcal{P}(k)$ denotes the *category of pairs of linear operators* acting on the same k-vector space, i.e. the category with objects the triples $\overline{U} = (U, \mathcal{A}, \mathcal{B})$, consisting of a k-vector space U and linear operators $\mathcal{A}, \mathcal{B}$ on U, and with morphisms from $\overline{U} = (U, \mathcal{A}, \mathcal{B})$ to $\overline{U'} = (U', \mathcal{A}', \mathcal{B}')$ the linear maps $X : U \to U'$ such that $\mathcal{A}X = X\mathcal{A}$, $\mathcal{B}X = X\mathcal{B}$. Since it is a Krull-Schmidt category, each object is uniquely determined by its direct summands. For natural numbers $n, m \geqslant 1$, denote by $\mathcal{P}^{\circ}_{k}(n, m)$ the full subcategory of $\mathcal{P}(k)$ consisting of all triples $(U, \mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ being n-potent, $\mathcal{B}$ being m-potent and $\mathcal{AB} = 0$. Our aim is to describe the type of every such category and to classify (up to isomorphism) the indecomposable objects in finite and tame cases.

Theorem 1. *A category $\mathcal{P}^{\circ}_{k}(n, m)$ is of*

- *finite type if $nm < n + m + 3$,*
- *tame type if $nm = n + m + 3$,*
- *wild type if $nm > n + m + 3$.*

With respect to the mentioned classification see section 2.

Note that from Theorems 3.1 and 3.2 of [2] it follows that without the relation $\mathcal{AB} = 0$ the corresponding overcategory $\mathcal{P}_{k}(n, m)$ is of tame type if $n = m = 2$ and of wild type otherwise (see more in 3.6 below).

1. Proof of the theorem

We first establish a connection between the categories $\mathcal{P}^{\circ}_{k}(n, m)$ and the categories of representations of quivers.

Recall the notion of representations of a quiver [3].

Let $Q = (Q_0, Q_1)$ be a finite quiver (directed graph), where Q_0 and Q_1 are the sets of its vertices and arrows, respectively. *A representation of the quiver $Q = (Q_0, Q_1)$ over a field K* is a pair $R = (V, \gamma)$ formed by a collection $V = \{V_x \mid x \in Q_0\}$ of K-vector spaces V_x and a collection $\gamma = \{\gamma_\alpha \mid \alpha : x \to y \text{ runs through } Q_1\}$ of linear maps $\gamma_\alpha : V_x \to V_y$. A morphism from $R = (V, \gamma)$ to $R' = (V', \gamma')$ is given by a collection $\overline{\lambda} = \{\lambda_x \mid x \in Q_0\}$ of linear maps $\lambda_x : V_x \to V'_x$, such that $\gamma_\alpha \lambda_y = \lambda_x {\gamma'}_\alpha$ for any arrow $\alpha : x \to y$. The category of representations of $Q = (Q_0, Q_1)$ over K will be denoted by $\operatorname{rep}_K Q$. It is a Krull-Schmidt category.

A quiver Q is said to be *of finite*, *tame* or *wild representation type over K* if the caregory $\operatorname{rep}_K Q$ has respectively finite, tame or wild type. By results of [3] (respectively, [4] and [5]), a connected quiver is of finite (respectively, tame) representation type if and only if it is a Dynkin (respectively, extended Dynkin) graph. Note that by a Dynkin graph we mean a Dynkin diagram with some orientation of edges, and for simplicity denote it in the same way as the Dynkin diagram (analogously for an extended Dynkin graph).

Now we proceed to investigate connections between categories of the forms $\mathcal{P}^\circ_k(n,m)$ and $\operatorname{rep}_k Q$.

We identify a linear map α of $U = U_1 \oplus \dots U_p$ into $V = V_1 \oplus \dots V_q$ with the matrix $(\alpha_{ij})_{i=1}^{p}{}_{j=1}^{q}$, where $\alpha_{ij} : U_i \to V_j$ are the linear maps induced by α; if $p = q$ and the matrix is diagonal, we write $\alpha = \oplus_{i=1}^{p} \alpha_i$. The identity linear operator on W is denoted by 1_W.

For natural numbers $n, m > 1$, denote by $Q(n,m)$ the quiver with set of vertices $Q_0(n,m) = \{1, 2 \dots, n+m\}$ and set of arrows $Q_1(n,m) = \{i \to j \mid j = 1, \dots, n, i = n+1, \dots, n+m\}$. The primitive root of unity of degree s is denoted by ε_s.

Define the functor G_{nm} from $\operatorname{rep}_k Q(n-1, m-1)$ to $\mathcal{P}^\circ_k(n,m)$ as follows. G_{nm} assigns to each object $(V, \gamma) \in \operatorname{rep}_k Q(n-1, m-1)$ the object $(V^\oplus, \mathcal{A}^\gamma, \mathcal{B}^\gamma) \in \mathcal{P}^\circ_k(n,m)$ where $V^\oplus = \oplus_{i=1}^{n+m-2} V_i$, $\mathcal{A}^\gamma_{ij} = \varepsilon^i_{n-1} 1_{V_i}$ if $i = j \leqslant n-1$ and $\mathcal{A}^\gamma_{ij} = 0$ if otherwise, $\mathcal{B}^\gamma_{n+i-1,n+i-1} = \varepsilon^i_{m-1} 1_{V_{n+i-1}}$ if $i \leqslant m-1$, $\mathcal{B}^\gamma_{n+i-1,j} = \gamma_{ij}$ if $i \leqslant m-1, j \leqslant n-1$, and $\mathcal{B}^\gamma_{pq} = 0$ in all other cases. G_{nm} assigns to each morphism λ of $\operatorname{rep}_k Q(m-1, n-1)$ the morphism $\oplus_{i=1}^{n+m-2} \lambda_i$ of $\mathcal{P}^\circ_k(n,m)$.

Proposition 1. *The functor G_{nm} is full and faithful.*

Proof. It is obvious that G_{nm} is faithful. Prove that it is full. Let δ be a morphism from $(V,\gamma)G_{nm} = (V^\oplus, \mathcal{A}^\gamma, \mathcal{B}^\gamma)$ to $(W,\sigma)G_{nm} = (W^\oplus, \mathcal{A}^\sigma, \mathcal{B}^\sigma)$. In other words, δ is a linear map of $V^\oplus$ into $W^\oplus$ such that $\mathcal{A}^\gamma \delta = \delta \mathcal{A}^\sigma$ and $\mathcal{B}^\gamma \delta = \delta \mathcal{B}^\sigma$. We consider these equalities as matrix ones (see the definition of $V^\oplus$), and the induced by them scalar equalities $(\mathcal{A}^\gamma \delta)_{ij} = (\delta \mathcal{A}^\sigma)_{ij}$ and $(\mathcal{B}^\gamma \delta)_{ij} = (\delta \mathcal{B}^\sigma)_{ij}$ denote, respectively, by $[a,i,j]$ and $[b,i,j]$.

Since $\varepsilon_{n-1}, \varepsilon^2_{n-1}, \dots, \varepsilon^{n-1}_{n-1}$ and 0 are pairwise different elements of the field k, it follows from the equalities $[a,i,j]$ with $i, j \in \{1, \dots, n-1\}$, $i \neq j$, $[a,i,j]$ with $i \in \{1, \dots, n-1\}, j \in \{n, \dots, n+m-2\}$ and $[a,i,j]$ with $i \in \{n, \dots, n+m-2\}, j \in \{1, \dots, n-1\}$ that the block $(\delta_{pq})_{p,q=1}^{n-1}$ of σ (as a matrix) is diagonal and the blocks $(\delta_{pq})_{p=1}^{n-1}{}_{q=n}^{n+m-2}$, $(\delta_{pq})_{p=n}^{n+m-2}{}_{q=1}^{n-1}$ are zero. Then analogously to above, it follows from the equalities $[b,i,j]$ with $i, j \in \{n, \dots, n+m-2\}$, $i \neq j$, that the block $(\delta_{pq})_{p,q=n}^{n+m-2}$ is diagonal. Thus σ (as a matrix) is diagonal, and it is easy to see that the equalities $[b,i,j]$ with $i \in \{n, \dots, n+m-2\}, j \in \{1, \dots, n-1\}$ means that $\overline{\sigma} =$

$(\sigma_1, \ldots, \sigma_{n+m-2})$ is a morphism between the objects (V, γ) and (W, σ) of the category $\mathrm{rep}_k\, Q(n-1, m-1)$. Since $\sigma = \overline{\sigma} G_{nm}$, the fullness of G_{nm} is proved. $\square$

Proposition 2. *Each object of $\mathcal{P}_k^\circ(n,m)$ is isomorphic to an object of the form $RG_{nm} \oplus (W, 0, 0)$, where R is an object of $\mathrm{rep}_k\, Q(n-1, m-1)$, W is a k-vector space of dimension $d \geqslant 0$.*

Proof. Let $T = (U, \mathcal{A}, \mathcal{B})$ be an objects of the category $\mathcal{P}_k^\circ(n,m)$. Since the roots $\varepsilon_{n-1}, \ldots, \varepsilon_{n-1}^{n-1}$ and 0 of the polynomial $x^n - x$ are pairwise different, we can assume (by the theorem on the Jordan canonical form) that $U = U_1 \oplus \ldots \oplus U_{n-1} \oplus U_0$ with $U_s = \mathrm{Ker}(\mathcal{A} - \varepsilon_{n-1}^s 1_U)$ and $U_0 = \mathrm{Ker}\, \mathcal{A}$; then $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{n-1} \oplus \mathcal{A}_0$ with $\mathcal{A}_s : U_s \to U_s$ to be the scalar operator $\varepsilon_{n-1}^s 1_{U_s}$ and $\mathcal{A}_0 : U_0 \to U_0$ to be zero (here $s = 1, \ldots, n-1$). From $\mathcal{AB} = 0$ it follows that $U_1 \oplus \ldots \oplus U_{n-1} \in \mathrm{Ker}\, \mathcal{B}$, and consequently we have (since $\mathcal{B}^m = \mathcal{B}$) that the operator $\mathcal{B}_0 : U_0 \to U_0$, induced by $\mathcal{B}$, satisfies the equality $\mathcal{B}_0^m = \mathcal{B}_0$. Then, analogously as above, $U_0 = U_n \oplus \ldots \oplus U_{n+m-2} \oplus W$ with $U_{n+s-1} = \mathrm{Ker}(\mathcal{B}_0 - \varepsilon_{m-1}^s 1_{U_0})$, $s = 1, \ldots, m-1$, and $W = \mathrm{Ker}\, \mathcal{B}_0$. Besides, it follows from $\mathcal{B}^m = \mathcal{B}$ that $W_0 \in \mathrm{Ker}\, \mathcal{B}$.

Thus, $U = U_1 \oplus \ldots \oplus U_{n+m-2} \oplus W$ and now the operators $\mathcal{A}, \mathcal{B}$ are uniquely defined by the maps $\mathcal{B}_{ij} : U_i \to U_j$ with i and j running from n to $n+m-2$ and from 1 to $n-1$, respectively. The representation R of the quiver $Q(n-1, m-1)$, corresponding to these maps, satisfies the required condition, i. e. $T = RG_{nm} \oplus (W, 0, 0)$. $\square$

Denote by $\widehat{\mathcal{P}}_k^\circ(n,m)$ the full subcategory of $\mathcal{P}_k^\circ(n,m)$ consisting of all objects that have no objects $(W, 0, 0)$, with $W \neq 0$, as direct summands.

We have as an immediate consequence of Propositions 1 and 2 the following statement.

Theorem 2. *The functor G_{nm}, viewed as a functor from the category $\mathrm{rep}_k\, Q(n-1, m-1)$ to the category $\widehat{\mathcal{P}}_k^\circ(n,m)$, is an equivalence of categories.*

Using this theorem it is easy to show by the standard method that the types of categories $\mathcal{P}_k^\circ(n,m)$ and $\mathrm{rep}_k\, Q(n-1, m-1)$ coincide.

Now Theorem 1 follows from the simple facts that $Q = Q(n-1, m-1)$ is a Dynkin graph iff either $n = 2, m = 2, 3, 4$, or vice versa, $n = 2, 3, 4, m = 2$ (then $Q = A_2, A_3, D_4$, respectively), and an extended Dynkin graph iff either $n = 2, m = 5$, or vice versa, $n = 5, m = 2$ (then $Q = \widetilde{D}_4$), or $n = m = 3$ (then $Q = \widetilde{A}_3$).

2. The classification of the indecomposable pairs of annihilating potent operators

The functor G_{nm} allows to obtain a classification of indecomposable objects (up to isomorphism) of any category $\mathcal{P}_k^{\circ}(n,m)$ of finite and tame types (see Theorem 2). To do this, it is need to take representatives of the classes of isomorphic indecomposable objects (one from each class) of the category $\operatorname{rep}_k Q$ with $Q = Q(n-1, m-1)$ and apply to them the functor G_{nm} (as a result we get all representatives of the classes of isomorphic indecomposable objects of $\mathcal{P}_k^{\circ}(n,m)$, except $(k,0,0)$). Such (of the most simple form) representatives are well-known: see [3] for $Q = A_2, A_3, D_4$ (our cases of finite type) and [4, 5] for $Q = \widetilde{A}_3, \widetilde{D}_4$ (our cases of tame type).

3. Remarks

3.1. All the above results are true if k is any field of characteristic 0 and $\varepsilon_n, \varepsilon_m \in k$.

3.2. All the above results are true if k is an algebraic closed field of characteristic $p \neq 0$, which does not divide nm.

3.3. All the above results are true if k is as in 3.2, but does not necessarily algebraically closed, and $\varepsilon_n, \varepsilon_m \in k$.

3.4. All the above results are true if k is an algebraic closed field of any characteristic and $\mathcal{A}, \mathcal{B}$ satisfy, respectively, polynomials $\varphi(x)$ and $\psi(x)$ of degrees n and m without multiple roots such that $\varphi(0) = 0$, $\psi(0) = 0$ (without the last condition the problem is trivial).

3.5. Theorem 1 is true if k is any field of any characteristic and $\mathcal{A}, \mathcal{B}$ satisfy, respectively, any fixed separable polynomials $\varphi(x)$ and $\psi(x)$ of degrees n and m such that $\varphi(0) = 0$, $\psi(0) = 0$ (see the definitions in [1]).

3.6. Classifying the pairs of idempotent operators. As the first author pointed out, the following classification of the pairs of idempotent operators (the objects of $\mathcal{P}_k(2,2)$) follows from [2, Section 3] and [4].

One will adhere to the matrix language. The field k is assumed to be any algebraic closed (otherwise, it is necessary to replace the below Jordan blocks in 1) by indecomposable Frobenius companion ones).

Let $J_m(\lambda)$ denotes the (upper) $m \times m$ Jordan block with diagonal entries λ, E_m the $m \times m$ identity matrix. Define ${}^0E_m$ (respectively, ${}_0E_m$) as E_m with added null first column (respectively, last row). For an $m \times m$ matrix X, put $X^+ = X$, $X^- = E_m - X$, and for a pair of $m \times m$ matrices $P = (X, Y)$ and $\mu, \nu \in \{+, -\}$, put $P^{\mu\nu} = (X^{\mu}, Y^{\nu})$. Finally, for matrices A, B with the same number of rows, introduce the squared matrices

$$F[A,B] = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \qquad S[A,B] = \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}.$$

Theorem 3. *The set of all pairs of matrices over k of the forms*

1) $P = (F[E_n, E_n], S[J_n(\lambda), E_n])$, $\lambda \in k \setminus 0$,

2) $P^{\mu\nu}$ *for* $P = (F[E_n, E_n], S[J_n(0), E_n])$ *and* $\mu, \nu \in \{+, -\}$,

3) $P^{\mu\nu}$ *for* $P = (F[E_n, {}_0E_{n-1}], S[{}^0E_{n-1}, E_{n-1}])$ *and* $\mu, \nu \in \{+, -\}$,

where n runs through the natural numbers, is a complete set of pairwise nonsimilar indecomposable pairs of idempotent matrices over k.

Note that this classification implies those of the pairs of involutory matrices (the representations of the infinite dihedral group) if $\operatorname{char} k \neq 2$.

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Normally ζ-reversible profinite groups

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Abstract. We examine (finitely generated) profinite groups in which two formal Dirichlet series, the normal subgroup zeta function and the normal probabilistic zeta function, coincide; we call these groups normally ζ-reversible. We conjecture that these groups are pronilpotent and we prove this conjecture if G is a normally ζ-reversible satisfying one of the following properties: G is prosoluble, G is perfect, all the nonabelian composition factors of G are alternating groups.

Assume that G is a profinite group with the property that for each positive integer n, G contains only finitely many open subgroups of index n. We denote by $\zeta_G(s)$ the Dirichlet generating function associated with the sequence counting the number of open subgroups of index n in G: so

$$\zeta_G(s) = \sum_{n\in\mathbb{N}} \frac{a_n(G)}{n^s}$$

where $a_n(G)$ is the number of open subgroups of G of index n and s is a complex variable. Another sequence of nonnegative integers can be associated to G by setting $b_n(G) = \sum_{|G:H|=n,\ H\leqslant_o G} \mu(H,G)$, where the Möbius function μ of the lattice of open subgroups of G is defined recursively by $\mu(G,G) = 1$ and $\sum_{H\leqslant K\leqslant_o G} \mu(K,G) = 0$ for any proper open subgroup $H <_o G$. Again we can consider the corresponding Dirichlet generating function

$$p_G(s) = \sum_{n\in\mathbb{N}} \frac{b_n(G)}{n^s}.$$

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The study of the subgroup sequence $\{a_n(G)\}_n$ and the corresponding zeta function $\zeta_G(s)$ started with [5]; since then there has been an intense research activity aiming at understanding analytical properties of subgroup zeta functions and their local factors for finitely generated nilpotent groups.

The formal inverse of $p_G(s)$ is the probabilistic zeta function which was first introduced and studied by A. Mann in [15] for finitely generated profinite groups and by N. Boston in [1] in the case of finite groups. A central role in the investigation of the properties of the probabilistic zeta function was played by the probabilistic meaning of $p_G(t)$ when G is a finite group and t is a positive integer: Hall in [9] showed that $p_G(t)$ is equal to the probability that t random elements of G generate G. In [15] Mann made a conjecture which implies that $p_G(s)$ has a similar probabilistic meaning for a wide class of profinite groups. More precisely, define $\mathrm{Prob}_G(t) = \mu(\Omega_G(t))$, where μ is the normalised Haar measure uniquely defined on the profinite group G^t and $\Omega_G(t)$ is the set of generating t-tuples in G (in the topological sense). We say that G is positively finitely generated if there exists a positive integer t such that $\mathrm{Prob}_G(t) > 0$. Mann considered the infinite sum

$$\sum_{H\leqslant G} \frac{\mu(H,G)}{|G:H|^s}.$$

As it stands, this is not well defined, but he conjectured that this sum is absolutely convergent if G is positively finitely generated. The Dirichlet series $p_G(s)$ can be obtained from this infinite sum, grouping together all terms with the same denominator so in particular Mann's conjecture implies that if G is positively finitely generated, then $p_G(s)$ converges in some right half-plane and $p_G(t) = \mathrm{Prob}_G(t)$, when $t \in \mathbb{N}$ is large enough. The second author proved in [13] that this is true if G is a profinite group with polynomial subgroup growth. But even when the convergence is not ensured, the formal Dirichlet series $p_G(s)$ encodes information about the lattice generated by the maximal subgroups of G and combinatorial properties of the probabilistic sequence $\{b_n(G)\}$ reflect on the structure of G. For example in [6] it is proved that a finitely generated profinite group G is prosoluble if and only if the sequence $\{b_n(G)\}$ is multiplicative.

One can ask whether and how the two formal Dirichlet series $\zeta_G(s)$ and $p_G(s)$ are related. The first example that it is usually presented is when $G = \widehat{\mathbb{Z}}$, the profinite completion of an infinite cyclic group. In this case $\zeta_{\widehat{\mathbb{Z}}}(s) = \sum_n 1/n^s$ is the Riemann zeta function, while $p_{\widehat{\mathbb{Z}}}(s) = \sum_n \mu(n)/n^s$ and an easy application of the Möbius Inversion Formula shows that

$p_{\widehat{\mathbb{Z}}}(s)$ and $\zeta_{\widehat{\mathbb{Z}}}(s)$ are one the multiplicative inverse of the other. A natural question is whether this is a particular coincidence or a more general phenomenon. Motivated by this question, in [4] it was introduced the notion of ζ-reversible profinite groups: a profinite group G is said to be ζ-reversible if and only if the formal identity $p_G(s)\zeta_G(s) = 1$ is satisfied. This definition can be introduced and studied independently of the convergence and possible analytic properties of $p_G(s)$ and $\zeta_G(s)$. Hence ζ-reversible only means that $\sum_{rs=n} a_r(G)b_s(G) = 0$ for each $n > 1$ while $a_1(G)b_1(G) = 1$. In [4] it is proved that, even when the convergence of the two series involved is not ensured, the information that G is ζ-reversible can have useful consequences. The results obtained in [4] indicate that ζ-reversibility is a strong property: a ζ-reversible group must have a sort of uniform subgroup structure, in the sense that the open subgroups, even when they are not all isomorphic, must have a comparable structure.

In this paper, our aim is to study a corresponding property, obtained by restricting the attention to the open normal subgroups of a profinite group G. We assume that G is a profinite group with the property that for each positive integer n, G contains only finitely many open normal subgroups of index n (a sufficient, but not necessary, condition for satisfying this property is that G is topologically finitely generated). For any $n \in \mathbb{N}$, let $a_n^{\triangleleft}(G)$ be the number of the open normal subgroups of G and let $b_n^{\triangleleft}(G) = \sum_{|G:H|=n, H\trianglelefteq_o G} \mu^{\triangleleft}(H,G)$, where $\mu^{\triangleleft}$ is the Möbius function in the lattice of the open normal subgroups of G. Again the properties of the sequences $\{a_n^{\triangleleft}(G)\}_{n\in\mathbb{N}}$ and $\{b_n^{\triangleleft}(G)\}_{n\in\mathbb{N}}$ can be encoded by the corresponding Dirichlet generating function

$$\zeta_G^{\triangleleft}(s) = \sum_{n\in\mathbb{N}} \frac{a_n^{\triangleleft}(G)}{n^s} \quad \text{and} \quad p_G^{\triangleleft}(s) = \sum_{n\in\mathbb{N}} \frac{b_n^{\triangleleft}(G)}{n^s}$$

called, respectively, the normal subgroup zeta function and the normal probabilistic zeta function of G. Again $p_G^{\triangleleft}(s)$ has a probabilistic meaning: if G is a finite group and $t \in \mathbb{N}$, then $p_G^{\triangleleft}(t)$ is the probability that t randomly chosen elements of G generate a subgroup whose normal closure is G (see [7, Section 3]). We will say that a profinite group G is normally ζ-reversible if $\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s) = 1$. We conjecture that a normally ζ-reversible profinite group is pronilpotent. An evidence for this conjecture will be given by the following theorem, which implies in particular that a prosoluble normally ζ-reversible profinite group is pronilpotent.

Theorem 1. *Assume that G is a normally ζ-reversible profinite group. If there is no open normal subgroup $N \lhd G$ such that G/N is a nonabelian simple group, then G is pronilpotent.*

Our main results are the following.

Theorem 2. *A non trivial normally ζ-reversible profinite group cannot be perfect.*

Theorem 3. *Let G be a normally ζ-reversible profinite group. If G is not pronilpotent, then G has as a composition factor a nonabelian simple group which is not an alternating group.*

The proofs of the previous two theorems rely on the following result (see Theorem 22): suppose that a normally ζ-reversible profinite group G admits a finite nonabelian simple group as an epimorphic image; then there exists a pair (H, T), where H is a finite epimorphic image of G and T is a finite nonabelin simple group, with the following properties:

1) $|H| = |T|^2$.
2) H contains a unique minimal normal sugroup N.
3) Either H/N is nilpotent, or there exists a finite nilpotent group X and a nonabelian simple group S such that $H/N \cong X \times S$. In the latter case $|T| \leqslant |S|$ and $\pi(S) = \pi(T)$.

With the help of the classification of the finite simple groups, we prove that there are no pairs (H, T) with these properties, under the additional assumption that either H is perfect or all the nonabelian composition factors of H are alternating groups.

1. Notations and general auxiliary results

Given an integer k and a set π of primes, k_π will be the greatest divisors of k whose prime divisors belong to π. In particular, with a little abuse of notation, if p is a prime we will call k_p the greatest power of p dividing k. Moreover we will say that k is a π-number if $k_\pi = k$.

Let $\mathcal{R}$ be the ring of formal Dirichlet series with integer coefficients. For every set π of prime number, we consider the ring endomorphism of $\mathcal{R}$ defined by:

$$F(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} \mapsto F_\pi(s) = \sum_{n \in \mathbb{N}} \frac{a_n^*}{n^s}$$

where $a_n^* = a_n$ if n is a π-number, $a_n^* = 0$ otherwise.

An element $F(s) = \sum_n a_n/n^s \in \mathcal{R}$ is said to be multiplicative if $a_{rs} = a_r a_s$ whenever $(r,s) = 1$ (equivalently $F(s)$ coincides with the infinite formal product $\prod_p F_p(s)$of its p-local factors). It can be easily proved that if $F(s)$ is multiplicative, then also the formal inverse $F(s)^{-1}$ is multiplicative.

During our proofs we will need information about the "prime gap". For our purpose the following result will suffice.

Lemma 4. *For every integer $n \geqslant 5$, $n \notin \{6, 10\}$, there exist two primes p, q such that $\frac{n}{2} < p < q \leqslant n$.*

This lemma is in fact a corollary of a more complete result, proved by Nagura in [14], stating that, if $n \geqslant 25$, then there is a prime p such that $n \leqslant p \leqslant 6n/5$.

We conclude this section by recalling some results concerning the finite nonabelian simple groups.

A crucial role in our proof will be played by the following result:

Theorem 5. *[11, Theorem 6.1] Let S and T be non-isomorphic finite simple groups. If $|S^a| = |T^b|$ for some natural numbers a and b, then $a = b$ and S and T either are $A_2(4)$ and $A_3(2)$ or are $B_n(q)$ and $C_n(q)$ for some $n \geqslant 3$ and some odd q.*

This result is a consequence of a collection of more general results obtained in [11] and leading to the conclusion that a finite simple group is in general uniquely determined by some partial information on its order encoded by some arithmetical invariants (called Artin invariants). We will make a large use of these results, so we recall here some related definitions.

Definition 6. Let n be a natural number and r one of its prime divisors. The greatest power of r dividing n is called the contribution of r to n and is denoted by n_r. Moreover, r is called the dominant prime if $n_r > n_q$ for every other prime q. Given a finite group G, we will call the dominant prime of G the dominant prime of its order. We will use the symbol $p(G)$ to denote the dominant prime of G.

Proposition 7. *[11, Theorem 3.3] The dominant prime of a simple group of Lie type coincides with its characteristic, apart from the following cases:*

1) *$A_1(q)$, where q is a Mersenne prime;*
2) *$A_1(q-1)$, where q is a Fermat prime;*
3) *$A_1(8)$, ${}^2A_2(3)$, ${}^2A_3(2)$.*

Definition 8. Let G be a finite group and $p = p(G)$ its dominant prime, then

$$\lambda(G) = \frac{\log(|G|_p)}{\log(|G|)}$$

is called the logarithmic proportion of G.

Proposition 9. *[11, Theorems 3.5, 3.6] Let $x = p^u$ be the contribution of the dominant prime of a finite simple group S of Lie type, then $x^2 < |G| < x^3$, that is*

$$\frac{1}{3} < \lambda(G) < \frac{1}{2}.$$

Definition 10. Let n be an integer which is not a prime power, let $p = p(n)$ be its dominant prime and p^l its contribution to n, then we define $\omega(n)$ as the largest order of p modulo a prime divisor p_1 of n/p^l. We will call such a p_1 a prominent prime in n.

Lemma 11. *[11, Lemma 4.2] Given n and $\alpha \in \mathbb{N}$, then $\omega(n^\alpha) = \omega(n)$. Furthermore, if p_1 is prominent in n with contribution $p_1^{l_1}$, then it is also prominent in n^α with contribution $p_1^{l_1 \alpha}$.*

Remark 12. Notice that, if a and b have the same prime divisors and the same dominant prime, then they have also the same prominent prime and $\omega(a) = \omega(b)$.

Let $S = L(q)$ be a finite simple group of Lie type, defined over a field of cardinality $q = p^r$, where p is a prime (which we will call the characteristic of S). We will factorize the order of a simple group $S = L(q)$ of Lie type in the form

$$|L(q)| = \frac{1}{d} q^h P(q),$$

where d, h and $P(q)$ are given in [11, Table L1]. In particular this order has the *cyclotomic factorization in terms of* p:

$$|L(q)| = \frac{1}{d} p^l \prod_m \Phi_m(p)^{e_m},$$

where $\Phi_m(x)$ is the m-th cyclotomic polynomial. Summing up [11, Proposition 4.5] and [11, Lemma 4.6], we obtain:

Theorem 13. *Let $S = L(q)$ be a simple group of Lie type with characteristic p and $q = p^r$. Then the cyclotomic factorization*

$$|S| = \frac{1}{d} p^{rh} \Phi_{\alpha_1}(p)\Phi_{\alpha_2}(p)\Phi_{\alpha_3}(p)\cdots\Phi_{\alpha_u}(p)$$

satisfies the following properties:
1) $\alpha_1 > \alpha_2$;
2) d *divides* $\Phi_{\alpha_3}(p)\cdots\Phi_{\alpha_u}(p)$ *unless* $S = A_1(q)$ *and* $r = 1$;
3) $\omega(|S|) = \alpha_1$ *unless* $p = 2$ *and* $\alpha_1 = 6$.

Definition 14. Let G be a group with dominant prime p_1, let $p_1^{n_1}$ be its contribution to the order of G. Suppose that p_i is a prime dividing the order of G and that $p_i^{n_i}$ is the contribution to the order. Then p_i is called a good contributor to G if $n_i \log(p_i)\log(3) > n_1 \log(p_1)\log(2)$.

The good contributors of the finite simple groups are classified in [2].

For later use we need to recall some definitions and results concerning Zsigmondy primes.

Definition 15. A prime number p is called a *primitive prime divisor* of $a^n - 1$ if it divides $a^n - 1$ but it does not divide $a^e - 1$ for any integer $1 \leqslant e \leqslant n-1$.

The following theorem is due to K. Zsigmondy [21]:

Theorem 16 (Zsigmondy's Theorem)**.** *Let a and n be integers greater than 1. There exists a primitive prime divisor of $a^n - 1$ except exactly in the following cases:*
1) $n = 2$, $a = 2^s - 1$ *(i.e. a is a Mersenne prime), where* $s \geqslant 2$.
2) $n = 6$, $a = 2$.

Primitive prime divisors have a close relation with the cyclotomic factorization described in Theorem 13: if r is a primitive prime divisor of $p^n - 1$, then n is the smallest positive integer with the property that r divides $\Phi_n(p)$.

2. A reduction to a question on finite groups

Assume that G is a profinite group and let $\mathcal{S}$ be the set of the open normal subgroups N of G with the property that $S_N := G/N$ is a nonabelian simple group. Let

$$A_G(s) = P_{G/G'}(s) \quad \text{and} \quad B_G(s) = \prod_{N\in\mathcal{S}} \left(1 - \frac{1}{|S_N|^s}\right).$$

We know from [7, Section 5] that

$$p_G^{\triangleleft}(s) = A_G(s)B_G(s). \tag{2.1}$$

Now consider the two series

$$\Gamma_G(s) := (A_G(s))^{-1} = \sum_n \frac{\gamma_n(G)}{n^s}, \quad \Delta_G(s) := (B_G(s))^{-1} = \sum_n \frac{\delta_n(G)}{n^s}.$$

Lemma 17. *If G is a normally zeta-reversible profinite group, then*

$$\Gamma_G(s) = \prod_p \Gamma_{G,p}(s) = \prod_p \zeta_{G,p}^{\triangleleft}(s).$$

Proof. Since G is normally ζ-reversible, we have

$$1 = (\zeta_G^{\triangleleft}(s)p_G^{\triangleleft}(s))_p = \zeta_{G,p}^{\triangleleft}(s)p_{G,p}^{\triangleleft}(s) = \zeta_{G,p}^{\triangleleft}(s)A_{G,p}(s)B_{G,p}(s).$$

Since $A_G(s)$ and $\Gamma_G(s)$ are multiplicative series, we deduce

$$\Gamma_G(s) = \prod_p \Gamma_{G,p}(s) = \prod_p A_{G,p}(s)^{-1} = \prod_p \zeta_{G,p}^{\triangleleft}(s)B_{G,p}(s),$$

but there are no nonabelian simple groups whose order is a prime power, thus $B_{G,p}(s) = 1$ for every prime p and we get $\Gamma_G(s) = \prod_p \zeta_{G,p}^{\triangleleft}(s)$. □

Lemma 18. *If G is a normally zeta-reversible profinite group, then for every $n \in \mathbb{N}$, $\gamma_n(G)$ coincides with the number of open normal subgroups N of G with the property that G/N is a nilpotent group of order n.*

Proof. For every $m \in \mathbb{N}$, let $\mathcal{N}_m$ be the set of the open normal subgroups N of G with the property that G/N is nilpotent of order m. Let $n \in \mathbb{N}$ and write $n = q_1 \cdots q_r$ as a product of powers of different primes. If $N_i \in \mathcal{N}_{q_i}$ for every $1 \leqslant i \leqslant r$, then $N = N_1 \cap \cdots \cap N_r \in \mathcal{N}_n$. Conversely every $N \in \mathcal{N}_n$ can be uniquely expressed in the form $N = N_1 \cap \cdots \cap N_r$, with $N_i \in \mathcal{N}_{q_i}$ for every $1 \leqslant i \leqslant r$. This implies that $|\mathcal{N}| = |\mathcal{N}_{q_1}| \cdots |\mathcal{N}_{q_r}|$. On the other hand if q is a prime power and N is an open normal subgroup of G of index q, then G/N, being a p-group, is nilpotent, hence $|\mathcal{N}_q| = a_q^{\triangleleft}(G)$; moreover $a_q^{\triangleleft}(G) = \gamma_q(G)$ by Lemma 17. Hence

$$\gamma_n(G) = \gamma_{q_1}(G) \cdots \gamma_{q_r}(G) = a_{q_1}^{\triangleleft}(G) \cdots a_{q_r}^{\triangleleft}(G) = |\mathcal{N}_{q_1}| \cdots |\mathcal{N}_{q_r}| = |\mathcal{N}|. \square$$

Proof of Theorem 1. If there is no open normal subgroup N of G such that G/N is a nonabelian simple group, then $B_G(s) = 1$, hence, by (2.1), we have $\Gamma_G(s) = A_G(s)^{-1} = p_G^{\triangleleft}(s)^{-1} = \zeta_G^{\triangleleft}(s)$, i.e. $\gamma_n(G) = a_n^{\triangleleft}(G)$ for every $n \in \mathbb{N}$. We conclude from Lemma 18 that G/N is nilpotent for every open normal subgroup N of G. □

Conjecture 1. If G is a normally ζ-reversible profinite group, then there is no open normal subgroup $N \lhd G$ such that G/N is a nonabelian simple group (and consequently G is pronilpotent).

For the remaining part of this section we will assume that G is a counterexample to the previous conjecture. We will denote with Σ_G the set of the finite nonabelian simple groups which are continuous epimorphic images of G. Take $T \in \Sigma_G$ with the property that the set $\pi = \pi(T)$ of the prime divisors of $|T|$ is minimal and let $M = O^{\pi}(G)$ be the intersection of the open normal subgroups N of G with the property that G/N is a π-group. It can be easily checked that G/M is a pro-π-group. Moreover $\zeta^{\lhd}_{G/M}(s) = \zeta^{\lhd}_{G,\pi}(s)$ and $p^{\lhd}_{G/M}(s) = p^{\lhd}_{G,\pi}(s)$. But then $\zeta^{\lhd}_{G/M}(s)p^{\lhd}_{G/M}(s) = \zeta^{\lhd}_{G,\pi}(s)p^{\lhd}_{G,\pi}(s) = (\zeta^{\lhd}_{G}(s)p^{\lhd}_{G}(s))_{\pi} = 1$, hence G/M is still a normally ζ-reversible profinite group and represents a counterexample to Conjecture 1. So we may assume that $M = 1$. With this assumption, if $S \in \Sigma_G$, then S is a π-group and, by the minimality property of T, $\pi \leqslant \pi(S)$. Hence $\pi(S) = \pi$ for every $S \in \Sigma_G$. There are only finitely many nonabelian simple groups S with $\pi(S) = \pi$, hence Σ_G is finite. Let $m = |T| = m_1 < m_2 < \cdots < m_u$ be the orders of the nonabelian simple in Σ_G and for $i \in \{1, \dots, u\}$ let t_i (with $t = t_1$) be the cardinality of the set of the open normal subgroups N of G such that G/N is a nonabelian simple group of order m_i. We must have:

$$\Delta_G(s) = \left(\prod_i \left(1 - \frac{1}{m_i^s}\right)^{t_i}\right)^{-1} = \prod_i \left(\sum_{j=0}^{\infty} \frac{1}{m_i^{s \cdot j}}\right)^{t_i}$$

and

$$\zeta^{\lhd}_G(s) = \Gamma_G(s)\Delta_G(s) = \Gamma_G(s) \prod_i \left(1 + \frac{1}{m_i^s} + \frac{1}{m_i^{2s}} + \cdots\right)^{t_i}.$$

We now want to collect information about the open normal subgroups N of G with $|G/N| \leqslant m^2$. Consider the series

$$\sum_n \frac{a_n^*}{n^s} := \Gamma_G(s) \left(1 + \frac{1}{m^s} + \frac{1}{m^{2s}}\right)^t \prod_{i=2}^{u} \left(1 + \frac{1}{m_i^s}\right)^{t_i}.$$

If $n \leqslant m^2$, then, as $n < m_i^2$ for $i \neq 1$, we have $a_n^{\lhd}(G) = a_n^*$.

Lemma 19. *Let N be an open normal subgroup of G. If $|G/N| < m^2$ then either G/N is nilpotent or $G/N \cong X_1 \times X_2$ where X_1 is nilpotent and X_2 is a nonabelian simple group.*

Proof. If $n < m^2$, then

$$a_n^{\lhd}(G) = a_n^* = \gamma_n(G) + \sum_{m_i r = n} t_i \gamma_r(G). \tag{2.2}$$

Let $\mathcal{N}_r$ be the set of the open normal subgroups N of G with the property that G/N is nilpotent of order r and let $\mathcal{S}_i$ be be the set of the open normal subgroups M of G with the property that G/M is a nonabelian simple group of order m_i. Suppose $m_i r = n$. If $N \in \mathcal{N}_r$ and $M \in \mathcal{S}_i$, then $G/(N \cap M) \cong G/N \times G/M$ (since the nilpotent group G/N and the simple group G/M have no common composition factor) and this is the unique way to obtain $N \cap M$ as intersection of two subgroups in $\mathcal{N}_{r^*}$ and $\mathcal{S}_{i^*}$, for some $r^* \leqslant n$ and $i^* \leqslant u$. Hence there are at least a_n^* open normal subgroups N of G of index n and with the property that G/N is either nilpotent or is the direct product of a nilpotent subgroup with a finite nonabelian simple group. Since, by (2.2), $a_n^{\lhd}(G) = a_n^*$ all the open normal subgroups of G of index n have this property. □

Let us consider now the set of open normal subgroups of index m^2 in G: in this case we have

$$a_{m^2}^{\lhd}(G) = a_{m^2}^* = \gamma_{m^2}(G) + \sum_{m_i r = m^2} t_i \gamma_r(G) + \binom{t}{2} + t. \tag{2.3}$$

With the same arguments used in the proof of the previous lemma, it can be easily noticed that:

Lemma 20. *The first three summands in the previous expression of $a_{m^2}^{\lhd}(G) = a_{m^2}^*$ have the following meaning:*

1) *$\gamma_{m^2}(G)$ is the number of the open normal subgroups N of index m^2 such that G/N is nilpotent;*
2) *$\sum_{m_i r = m^2} t_i \gamma_r(G)$ is the number of the open normal subgroups N of index m^2 such that G/N is a direct product of a nilpotent group and a nonabelian simple group.*
3) *$\binom{t}{2}$ is the number of the open normal subgroups N of index m^2 such that G/N is the direct product of two nonabelian simple groups of order m.*

Notice that the last summand in equation (2.3) consists of t open normal subgroups of index m^2 that does not fill in any of the three classes described in Lemma 20: let M be one of these normal subgroups and let $H = G/M$.

Lemma 21. *H has a unique minimal normal subgroup.*

Proof. Suppose by contradiction that H has two different minimal normal subgroups N_1, N_2. By Lemma 19, there exists two finite nilpotent groups X_1, X_2 and two finite groups Y_1 and Y_2 that are either trivial or nonabelian and simple such that $G/N_1 \cong X_1 \times Y_1$ and $G/N_2 \cong X_2 \cap Y_2$. Since $N_1 \cap N_2 = 1$, H is a subdirect product of $X_1 \times X_2 \times Y_1 \times Y_2$, However this implies that H is either nilpotent, or it is the direct product of two nonabelian simple groups of order m, or it is the direct product of a simple nonabelian group with a nilpotent group; but then M fills in one of the three family of open normal subgroups described in Lemma 20, a contradiction. □

We may summarize the conclusions of this section in the following statement.

Theorem 22. *If Conjecture 1 is false, then there exists a finite nonabelian simple group T and a finite group H with the following properties:*

1) $|H| = |T|^2$.
2) *H contains a unique minimal normal sugroup N.*
3) *Either H/N is nilpotent, or there exists a finite nilpotent group X and a nonabelian simple group S such that $H/N \cong X \times S$. In the latter case $|T| \leqslant |S|$ and $\pi(S) = \pi(T)$.*

3. Perfect profinite groups

In this section we concentrate our attention on the case of perfect profinite groups. Our aim is to prove that a perfect profinite group cannot be normally ζ-reversible.

It follows immediately from Theorem 22 that:

Proposition 23. *If there exists a perfect normally ζ-reversible profinite group, then exist there a finite nonabelian simple group T and a finite perfect group H with the following properties:*

1) $|H| = |T|^2$.
2) *H contains a unique minimal normal sugroup N.*
3) *There exists a finite nonabelian simple group S such that $H/N \cong S$. Moreover $|T| \leqslant |S|$ and $\pi(S) = \pi(T)$.*

Lemma 24. *If H is a finite group satisfying the statement of Proposition 23, then $N = \operatorname{soc} H$ is abelian.*

Proof. Suppose by contradition that N is nonabelian: there exist a nonabelian simple group L and a positive integer u such that $N = L_1 \times \cdots \times L_u$, with $L_i \cong L$ for all i. It must be $u \neq 1$ (otherwise, by the Schreier conjecture, H/N would be soluble). The conjugation action on $\{L_1, \cdots, L_u\}$ induces a homomorphism $\psi : H \to \mathrm{Sym}(u)$ and $\psi(H)$ is a transitive subgroup of $\mathrm{Sym}(u)$. The kernel of this action coincides with N so $S \cong H/N \cong \psi(H)$. In particolar S contains a subgroup of index u. We have two cases:

1) $S \cong \mathrm{Alt}(n)$ for some n. We must have $n \leqslant u$. Moreover, by Lemma 4, there exists a prime number r such that $n/2 < r \leqslant n$, in particular r divides $|S|$ with multiplicity 1. On the other hand $|H| = |T|^2 = |S||N| = |S||L|^u$, hence $r|\,|L|$. Since finite nonabelian simple groups have even order, we deduce that $2r$ divides $|L|$ and $(2r)^u$ divides $|N|$, thus

$$\frac{|T|^2}{|S|} = |N| \geqslant (2r)^u \geqslant n^u \geqslant n^n > \frac{n!}{2} = \left|\frac{H}{N}\right| = |S|,$$

but then $|T| > |S|$, against Proposition 23.

2) S is not an alternating group and has a (faithful) transitive action of degree u. In particular S has a primitive action of degree $v \leqslant u$, hence, by [16], $|S| \leqslant 4^v \leqslant 4^u$. By Proposition 23, $|T| \leqslant |S|$, hence

$$|L|^u = |N| = \frac{|T|^2}{|S|} \leqslant |S| \leqslant 4^u,$$

but then $|L| \leqslant 4$, contradiction. □

Corollary 25. *If there exists a perfect normally ζ-reversible profinite group, then there exists a triples (S, T, V) with the following properties:*

1) *T and S are finite nonabelian simple groups;*
2) *V is an irreducible S-module of dimension a over the field with p elements;*
3) $|T|^2 = |S| \cdot |V| = |S| \cdot p^a$;
4) $|V| < |T| < |S|$;
5) $p \in \pi(T) = \pi(S)$;
6) *if $a = 1$, then p divides the order of the Schur multiplier $M(S)$ of S and divides $|S|$ with multiplicity at least 3.*

Proof. The first five statements follow immediately from Proposition 23, taking $V = \mathrm{soc}(H)$ (we cannot have $|S| = |T|$, since this would imply $|T| = p^a$). We have only to prove (6). A faithful irreducible representation of a nonabelian simple group cannot have degree 1; thus, if $a = 1$, then V is a central S-module: in particular $H = V.S$ is a central perfect extension of S and, consequently, $|V| = p$ divides $|M(S)|$. Moreover, if $a = 1$ then, by (3), p must divide $|S|$ with odd multiplicity. Now suppose that $a = 1$ and p divides $|S|$ with multiplicity 1: then a Sylow p-subgroup of H, having order p^2, is abelian. We apply [10, Proposition 5.6] stating that, if a group J has an abelian Sylow p-subgroup, then p does not divide $|J' \cap Z(J)|$: since $H' = H$ and $Z(H) = \operatorname{soc} H \cong V$, we would have that p does not divide $|V| = p$, a contradiction. □

In the remaining part of this section, we will prove that there is no triple (S, T, V) satisfying the properties listed in the previous corollary. Suppose by contradiction that such a triple (S, T, V) exists.

Remark 26. Since $|S| \cdot p^a = |T|^2$, every prime divisor of $|S|$ different from p divides $|S|$ with even multiplicity.

Proposition 27. *S is a simple group of Lie type.*

Proof. By Remark 26, it suffices to prove that, if S is alternating or sporadic, then there are at least two primes dividing $|S|$ with odd multiplicity. This can be directly verified for the sporadic groups and for the alternating groups $\mathrm{Alt}(n)$ when $n \leqslant 10$. For the remaining alternating groups, we deduce from Lemma 4 that there are at least two primes p, q dividing $|\mathrm{Alt}(n)| = n!/2$ with multiplicity exactly one. □

Proposition 28. *If $a \neq 1$, then p is the characteristic of S.*

Proof. If $a \neq 1$, then a is the degree of a faithful irreducible representation of S over the field of order p. Assume, by contradiction, that p does not coincide with the characteristic of S. We must have $a \geqslant \delta(S)$, denoting by $\delta(S)$ the smallest degree of a nontrivial irreducible representation of S in cross characteristic. Lower bounds for the degree of irreducible representations of finite groups of Lie type in cross characteristic were found by Landazuri and Seitz [12] and improved later by Seitz and Zalesskii [17] and Tiep [18]. It turns out that $\delta(S)$ is quite large, and, apart from finitely many exceptions, we have $p^{\delta(S)} > |S|$, in contradiction with $|S| > p^a \geqslant p^{\delta(S)}$. The few exceptions can be easily excluded, proving directly that, for these particular choices of S, there are no T and V with $|T^2| = |S| \cdot |V|$. For example, if $S = A_n(q)$ with $n \geqslant 2$, then $|S| < q^{n^2+2n}$ and, except in the exceptional cases $(n, q) = (2, 2), (2, 4), (3, 2), (3, 3)$, we have $\delta(S) \geqslant \frac{q^{n+1}-q}{q-1} - 1$ [18, Table II], which implies that either $p^{\delta}(S) > |S|$ or $(n, q) = (2, 3)$. On the other hand, if $(n,q) = (2,2), (2,3), (2,4), (3,2), (3,3)$, then there are at least two primes dividing $|S| = |A_n(q)|$ with odd multiplicites, so these cases must be excluded by Remark 26. The other families of finite simple groups of Lie type can be discussed with similar arguments. □

Proposition 29. *The dominant prime of S coincides with the characteristic of S.*

Proof. By Proposition 7, if the dominant prime of S does not coincide with the characteristic of S, then one of the following three cases occurs.

1) $S = A_1(q)$, with $q = 2^t - 1$ a Mersenne prime. We must have that t is an odd prime but then 2 and q divide $|S| = (q-1) \cdot q \cdot (q+1)/2$ with odd multiplicity, against Remark 26.
2) $S = A_1(q-1)$ with $q = 2^{2^k} + 1$ a Fermat prime. Since

$$|T|^2 = (q-2) \cdot (q-1) \cdot q \cdot p^a$$

we have that $p = q$, a is odd and $|T|^2 = \left(2^{2^k}+1\right)^{a+1} 2^{2^k} \left(2^{2^k}-1\right)$; this would imply that $2^{2^k} - 1$ is a square too, which is impossible.
3) $S \in \{A_1(8), {}^2A_2(3), {}^2A_3(2)\}$. The orders $|A_1(8)|$ and $|{}^2A_2(3)|$ are divisible by at least two different primes with odd multiplicity, so these two cases must be excluded. If $S = {}^2A_3(2)$, then $|T|^2 = |S| \cdot p^a = 2^6 \cdot 3^4 \cdot 5 \cdot p^a$, hence $p = 5$, a is odd and the condition $|T| < |S|$ implies $a = 1, 3, 5$; however it cannot be $a = 1$ since 5 does not divide the order of the Schur multiplier of ${}^2A_3(2)$, and it cannot be $a = 3, 5$ since there exists no simple group of order $2^3 \cdot 3^2 \cdot 5^2$ or $2^3 \cdot 3^2 \cdot 5^3$. □

Corollary 30. *If $a \neq 1$, then p is the dominant prime of S and T.*

Proof. Suppose $a \neq 1$. By Propositions 28 and 29, p is the characteristic and the dominant prime of S. Since $|T|^2 = |S| \cdot p^a$, p is also the dominant prime of T. □

Proposition 31. *T is not an alternating group.*

Proof. Let $T = \mathrm{Alt}(m)$, $m \geqslant 5$. First assume $m \leqslant 9$. We use [3, p. 239–242] to check that if $|S|$ is a finite simple group with $\pi(S) = \pi(T)$ and $|T|^2 = |S| \cdot p^a$ for some prime power p^a, then $m = 6$, $p = 5$, $a = 1$ and $S = {}^2A_3(2)$; however we must exclude this possibility, since 5 does not divide the order of the Schur multiplier of ${}^2A_3(2)$.

So from now on we will assume $m \geqslant 10$. This implies that 2 is the dominant prime of T [11, Table L.4]. We will prove that the dominant prime of S is 2 too. Suppose, by contradiction, that the dominant prime q of S is not 2. Then, being $|T|^2 = |S|p^a$, we must have $p = 2$ and, by Corollary 30, $a = 1$, so

$$|T|^2 = 2\,|S|\,. \tag{3.1}$$

Let $|T|_2 = 2^t$, $|T|_q = q^h$, then $2^t > q^h$ (as 2 is the dominant prime of T) and, by (3.1), $q^{2h} > 2^{2t-1}$ (as q is the dominant prime of S). Joining these inequalities we get $q^h < 2^t < q^{h+1/2}$, whence $h\log(q) < t\log(2) < \left(h + \frac{1}{2}\right)\log(q)$, and so

$$1 < \frac{t\log(2)}{h\log(q)} < 1 + \frac{1}{2h} \leqslant \frac{3}{2} < \frac{\log(3)}{\log(2)}. \tag{3.2}$$

By Equation (3.2), q is a good contributor to T, but [2, Theorem 3.8] enlists all good contributors to alternating groups, and for $m \geqslant 10$, it must be

$$\begin{cases} q = 3 & \text{or} \\ q = 5 & \text{and } m \in \{10, 11, 15, 25, 26, 30\}. \end{cases}$$

Moreover [2, 3.2] gives some useful lower and upper bounds for t, h as linear functions on m. Using these bounds and some direct computations for the small values of m, it can be easily proved that the only case in which we really have $q^{2h} > 2^{2t-1}$ is when $q = 3$ and $m = 15$; however, we can again use [3, p. 239–242] to see that there is no simple group S with $2 \cdot |S| = |\operatorname{Alt}(15)|^2$, against (3.1).

Now we claim that $p \neq 2$. Indeed, assume by contradiction, $p = 2$. By Corollary 30, it must be $a = 1$. If $m = 10$, then we would have $\lambda(S) < 1/3$, in contradiction with Proposition 9. For $m \geqslant 11$ we have $\lambda(\operatorname{Alt}(m)) < 1/3$ (see [11, Table L.4]), hence

$$\frac{1}{3} > \frac{\log(|T|_2^2)}{\log(|T|^2)} = \frac{\log(|S|_2) + \log(2)}{\log(|S|) + \log(2)} > \frac{\log(|S|_2)}{\log(|S|)}$$

contradicting again Proposition 9.

Thus S and T both have dominant prime 2 and p is odd. By Proposition 9

$$\left(\frac{m}{e}\right)^m < \frac{m!}{2} = |T| < |S| \leqslant |S|_2^3 \leqslant |T|_2^6\,. \tag{3.3}$$

Let $|T|_2 = 2^l$, then we can estimate l by

$$l = \sum_{i=1}^{\infty} \left[\frac{m}{2^i}\right] - 1 < \sum_{i=1}^{\infty} \frac{m}{2^i} - 1 = m - 1.$$

This result, joined with (3.3), gives $m < e \cdot 2^{6-12/m}$; in particular $m \leqslant 165$.

Since $p \neq 2$, we have $|S|_2 = |T|_2^2$ and, by Proposition 9,

$$\frac{1}{3} \leqslant \frac{\log(|S|_2)}{\log(|S|)} = \frac{\log(|T|_2^2)}{\log(|T|)^2 - a\log(p)}. \tag{3.4}$$

Moreover 3 is dominant prime of $|\mathrm{Alt}(m)|_{2'}$ for every $m \geqslant 10$ (see [2, Theorem 3.7 (b)]), so

$$p^a \leqslant \frac{|T|_3^2}{3}. \tag{3.5}$$

From Equations (3.4) and (3.5) we finally get

$$\frac{1}{3} \leqslant \frac{\log(|T|_2^2)}{\log(|T|)^2 - \log(|T|_3^2) + \log(3)} = \frac{\log(|T|_2)}{\log(|T|_{3'}) + \log(3)/2} \tag{3.6}$$

and it is easy to verify that, in the given range $10 \leqslant m \leqslant 165$, (3.6) is true only for $10 \leqslant m \leqslant 14$ or $16 \leqslant m \leqslant 21$ or $m = 24$. In all these cases, S should be a simple group of Lie type of characteristic 2 with the property that $|S| = |\,\mathrm{Alt}(m)|^2 \cdot p^a$ for some odd prime prime $p \leqslant m$ and some positive integer a. A boring but elementary check shows that there is no simple group S with these properties. □

Proposition 32. *T is not a sporadic simple group.*

Proof. At first, we will prove that S and T have the same dominant prime. Suppose by contradiction that the dominant primes do not coincide: then, since $|T|^2 = |S|p^a$, p coincides with the dominant prime of T and, by Corollary 30, $a = 1$. So we have

$$|T|^2 = p \cdot |S|. \tag{3.7}$$

Let q be the dominant prime of $|T|_{p'}$, necessarily it is the dominant prime of S. Let $|T|_p = p^t$, $|T|_q = q^h$, then $p^t > q^h$ and, by (3.7), $q^{2h} > p^{2t-1}$, so we get

$$q^h < p^t < q^{ht/(t-1/2)}.$$

By Corollary 25 (6), it must be $t > 1$ so

$$1 < \frac{t\log(p)}{h\log(q)} < \frac{t}{t-1/2} < \frac{\log(3)}{\log(2)}. \tag{3.8}$$

This implies that q is a good contributor to T. The good contributors to sporadic simple groups are listed in [2, Theorem 1]: it is easy to verify that these good contributors does not satisfy (3.8), apart from the cases $T = F_5$ and $T = J_1$. However

$$\begin{cases} T = F_5 \Rightarrow |S| = |F_5|^2/2 \Rightarrow \lambda(S) < 1/3 \\ T = J_1 \Rightarrow |S| = |J_1|^2/19 \Rightarrow \lambda(S) < 1/3. \end{cases}$$

contradicting Proposition 9.

Thus, we know that S and T have the same dominant prime $p(S)$

Now suppose $a \neq 1$. Then $p = p(S)$ by Corollary 30 and $\lambda(S) > 1/3$ by Proposition 9, so

$$\frac{1}{3} < \frac{2\log(|T|_p) - a\log(p)}{2\log(|T|) - a\log(p)}$$

whence

$$2 \leqslant a \leqslant \left\lfloor \frac{3\log(|T|_p) - \log(|T|)}{\log(p)} \right\rfloor = a_*(T). \tag{3.9}$$

It can be easily checked that Equation (3.9) is satisfied only if

$$T \in \{B, Fi_{22}, Co_2, Ru, M_{24}, M_{22}, {}^2F_4(2)'\}.$$

All these groups have dominant prime 2, so $p = p(S) = p(T) = 2$ and S should be a simple group of Lie type of characteristic 2 with $|T|^2 = |S| \cdot 2^a$ and $2 \leqslant a \leqslant a_*(T)$. It can be checked that no simple group S satisfies these conditions.

Thus, $a = 1$. In particular, $|S| = |T|^2/p$. A direct computation shows that that, for every possible choice of a sporadic simple group T and every prime divisor p of its order, there is no simple group of Lie type satisfying this condition (many possibilities can be excluded since they are not compatible with the condition $\lambda(S) > 1/3$). □

So from now on we may assume that both S and T are simple groups of Lie type.

Lemma 33. *If p is the dominant prime of S, then p coincides with the characteristic of T.*

Proof. Suppose that p is the dominant prime of S. Since $|T|^2 = |S| \cdot p^a$, p is also the dominant prime of T. By Proposition 7, if p does not coincide with the characteristic of T, then one of the following cases occurs.

1) $T = A_1(q)$, where $q = 2^k - 1$ is a Mersenne prime (so in particular k is prime). The dominant prime of T is 2. So $p = 2$ and, by Proposition 29, it also coincides with the characteristic of S. The order of $|S|$ has a cyclotomic factorization in term of 2 as it is described in the statement of Theorem 13. We have

$$|S| = \frac{|T|^2}{2^a} = 2^{2k-a} \cdot (2^k - 1)^2 \cdot (2^{k-1} - 1)^2 = \frac{2^b \cdot \Phi_{\alpha_1}(2) \cdots \Phi_{\alpha_u}(2)}{d}.$$

We must have $\alpha_1 = k$. Moreover $\Phi_k(2) = 2^k - 1 = q$, and the multiplicity of $\Phi_k(2)$ in the factorization of $|S|$ is 2, so $\alpha_2 = \alpha_1$, contradicting Theorem 13 (1).

2) $T = A_1(q-1)$, where $q = 2^{2^k} + 1$ is a Fermat prime. Then q is the dominant prime of T, whence $q = p$ and $(q\cdot(q-1)\cdot(q-2))^2 = |S|\cdot q^a$, in particular $q^2 = q^a \cdot |S|_q$. As $|S|$ and $|T|$ have the same prime divisors, q must divide $|S|$, so $a = 1$, but then $|S| = q\cdot(q-1)^2\cdot(q-2)^2$ and

$$|S|_2 = (q-1)^2 = 2^{2^{k+1}} > 2^{2^k} + 1 = q = |S|_q\,,$$

thus q cannot be the dominant prime for S, a contradiction.

3) $T = A_1(8)$. Then $|T| = 2^3 \cdot 3^2 \cdot 7$, $p = 3$ and $2^6 \cdot 3^4 \cdot 7^2 = |S| \cdot 3^a$ for $a \geqslant 1$, whence $|S|_3 \leqslant 3^3 < 2^6 = |S|_2$, a contradiction.

4) $T = {}^2A_2(3)$. Then $|T| = 2^5 \cdot 3^3 \cdot 7$, $p = 2$ and $2^{10} \cdot 3^6 \cdot 7^2 = |S| \cdot 2^a$ for $a \geqslant 1$, whence $|S|_2 \leqslant 2^9 < 3^6 = |S|_3$, a contradiction.

5) $T = {}^2A_3(2)$. Then $|T| = 2^6 \cdot 3^4 \cdot 5$, $p = 3$ and $2^{12} \cdot 3^8 \cdot 5^2 = |S| \cdot 3^a$ for $a \geqslant 1$, whence $|S|_3 \leqslant 3^7 < 2^{12} = |S|_2$, a contradiction. □

From Lemma 33, Proposition 28 and Proposition 29, it follows:

Corollary 34. *If $a \neq 1$, then p coincides with the characteristic and dominant primes of S and T.*

Lemma 35. *Let $\alpha_1(T), \alpha_1(S)$ be the greatest indexes in the cyclotomic decompositions of $|T|$ and $|S|$ described in Theorem 13. Then*

$$\alpha_1(T),\ \alpha_1(S) \geqslant 2$$

and, denoting by p_T and p_S the characteristics of S and T, we have

$$(p_T, \alpha_1(T)),\ (p_S, \alpha_1(S)) \notin \{(2,6), (2^k - 1, 2) | k \in \mathbb{N}\}.$$

Proof. First notice that $\alpha_1(T), \alpha_1(S) \geqslant 2$ from Theorem 13.

If R is a simple group of Lie type with $p_R = 2^k - 1$ and $\alpha_1(R) = 2$, then $R = A_1(2^k - 1)$. We can exclude $(p_S, \alpha_1(S)) = (2^k - 1, 2)$ by Proposition 29 and $(p_T, \alpha_1(T)) = (2^k - 1, 2)$ by Lemma 33. Suppose now $(p_S, \alpha_1(S)) = (2, 6)$. Then $S \in \Sigma = \{A_5(2), A_2(2^2), A_1(2^3), B_3(2), D_4(2)\}$, but in these cases $|S|$ is divisible with odd multiplicity by at least two primes, contradicting Remark 26. Finally assume $(p_T, \alpha_1(T)) = (2, 6)$. Then $T \in \Sigma$. We may exclude $T = A_1(2^3)$, since there is no simple group S with $|S| \cdot p^a = |T|^2$ for some prime power p^a. In the remaining cases, 2 is the dominant prime of $|T|$ and also of $|T|/2$ and this implies that 2 is also the dominant prime of S (if $a \neq 1$ this follows from Corollary 30, while if $a = 1$ it suffices to recall that $|S| = |T|^2/p$). Hence the characteristic of S is 2 too, moreover $\alpha_1(S) \leqslant 6$, as $|S|$ cannot have primitive prime divisors not dividing $|T|$. We have already proved that $\alpha_1(S) \neq 6$. It is easy to verify that if S is a simple group of Lie type with characteristic 2 and satisfying $\alpha_1(S) \leqslant 5$ then the condition $|T|^2 = |S| \cdot p^a$ cannot be verified. □

Lemma 36. *The characteristic p_S of S does not coincide with the prime p.*

Proof. Suppose $p = p_S$. By Proposition 29, p coincides with the dominant prime of S, and consequently, since $|S| = |T|^2 \cdot p^a$, with the dominant prime of T; but then, by Lemma 33, p coincides also with the characteristic of T. By Lemma 35 and Theorem 13 (3), we get that $\alpha_1(T) = \omega(|T|)$ and $\alpha_1(S) = \omega(|S|)$. By Remark 12, $\omega(|S|) = \omega(|T|)$, so we conclude that $\alpha_1(T) = \alpha_1(S)$. Again by Lemma 35, we can use Zsigmondy's Theorem to find a primitive prime divisor t of $p^{\alpha_1(T)} - 1$. The multiplicity of t in $|T|$ coincides with the multiplicity of t in $\Phi_{\alpha_T}(p_T) = \Phi_{\alpha_S}(p_S)$, which is equal to the multiplicity of t in $|S|$, thus contradicting $|T|^2 = |S| \cdot p^a$. □

Proposition 37. $a = 1$.

Proof. Suppose $a \neq 1$: then, by Corollary 34, p is the characteristic and dominant prime of both S and T, contradicting Lemma 36. □

We remain with the possibility that $a = 1$ and consequently $|T|^2 = |S| \cdot p$ where p divides the order of the Schur multiplier $M(S)$. Moreover, the Schur multiplier can be decomposed as $M(S) = R \times P$, where P is a p_S-group and R a p'_S-group whose order coincides with the denominator d_S of the cyclotomic factorization of the order of S (see [8, Table 4.1]). By Lemma 36, $p \neq p_S$, thus p divides d_S.

Lemma 38. *If S, T have the same dominant prime u and $u \neq p$, then u coincides with the characteristic of T.*

Proof. By Proposition 7, if u does not coincide with the characteristic of T, then one of the following cases occurs.

1) $T = A_1(q)$, where $q = 2^k - 1$ is a Mersenne prime. Then $u = 2$ and

$$((2^k - 1) \cdot 2^k \cdot (2^{k-1} - 1))^2 = |S| \cdot p.$$

By Proposition 29, the characteristic of S coincides with $u = 2$, hence, considering the cyclotomic factorization of $|S|$ described in Theorem 13, we have $\alpha_1(S) = k$ and $\Phi_k(2) = 2^k - 1 = q$. By Theorem 13 (1), $\Phi_k(2)$ divides $|S|$ with multiplicity 1, so necessarily $p = q$ by Remark 26. On the other hand, p divides d_S and, by Theorem 13 (2), d_S divides $\Phi_{\alpha_3}(2) \cdots \Phi_{\alpha_u}(2) = (2^{k-1} - 1)^2 / \Phi_{\alpha_2}(2)$, thus p divides $(2^{k-1} - 1)$, whence $p \leqslant 2^{k-1} - 1 < 2^k - 1 = q = p$, a contradiction.

2) $T = A_1(q - 1)$, where $q = 2^{2^k} + 1$ is a Fermat prime. Then $u = q$ and

$$q^2 \cdot (q - 1)^2 \cdot (q - 2)^2 = |S| \cdot p.$$

By Proposition 29, the characteristic of S coincides with $u = q$, in particular the characteristic of S divides $|S|$ with multiplicity 2 and it is easy to check that the only group satisfying this condition is $S = A_1(q^2)$, but then $d_S = 2$ whence $p = 2$. Hence

$$q^2 \cdot (q - 1)^2 \cdot (q - 2)^2 = |A_1(q^2)| \cdot 2 = q^2 \cdot (q^2 - 1) \cdot (q^2 + 1),$$

whence $(q - 1) \cdot (q - 2)^2 = (q + 1) \cdot (q^2 + 1)$, but this is false.

3) $T = A_1(2^3)$. Then $|T| = 2^3 \cdot 3^2 \cdot 7$, $u = 3$, $p = 2$ and $|S| = 2^5 \cdot 3^4 \cdot 7^2$, however there is no simple group of Lie type S with this order.

4) $T = {}^2A_2(3)$. Then $|T| = 2^5 \cdot 3^3 \cdot 7$, $u = 2$, $p = 3$ and $|S| = 2^{10} \cdot 3^5 \cdot 7^2$, however there is no simple group of Lie type S with this order.

5) $T = {}^2A_3(2)$. Then $|T| = 2^6 \cdot 3^4 \cdot 5$, $u = 3$, $p = 2$ and $|S| = 2^{11} \cdot 3^8 \cdot 5^2$, however there is no simple group of Lie type S with this order. □

Lemma 39. *S and T have different dominant primes.*

Proof. Suppose that r is the dominant prime of S and T. Then, by Lemma 36, $r \neq p$ and therefore $|T|_r^2 = |S|_r$ and, by Remark 12, $\omega(S) = \omega(T)$. Moreover, by Lemma 35 and Theorem 13 (3), $\alpha_1(S) = \omega(S)$ and $\alpha_1(T) = \omega(T)$, whence $\alpha_1(S) = \alpha_1(T) = \alpha$. By Proposition 29 and

Lemma 38, r is also the characteristic of both S and T. Again by Lemma 35, we can apply Zsigmondy's Theorem and consider a primitive prime divisor u dividing of $r^\alpha - 1$. This prime u divides $|S|$ and $|T|$ with the same multiplicity (coinciding with the multiplicity of u in $\Phi_\alpha(r)$). On the other hand $|S| \cdot p = |T|^2$, so we must have that $r = p$ and that p divides $|S|$ with multiplicity 1, in contradiction with Corollary 25 (6). □

Now we are ready to conclude our proof. We have reduced to the case $|T|^2 = p \cdot |S|$, where the dominant prime of T and S (which coincide with their characteristic) are different, and consequently p is the dominant prime of T. Let r be the dominant prime of S and let p^t, r^h be the contributions of p and r to $|S|$. We have

$$p^t < r^h < p^{t+1}, \tag{3.10}$$

and consequently, since $t > 1$ by Corollary 25 (6),

$$1 < \frac{h \log(r)}{t \log(p)} < 1 + \frac{1}{t} < \frac{\log(3)}{\log(2)}.$$

thus p is a good contributor of S. By [2, Theorem 4.1] S is one of following groups:

1) $A_3(3)$, ${}^2A_3(3)$, ${}^2A_3(7)$, ${}^2A_4(3)$,$B_2(3)$, $B_2(5)$, $B_2(7)$, $B_2(9)$, $B_3(3)$, $C_3(3)$, $D_4(3)$, $G_2(3)$ (and $p = 2$);
2) ${}^2A_3(2)$, ${}^2A_4(2)$, ${}^2A_5(2)$, $B_3(2)$, $D_4(2)$ (and $p = 3$);
3) $A_1(r)$, $A_2(r)$, ${}^2A_2(r)$.

The possibilities listed in (1) and (2) can be immediately excluded noticing that either p does not divide $|M(S)|$, or there exists a prime different from p dividing $|S|$ with odd multiplicity, or $|S| \cdot p$ is not a square.

The only cases that remain to be discussed are thus $A_1(r)$, $A_2(r)$, ${}^2A_2(r)$: we have $|S| = r^\epsilon \cdot u$ where ϵ is odd and $(u, r) = 1$, so, by Remark 26, $r = v^2$ for some integer v. If $S = {}^2A_2(v^2)$, then $|M(S)| = (3, v^2 + 1) = 1$, a contradiction. Suppose $S = A_1(v^2)$. We have already excluded the possibilities $S \cong A_1(4) \cong \mathrm{Alt}(5)$ and $S \cong A_1(9) \cong \mathrm{Alt}(6)$, so we have $|M(S)| = (2, v^2 - 1)$ and consequently $p = 2$ and v is odd. In particular

$$|S|_2 = \frac{(v^4 - 1)_2}{2} = \frac{(v^2 - 1)_2(v^2 + 1)_2}{2} = (v^2 - 1)_2$$

and from (3.10) we deduce $(v^2 - 1)_2 < v^2 < 2(v^2 - 1)_2$: the only possibility is $v = 3$, but we have already excluded this case. Finally, suppose $S = A_2(v^2)$. We may exclude $S = A_2(4)$ since in this case 5 and 7 divide $|S|$

with multiplicity 1. In the remaining case $M(S) = (3, v^2 - 1)$, so it must $v^2 - 1 = 0 \mod 3$ and $p = 3$. But then $v^4 + v^2 + 1 = (v^2 - 1)^2 + 3v^2 \cong 3v^2 \cong 3 \mod 9$, thus

$$|S|_3 = \frac{(v^2+1)_3(v^2-1)_3^2(v^4+v^2+1)_3}{3} = (v^2-1)_3^2$$

and by (3.10) we have $(v^2-1)_3^2 < v^6 < 3(v^2-1)_3^2$, whence $v^6 < 3(v^2-1)^2$, a contradiction.

4. Proof of Theorem 3

It follows immediately from Theorem 22 that:

Proposition 40. *If there exists a non-pronilpotent normally ζ-reversible profinite group all of whose composition factors are of alternating type, then there exist a positive integer m and a finite group H with the following properties:*

1) $|H| = |\operatorname{Alt}(m)|^2$.
2) *H contains a unique minimal normal sugroup N.*
3) *Either H/N is nilpotent or there exist a nilpotent group X and a positive integer $n \geqslant m$ such that $H/N \cong X \times \operatorname{Alt}(n)$; in the latter case $\pi(m!) = \pi(n!)$ i.e. there is no prime q with $m < q \leqslant n$.*
4) *Either N is abelian or $N \cong \operatorname{Alt}(u)^t$ for some u and $t \in \mathbb{N}$.*

In this section we will prove that there is no pair (m, H) satisfying the condition requested by the previous proposition. We will assume, by contradiction, that (m, H) is one of these pairs and we will prove a series of restrictions that will lead to a finale contradiction.

Lemma 41. *H is not soluble.*

Proof. If H is soluble, then H is a finite soluble group which is not nilpotent but all of whose proper quotients are nilpotent. This implies that $H = N \rtimes A$, where N is an elementary abelian p-group and A is a nilpotent p'-subgroup of $\operatorname{Aut} N$. By [20, Theorem 1.6], $|A| \leqslant |N|^\beta/2$ with $\beta = \log(32)/\log(9)$ so

$$\frac{\log(|H|)}{\log(|N|)} < \frac{\log(288)}{\log(9)}. \tag{4.1}$$

On the other hand, since $|H| = |\operatorname{Alt}(m)|^2$, we have

$$\log(|H|)/\log(|N|) \geqslant (\lambda(\operatorname{Alt}(m)))^{-1}.$$

The values of the logarithmic proportion of alternating groups are listed in [11, Tables $L.3$ and $L.4$] and it can be easily seen that

$$\frac{\log(|H|)}{\log(|N|)} \geqslant (\lambda(\mathrm{Alt}(m)))^{-1} > \frac{\log(288)}{\log(9)} \quad \text{for} \quad m \notin \{5,8\}$$

contradicting (4.1). Direct computations show that (4.1) is false also when $m \in \{5,8\}$. □

Lemma 42. *$N = \operatorname{soc} H$ is abelian.*

Proof. Suppose by contradiction that N is nonabelian: then there exist positive integers $u \geqslant 5$ and t such that $N = L_1 \times \cdots \times L_t$, with $L_i \cong L = \mathrm{Alt}(u)$ for all i. In particular

$$L^t \cong N \trianglelefteq H \leqslant \mathrm{Aut}(N) \cong \mathrm{Aut}(L) \wr \mathrm{Sym}(t).$$

If $t = 1$, then $|\mathrm{Alt}(m)|^2 = |H| = 2^j \cdot u!$ for some $j \in \mathbb{Z}$, however by Lemma 4 there exists an odd prime dividing $u!$ with multiplicity 1, a contradiction. If $t = 2$, then from $|\mathrm{Alt}(m)|^2 = H$, we would deduce $(m!)^2 = (u!)^2 2^j$ for some positive integer $j \in \{1,2,3,4,5\}$, but this is impossible. So we can assume $t \geqslant 3$. By Proposition 40 we can write $H/N = X_1/N \times X_2/N$, where X_1/N is nilpotent and either $X_2/N = 1$ or $X_2/N \cong \mathrm{Alt}(n)$ for some $n \geqslant m$. First suppose that either $X_2/N = 1$ and $m \notin \{6,10\}$, or $X_2/N \cong \mathrm{Alt}(n)$ with $n \notin \{6,10\}$. Then, by Lemma 4, we can find two primes p, q as follows:

$$\begin{cases} \frac{n}{2} < p < q \leqslant n & \text{if } X_2/N \cong \mathrm{Alt}(n), \\ \frac{m}{2} < p < q \leqslant m & \text{if } X_2/N = 1. \end{cases}$$

We claim that p, q both divide the order of $\mathrm{Alt}(m)$ with multiplicity 1: this is clear if X_2/N is trivial, while if $X_2/N \cong \mathrm{Alt}(n)$ it follows from the fact that $m/2 \leqslant n/2 < p < q \leqslant m$. So p and q divide $|H| = (m!/2)^2$ with multiplicity exactly 2: as $L^t \leqslant H$ and $t > 2$, they cannot divide $|L|$, so they divide $|H/N| = |X_1/N||X_2/N|$ with multiplicity 2. On the other hand, by the way in which they have been defined, they divide $|X_2/N|$ with multiplicity at most 1, so $p \cdot q$ must divide order of the nilpotent group X_1/N. This implies that the transitive permutation group induced by the conjugacy action of H on the t direct factors $L_1, \ldots, L_t$ contains a central element of order $p \cdot q$. In particular $t \geqslant p \cdot q$ and consequently,

$$60^{\frac{m^2}{4}} \leqslant 60^{p \cdot q} \leqslant |L|^t \leqslant |H| = (m!)^2 \leqslant m^{2m}$$

but this is false for all $m \geqslant 5$. We have still to consider the two cases $X_2/N = 1$ and $m \in \{6, 10\}$ or $X_2 \cong \mathrm{Alt}(n)$ with $n \in \{6, 10\}$. If $m = 6$ or $n = 6$ (and consequently $m \leqslant 6$), then $|\,\mathrm{Alt}(u)|^3$ divides $|\,\mathrm{Alt}(6)|^2$, hence 5^3 divides $(6!)^2$, a contradiction. If $m = 10$ or $n = 10$, then 7 divides $|H|$ with multiplicity at most 2; as a consequence $|H/N|$ is divisible by 7 and $t \geqslant 7$; but then

$$7 \cdot 60^7 \leqslant |H/N| \cdot |N| = |H| \leqslant (10!)^2$$

which leads again to a contradiction. □

Combining Proposition 40 with Lemma 41 and Lemma 42, we can conclude that there exist two subgroups X_1 and X_2 of H such that

1) $H/N = X_1/N \times X_2/N$;
2) X_1/N is nilpotent;
3) $X_2/N \cong \mathrm{Alt}(n)$.
4) N is an elementary abelian p-group.

Lemma 43. *N is not central in X_2.*

Proof. Assume, by contradiction, $N \leqslant Z(X_2)$. Notice that $\mathrm{Frat}(X_2)$ is a nilpotent normal subgroup of H, so either $\mathrm{Frat}(X_2) = 1$ or $\mathrm{Frat}(X_2) = N$. In the first case, we would have $X_2 = N \times S$, with $S \cong \mathrm{Alt}(n)$. But then S would be normal in H, against the fact that N is the unique minimal normal subgroup of G. If $\mathrm{Frat}(X_2) = N$, then X_2 is a perfect central extension of N, so in particular $|N|$ divides the order of the Schur multiplier of $\mathrm{Alt}(n)$, hence $|N| \in \{2, 3\}$. This implies that X_1 is a $\{2, 3\}$-group (if a prime $q > 3$ would divide $|X_1|$, then a Sylow q-subgroup of X_1 would coincide with $O^q(C_{X_1}(N))$ and would be normal in H). From $|H| = |X_1/N| \cdot |X_2|$, we deduce

$$(m!)^2 = n! \cdot 2^\alpha \cdot 3^\beta$$

for some positive integers α, β, in contradiction with the fact that, by Lemma 4, there exists a prime dividing $n!$ with multiplicity 1. □

The previous result, combined with Clifford's theory, implies that N contains a nontrivial irreducible $\mathrm{Alt}(n)$-modulo, say M.

Lemma 44. *$n \leqslant 8$.*

Proof. Suppose $n \geqslant 9$. By [19, Theorem 1.1], the dimension of a nontrivial irreducible $\mathrm{Alt}(n)$-module is at least $n-2$, so $|N| \geqslant |M| \geqslant p^{n-2}$. But then, from $|\mathrm{Alt}(m)|^2 = |H| \geqslant |N| \cdot |\mathrm{Alt}(n)|$, we get

$$((m!/2))_p^2 \geqslant p^{n-2} \cdot (n!/2)_p.$$

Let now $a = m - n \geqslant 0$ and $\eta_p = 0$ is p is odd, $\eta_2 = 1$ if $p = 2$; since $(m!)_p < p^{m/(p-1)}$, we have

$$p^{m/(p-1)-\eta_p} > (m!/2)_p \geqslant (m+1)_p \cdots (m+a)_p \cdot p^{m+a-2} \geqslant p^{m+a-2}.$$

This implies

$$p = 2, \quad n = m, \quad |N| = |M| = 2^{n-2} = (n!/2)_2.$$

Since

$$|H| = \left(\frac{n!}{2}\right)^2 = \frac{|X_1||X_2|}{|N|} = \frac{n!|X_1|}{2} \quad \text{and} \quad 2^{n-2} = (n!/2)_2,$$

we must have that $X_1 = N \rtimes K$, where N is an elementary abelian 2-group and K is a nilpotent group of odd order; more precisely $|K| = (n!)_{2'}$. Moreover, the fact that N is the unique minimal normal subgroup of H implies $C_K(N) = 1$, hence K is a completely reducible subgroup of $\mathrm{Aut}\, N$. In particular

$$|K| \leqslant \frac{|N|^\beta}{2} = 2^{\beta(n-2)-1} \text{ with } \beta = \frac{\log(32)}{\log(9)}$$

whence

$$n! = (n!)_{2'} \cdot (n!)_2 = |K| \cdot (n!)_2 \leqslant 2^{\beta(n-2)-1} \cdot 2^{n-1} = 2^{n(\beta+1)-2\beta-2}$$

which is false for $n \geqslant 9$. □

We remain with the the cases $5 \leqslant m \leqslant n \leqslant 8$. Recall that $\pi(n!) = \pi(m!)$ and that $|N| \cdot |\mathrm{Alt}(n)|$ divides $|H| = \left(\frac{m!}{2}\right)^2$ (i.e. $2|N|n!$ divides $(m!)^2$). This means that N is a completely reducible $\mathrm{Alt}(n)$-module of relatively small order. Looking to the irreducible representations of small degree of $\mathrm{Alt}(n)$ over the field with p elements when $5 \leqslant n \leqslant 8$ and $p \leqslant n$, we easily conclude that the only possibilities are: $m = n = 8$ and N is an irreducible $\mathrm{Alt}(8)$-module with $|N| \in \{2^4, 2^6\}$. In both these cases, a $2'$-Hall subgroup K of X_1 would be nilpotent and of order $3^2 \cdot 5 \cdot 7$. Moreover $C_K(N) = 1$ (otherwise we would have $N \neq \mathrm{soc}\, H$) and $\mathrm{Aut}(N)$ would contain an element of order $3^2 \cdot 5 \cdot 7$, which is false.

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An outer measure on a commutative ring

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Abstract. We show how to produce a reasonable outer measure on a commutative ring from a given measure on a family of prime ideals of this ring. We provide a few examples and prove several properties of such outer measures.

Introduction

Throughout the present paper, R is a nonzero commutative ring with identity. We denote by $\mathrm{Spec}(R)$ the family of all the prime ideals of R. (Notice that, by definition, every prime ideal is proper).

It is well known [1] that topological properties of $\mathrm{Spec}(R)$ equipped with the Zariski topology reflect algebraic properties of R. But are there useful relationships between algebraic or geometric properties of R and measures on $\mathrm{Spec}(R)$? This question seems to be quite interesting and not worked out in the specialist literature. The present paper provides some basic remarks concerning the question and, hopefully, is a starting point for further study.

In the paper, we will show that an arbitrary measure on a suitable subfamily of $\mathrm{Spec}(R)$ induces an outer measure on R with good multiplicative properties. We will also discuss a few elementary examples of such outer measures.

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By "measure" we mean a "non-negative σ-additive measure". We denote by 2^X the power set of a set X. We define

$$|X| = \begin{cases} \text{the cardinality of } X, & \text{if } X \text{ is finite,} \\ +\infty, & \text{otherwise.} \end{cases}$$

By $R^\times$ we denote the set of invertible elements of R. Notice that $\wp \cap R^\times = \varnothing$ whenever $\wp \in \operatorname{Spec}(R)$. We define $\operatorname{Max}(R)$ to be the family of all the maximal ideals of R. One can prove that $\operatorname{Max}(R) \subseteq \operatorname{Spec}(R)$ and $\bigcup \operatorname{Max}(R) = R \setminus R^\times$.

We refer to [1] for more information about commutative rings and to [2] for elements of measure theory.

1. Construction

We will use the definition of outer measure taken from [2].

Definition 1. We say that $\mu^* : 2^X \longrightarrow [0, +\infty]$ is an outer measure on a set X, if the following conditions are satisfied:

(1) $\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(B_n)$ for every sequence $\{B_n\}_{n=1}^{\infty}$ of subsets of X and every $A \subseteq \bigcup_{n=1}^{\infty} B_n$,

(2) $\mu^*(\varnothing) = 0$.

Let $\mathcal{P} \subseteq \operatorname{Spec}(R)$ be such that $\bigcup \mathcal{P} = R \setminus R^\times$, and let $\mathfrak{M}$ be a σ-algebra of subsets of $\mathcal{P}$. For a set $A \subseteq R$ we define

$$\Omega(A) = \left\{ \mathcal{S} \in \mathfrak{M} : \bigcup \mathcal{S} \supseteq A \setminus R^\times \right\}.$$

Proposition 1. *Suppose that $\mu : \mathfrak{M} \longrightarrow [0, +\infty]$ is a measure. Then the function $\mu^* : 2^R \longrightarrow [0, +\infty]$ defined by*

$$\mu^*(A) = \inf_{\mathcal{S} \in \Omega(A)} \mu(\mathcal{S})$$

is an outer measure on R. (This outer measure will be referred to as the outer measure induced by μ).

Proof. It is obvious that $\mu^*(\varnothing) = 0$. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of subsets of R and let ε be an arbitrary positive real number. Observe that

$$\forall\, n \in \mathbb{N} \setminus \{0\}\, \exists\, \mathcal{S}_n \in \Omega(B_n) : \; \mu(\mathcal{S}_n) \leqslant \mu^*(B_n) + \frac{\varepsilon}{2^n}.$$

If $A \subseteq \bigcup_{n=1}^{\infty} B_n$, then $\bigcup_{n=1}^{\infty} \mathcal{S}_n \in \Omega(A)$ and hence

$$\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu(\mathcal{S}_n) \leqslant \varepsilon + \sum_{n=1}^{\infty} \mu^*(B_n).$$

Since ε is arbitrary, the above inequalities yield $\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(B_n)$. $\square$

The outer measure induced by a measure on a family of prime ideals is a slight modification of a well known measure-theoretical construction. In the next section we give examples that illustrate and motivate this modification.

2. Examples

We denote by (a) the principal ideal generated by an element $a \in R$. Consider a further example of a "covering by prime ideals".

Example 1. We assume that R is a unique factorization domain and define $\mathcal{P}_{\text{irr}}(R) = \{(0)\} \cup \{(a) : a \in R, a \text{ is irreducible}\}$. Observe that $\mathcal{P}_{\text{irr}}(R) \subseteq \operatorname{Spec}(R)$ and $\bigcup \mathcal{P}_{\text{irr}}(R) = R \setminus R^{\times}$. Moreover, if $n \in \mathbb{N} \setminus \{0, 1\}$ and $R = \mathbb{C}[x_1, \ldots, x_n]$, then $\mathcal{P}_{\text{irr}}(R) \cap \operatorname{Max}(R) = \varnothing$.

Recall that for every ideal I of the ring of integers there exists exactly one $m \in \mathbb{N} \cup \{0\}$ such that $I = (m)$. Notice also that $\operatorname{Max}(\mathbb{Z}) = \{(p) : p \in \mathbb{P}\}$, where $\mathbb{P}$ stands for the set of prime numbers.

Proposition 2. *Let $\mu^* : 2^{\mathbb{Z}} \longrightarrow [0, +\infty]$ be the outer measure induced by the counting measure on $\operatorname{Max}(\mathbb{Z})$, and let $A \subseteq \mathbb{Z}$ be such that $A \setminus \{-1, 1\} \neq \varnothing$. Then*

(i) $\mu^*(\{-1, 1\}) = 0$,

(ii) $\mu^*(A) = 1$ *if and only if*

$$\exists\, d \in \mathbb{N} \setminus \{0, 1\}\, \forall\, k \in A \setminus \{-1, 1\} : d \mid k$$

(in particular, $\mu^*(A) = 1$ whenever A is a singleton or a proper ideal of $\mathbb{Z}$),

(iii) $\mu^*(A) \leqslant |A|$.

Moreover, in the case where $A \cap \{-1, 1\} = \varnothing$ and A is a finite set, $\mu^*(A) = |A|$ if and only if the elements of A are pairwise relatively prime.

Proof. Since $\{-1,1\} = \mathbb{Z}^\times$, we have $\varnothing \in \Omega(\{-1,1\})$. Equality (i) follows.

By the above characterization of $\mathrm{Max}(\mathbb{Z})$ and the definition of counting measure, $\mu^*(A) = 1$ if and only if $A \setminus \{-1,1\} \subseteq (p_1)$ for a prime number p_1. The latter condition means precisely that

$$\exists\, p_1 \in \mathbb{P}\, \forall\, k \in A \setminus \{-1,1\} :\, p_1 \mid k.$$

Finally, if $d \in \mathbb{N} \setminus \{0,1\}$, $k \in \mathbb{Z}$ and $d \mid k$, then k is divisible by every prime factor of d. Property (ii) follows.

Property (iii) is an immediate consequence of the definition of outer measure and the fact that $\mu^*(\{k\}) \leqslant 1$ for all $k \in \mathbb{Z}$.

Assume that $A \cap \{-1,1\} = \varnothing$ and A is a finite set. Let us define $\ell = |A|$. Observe that $\mu^*(A) \neq |A|$ if and only if

$$\exists\, \mathcal{S} \in \Omega(A) :\, |\mathcal{S}| \leqslant \ell - 1.$$

Since the cardinality of A is greater than the cardinality of $\mathcal{S}$, the latter condition holds true if and only if

$$\exists\, s,t \in A\, \exists\, p_2 \in \mathbb{P} : \begin{cases} s \neq t, \\ s,t \in (p_2), \end{cases}$$

and this means precisely that there exist two distinct elements of A which are not relatively prime. □

Let $n \in \mathbb{N} \setminus \{0\}$. Consider a σ-algebra $\mathfrak{N}$ of subsets of $\mathbb{C}^n$, a measure $\lambda : \mathfrak{N} \longrightarrow [0,+\infty]$, and the map

$$\Phi : \mathbb{C}^n \ni z \longmapsto \{f \in \mathbb{C}[x_1,\dots,x_n] :\, f(z) = 0\} \in \mathrm{Max}(\mathbb{C}[x_1,\dots,x_n]).$$

The family $\mathfrak{M} = \{\mathcal{S} \subseteq \mathrm{Max}(\mathbb{C}[x_1,\dots,x_n]) :\, \Phi^{-1}(\mathcal{S}) \in \mathfrak{N}\}$ is a σ-algebra of subsets of $\mathrm{Max}(\mathbb{C}[x_1,\dots,x_n])$. The function $\eta : \mathfrak{M} \ni \mathcal{S} \mapsto \lambda(\Phi^{-1}(\mathcal{S})) \in [0,+\infty]$ is a measure.

Let us define $U = \mathbb{C}[x_1,\dots,x_n]^\times$. (Obviously, $U = \mathbb{C} \setminus \{0\}$).

Proposition 3. *If $\eta^* : 2^{\mathbb{C}[x_1,\dots,x_n]} \longrightarrow [0,+\infty]$ is the outer measure induced by η and $A \subseteq \mathbb{C}[x_1,\dots,x_n]$ is such that $A \setminus U \neq \{0\}$, then*

$$\eta^*(A) = \inf\{\lambda(Z) :\, Z \in \mathfrak{N},\ Z \cap f^{-1}(0) \neq \varnothing \text{ for every } f \in A \setminus \mathbb{C}\}.$$

Proof. If $A \subseteq U$, then $\{Z \in \mathfrak{N} :\, Z \cap f^{-1}(0) \neq \varnothing$ for every $f \in A \setminus \mathbb{C}\} = \mathfrak{N}$, and hence

$$\inf\{\lambda(Z) :\, Z \in \mathfrak{N},\ Z \cap f^{-1}(0) \neq \varnothing \text{ for every } f \in A \setminus \mathbb{C}\} = 0 = \eta^*(A).$$

By Hilbert's Nullstellensatz, the map Φ is bijective. Consequently, $\mathfrak{M} = \{\Phi(Z) : Z \in \mathfrak{N}\}$. Suppose that $A \setminus \mathbb{C} \neq \varnothing$. Then for any $Z \in \mathfrak{N}$ the following equivalences hold true:

$$\Phi(Z) \in \Omega(A) \iff (\forall\, f \in A \setminus U\, \exists\, \wp \in \Phi(Z) :\, f \in \wp) \iff$$

$$(\forall\, f \in A \setminus \mathbb{C}\, \exists\, z \in Z :\, f(z) = 0) \iff \Big(\forall\, f \in A \setminus \mathbb{C} :\, Z \cap f^{-1}(0) \neq \varnothing\Big).$$

(The second equivalence holds because 0 belongs to every ideal). Therefore,

$$\eta^*(A) = \inf_{\mathcal{S} \in \Omega(A)} \eta(\mathcal{S}) =$$

$$= \inf\{\lambda(\Phi^{-1}(\Phi(Z))) :\, Z \in \mathfrak{N},\, Z \cap f^{-1}(0) \neq \varnothing \text{ for every } f \in A \setminus \mathbb{C}\},$$

which completes the proof. □

Example 2. Let $\eta^* : 2^{\mathbb{C}[x,y]} \longrightarrow [0, +\infty]$ be the outer measure induced by the counting measure on $\mathrm{Max}(\mathbb{C}[x, y])$. Consider the set $E = \{f, g, h, k\} \subset \mathbb{C}[x, y]$, where

$$f(x, y) = x^2 - y + 1,\ g(x, y) = y^2,\ h(x, y) = xy - 1,\ k(x, y) = xy + 1.$$

Since $f^{-1}(0) \cap g^{-1}(0) \cap h^{-1}(0) \cap k^{-1}(0) = \varnothing$ and $f^{-1}(0) \cap g^{-1}(0) \neq \varnothing$, we have $\eta^*(E) \in \{2, 3\}$. Observe that $\{f, g\}$, $\{f, h\}$ and $\{f, k\}$ are the only two-element subsets of E which have a common zero. Consequently, no three-element subset of E has a common zero. It follows, therefore, that $\eta^*(E) = 3$.

Notice that in the example above, if I is a proper ideal of $\mathbb{C}[x, y]$, then $I \subseteq \wp$ for an ideal $\wp \in \mathrm{Max}(\mathbb{C}[x, y])$ and hence $\eta^*(I) = 1$.

Let K be a nonempty compact subset of $\mathbb{R}^n$ and let $\mathcal{C}(K, \mathbb{R})$ stand for the ring of all the continuous functions $f : K \longrightarrow \mathbb{R}$. Recall that $\mathcal{C}(K, \mathbb{R})^{\times} = \{f \in \mathcal{C}(K, \mathbb{R}) :\, f(x) \neq 0 \text{ for all } x \in K\}$. The map

$$\Psi : K \ni x \longmapsto \{f \in \mathcal{C}(K, \mathbb{R}) :\, f(x) = 0\} \in \mathrm{Max}(\mathcal{C}(K, \mathbb{R}))$$

is well known to be a bijection [3]. Consequently, if $\mathfrak{B}$ is a σ-algebra of subsets of K and $\xi : \mathfrak{B} \longrightarrow [0, +\infty]$ is a measure, then $\mathfrak{M} = \{\Psi(Z) :\, Z \in \mathfrak{B}\}$ is a σ-algebra of subsets of $\mathrm{Max}(\mathcal{C}(K, \mathbb{R}))$ and

$$\eta : \mathfrak{M} \ni \mathcal{S} \mapsto \xi(\Psi^{-1}(\mathcal{S})) \in [0, +\infty]$$

is a measure. The obvious counterpart of Proposition 3 remains true.

Example 3. Let $\eta^* : 2^{\mathcal{C}(K,\mathbb{R})} \longrightarrow [0,+\infty]$ be the outer measure induced by η. We will denote by W the set of all the polynomial functions $f : K \longrightarrow \mathbb{R}$. Since

$$\forall\, x \in K\, \exists\, f \in W :\ f^{-1}(0) = \{x\},$$

we have $\eta^*(W) = \eta^*(\mathcal{C}(K,\mathbb{R})) = \xi(K)$.

Now, suppose that K is the Euclidean closed unit ball and ξ is the n-dimensional Lebesgue measure. If E stands for the set of all the radially symmetric functions belonging to $\mathcal{C}(K,\mathbb{R})$ and L is the straight line segment that joins the origin to a boundary point of K, then

$$\forall\, f \in E \setminus \mathcal{C}(K,\mathbb{R})^{\times} :\ L \cap f^{-1}(0) \neq \varnothing.$$

Consequently, $\eta^*(E) = \xi(L) = 0$ whenever $n \geqslant 2$. It is easy to see that if $n = 1$, then $\eta^*(E) = 1$.

3. General properties

In the theorem below (it is the main result of the paper) we use the notations and assumptions of Proposition 1. For $n \in \mathbb{N} \setminus \{0\}$ and $A_1, \ldots, A_n \subseteq R$ we define $A_1 \ldots A_n = \{a_1 \ldots a_n : a_1 \in A_1, \ldots, a_n \in A_n\}$. Moreover, if $A \subseteq R$, then $A^n = \{a^n : a \in A\}$ and $A^{\bullet n} = \underbrace{A \ldots A}_{n}$.

Theorem 1. *Let $A, B \subseteq R$ and let C be a nonempty subset of $R^{\times}$. Then*

(i) $\mu^*(R^{\times}) = 0$,

(ii) $\mu^*(A) = \mu^*(A \setminus R^{\times})$,

(iii) $\mu^*(\{0\}) = \min\{\mu^*(E) : E \subseteq R,\ E \setminus R^{\times} \neq \varnothing\}$,

(iv) $\forall\, n \in \mathbb{N} \setminus \{0\} :\ \mu^*(A^n) = \mu^*(A^{\bullet n}) = \mu^*(A)$,

(v) $\mu^*(AC) = \mu^*(A)$,

(vi) $\mu^*(AB) \geqslant \max\{\mu^*(A), \mu^*(B)\}$ *whenever* $A \cap R^{\times} \neq \varnothing$ *and* $B \cap R^{\times} \neq \varnothing$,

(vii) $\mu^*(AB) \leqslant \mu^*(A) + \mu^*(B)$,

(viii) $\mu^*(AB) = \mu^*(A)$ *whenever* $A \cap R^{\times} = \varnothing$ *and* $B \cap R^{\times} \neq \varnothing$,

(ix) $\mu^*(AB) = \min\{\mu^*(A), \mu^*(B)\}$ *whenever* $A \cap R^{\times} = \varnothing$ *and* $B \cap R^{\times} = \varnothing$.

Proof. Properties (i) and (ii) are obvious.

Property (iii) follows from the facts that $0 \notin R^{\times}$ and 0 belongs to every ideal of R.

Fix a positive integer n. Let $a_1, \ldots, a_n \in R$. The product $a_1 \ldots a_n$ is not invertible if and only if there exists an index $i \in \{1, \ldots, n\}$ such that

a_i is not invertible. Similarly, $a_1 \ldots a_n \in \wp$ for an ideal $\wp \in \operatorname{Spec}(R)$ if and only if there exists an index $i \in \{1, \ldots, n\}$ such that $a_i \in \wp$. Therefore, $\Omega(A^n) = \Omega(A^{\bullet n}) = \Omega(A)$. Property (iv) follows.

Let $a \in R$ and $c \in R^{\times}$. Observe that $ac \notin R^{\times}$ if and only if $a \notin R^{\times}$. Moreover,

$$\forall \wp \in \operatorname{Spec}(R) : \, ac \in \wp \Leftrightarrow a \in \wp.$$

Consequently, $\Omega(AC) = \Omega(A)$.

Suppose that $C_1 = A \cap R^{\times} \neq \varnothing$ and $C_2 = B \cap R^{\times} \neq \varnothing$. Since $AC_2 \cup BC_1 \subseteq AB$, we have $\max\{\mu^*(AC_2), \mu^*(BC_1)\} \leqslant \mu^*(AB)$. Property (v) yields $\mu^*(AC_2) = \mu^*(A)$ and $\mu^*(BC_1) = \mu^*(B)$. This completes the proof of (vi).

Let $\mathcal{S} \in \Omega(A)$ and $\mathcal{T} \in \Omega(B)$. Suppose that $ab \notin R^{\times}$ for some $a \in A$ and $b \in B$. Then $a \notin R^{\times}$ or $b \notin R^{\times}$. By the definition of ideal, we get therefore

$$ab \in \bigcup \mathcal{S} \cup \bigcup \mathcal{T}.$$

Consequently, $\mathcal{S} \cup \mathcal{T} \in \Omega(AB)$ and hence $\mu^*(AB) \leqslant \mu(\mathcal{S}) + \mu(\mathcal{T})$. Since $\mathcal{S}$ and $\mathcal{T}$ are arbitrarily chosen, it follows that $\mu^*(AB) \leqslant \mu^*(A) + \mu^*(B)$.

Assume that $A \cap R^{\times} = \varnothing$ and $C_2 = B \cap R^{\times} \neq \varnothing$. Then, by the definition of ideal, $\Omega(A) \subseteq \Omega(AB)$ which implies that $\mu^*(AB) \leqslant \mu^*(A)$. On the other hand, by (v), we have $\mu^*(A) = \mu^*(AC_2) \leqslant \mu^*(AB)$. Therefore, $\mu^*(AB) = \mu^*(A)$.

Finally, assume that $A \cap R^{\times} = \varnothing$ and $B \cap R^{\times} = \varnothing$. Then $\mu^*(AB) \leqslant \min\{\mu^*(A), \mu^*(B)\}$ (cf. the proof of property (viii)). Suppose now that $\mu^*(AB) < \min\{\mu^*(A), \mu^*(B)\}$. Then

$$\exists \mathcal{U} \in \Omega(AB) : \begin{cases} \mu^*(\bigcup \mathcal{U}) < \mu^*(A), \\ \mu^*(\bigcup \mathcal{U}) < \mu^*(B). \end{cases}$$

(Notice that $\mu^*(\bigcup \mathcal{U}) \leqslant \mu(\mathcal{U})$). Consequently,

$$\mu^*(A \setminus \bigcup \mathcal{U}) \geqslant \mu^*(A) - \mu^*(A \cap \bigcup \mathcal{U}) \geqslant \mu^*(A) - \mu^*(\bigcup \mathcal{U}) > 0$$

and, in the same way, $\mu^*(B \setminus \bigcup \mathcal{U}) > 0$. Since $AB \cap R^{\times} = \varnothing$ and therefore $AB \subseteq \bigcup \mathcal{U}$, we get

$$\exists \, a \in A \, \exists \, b \in B \, \exists \, \wp \in \mathcal{U} \subseteq \operatorname{Spec}(R) : \begin{cases} ab \in \wp, \\ a \notin \wp, \, b \notin \wp, \end{cases}$$

a contradiction. Property (ix) follows. □

We will conclude the paper with an example illustrating the behavior of $\mu^*(AB)$ in the case where A and B both contain invertible elements.

Example 4. Let $\mu^* : 2^{\mathbb{Z}} \longrightarrow [0, +\infty]$ be the outer measure induced by the counting measure on $\mathrm{Max}(\mathbb{Z})$. If $A = \{1, 2, 3\}$, $B_1 = \{1, 2, 3, 5\}$, $B_2 = \{1, 2, 5, 7\}$ and $B_3 = \{1, 5, 7, 11\}$, then $\mu^*(A) = 2$, $\mu^*(B_1) = \mu^*(B_2) = \mu^*(B_3) = 3$, $\mu^*(AB_1) = 3$, $\mu^*(AB_2) = 4$ and $\mu^*(AB_3) = 5$.

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Construction of self-dual binary $[2^{2k}, 2^{2k-1}, 2^k]$-codes

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Abstract. The binary Reed-Muller code RM$(m-k, m)$ corresponds to the k-th power of the radical of $GF(2)[G]$, where G is an elementary abelian group of order 2^m (see [2]). Self-dual RM-codes (i.e. some powers of the radical of the previously mentioned group algebra) exist only for odd m.

The group algebra approach enables us to find a self-dual code for even $m = 2k$ in the radical of the previously mentioned group algebra with similarly good parameters as the self-dual RM codes.

In the group algebra

$$GF(2)[G] \cong GF(2)[x_1, x_2, \ldots, x_m]/(x_1^2 - 1, x_2^2 - 1, \ldots x_m^2 - 1)$$

we construct self-dual binary $C = [2^{2k}, 2^{2k-1}, 2^k]$ codes with property

$$\mathrm{RM}(k-1, 2k) \subset C \subset \mathrm{RM}(k, 2k)$$

for an arbitrary integer k.

In some cases these codes can be obtained as the direct product of two copies of RM$(k-1, k)$-codes. For $k \geqslant 2$ the codes constructed are doubly even and for $k = 2$ we get two non-isomorphic $[16, 8, 4]$-codes. If $k > 2$ we have some self-dual codes with good parameters which have not been described yet.

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Introduction and Notation

Let K be a finite field of characteristic p and let V be a vector space over K, and C be a subspace of V. Then C is called a *linear code*. Let $x, y \in C$, then the Hamming weight of x is the number of its non-zero coordinates and the *Hamming distance* of x and y is the weight of $x - y$. The Hamming distance (or weight) of a linear code C is the minimum of all Hamming distances of its codewords.

In the study of binary codes $C \subseteq V$ it is convenient that the space V has an additional algebraic structure. For example, if V is a group algebra $K[G]$, where G is a finite abelian p-group and C is an ideal of such a group algebra, then C is called an *abelian group code*.

The Hamming distance of a linear code determines the ability of error-correcting property of the code. The authors in [6] proved that for any $1 \leqslant d \leqslant \left[\frac{m+1}{2}\right]$ there exists an Abelian 2-group G_d that a power of the radical is a self-dual code with parameters $(2^m, 2^{m-1}, 2^d)$. These codes are ideals in the group algebra $GF(2)[G_d]$ and they are "monomial codes" in the sense of [5] as defined below.

Throughout, p will denote a prime and K a field of p elements. Let $G = \langle g_1 \rangle \times \cdots \times \langle g_m \rangle \cong C_p^m$ be an elementary abelian p-group of order p^m i.e. $K[G]$ is a modular group algebra, then the group algebra $K[G]$ and K^n are isomorphic as vector spaces by the mapping

$$\varphi : K[G] \mapsto K^n, \text{ where } \varphi\left(\sum_{i=1}^{n} a_i g_i\right) \mapsto (a_1, a_2, \ldots, a_n) := \mathbf{c} \in C.$$

Reed-Muller (RM) binary codes were introduced in [12] as binary functions. These codes are frequently used in applications and have good error correcting properties. Now we are looking for self-dual codes in the radical of $K[G]$ with similarly good parameters as the RM codes.
If K is a field of characteristic 2 Berman [2] and in the general case Charpin [3] proved that all Generalized Reed-Muller (GRM) codes coincide with powers of the radical of the modular group algebra of $K[G]$, where G is an elementary abelian p-group. This group algebra is clearly isomorphic with the quotient algebra

$$GF(p)[x_1, x_2, \ldots x_m]/(x_1^p - 1, \ldots x_m^p - 1).$$

Self-dual RM-codes (i.e. some power of the radical of the group algebra $GF(2)[G]$) exist only for odd m. They are $(2^m, 2^{m-1}, 2^{\frac{m+1}{2}})$-codes.

For any basis $\{g_1, g_2, \dots g_m\}$ of such a group G consider the algebra isomorphism μ mapping $g_j \mapsto x_j \ \ (1 \leqslant j \leqslant m)$, and therefore we have the algebra isomorphism

$$\mathcal{A}_{p,m} \cong GF(p)[x_1, x_2, \dots, x_m]/(x_1^p - 1, x_2^p - 1, \dots x_m^p - 1),$$

where $GF(p)[x_1, x_2, \dots, x_m]$ denotes the algebra of polynomials in m variables with coefficients in $GF(p)$.

It is known ([7]) that the set of monomial functions ($k_i \in \mathbb{N} \cup 0$)

$$\left\{ \prod_{i=1}^{m} (x_i - 1)^{k_i} \text{ where } 0 \leqslant k_i < p \right\}$$

form a linear basis of the radical $\mathcal{J}_{p,m}$. Clearly the nilpotency index of $\mathcal{J}_{p,m}$ (i.e. the smallest positive integer t, such that $\mathcal{J}_{p,m}^t = 0$) is equal to $t = m(p-1) + 1$.

Introducing the notation

$$X_i = x_i - 1, \ (1 \leqslant i \leqslant m)$$

(which will be used from now on) we have the following isomorphism

$$\mathcal{J}_{p,m} \simeq GF(p)[X_1, X_2, \dots, X_m]/(X_1^p, X_2^p, \dots X_m^p). \tag{1}$$

The k-th power of the radical consists of reduced m-variable (non-constant) polynomials of degree at least k, where $0 \leqslant k \leqslant t-1$, where $t = m(p-1) + 1$.

$$\mathcal{J}_{p,m}^k = \mathrm{GRM}(t-1-k, m) = \langle \prod_{i=1}^{m} (X_i)^{k_i} \mid \sum_{i=1}^{m} k_i \geqslant k \ (0 \leqslant k_i < p) \rangle. \tag{2}$$

Such a basis was exploited by Jennings [7].

By (2) the quotient space $\mathcal{J}_{p,m}^k / \mathcal{J}_{p,m}^{k+1}$ has a basis

$$\left\{ \prod_{i=1}^{m} X_i^{k_i} + \mathcal{J}_{p,m}^{k+1}, \text{ where } 0 \leqslant k_i < p \text{ and } \sum_{i=1}^{m} k_i = k \right\}. \tag{3}$$

Remark 1. It is known [15] that the dual code $C^\perp$ of an ideal C in $\mathcal{A}_{p,m}$ coincides with the annihilator of C^*, where C^* is the image of C by the involution $*$ defined on $\mathcal{A}_{p,m}$ by

$$* : \ g \mapsto g^{-1} \text{ for all } g \in G \text{ from } \mathcal{A}_{p,m} \text{ to itself.}$$

The annihilator of $\mathcal{J}_{p,m}^k$ is obviously $\mathcal{J}_{p,m}^{m(p-1)+1-k}$. Thus the dual codes of GRM-codes are GRM-codes and

$$\mathrm{GRM}(k, m)^\perp = \mathrm{GRM}(m(p-1) - k - 1, m).$$

It follows that for $m = 2k + 1$ and $p = 2$ the code $\mathrm{GRM}(k, m)$ is self-dual.

1. Construction of binary self-dual codes

Let us consider the group algebra

$$\mathcal{A}_{2,m} = GF(2)[x_1, \dots x_m]/(x_1^2 - 1, x_2^2 - 1, \dots x_m^2 - 1) \simeq GF(2)[C_2^m]$$

as a vector space with basis

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}, \ a_i \in \{0, 1\}. \tag{4}$$

It is known ([7]) that the radical $\mathcal{J}_{2,m}$ of this group algebra is generated by the monomials $X_i = x_i - 1 = x_i + 1$.

Definition 1 ([5]). The code C in $\mathcal{J}_{2,m}$ (see (1)) is said to be a *monomial code* if it is an ideal in $\mathcal{A}_{2,m}$ generated by some monomials of the form

$$X_1^{k_1} X_2^{k_2} \dots X_m^{k_m}, \text{ where } 0 \leqslant k_i \leqslant 1 \tag{5}$$

The codes we intend to study are monomial codes.

For $p = 2$ using the usual polynomial product in the Boolean monomial $X_1^{k_1} X_2^{k_2} \dots X_m^{k_m}$ $(k_i \in \{0, 1\})$ we have

$$X_1^{k_1} X_2^{k_2} \dots X_m^{k_m} = (x_1 + 1)^{k_1} (x_2 + 1)^{k_2} \dots (x_m + 1)^{k_m}$$

and the Hamming weight in the basis (4) of this monomial equals $\prod_{i=1}^{m}(1+k_i)$.

Example. Let G be an elementary abelian group of order 2^m, $m \geqslant 2$. Define the codes C_j as ideals in $K[G]$ generated by $X_j = x_j - 1$. These codes are binary self-dual $[2^m, 2^{m-1}, 2]$ codes and they are self-dual since $C_j = C_j^{\perp} = \langle X_j \rangle$. Further, this code is a direct sum of $[2, 1, 2]$-codes. The dimension of the code C_j is 2^{m-1}, the same as the dimension of the radical of the group algebra $GF(2)[H]$, where H is an elementary abelian 2-group of rank $m - 1$. The minimal distance of C_j is $d = 2$. This follows from the fact that the element $X_j = x_j + 1$ is included in the basis of C_j. Thus, C_j is a self-dual $[2^m, 2^{m-1}, 2]$-code.

By Remark 1 one can see that a power of the radical of a modular group algebra is self-dual if and only if the nilpotency index of the radical is even. In our case (when G is elementary abelian of order p^m) the nilpotency index is even if and only if $p = 2$ and m is odd.

If m is odd, the binary RM-codes with parameters $[2^m, 2^{m-1}, 2^{\frac{m+1}{2}}]$ are self-dual and they are the $\frac{m+1}{2}$-th powers of the radical $\mathcal{A}_{2,m}$.

For $m = 2k$ where k is an arbitrary integer, we have a new method to construct a doubly-even class of binary self-dual C codes with parameters $[2^m, 2^{m-1}, 2^k]$. For this code C we have $\mathrm{RM}(k-1, 2k) \subset C \subset \mathrm{RM}(k, 2k)$. In the case of $m = 4$, we get two known extremal $[16, 8, 4]$ codes (listed in [14]) and for $m > 4$ these codes are not extremal. A doubly-even (i.e. its minimum distance is divisible by 4) self-dual code is called extremal, if we have for its minimum distance $d = 4\left[\frac{n}{24}\right] + 4$, where n denotes the code length (see Definition 39 and Lemma 40 in [8]).

To abbreviate the description of our codes, we shall refer to the monomial $X_1^{k_1} \dots X_m^{k_m}$ as the m-tuple $(k_1, k_2, \dots, k_m) \in \{0, 1, \dots, p-1\}^m$ of exponents.

Using Plotkin's construction of RM-codes (see Theorem 2 [13], Ch. 13, § 3) we obtain the following property of RM-codes.

Lemma 1. *If m is even and $m = 2k$, then* $\mathrm{RM}(k-1, m) = \mathcal{J}_{2,m}^{k+1}$ *contains a proper subspace which is isomorphic to* $\mathrm{RM}(k-1, m-1)$.

Proof. Recall, that the set of monomials in the basis (2) of $\mathcal{J}_{2,m}^{k+1}$ is invariant under the permutations of the variables X_i, i.e. the set of binary m-tuples $(k_1, k_2, \dots, k_m)$ assigned to the basis (2) is invariant under the permutation of all elements of the symmetric group S_m. Take the basis elements with $k_m = 1$. Then the monomials $X_1^{k_1} \dots X_m^{k_m}$ of degree m can be projected by $\pi : (k_1, k_2, \dots, k_{m-1}, 1) \mapsto (k_1, k_2, \dots, k_{m-1})$. In this way we get a basis of $\mathcal{J}_{2,m-1}^{k} \cong \mathrm{RM}(k-1, m-1)$. □

For $m = 2k$ denote the set of all k-subsets of $\{1, 2, \dots, 2k\}$ by X. The elements of X can be described by binary sequences $(k_1, k_2, \dots, k_m)$ consisting of k '0'-s and k '1'-s in any order. Clearly, the cardinality of the set X is $\binom{2k}{k}$.

We say that the subset Y of binary m-tuples in X is *complement free* if $y \in Y$ implies $\mathbf{1} - y \notin Y$, where $\mathbf{1} = (1, 1, \dots, 1)$. Denote the set of monomials corresponding to the set of exponents in X by $\mathcal{X}$. Denote the set with maximum number of pairwise orthogonal monomials in $\mathcal{X}$ by $\mathcal{Y}$ and their corresponding exponents in X by Y.

Example. For $m = 6$ the quotient space $\mathcal{J}_{2,m}^3 / \mathcal{J}_{2,m}^4$ has a basis with $\binom{6}{3} = 20$ elements, where the binary 6-tuples corresponding to the coset representative monomials (the set X) are listed in pairs of complements:

$$
\begin{array}{ll}
(1,1,1,0,0,0) & (0,0,0,1,1,1)\\
(1,1,0,1,0,0) & (0,0,1,0,1,1)\\
(1,1,0,0,1,0) & (0,0,1,1,0,1)\\
(1,1,0,0,0,1) & (0,0,1,1,1,0)\\
(1,0,1,1,0,0) & (0,1,0,0,1,1)\\
(1,0,1,0,1,0) & (0,1,0,1,0,1)\\
(1,0,1,0,0,1) & (0,1,0,1,1,0)\\
(1,0,0,1,1,0) & (0,1,1,0,0,1)\\
(1,0,0,1,0,1) & (0,1,1,0,1,0)\\
(1,0,0,0,1,1) & (0,1,1,1,0,0)
\end{array}
$$

and we have $2^{\frac{1}{2}\binom{6}{3}} = 2^{10}$ complement-free sets. For example the following complement free sets Y and $\mathcal{Y}$ of 10 elements:

Y	$\mathcal{Y}$
$(1,1,1,0,0,0)$,	$X_1X_2X_3$
$(0,0,1,0,1,1)$,	$X_3X_5X_6$
$(1,1,0,0,1,0)$,	$X_1X_2X_5$
$(0,0,1,1,1,0)$,	$X_3X_4X_5$
$(1,0,1,1,0,0)$,	$X_1X_3X_4$
$(0,1,0,1,0,1)$,	$X_2X_4X_6$
$(0,1,0,1,1,0)$,	$X_2X_4X_5$
$(0,1,1,0,0,1)$,	$X_2X_3X_6$
$(1,0,0,1,0,1)$,	$X_1X_4X_6$
$(1,0,0,0,1,1)$,	$X_1X_5X_6$

Theorem 1. *Let C be a binary code with $\mathrm{RM}(k-1,2k) \subset C \subset \mathrm{RM}(k,2k)$ with the following basis of the factorspace $C/\mathrm{RM}(k-1,2k)$*

$$\left\{\prod_{i=1}^{m} X_i^{k_i} + \mathrm{RM}(k-1,2k),\ \text{where } k_i \in \{0,1\} \text{ and } \sum_{i=1}^{m} k_i = k\right\}, \qquad (6)$$

where the set of the exponents $(k_1, k_2, \ldots, k_m)$ is a maximal (with cardinality $2^{\frac{1}{2}\binom{2k}{k}}$) complement free subset of X. Then C forms a $[2^{2k}, 2^{2k-1}, 2^k]$ self-dual doubly-even code.

Proof. Let G be an elementary abelian group of order 2^m, where $m = 2k,\ k \geqslant 2$. By the group algebra representation of RM-codes and the definition of C we have the relation $\mathcal{J}_{2,m}^{k+1} \subset C \subset \mathcal{J}_{2,m}^{k}$.

For $m = 2k$ the set $\mathcal{X}$ is the set of coset representatives of the quotient space $\mathcal{J}_{2,m}^{k}/\mathcal{J}_{2,m}^{k+1}$, i.e. the set of monomials satisfying (6).

Clearly, two monomials $X_1^{k_1}X_2^{k_2}\dots X_m^{k_m}$ and $X_1^{l_1}X_2^{l_2}\dots X_m^{l_m}$ are orthogonal, i.e. their product is zero, if for some $i : 1 \leqslant i \leqslant m$ we have $k_i = l_i$.

Thus, the elements in the radical corresponding to these monomials are orthogonal if their exponent m-tuples belong to a complement free set.

The m-tuples $(k_1, k_2 \dots k_m)$ have to be complement free in Y, otherwise the corresponding monomials in $\mathcal{Y}$ are not orthogonal. Clearly Y is a complement free subset of X (given by (4)) with cardinality $\frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$.

By definition, $C = \langle\, \mathcal{J}_{2,m}^{k+1} \cup \mathcal{Y}\,\rangle$ is a subspace of the radical $\mathcal{J}_{2,m}$ of the group algebra $\mathcal{A}_{2,m}$ generated by the union of $\mathcal{J}_{2,m}^{k+1}$ and $\mathcal{Y}$. For the dimension of C we have

$$\dim(C) = \dim(\mathrm{RM}(k-1, m)) + \frac{1}{2}\binom{2k}{k} = 1 + \sum_{i=1}^{k-1}\binom{2k}{i} + \frac{1}{2}\binom{2k}{k} = 2^{2k-1}.$$

It follows that C is self-dual. Since a binary self-dual code contains a word of weight 2 if and only if the generator matrix has two equal columns, we have our self-dual code to be doubly-even.

Each monomial in $\mathcal{Y}$ has the same weight 2^k, that is the minimal distance of C. Using the identities for the monomials involved in the basis of our codes

$$x_i(x_j + 1) = (x_i + 1)(x_j + 1) + (x_j + 1) \text{ and } (x_i + 1)^2 = 0,$$

we easily obtain that C (which is subspace of $\mathcal{J}_{2,m}$) is an ideal in the group algebra $GF(2)[G]$. □

Theorem 2. *Let Y and $\mathcal{Y}$ be sets defined above and let C be the code defined in Theorem 1. Suppose that $k_i = 0$ for some $i : 1 \leqslant i \leqslant m$ in each element of the subset Y, (i.e. the variable X_i is missing in each monomial of $\mathcal{Y}$). Then we have the isomorphism*

$$C \simeq \mathrm{RM}(k-1, 2k-1) \oplus \mathrm{RM}(k-1, 2k-1).$$

Proof. The elements of $\mathcal{Y}$ are of the form

$$X_1^{k_1}\dots X_m^{k_m} = (x_1 + 1)^{k_1}(x_2 + 1)^{k_2}\dots(x_m + 1)^{k_m}, \text{ where } \sum_{i=1}^{m} k_i = k$$

and their weight is 2^k. Project the set of monomials with $k_i = 0$ in $C = \langle\, \mathcal{J}_{2,m}^{k+1} \bigcup \mathcal{Y} \,\rangle$ onto the monomials $X_1^{k_1}, \ldots, X_{i-1}^{k_{i-1}}, X_{i+1}^{k_{i+1}}, \ldots, X_m^{k_m}$. The image C_1 of this projection is a self-dual RM$(k-1, 2k-1)$-code with parameters $[2^{2k-1}, 2^{2k-2}, 2^k]$.

By Lemma 1 the elements of the basis of $J_{2,m}^{k+1}$ with $k_i = 1$ generate a subspace C_2 which is isomorphic to RM$(k-1, 2k-1)$. The intersection of C_1 and C_2 is empty. Therefore $C \simeq C_1 \oplus C_2$ and the statement follows. □

Remark 2. In particular, by Theorem 1 we get $[16, 8, 4]$ self-dual codes for $m = 4$. These codes are extremal doubly-even codes. Using the SAGE computer algebra software we may check easily the classification of binary self-dual codes listed in [14].

There are two cases:

1) If $k_i = 0$ for some $i : 1 \leqslant i \leqslant m$ in each element of the set Y, then we get the direct sum $E_8 \oplus E_8$, where E_8 is the extended Hamming code.
2) otherwise we get an indecomposable $[16, 8, 4]$ code (which is denoted by E_{16} in [14]).

These codes are formally self-dual. Both classes have the following weight function:

$$z^{16} + 28z^{12} + 198z^8 + 28z^4 + 1$$

Remark 3. It is known that for each odd $m > 1$ there exists a self-dual affine-invariant code of length 2^m over $GF(2)$, which is not a self-dual RM-code [4].
The factor space $\mathcal{J}_{p,m}^{k} / \mathcal{J}_{p,m}^{k+1}$ is an irreducible $AGL(m, GF(p))$ module. Thus the code C is not affine invariant (see [1] Theorem 4.17) as the powers of the radical of $\mathcal{A}_{p,m}$ are. The code C cannot be an extended cyclic code by Corollary 1 in [4].

Remark 4. Using the inclusion-exclusion principle a formula can be given for the dimension of the $RM(k+1, m)$-code (see for example in [1] Theorem 5.5). If $p = 2$ and $0 \leqslant k \leqslant m$, then we have

$$\dim\, C = \tfrac{1}{2}\tbinom{2k}{k} + \sum_{i=k+1}^{m} \sum_{j=0}^{2k} (-1)^j \tbinom{2k}{j} \tbinom{2k-2j+i-1}{i-2j} = \sum_{i=k+1}^{m} \tbinom{2k}{i} + \tfrac{1}{2}\tbinom{2k}{k},$$

where $i - 2j \geqslant 0$.

The codes constructed in the current paper are worth to be studied further. Already for $k = 2$ we get two non-isomorphic codes with the same parameters. It would be interesting to determine all classes of codes up to isomorphism for each arbitrary integer k and to determine their automorphism group. The code C in Theorem 1 is not affine-invariant and first computations show that the automorphism group of C with $k_i = 0$ differs from the automorphism group of C with $k_i = 1$ for some $1 \leqslant i \leqslant m$.

We can formulate the following open questions about the code C of Theorem 1:

1) Does there exist a classification for all complement-free sets for arbitrary even m?
2) How many non-equivalent (in any sense) self-dual binary codes exist for fixed m and p?
3) Compare the automorphism groups of the codes C defined in Theorem 1 with the automorphism group of RM-codes.
4) Find decoding algorithms for C.

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SURVEY ARTICLE

A survey of results on radicals and torsions in modules

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Abstract. In this work basic results of the author on radicals in module categories are presented in a short form. Principal topics are: types of preradicals and their characterizations; classes of R-modules and sets of left ideals of R; notions and constructions associated to radicals; rings of quotients and localizations; preradicals in adjoint situation; torsions in Morita contexts; duality between localizations and colocalizations; principal functors and preradicals; special classes of modules; preradicals and operations in the lattices of submodules; closure operators and preradicals.

The present review contains the formulations of basic results of the author in the theory of radicals and torsions in modules. We preserve the chronological order, as well as the terminology and notations of surveyed works of *References* [1–66]. For convenience the article is divided into sections, dedicated to cycles of works with close subjects.

1. Radicals in modules: general questions ([1–6])

The theory of radicals and torsions in modules has its source in works of P. Gabriel, S.E. Dickson, J.P. Jans, J.-M. Maranda, K. Morita, J. Lambek, O. Goldman and many other algebraists. Fundamental books in this field were written by L.A. Skornyakov, A.P. Mishina (1969), J.S. Golan (1976), B. Stenström (1975), L. Bican, T. Kepka, P. Nemec (1982).

In the article [1] some general questions on radicals in modules are discussed, the characterizations of hereditary and cohereditary radicals

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are shown by suitable conditions on torsion or torsion free classes of modules. The upper and lower radicals over the given class of modules, as well as special and cospecial radicals are studied. The filters of left ideals corresponding to special radicals are described. In particular, the following theorem is proved.

Theorem 1.1. *A radical filter $\mathcal{F}$ is special if and only if the following condition holds:*

(∗) *If $I \in \mathbb{L}({}_RR)$, $I \subseteq J$, J is an irreducible left ideal and $(J : \lambda) \in \mathcal{F}$ for some $\lambda \notin J$, then $I \in \mathcal{F}$.*

In the paper [2] the axiomatic basis of torsions in R-modules is indicated in the terms of left ideals of R: technique of the work by classes of R-modules is adapted to the sets of left ideals of R. In particular, the approach of S.E. Dickson to torsion theories by two classes of modules is transferred in the terms of ring R by two sets $(\mathcal{F}_1, \mathcal{F}_2)$ of left ideals of R. Properties of sets $\mathcal{F}_1$ and $\mathcal{F}_2$, as well as the relations between them, are shown.

Similar ideas are developed in the work [4], where the complete description of relations between the classes of modules and the sets of left ideals of R is obtained. These relations are expressed by mappings Φ and Ψ, where: $\Phi(\mathcal{M}) = \bigcup_{M\in\mathcal{M}} \pi(M)$, $\mathcal{M}$ is a class of modules and $\pi(M) = \{(0 : m) \mid m \in M\}$; $\Psi(\mathcal{F}) = \{M \in R\text{-Mod} \mid \pi(M) \subseteq \mathcal{F}\}$, $\mathcal{F} \subseteq \mathbb{L}({}_RR)$.

A class $\mathcal{M} \subseteq R$-Mod is called *closed* if $\mathcal{M} = \Psi\Phi(\mathcal{M})$; the set $\mathcal{F} \subseteq \mathbb{L}({}_RR)$ is called *closed* if $\mathcal{F} = \Phi\Psi(\mathcal{F})$. The descriptions of closed classes and closed sets are obtained. Namely, the class $\mathcal{M}$ is closed if and only if the following condition holds:

(A_1) $M \in \mathcal{M} \Leftrightarrow Rm \in \mathcal{M}$ for every $m \in M$.

The set $\mathcal{F}$ is closed if and only if it satisfies the condition:

(a_1) If $I \in \mathcal{F}$, then $(I : a) \in \mathcal{F}$ for every $a \in R$.

Further, the properties of the class $\mathcal{M}$ are considered to be closed under: (A_2) homomorphic images; (A_3) direct sums; (A_4) direct products; (A_5) extensions; (A_6) essential extensions. In parallel, for arbitrary set $\mathcal{F}$ of left ideals of R special conditions $(a_2), (a_3), \ldots, (a_6)$ are considered.

Theorem 1.2. *The mappings Φ and Ψ define an isotone bijection between closed classes of R-modules and closed sets of left ideals of R.*

Theorem 1.3. *Let $\mathcal{M}$ and $\mathcal{F}$ correspond each to other in the sense of Theorem 1.2. The class $\mathcal{M}$ satisfies the condition (A_n) if and only if the set $\mathcal{F}$ satisfies the condition (a_n), where $n = 2, 3, \ldots, 6$.*

So basic closure conditions on the class $\mathcal{M}$ are "translated" in the terms of left ideals. Combining suitable conditions, this result gives us the descriptions by left ideals of many types of preradicals, such as pretorsions, torsions, cotorsions, special radicals, etc.

In the article [3] some constructions of radicals of special types using the properties of elements are shown. Some concrete examples are indicated, in particular the multiplicative closed systems S of elements of the ring R are considered. It is proved that the set of S-torsion elements of each module M is a submodule of M if and only if S satisfies the left Ore condition. Moreover, the condition is indicated when the associated to S radical r_S is a torsion.

The works [5] *and* [6] represent the thesis for a candidate's degree and its short exposition.

2. Radical closures ([7, 9, 20])

Studying idempotent radicals of R-Mod, their relations with closure operators of special type were observed. If r is an idempotent radical of R-Mod, then for every pair $N \subseteq M$, where $N \in \mathbb{L}({}_R M)$, denoting by $\overline{N}$ such submodule of M that $\overline{N}/N = r(M/N)$, we obtain a closure operator $N \longmapsto \overline{N}$ in $\mathbb{L}({}_R M)$ for every $M \in R$-Mod.

In the paper [7] the notion of *radical closure* of R-Mod is introduced and studied.

Theorem 2.1. *There exists an isotone bijection between idempotent radicals of* R-Mod *and radical closure of this category.*

In continuation it is shown that radical closures of R-Mod can be described both by dense submodules and by closed submodules. For this purpose properties of the functions $\mathcal{F}_1^t$ and $\mathcal{F}_2^t$ are shown, which in every module M distinguish the set of dense submodules $\mathcal{F}_1^t(M)$ and the set of closed submodules $\mathcal{F}_2^t(M)$.

Theorem 2.2. *There exists a bijection between radical closures of* R-Mod *and functions* $\mathcal{F}$ *of the type* $\mathcal{F}_1^t$ *(as well as functions* $\mathcal{F}$ *of the type* $\mathcal{F}_2^t$*).*

The application of this result to torsions is indicated: the *hereditary* radical closures are obtained which uniquely correspond to the torsions of R-Mod. Such a radical closure can be reduced to a closure operator t in $\mathbb{L}({}_R R)$ with the condition: $t(I : a) = \big(t(I) : a\big)$ for every $a \in R$.

Some types of radicals can be described by radical closures.

In the work [9] the characterizations of torsions and of stable radicals are obtained in terms of radical closures.

Theorem 2.3. *For every idempotent radical r the following conditions are equivalent:*
1) *r is a torsion;*
2) $\{\mathcal{F}_1(A)\} \cap B \subseteq \mathcal{F}_1(B)$ *for every pair* $A \subseteq B$*;*
3) $t_r(B \cap C, A) = t_r(B, A) \cap t_r(C, A)$ *for every* $B, C \in \mathbb{L}(A)$*;*
4) *if* $B, C \in \mathcal{F}_1(A)$ *then* $B \cap C \in \mathcal{F}_1(A)$ *for every module A.*

Dually the description of stable radicals is obtained.

3. Divisibility, generators, cogenerators ([8, 10, 11, 20])

Some notions and constructions closely related to radicals of modules are studied. In particular, some known notions are generalized with respect to radicals or torsions.

In the article [8] *r-divisible* modules are investigated. They represent a generalization of injectivity with respect to an idempotent radical r. Some characterizations of r-divisible modules are indicated. Moreover, the r-divisible envelope $E_r(A)$ of a module A is constructed. $E_r(A)$ exists for every module A and is unique up to an isomorphism.

Theorem 3.1. *Let* $B \subseteq A$, $A \in R$-Mod *and r be an idempotent radical. The following conditions are equivalent:*
1) *A is a maximal r-essential extension of B;*
2) *A is a minimal r-divisible module containing B;*
3) *A is an r-divisible and r-essential extension of B.*

As an application, using r-divisible envelope $E_r({}_RR)$ of the module ${}_RR$, an analogue of ring of quotients in the sense of Y. Utumi is constructed.

In the paper [11] some modifications of the notions of generator and cogenerator with respect to some preradicals are studied. Any class of modules $\mathcal{M}$ defines in R-Mod preradicals $r^{\mathcal{M}}$ and $r_{\mathcal{M}}$ by the rules:

$$r^{\mathcal{M}}(X) = \sum\{\operatorname{Im} f \mid f\colon M \to X, M \in \mathcal{M}\},$$
$$r_{\mathcal{M}}(X) = \bigcap\{\operatorname{Ker} g \mid g\colon X \to M, M \in \mathcal{M}\}.$$

If $\mathcal{M} = \{M\}$, we have the preradicals r^M and r_M.

The class $\mathcal{R}(r^{\mathcal{M}})$ contains all modules for which $\mathcal{M}$ is a generator class, and similarly $\mathcal{P}(r_{\mathcal{M}})$ contains all modules for which $\mathcal{M}$ is a cogenerator class. A module M is a *generator* (*cogenerator*) of a preradical r if $r = r^M$ ($r = r_M$). Every pretorsion has a generator module and every torsion has an injective cogenerator. Generators or cogenerators for some concrete preradicals are indicated.

Theorem 3.2. 1) *The module M is a cogenerator of the radical r_R if and only if M is faithful and torsion free in the sense of H. Bass.*

2) *The module M is a generator of the preradical $r^{E(R)}$ if and only if it is a faithful, fully divisible and endofinite module.*

3) *The module M is a cogenerator of a Lambek's torsion $r_{E(R)}$ if and only if M is $r_{E(R)}$-torsion free and contains a faithful, fully divisible module.*

For every $M \in R$-Mod the radical r_M, where $r_M(X) = \bigcap\{\operatorname{Ker} f \mid f\colon X \to M\}$, is the greatest between the radicals r such that $r(M) = 0$.

In the article [10] the following question is discussed: for which modules M the radical r_M is idempotent or it is a torsion.

Theorem 3.3. *For every module $M \in R$-*Mod *the following conditions are equivalent:*

1) *r_M is a torsion;*
2) *the class $\mathcal{P}(r_M)$ is closed under extensions and $\mathcal{R}(r_M)$ is hereditary;*
3) *$r_M = r_{E(M)}$, where $E(M)$ is the injective envelope of M;*
4) *M is a pseudo-injective module (i.e. $E(M) \in \mathcal{P}(r_M)$).*

Some applications in the particular case $M = {}_RR$ are considered.

4. Rings of quotients as bicommutators ([12, 13, 35, 36])

A problem of special interest is to determine when the ring of quotients with respect to a torsion has a simple form, in particular when it can be expressed by some known constructions. One of the most convenient forms of representation is the bicommutator of a suitable module.

If $M_R \in$ Mod-R and $E = \operatorname{Hom}_R(M, M)$ is the ring of endomorphisms, then M is a left E-module and the ring $S = \operatorname{Hom}_E({}_EM, {}_EM)$ is called the *bicommutator* of M_R. Then we have the canonical homomorphism $h\colon R \to S$, defined by the rule $(x)[h(r)] = xr$, where $r \in R$, $x \in M_R$ and $h(r)\colon {}_EM \to {}_EM$.

Let M_R be a pseudo-injective module. Then the radical r_M of Mod-R is a torsion, so it defines a localization functor Q_{r_M}. In particular, the ring of quotients $Q_{r_M}(R_R)$ with canonical homomorphism $\sigma\colon R \to Q_{r_M}(R_R)$ is defined.

In the paper [12] the necessary and sufficient conditions for coincidence of ring of quotients $Q_{r_M}(R_R)$ with the bicommutator of M_R are shown. We denote by $\mathcal{F}(r_M)$ the radical filter of the torsion r_M.

Theorem 4.1. *The bicommutator S of the pseudo-injective module M_R is the right ring of quotients of R with respect to r_M if and only if the following conditions hold:*

(A) *for every homomorphism $f\colon S_R \to M_R$ there exists $x \in M$ such that $f(s) = xs$ for every $s \in S$;*

(B) *if $K \in \mathcal{F}(r_M)$, then every homomorphism from $\mathrm{Hom}_R(K_R, M_R)$ of the form φ_x can be extended to $\overline{\varphi}_x\colon R_R \to M_R$, where $\varphi \in \mathrm{Hom}_R(K_R, S_R)$, $x \in M$ and φ_x acts by the rule $\varphi_x(k) = x\varphi(k)$, $k \in K$.*

From this theorem some results of J. Lambek (1971), K. Morita (1971) and H.H. Storrer (1971) follow as particular cases.

The similar question on the coincidence of ring of quotients with the bicommutator of suitable module is discussed *in the work [13]*. The situation is studied when by the module M_R the ring $Q_M(R)$ can be constructed as the r_M-closure of R-module R/K in $E(R/K)$, where $K = (0 : M)$. The main result is the following.

Theorem 4.2. *Let M_R be a K-fully divisible module, where $K = (0 : M)$ is a torsion ideal of R. The bicommutator S of M_R coincides with the ring of quotients $Q_M(R)$ if and only if M_R is a module of type F_h (in the sense of K. Morita) and the canonical homomorphism $h\colon R \to S$ is essential.*

We obtain as corollaries the following statements:

1) If M_R is a cofaithful and fully divisible module, then $Q_M(R) \cong S$;
2) If M_R is an injective and endofinite module, then $Q_M(R) \cong S$;
3) If M_R is injective, then $Q_M(R) \cong S$ if and only if M_R is a module of type F_h.

5. Preradicals and adjointness ([14, 15, 20, 35, 36])

Further investigations of radicals in modules require the most intensive utilization of categorical methods, in particular, of adjoint functors and their properties.

In the article [14] preradicals associated to the pair of adjoint functors $R\text{-Mod} \underset{S}{\overset{T}{\rightleftarrows}} \mathfrak{B}$ are studied, where $\mathfrak{B}$ is an abelian category and T is left adjoint to S. Then there exist associated natural transformations $\Phi\colon 1_{R\text{-Mod}} \to ST$ and $\Psi\colon TS \to 1_{\mathfrak{B}}$. Preradicals generated by this situation are studied, the relations between them are elucidated and also criteria of their coincidence are shown.

In particular, the radical r is defined by the rule $r(M) = \operatorname{Ker}\Phi_M$ and if the functor T is exact, then r is a torsion. Therefore r defines a localization functor $\mathbb{L}_r$.

Furthermore, the functor $Q_r\colon R\text{-Mod} \to R\text{-Mod}$ is considered, where $Q_r(M)$ is the r-closure of $\operatorname{Im}\Phi_M$ in $ST(M)$. The question when these functors ($\mathbb{L}_r$ and Q_r) coincide is studied.

Theorem 5.1. *Let T be a selfexact functor and r be a torsion. Then for every module $M \in R$-Mod the module $Q_r(M)$ coincides with the module of quotients of M with respect to r (i.e. $Q_r = \mathbb{L}_r$).*

Theorem 5.2. *Let T be a selfexact functor and r be a torsion. Then the following conditions are equivalent:*
1) $Q_r(M) = \operatorname{Im}\Phi_M$*;*
2) $\operatorname{Im}\Phi_M$ *is an r_M-injective module.*

These statements generalize some results of K. Morita (1971), J. Lambek (1971) and J.A. Beachy (1974).

In the paper [15] the same adjoint situation (T, S) is considered and the question on the correspondences between preradicals (torsions) of the categories R-Mod and $\mathfrak{B}$ is studied. Some methods of transition from preradicals of R-Mod to preradicals of $\mathfrak{B}$ and inversely are indicated.

In particular, if r is a preradical of R-Mod, then the preradical r^* of $\mathfrak{B}$ is defined by the rule:

$$r^*(B) = \operatorname{Im}\left(\Psi_B \cdot T(i_B)\right),$$

where $B \in \mathfrak{B},\ i_B\colon r\left(S(B)\right) \to S(B)$ is the inclusion and the right part is the image of composition $T\big(r\left(S(B)\right)\big) \xrightarrow{T(i_B)} TS(B) \xrightarrow{\Psi_B} B$.

Similarly the inverse transition $s \longmapsto s^*$ is defined for an arbitrary preradical s of $\mathfrak{B}$.

Theorem 5.3. *The functions r^* and s^* are preradicals. The operators $r \longmapsto r^*$ and $s \longmapsto s^*$ preserve the order of preradicals. Moreover, the following relations hold:*

$$r \leqslant r^{**}, \qquad (r_1 \vee r_2)^* = r_1^* \vee r_2^*, \qquad (r_1 \cdot r_2)^* \leqslant r_1^* \cdot r_2^*,$$
$$s \geqslant s^{**}, \qquad (s_1 \wedge s_2)^* = s_1^* \wedge s_2^*, \qquad (s_1 \cdot s_2)^* \geqslant s_1^* \cdot s_2^*.$$

Theorem 5.4. 1) *The operator $r \longmapsto r^*$ preserves the idempotence of preradicals;*

2) *If T is exact and Ψ is an equivalence, then the operator $r \longmapsto r^*$ preserves the hereditary property;*

3) *If S is exact, then the operator $r \longmapsto r^*$ preserves the cohereditary property.*

Some similar results are obtained for the operator $s \longmapsto s^*$. This situation is analyzed more detailed in the case when T is exact and Ψ is a natural equivalence. The main result is the following.

Theorem 5.5. *Let T be an exact functor and Ψ be a natural equivalence. Then the operators $r \longmapsto r^*$ and $s \longmapsto s^*$ establish an isotone bijection between torsions r of R-*Mod *such that $r \geqslant r_T$, and all torsions of $\mathfrak{B}$, where $r_T(M) = \operatorname{Ker} \Phi_M$.*

6. Torsions in Morita contexts ([16–18, 27])

Morita context is an important construction with a considerable role in studying the equivalence of module categories (Morita theorems). We use Morita contexts for the investigation of relations between torsions of two module categories. It turned out that in this case there exists a remarkable isomorphism between two parts of the lattices of torsions.

In the article [17] an arbitrary Morita context $(R, {}_RV_S, {}_SW_R, S)$ is considered with bimodule homomorphisms $(\,,\,)\colon V \otimes_S W \to R$ and $[\,,\,]\colon W \otimes_R V \to S$. The following functors are studied:

$$R\text{-Mod} \underset{H^* = \operatorname{Hom}_S(W,-)}{\overset{H = \operatorname{Hom}_R(V,-)}{\rightleftarrows}} S\text{-Mod}.$$

The trace-ideals $T = \operatorname{Im}(\,,\,)$ and $L = \operatorname{Im}[\,,\,]$ generate torsions r_0 in R-Mod and s_0 in S-Mod such that:

$$\mathcal{P}(r_0) = \{{}_RM \mid Tm = 0 \Rightarrow m = 0\}, \quad \mathcal{P}(s_0) = \{{}_SN \mid Ln = 0 \Rightarrow n = 0\}.$$

We use the notations:

$$\mathcal{L}(R)\ (\mathcal{L}(S)) \text{ is the lattice of torsions of } R\text{-Mod } (S\text{-Mod}),$$
$$\mathcal{L}_0(R) = \{r \in \mathcal{L}(R) \mid r \geqslant r_0\}, \quad \mathcal{L}_0(S) = \{s \in \mathcal{L}(S) \mid s \geqslant s_0\}.$$

Theorem 6.1. *The functors H and H^* determine an isotone bijection between torsions r of R-*Mod *such that $r \geqslant r_0$ and torsions s of S-*Mod *such that $s \geqslant s_0$, i.e. $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$.*

This is a key result and it will be repeatedly used in continuation. The particular cases are:

1) If ${}_RV$ is a generator of R-Mod, then $r_0 = 0$, therefore $\mathcal{L}(R) \cong \mathcal{L}_0(S)$;

2) If ${}_RV$ if finitely generated and projective, then $s_0{=}0$, so $\mathcal{L}_0(R){\cong}\mathcal{L}(S)$;

3) If the rings R and S are Morita equivalent, then $\mathcal{L}(R) \cong \mathcal{L}(S)$.

We remark that the bijection of Theorem 6.1 is obtained acting by functors H and H^* on the injective cogenerators of corresponding torsions.

In the papers [16] *and* [18] the question about the preservation of properties of torsions under the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ is investigated. The torsion $r \in \mathcal{L}(R)$ is called *faithful*, if $r({}_RR) = 0$. In [16] it is shown under which conditions on the Morita context the isomorphism of Theorem 6.1 preserves the faithfulness of torsions. In particular the following theorem is true.

Theorem 6.2. *Let V_S and ${}_RV$ be faithful, ${}_RV$ be torsion free in the sense of Bass and $[W, v] = 0$ implies $v = 0$. Then the torsion r is faithful if and only if the torsion s is faithful, where (r, s) is a pair of corresponding torsions.*

Some applications of obtained results to the standard Morita context $(R, {}_RV_S, {}_SV^*_R, S)$ are shown, where $S = \mathrm{End}({}_RV)$ and $V^* = \mathrm{Hom}_R(V, R)$.

In [18] the similar question is studied for two classes of torsions: jansian and ideal torsions. A torsion r is *jansian* if its radical filter $\mathcal{E}_r$ has the smallest element, the ideal I_r of R. A torsion r_I is called *the ideal torsion*, defined by ideal I, if $\mathcal{E}_{r_I}$ is the smallest radical filter containing I.

Theorem 6.3. *A torsion $r \in \mathcal{L}_0(R)$ is jansian if and only if the corresponding torsion $s \in \mathcal{L}_0(S)$ is jansian. In this case the ideals I_r and J_s, defining r and s, are related by the rules:*

$$J_s = [W, I_rV], \qquad I_r = (V, J_sW).$$

Similar methods are used also in the case of ideal torsions.

Theorem 6.4. *Let r and s be the corresponding torsions in the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$. Then r is an ideal torsion if and only if s is an ideal torsion. If $r = r_I$ and $s = r_J$, then the ideals I and J are related by the rules:*

$$J = [W, IV], \qquad I = (V, JW).$$

In the notice [27] one more application of the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ is shown. For the torsion r a submodule $N \subseteq M$ is called *r-closed* in M if $M/N \in \mathcal{P}(r)$. For the Morita context $(R, {}_RU_S, {}_SV_R, S)$ with trace ideals I and J the pair (r, s) of corresponding torsions is considered, where $r \in \mathcal{L}_0(R)$ and $s \in \mathcal{L}_0(S)$. Denote by $\mathbb{L}^r({}_RU)$ the lattice of r-closed submodules of ${}_RU$ and by $\mathbb{L}^s({}_SS)$ the lattice of s-closed left ideals of R.

Theorem 6.5. *The lattices* $\mathbb{L}^r({}_RU)$ *and* $\mathbb{L}^s({}_SS)$ *are isomorphic.*

In this case to every submodule $U' \subseteq U$ the annihilator $\{s' \in S \mid Us' \subseteq U'\}$ corresponds, and to every left ideal $K \subseteq {}_SS$ the submodule $\{u \in U \mid [V, u] \subseteq K\}$ is associated. Some results of B.J. Müller (1974) and S.M. Khuri (1984) follow as particular cases.

7. Adjointness and localizations ([19–22, 24])

Further investigations aim to compare the localizations (colocalizations) of modules with the canonical homomorphisms of adjoint situation, as well as to search for criteria of their coincidence.

In the article [19] the adjointness determined by a bimodule ${}_RU_S$ is considered:

$$S\text{-Mod} \underset{H=\mathrm{Hom}_R(U,-)}{\overset{T=U\otimes_S -}{\rightleftarrows}} R\text{-Mod}$$

with the natural transformations $\Psi\colon 1_{S\text{-Mod}} \to HT$ and $\Phi\colon TH \to 1_{R\text{-Mod}}$. The torsion class $\mathrm{Ker}\, T$ defines in S-Mod the idempotent radical s such that $\mathcal{R}(s) = \mathrm{Ker}\, T$; similarly, the torsion free class $\mathrm{Ker}\, H$ determines in R-Mod the idempotent radical r such that $\mathcal{P}(r) = \mathrm{Ker}\, H$. The following questions are studied:

1) when the homomorphism $\Psi_N: N \to HT(N)$ is the s-localization of N for every $N \in S$-Mod?
2) when the homomorphism $\Phi_M: TH(M) \to M$ is the r-colocalization of M for every $M \in R$-Mod?

To answer these questions the requirements from definitions of localizations and colocalizations are analyzed separately, indicating for each of them some equivalent conditions.

Theorem 7.1. *The following conditions are equivalent:*

1) Ψ_N *is the* s*-localization of* N *for every* $N \in S$-Mod*;*
2) T *is left* Ψ*-exact, full on* $\mathrm{Im}\, H$ *and left selfexact;*
3) HT *is left exact and the pair* (T, H) *is idempotent* (*i.e. the associated triple* $\mathbb{F}$ *is a localization triple*)*;*

4) *the class* $\mathcal{L} = \{{}_S N \mid \Psi_N$ *is an isomorphism*$\}$ *is a Giraud subcategory of* S-Mod*, whose reflector is induced by* HT*.*

The dual result which shows when Φ_M is the r-colocalization of M for every $M \in R$-mod is also proved. Some applications and particular cases are indicated. Some results of T. Kato (1978), R.S. Cunnigham, etc. (1972), K. Morita (1970), R.J. McMaster (1975) are obtained as corollaries.

The continuation of these investigations *is the article* [21], in which the similar questions are studied for a pair of adjoint *contravariant* functors. For a bimodule ${}_S V_R$ the following functors are considered:

$$S\text{-Mod} \underset{H'=\mathrm{Hom}_R(-,V)}{\overset{H=\mathrm{Hom}_S(-,V)}{\rightleftarrows}} \text{Mod-}R$$

with the natural transformations $\Psi\colon 1_{S\text{-Mod}} \to H'H$ and $\Phi\colon 1_{\text{Mod-}R} \to HH'$. In this situation all the facts which take place in S-Mod have analogous statements in Mod-R, so it is sufficient to study one of these categories, for example S-Mod. We have the idempotent radical s in S-Mod, defined by the class $\mathrm{Ker}\, H = \mathcal{R}(s)$.

Conditions are searched under which Ψ_N is the s-localization of N for every $N \in S$-Mod. The analogue of Theorem 7.1 is proved and some applications are shown.

A slightly different approach to these questions is applied *in the paper* [22]: criteria of coincidence of localizations (colocalizations) of modules with some simple *modifications* of canonical homomorphism of adjointness are searched. For the bimodule ${}_R U_S$ the following functors are considered:

$$R\text{-Mod} \underset{H=\mathrm{Hom}_S(U,-)}{\overset{T=U\otimes_R -}{\rightleftarrows}} S\text{-Mod}$$

with natural transformations $\Phi\colon 1_{R\text{-Mod}} \to HT$ and $\Psi\colon TH \to 1_{S\text{-Mod}}$.

The kernels of the functors T and H define the idempotent radicals r_0 and s_0. For any $M \in R$-Mod we have the canonical homomorphism $\Phi_M\colon M \to HTM$ and we consider its modification Φ'_M, denoting by $Q(M)$ the r_0-closure of $\mathrm{Im}\, \Phi_M$ in $HT(M)$ and representing Φ_M as the decomposition $M \xrightarrow{\Phi'_M} Q(M) \xrightarrow{\subseteq} HT(M)$.

Theorem 7.2. *If the functor* T *is exact, then for every* $M \in R$-Mod *the homomorphism* $\Phi'_M\colon M \to Q(M)$ *is a localization of* M *with respect to the torsion* r_0*.*

The dual result about s_0-colocalizations in S-Mod is also true. As corollaries we obtain the following statements:

1) if T is exact, then Φ_M is the r_0-localization of M if and only if $\Psi_{T(M)}$ is an isomorphism;
2) if H is exact, then Ψ_N is the s_0-colocalization of N if and only if $\Phi_{H(N)}$ is an isomorphism.

To the same cycle of works we can attribute *the article* [24], in which so called *double localizations* are defined and studied. This notion generalizes ordinary localizations and is defined by a pair (r, s), where r is a torsion and s is an idempotent radical of R-Mod (if $r = s$, then the ordinary localization is obtained).

Let $\mathcal{T}_r$ ($\mathcal{T}_s$) be the class of r-torsion (s-torsion) modules and $\mathcal{L}_{rs}$ be the class of r-torsion free and s-injective modules. The homomorphism $\varphi\colon M \to L$ is called (r, s)-*localization* of M if $\operatorname{Ker}\varphi \in \mathcal{T}_r$, $\operatorname{Coker}\varphi \in \mathcal{T}_s$ and $L \in \mathcal{L}_{rs}$.

The uniqueness and the existence of (r, s)-localization are proved for every module M in the case $r \geqslant s$. Then we have the functor of (r, s)-localization $L_{rs}\colon R\text{-Mod} \to R\text{-Mod}$ with the natural transformation $\varphi\colon 1_{R\text{-Mod}} \to L_{rs}$.

Theorem 7.3. *The module $L_{rs}({}_RR)$ can be transformed into a ring and $\varphi_R\colon {}_RR \to {}_RL_{rs}({}_RR)$ can be improved to a ring homomorphism. Every module $H \in \mathcal{L}_{rs}$ is an $L_{rs}(R)$-module, every R-homomorphism $f\colon {}_RH \to {}_RK$, where $H, K \in \mathcal{L}_{rs}$, is an $L_{rs}(R)$-homomorphism.*

The connections between the localization functors L_r, L_s and L_{rs} are indicated. The existence of a close relation between (r, s)-localizations and reflective subcategories of special type is also shown. As a consequence we obtain a bijection between the torsions of R-Mod and the Giraud subcategories of R-Mod. In this way some results of L. Fuchs and K. Messa (1980) are generalized.

In the book [20] both the foundations of radical theory in modules and some special related questions are expounded: localizations and colocalizations; modules and rings of quotients; torsions in diverse situations; Giraud subcategories; torsions and triples (monads); duality between localizations and colocalizations; lattice of torsions of R-Mod, etc. The diversity of possible approaches to radicals and torsions in modules is elucidated.

8. Idempotent radicals and adjointness ([23, 26, 28])

In the article [26] the connection between idempotent radicals of two module categories in the adjoint situation is studied. The adjoint functors defined by a bimodule ${}_SU_R$ are considered:

$$R\text{-Mod} \underset{H=\mathrm{Hom}_S(U,\text{-})}{\overset{T=U\otimes_R\text{-}}{\rightleftarrows}} S\text{-Mod}.$$

Let $\mathcal{I}(R)$ $(\mathcal{I}(S))$ be the class of all idempotent radicals of R-Mod (S-Mod). The following mappings are defined:

$$\mathcal{I}(R) \underset{\alpha}{\overset{\alpha'}{\rightleftarrows}} \mathcal{I}(S),$$

where $\mathcal{R}(\alpha(s)) = T^{-1}(\mathcal{R}(s))$ and $\mathcal{P}(\alpha'(r)) = H^{-1}(\mathcal{P}(r))$ for every $r \in \mathcal{I}(R)$ and $s \in \mathcal{I}(S)$. The operators of orthogonality $(\ )^{\uparrow}$ and $(\ )^{\downarrow}$ on the classes of modules determine the transition from the torsion to the torsion free classes and inversely. By these operators the mappings α and α' can be expressed as follows:

$$\mathcal{P}(\alpha(s)) = [H(\mathcal{P}(s))]^{\uparrow\downarrow}, \qquad \mathcal{R}(\alpha'(r)) = [T(\mathcal{R}(r))]^{\downarrow\uparrow}.$$

Some properties of the mappings α and α' are shown. In particular, α preserves the intersection, while α' preserves the sum of idempotent radicals. Furthermore,

$$\alpha(s) = \alpha\alpha'\alpha(s), \qquad \alpha'(r) = \alpha'\alpha\alpha'(r)$$

for every $s \in \mathcal{I}(S)$ and $r \in \mathcal{I}(R)$. The necessary and sufficient conditions are found for the relations $s = \alpha'\alpha(s)$ and $r = \alpha\alpha'(r)$. Such idempotent radicals are called *U-closed.*

Theorem 8.1. *The mappings α and α' define an isotone bijection between U-closed idempotent radicals of R-*Mod *and U-closed idempotent radicals of S-*Mod*.*

The conditions under which the mappings α and α' define the isomorphism $\mathcal{I}(R) \cong \mathcal{I}(S)$ are indicated. The dual results are proved for adjoint contravariant functors.

The continuation of these investigations is contained *in the article [28]*, where the action of the mappings α and α' on torsions and cotorsions of given categories is studied. The question is: under which conditions α preserves torsions or α' preserves cotorsions. Furthermore, the question when H preserves localizations or T preserves colocalizations of modules is studied.

Theorem 8.2. *The following conditions are equivalent:*

1) *α preserves torsions;*
2) *H transfers injective modules in up-hereditary modules;*
3) *H preserves up-hereditary property;*
4) *for every monomorphism $i\colon N' \to N$ of S-Mod the relation $\operatorname{Ker} T(i) \in \mathcal{R}(T(N))$ (the smallest torsion class containing $T(N)$) is true.*

Theorem 8.3. *Let T be an exact functor and r be a torsion of R-Mod. The following conditions are equivalent:*

1) *H preserves r-localizations;*
2) *H is r-full and r-exact.*

The dual results are also proved: conditions when α' preserves cotorsions and the functor T preserves colocalizations of modules are shown.

The preprint [23] contains the detailed exposition of all results on the mappings α and α', and also of similar facts on adjoint *contravariant* functors.

9. Classes of modules and localizations in Morita contexts ([29, 30])

If a Morita context $(R, {}_RU_S, {}_SV_R, S)$ with the bimodule homomorphisms $(\,,\,)$ and $[\,,\,]$ is given, then in the categories R-Mod and S-Mod quite a number of classes of modules with diverse closure properties (under homomorphic images, submodules, extensions, etc.) appear in a natural way. Therefore these classes of modules determine preradicals of various types (idempotent radicals, torsions, etc.).

In the article [29] the most important classes of modules determined by a Morita context, as well as the corresponding preradicals are investigated. Properties of these classes and connections between them are shown. Furthermore, criteria of the coincidence of some “near” preradicals are obtained.

For example, the pairs of adjoint functors (T^U, H^U) and (T^V, H^V) lead to the classes $\operatorname{Ker} T^U$ and $\operatorname{Ker} T^V$, which are torsion classes, and also to the classes $\operatorname{Ker} H^U$ and $\operatorname{Ker} H^V$, which are torsion free classes. An important role is played by the classes $\mathrm{Gen}({}_RU)$ and $\mathrm{Gen}({}_SV)$ of modules generated by ${}_RU$ or ${}_SV$, and also by the dual classes $\mathrm{Cog}({}_RV^*)$ and $\mathrm{Cog}({}_SU^*)$.

In the same time, the classes of modules are considered which are defined by trace ideals I and J of a given Morita context. For example, the ideal I determines the classes of modules:

$$ {}_I\mathcal{T} = \{{}_RM \mid IM = M\}, \qquad {}_I\mathcal{F} = \{{}_RM \mid \mathrm{Im} = 0 \Rightarrow m = 0\}, $$
$$ \mathcal{A}(I) = \{{}_RM \mid IM = 0\}, $$

and similarly for ideal J. The classes ${}_I\mathcal{T}$ and ${}_J\mathcal{T}$ are torsion classes, while ${}_I\mathcal{F}$ and ${}_J\mathcal{F}$ are torsion free classes. Various closure properties of the classes $\mathcal{A}(I)$ and $\mathcal{A}(J)$ lead to quite a number of associated preradicals.

Some relations between the studied classes of modules are shown. In particular, is true the following theorem.

Theorem 9.1. 1) $\mathcal{A}(I)^{\uparrow} = {}_I\mathcal{T}$, $\mathcal{A}(J)^{\uparrow} = {}_J\mathcal{T}$; $\mathcal{A}(I)^{\downarrow} = {}_I\mathcal{F}$, $\mathcal{A}(J)^{\downarrow} = {}_J\mathcal{F}$;
2) $\mathrm{Ker}\, T^V \subseteq \mathcal{A}(I)$, $\mathrm{Ker}\, T^U \subseteq \mathcal{A}(J)$; $\mathrm{Ker}\, H^U \subseteq \mathcal{A}(I)$, $\mathrm{Ker}\, H^V \subseteq \mathcal{A}(J)$;
3) $\mathcal{A}(I) \subseteq {}_I\mathcal{T}^{\downarrow}$, $\mathcal{A}(J) \subseteq {}_J\mathcal{T}^{\downarrow}$; $\mathcal{A}(I) \subseteq {}_I\mathcal{F}^{\uparrow}$, $\mathcal{A}(J) \subseteq {}_J\mathcal{F}^{\uparrow}$.

Further, preradicals of diverse types, which are defined by these classes of modules are considered. Some connections between them are indicated and conditions under which some preradicals coincide are found. These results are closely related to the investigations of T. Kato (1978), K. Ohtake (1980, 1982), etc.

The article [30] is devoted to the study of localizations in the Morita context $(R, {}_RU_S, {}_SV_R, S)$, which define the functors:

$$ R\text{-Mod} \underset{H^V = \mathrm{Hom}_S(V, -)}{\overset{H^U = \mathrm{Hom}_R(U, -)}{\rightleftarrows}} S\text{-Mod}. $$

The trace ideal $I = (U, V)$ of R leads to the natural transformation $\varphi\colon 1_{R\text{-Mod}} \to H^V H^U$, where $\varphi_M\colon {}_RM \to {}_RH^V H^U(M)$ acts by the rule $u(v(m\varphi_M)) = (u, v)m$. Furthermore, the ideal I determines in R-Mod the torsion r_I such that $\mathcal{P}(r_I) ={}_I \mathcal{F} = \{{}_RM \mid \mathrm{Im} = 0 \Rightarrow m = 0\}$. The goal of investigation: to find necessary and sufficient conditions under which the homomorphism φ_M is an r_I-localization of M for every $M \in R$-Mod.

Theorem 9.2. *The following conditions are equivalent:*
1) *φ_M is the r_I-localization of M for every $M \in R$-*Mod*;*
2) *$I^2 = I$ and the module ${}_R(U \otimes {}_SV)$ is fully r_I-projective;*
3) *$I^2 = I, I(U \otimes {}_SV) = U \otimes {}_SV$ and ${}_R(U \otimes {}_SV)$ is projective relative to the epimorphisms $\pi_M\colon M \to M/_{r_I}(M)$, $M \in R$-*Mod*.*

In particular, if ${}_R(U \otimes {}_SV)$ is projective with the trace I, then the conditions of this theorem hold. The connection with some results of K. Ohtake (1980, 1982), T. Kato (1978) and B.J. Muller (1974) is indicated.

10. Principal functors and preradicals ([25, 35, 36, 43])

The investigation of general questions on connections between preradicals of two module categories is continued (see Section 5).

In the preprint [25] (see also [35, 36, 43]) the pair of adjoint functors defined by the bimodule ${}_RU_S$ is considered:

$$R\text{-Mod} \underset{T=U\otimes_S -}{\overset{H=\mathrm{Hom}_R(U,-)}{\rightleftarrows}} S\text{-Mod}.$$

Let $\Phi\colon TH \to 1_{R\text{-Mod}}$ and $\Psi\colon 1_{S\text{-Mod}} \to HT$ be the associated natural transformations. These functors permit to define some mappings between preradicals of diverse types of the categories R-Mod and S-Mod on different "levels":

1) for *preradicals*, with the help of Φ and Ψ;
2) for *radicals* and *idempotent preradicals*, by functors T and H;
3) for *idempotent radicals*, applying T and H to torsion or torsion free classes (see Section 8).

In general case of preradicals the "star" mappings $r \longmapsto r^*$ and $s \longmapsto s^*$ are considered (see Section 5). For radicals other method is used: acting by T and H on the generating or cogenerating classes, $r_{\mathcal{K}} \longmapsto r_{H(\mathcal{K})}$, $r^{\mathcal{K}} \longmapsto r^{T(\mathcal{K})}$. Properties of these mappings are indicated and also their relation with the "star" mappings is shown. On the next "level" of idempotent radicals the mappings α and α' are used (see Section 8). It is proved that $\alpha(r)$ is the greatest idempotent radical contained in r^* and similarly for α'. If the functors H or T are exact, then α and α' coincide with the "star" mappings and in this case they preserve torsions or cotorsions.

The next step consists in the comparison of localizations of modules with special *modifications* of canonical homomorphisms for every torsion r of R-Mod (the particular case $r = r_0$ is considered in [22]).

By functors (T, H) and a torsion r of R-Mod for every $N \in S$-Mod we consider the homomorphism:

$$\Psi_N^r\colon N \xrightarrow{\Psi_N} HT(N) \xrightarrow{H(\pi_{T(N)})} (H \cdot 1/r \cdot T)(N),$$

where $\pi_{T(N)}$ is the natural homomorphism. In such a way we obtain the natural transformation $\Psi^r\colon 1_{S\text{-Mod}} \to H \cdot 1/r \cdot T$. Furthermore, in S-Mod we have the idempotent radical $s = \alpha(r)$, where $\mathcal{R}(s) = T^{-1}(\mathcal{R}(s))$. The problem is to find necessary and sufficient conditions under which Ψ^r_N is the s-localization of N for every $N \in S$-Mod. The torsion r defines a new pair of adjoint functors:

$$\mathcal{P}(r) \underset{T^r}{\overset{H^r}{\rightleftarrows}} S\text{-Mod}$$

(closely related to (H, T)) with the natural transformations Ψ^r and Φ^r. To this pair the triple $\mathbb{F}^r$ is associated.

Theorem 10.1. *The following conditions are equivalent:*

1) Ψ^r_N *is the s-localization of N for every $N \in S$-*Mod*;*
2) T^r *is left Ψ^r-exact, full on* $\operatorname{Im} H^r$ *and left selfexact;*
3) $\mathbb{F}^r$ *is a localization triple;*
4) $\mathcal{P}(s) = \mathcal{P}(r^*)$ *and* $\operatorname{Im} H^r = \operatorname{Fix} \Psi^r = \mathcal{L}_s$*;*
5) $\operatorname{Fix} \Psi^r$ *is a Giraud subcategory of S-*Mod.

In the same situation the colocalizations of modules with canonical homomorphisms are compared for some cotorsion s of S-Mod and $r = \alpha'(s)$. The modification Φ^s_M of Φ_M is considered and conditions under whichΦ^s_M is the r-colocalization of M for any $M \in R$-Mod (the analogue of Theorem 10.1) are shown. In addition, the work [25] contains the exposition of the case of *contravariant* functors with dual results.

11. Principal functors and lattices of submodules ([31–37, 39, 43])

The problem of the influence of principal functors of module categories on the lattices of submodules is of considerable interest. More exactly, if $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a functor between module categories, then the question on the relation between the lattices of submodules $\mathbb{L}(X)$ and $\mathbb{L}(F(X))$ is considered, where $X \in \mathcal{M}_1$ and $F(X) \in \mathcal{M}_2$. This problem is studied for the principal functors:

$$H = \operatorname{Hom}_R(U, \text{-}), \quad T = U \otimes {}_S\text{-}, \qquad H_1 = \operatorname{Hom}_R(\text{-}\,, U)$$

for a bimodule ${}_RU_S$. The mainly used method is the transition from the lattices of all submodules to the lattices of special submodules determined by associated preradicals r and s. To these questions the cycle of works [31–34] is dedicated (see also [35, 36, 43]).

Further we expose shortly the basic results for every principal functor.

Functor $H = \mathrm{Hom}_R(U, \text{-}) \colon R\text{-Mod} \to S\text{-Mod}$

The bimodule ${}_RU_S$ defines the pair (T, H) of adjoint functors with the natural transformations $\Phi\colon TH \to 1_{R\text{-Mod}}$ and $\Psi\colon 1_{S\text{-Mod}} \to HT$. Then we have preradicals r of R-Mod and s of S-Mod such that:

$$r({}_RM) = \mathrm{Im}\,\Phi_M, \qquad s({}_SN) = \mathrm{Ker}\,\Psi_N.$$

In the lattices of submodules $\mathbb{L}({}_RX)$ and $\mathbb{L}({}_SY)$ the following sublattices are defined:

$$\mathcal{L}^r({}_RX) = \{X' \subseteq X \mid r(X') = X'\}, \qquad \mathcal{L}_s({}_SY) = \{Y' \subseteq Y \mid s(Y/Y') = 0\}.$$

For every $X \in R$-Mod the following mappings are defined:

$$\mathbb{L}({}_RX) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}({}_SH(X)),$$

where

$$\alpha(X') = \{f\colon {}_RU \to {}_RX \mid \mathrm{Im}\, f \subseteq X'\}, \quad X' \subseteq X,$$
$$\beta(Y') = \sum\{\mathrm{Im}\, f' \mid f' \in Y'\} \ (= UY'), \quad Y' \subseteq H(X).$$

Some properties of the mappings α and β are shown. In particular, $\alpha(X') \in \mathcal{L}_s(H(X))$ and $\beta(Y') \in \mathcal{L}^r(X)$. The lattices $\mathbb{L}({}_RX)$ and $\mathbb{L}({}_SH(X))$ are called *canonically isomorphic* if α and β determine the isomorphism of these lattices, which is equivalent to the conditions:

I) $\mathbb{L}({}_RX) = \mathcal{L}^r({}_RX)$;
II) $\mathbb{L}({}_SH(X)) = \mathcal{L}_s({}_SH(X))$;
III) $\mathcal{L}^r({}_RX) \cong \mathcal{L}_s({}_SH(X))$.

Further, every of these conditions is investigated in detail, the third being the most nontrivial. For its fulfilment the key question is when the relation $Y' = \alpha\beta(Y')$ is true for $Y' \subseteq H(X)$. Necessary and sufficient conditions for this relation being satisfied are shown. We mention some results.

Theorem 11.1. *If ${}_RU$ is a projective module and $S = \mathrm{End}({}_RU)$, then the lattices $\mathcal{L}^r({}_RX)$ and $\mathcal{L}_s({}_SH(X))$ are canonically isomorphic for every $X \in R$-Mod.*

Theorem 11.2. *The following conditions are equivalent:*
1) *the lattices $\mathcal{L}^r({}_RX)$ and $\mathcal{L}({}_SH(X))$ are canonically isomorphic;*
2) *the module ${}_RU$ is inner X-projective and X-compact.*

Some results of F.L. Sandomierski (1972), G.M. Brodskii (1983), A.K. Gupta, K. Varadarajan (1980) are obtained as corollaries.

Functor $T = U \otimes_S$ -: S-Mod $\to$ R-Mod

Similar questions for the functor of tensor multiplication T are investigated. For the adjoint situation generated by ${}_RU_S$ and for every module $Y \in S$-Mod the following mappings are considered:

$$\mathbb{L}\big({}_RT(Y)\big) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \mathbb{L}({}_SY),$$

where

$$\alpha'(Y') = \operatorname{Im} T(j), \quad j\colon Y' \xrightarrow{\subseteq} Y,$$
$$\beta'(X') = \{y \in Y \mid U \otimes {}_Sy \subseteq X'\}, \quad X' \subseteq T(Y).$$

Then $\alpha'(Y') \in \mathcal{L}^r\big({}_RT(Y)\big)$ and $\beta'(X') \in \mathcal{L}_s({}_SY)$. Therefore the canonical isomorphism $\mathbb{L}({}_SY) \cong \mathbb{L}\big({}_RT(Y)\big)$ holds if and only if:

I) $\mathbb{L}({}_SY) = \mathcal{L}_s({}_SY)$;

II) $\mathbb{L}\big({}_RT(Y)\big) = \mathcal{L}^r\big({}_RT(Y)\big)$;

III) $\mathcal{L}_s({}_SY) \cong \mathcal{L}^r\big({}_RT(Y)\big)$.

The analysis of these conditions (the condition III) is basic) elucidates the situation when the required isomorphism takes place. The main question is when the equality $X' = \alpha'\beta'(X')$ holds for $X' \subseteq {}_RT(Y)$. In particular, is proved the following theorem.

Theorem 11.3. *If U_S is flat and the pair (T, H) is idempotent, then $\mathcal{L}^r\big({}_R(T(Y)\big) \cong \mathcal{L}_s({}_SY)$ for every $Y \in S$-*Mod.

One of basic results in this case is the following.

Theorem 11.4. *Let ${}_RC$ be a cogenerator of R-*Mod *which is finitely generated and injective. For every $AB5^*$-module $Y \in S$-*Mod *the following conditions are equivalent:*

1) *the lattices $\mathbb{L}\big({}_RT(Y)\big)$ and $\mathbb{L}({}_SY)$ are canonically isomorphic;*
2) *the module ${}_SH(C)$ is inner Y-injective, Y-cofinitely cogenerated and Y-cogenerating.*

Functor $H_1 = \mathrm{Hom}_R(\text{-}, U)$: R-Mod $\to$ Mod-S

The question of the impact of functors on the lattices of submodules is investigated by similar methods for *contravariant* functor H_1 ([31, 34–36, 43]). The bimodule ${}_RU_S$ determines the adjoint functors:

$$R\text{-Mod} \underset{H_2=\mathrm{Hom}_S(\text{-},U)}{\overset{H_1=\mathrm{Hom}_R(\text{-},U)}{\rightleftarrows}} \text{Mod-}S$$

with the natural transformations $\Phi\colon 1_{R\text{-Mod}} \to H_2H_1$ and $\Psi\colon 1_{\text{Mod-}S} \to H_1H_2$. In view of the full symmetry, it is sufficient to study one of these functors, for example H_1.

For every $X \in R$-Mod the following mappings are defined:

$$\mathbb{L}({}_RX) \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} \mathbb{L}(H_1(X)_S),$$

where

$$\alpha^*(X') = \{f \in H_1(X) \mid \operatorname{Ker} f \supseteq X'\}, \quad X' \subseteq {}_RX;$$
$$\beta^*(Y') = \cap\{\operatorname{Ker} f' \mid f' \in Y'\}, \quad Y' \subseteq H_1(X)_S.$$

Further, preradicals r of R-Mod and s of Mod-S are used, where $r({}_RX) = \operatorname{Ker}\Phi_X$ and $s(Y_S) = \operatorname{Ker}\Psi_Y$. For every ${}_RX \in R$-Mod and $Y_S \in$ Mod-S the following lattices of submodules are considered:

$$\mathcal{L}_r({}_RX) = \{X' \subseteq X \mid r(X/X') = 0\}, \qquad \mathcal{L}_s(Y_S) = \{Y' \subseteq Y \mid s(Y/Y') = 0\}.$$

The problem is to find conditions under which α^* and β^* determine an antiisomorphism of lattices $\mathbb{L}({}_RX)$ and $\mathbb{L}(H_1(X)_S)$, which is equivalent to the conditions:

I) $\mathbb{L}({}_RX) = \mathcal{L}_r({}_RX)$;

II) $\mathbb{L}(H_1(X)_S) = \mathcal{L}_s(H_1(X)_S)$;

III) $\mathcal{L}_r({}_RX) \approx \mathcal{L}_s(H_1(X)_S)$.

The main question here is when the relation $Y' = \alpha^*\beta^*(Y')$ holds for $Y' \subseteq H_1(X)_S$. A series of equivalent conditions, which ensures this relation, is found. From the basic results we mention the following.

Theorem 11.5. *If ${}_RU$ is injective and $H_1(\Phi_X)$ is a monomorphism, then $\mathcal{L}_r({}_R(X)) \approx \mathcal{L}_s(H_1(X)_S)$.*

Theorem 11.6. *For every $X \in R$-Mod the following conditions are equivalent:*

1) *$\mathcal{L}_r({}_RX) \approx \mathbb{L}(H_1(X)_S)$;*

2) *${}_RU$ is inner X-injective and X-cocompact.*

As a consequence it is obtained that the antiisomorphism $\mathbb{L}({}_RX) \approx \mathbb{L}(H_1(X)_S)$ is equivalent to the conditions that ${}_RU$ is inner X-injective, X-cocompact and X-cogenerator.

From these results all the earlier known facts on the antiisomorphism of lattices of submodules follow, in particular those of K.R. Fuller (1974), F.L. Sandomierski (1972), C. Năstăsescu (1979), A.K. Gupta, K. Varadarajan (1980), G.M. Brodskii (1983).

A combined investigation of mappings between the lattices of submodules for all principal functors is realized *in the paper* [37] (see also [39]). For a bimodule ${}_RU_S$ the following functors are considered:

$$S\text{-Mod} \underset{T=U\otimes_S -}{\overset{H=\mathrm{Hom}_R(U,-)}{\rightleftarrows}} R\text{-Mod} \underset{H_2=\mathrm{Hom}_S(-,U)}{\overset{H_1=\mathrm{Hom}_R(-,U)}{\rightleftarrows}} \text{Mod-}S$$

Every module $M \in R$-Mod defines the mappings:

$$\mathbb{L}({}_SH(M)) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \mathbb{L}({}_RM) \underset{\beta_M^*}{\overset{\alpha_M^*}{\rightleftarrows}} \mathbb{L}(H_1(M)_S),$$

$$\mathbb{L}({}_SH(M)) \underset{\mathcal{L}_M}{\overset{\mathcal{R}_M}{\rightleftarrows}} \mathbb{L}(H_1(M)_S),$$

where

$$\alpha_M(M') = \{f \in H(M) \mid \mathrm{Im}\, f \subseteq M'\}, \qquad \beta_M(N') = \sum\{\mathrm{Im}\, f' \mid f' \in N'\};$$

$$\alpha_M^*(M') = \{f \in H_1(M) \mid \mathrm{Ker}\, f \supseteq M'\}, \quad \beta_M^*(N') = \bigcap\{\mathrm{Ker}\, f' \mid f' \in N'\};$$

$$\mathcal{R}_M(N') = \{g\colon {}_RM \to {}_RU \mid fg = 0 \ \forall f \in N'\},$$

$$\mathcal{L}_M(L') = \{f\colon {}_RU \to {}_RM \mid fg = 0 \ \forall g \in L'\}.$$

These pairs of mappings constitute the “triangular Galois theory”. They are combined with the pairs of mappings defined by the natural transformations $\Psi\colon 1_{R\text{-Mod}} \to HT$ and $\Phi\colon TH \to 1_{R\text{-Mod}}$. The commutativity of the resulting diagram is studied (it contains 12 pairs of mappings) and for the obtained Galois connections the accompanying projectivities (isotone bijections) or dualities (antiisotone bijections) are indicated. For example, is true the next theorem.

Theorem 11.7. *For any module $M \in R$-Mod the restrictions of the mappings α_M^* and β_M^* define a duality* $\mathrm{Im}\,\mathcal{R}_M \underset{\beta_M^*}{\overset{\alpha_M^*}{\rightleftarrows}} \mathcal{S}_1$, *where* $\mathcal{S}_1 = \{X \subseteq M \mid \beta_M^*\alpha_M^*\beta_M\alpha_M(X) = X\}$*. The restrictions of the mappings α_M and $\beta_M^*\mathcal{R}_M$ define the projectivity* $\mathrm{Im}\,\mathcal{L}_M \underset{\beta_M^*\mathcal{R}_M}{\overset{\alpha_M}{\rightleftarrows}} \mathcal{S}_1$*. The symmetrical statement is also true.*

These facts generalize some results of G.M. Brodskii (1983) and S.M. Khuri (1989).

12. Morita contexts and lattices of submodules ([38, 40, 42–45, 50])

Ample opportunities for the investigation of relations between the lattices of submodules are provided by Morita contexts. Some mappings between the lattices of submodules in this case are considered *in the work* [38]. A Morita context $(R, {}_RM_S, {}_SN_R, S)$ with the bimodule homomorphisms $(\,,\,)\colon M \otimes {}_SN \to R$ and $[\,,\,]\colon N \otimes {}_RM \to S$ and with trace ideals $I_0 = (M, N)$ and $J_0 = [N, M]$ determines some pairs of mappings between the lattices of submodules, in particular:

$$\mathbb{L}({}_RM) \; \overset{r'}{\underset{l'}{\rightleftarrows}} \; \mathbb{L}(N_R), \qquad \mathbb{L}({}_RM) \; \overset{P_M}{\underset{f_M}{\rightleftarrows}} \; \mathbb{L}({}_SS),$$

$$\mathbb{L}({}_SS) \; \overset{G_N}{\underset{Q_N}{\leftrightarrows}} \; \mathbb{L}(N_R),$$

where

$$\begin{aligned} &r'(K) = \{n \in N \mid (K, n) = 0\}, && l'(L) = \{m \in M \mid (m, L) = 0\};\\ &p_M(J) = N^{-1}J, && f_M(K) = (N, K);\\ &G_N(L) = \operatorname{ann}_S(L), && Q_N(J) = \operatorname{ann}_N(J). \end{aligned}$$

Properties of these mappings, as well as connections between them are shown. Conditions under which the restrictions of these mappings determine projectivities or dualities are obtained. In particular, is proved the following theorem.

Theorem 12.1. *Let N_R be a faithful module and $[N, M] = S$. Then the pair (p_M, f_M) defines a projectivity between $\mathcal{C} = \{J \subseteq {}_SS \mid J = G_NQ_N(J)\}$ and $\mathcal{L}' = \{K \subseteq {}_RM \mid K = l'r'(K)\}$; also, the pair (G_N, Q_N) determines the duality between $\mathcal{C}$ and $\mathcal{R}' = \{L \subseteq N_R \mid L = l'r'(L)\}$.*

Together with dual results the "quadrangular Galois theory" is obtained, which consists in five bijections (projectivities and dualities). These facts generalize some results of S. Kyuno, M.-S.B. Smith (1989), J.J. Hutchinson (1987), G.M. Brodskii (1983).

A rather full picture of the relation between the lattices of submodules in Morita contexts is exposed *in the article* [40], using preradicals determined by trace ideals $I = (M, N)$ and $J = [N, M]$ of the Morita context $(R, {}_RM_S, {}_SN_R, S)$. Two types of mappings between the lattices of submodules are distinguished.

Mappings of the first type are defined using the idempotent radicals r^I in R-Mod and r^J in S-Mod, where $\mathcal{R}(r^I) = \{{}_RX \mid IX = X\}$ and $\mathcal{R}(r^J) = \{{}_SY \mid JY = Y\}$. In the lattices of submodules $\mathbb{L}({}_RX)$ and $\mathbb{L}({}_SS)$ the following sublattices are considered:

$$\mathcal{L}^{r^I}({}_RX) = \{X' \subseteq {}_RX \mid IX' = X'\}, \qquad \mathcal{L}^{r^J}({}_SS) = \{A \subseteq {}_SS \mid A = JA\}.$$

The following mappings are studied:

$$\mathbb{L}({}_RM) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \mathbb{L}({}_SS), \quad \alpha_M({}_RK) = [N, K], \quad \beta_M({}_SA) = MA;$$

$$\mathbb{L}({}_SN) \underset{\beta_N}{\overset{\alpha_N}{\rightleftarrows}} \mathbb{L}({}_RR), \quad \alpha_N({}_SL) = (M, L), \quad \beta_N({}_RB) = NB.$$

Theorem 12.2. *The pair of mappings (α_M, β_M) defines the lattice isomorphism $\mathcal{L}^{r^I}({}_RM) \cong \mathcal{L}^{r^J}({}_SS)$ and the pair (α_N, β_N) defines the lattice isomorphism $\mathcal{L}^{r^J}({}_SN) \cong \mathcal{L}^{r^I}({}_RR)$. Right variants of these statements also hold: $\mathcal{L}^{r^J}(M_S) \cong \mathcal{L}^{r^I}(R_R)$, $\mathcal{L}^{r^I}(N_R) \cong \mathcal{L}^{r^J}(S_S)$.*

It is interesting that all considered mappings can be restricted to *subbimodules* and then isomorphisms of lattices of subbimodules are obtained. In particular, we have the mappings

$$\mathbb{L}({}_RM_S) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}({}_SN_R), \qquad \mathbb{L}({}_SS_S) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \mathbb{L}({}_RR_R),$$

whose restrictions lead to lattice isomorphisms.

Mappings of the second type are defined by the torsions r_I of R-Mod and r_J of S-Mod (the ideal torsions determined by I and J). For every module ${}_RX$ and every torsion r of R-Mod the lattice of r-closed submodules $\{X' \subseteq X \mid r(X/X') = 0\}$ is denoted by $\mathcal{L}_r({}_RX)$.

The following mappings are considered:

$$\mathbb{L}({}_RM) \underset{\delta_M}{\overset{\gamma_M}{\rightleftarrows}} \mathbb{L}({}_SS), \qquad \mathbb{L}({}_SN) \underset{\delta_N}{\overset{\gamma_N}{\rightleftarrows}} \mathbb{L}({}_RR),$$

where $\gamma_M({}_RK) = \{s \in S \mid Ms \subseteq K\}$, $\delta_M({}_SA) = \{m \in M \mid [N, m] \subseteq A\}$, and similarly for γ_N and δ_N.

Theorem 12.3. *The mappings γ_M and δ_M define the lattice isomorphism $\mathcal{L}_{r_I}({}_RM) \cong \mathcal{L}_{r_J}({}_SS)$; the mappings γ_N and δ_N define the lattice isomorphism $\mathcal{L}_{r_J}({}_SN) \cong \mathcal{L}_{r_I}({}_RR)$. Right variants of these statements also hold.*

As in the first case, these mappings can be restricted to the *subbimodules* and isomorphisms of lattices of subbimodules are obtained.

These results are closely related to the investigations of S. Kyuno, M.-S.B. Smith (1989), S.M. Khuri (1986), B.J. Müller (1974).

In the paper [42] these investigations are continued: the pair of torsions (r_I, r_J) of the Theorem 12.3 is substituted by an arbitrary pair (r, s) of torsions which correspond each to other in the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ (see Theorem 6.1). For a Morita context $(R, {}_RU_S, {}_SV_R, S)$ with the trace ideals I and J, the following mappings are considered:

$$\mathbb{L}({}_RU) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}({}_SS) \underset{L_S}{\overset{R_S}{\rightleftarrows}} \mathbb{L}(S_S),$$

$$\mathbb{L}({}_RU) \underset{Q}{\overset{G}{\rightleftarrows}} \mathbb{L}(S_S),$$

where

$$\alpha(U') = \{s' \in S \mid Us' \subseteq U'\}, \qquad \beta({}_SA) = \{u \in U \mid [V, u] \subseteq A\};$$
$$R_S({}_SX) = \{s' \in S \mid s'X \subseteq s({}_SS)\}, \quad L_S(X) = \{s' \in S \mid s'X \subseteq s({}_SS)\};$$
$$G(U') = \{s' \in S \mid U's' \subseteq r({}_RU)\}, \quad Q(X) = \{u \in U \mid uX \subseteq r({}_RU)\}.$$

Theorem 12.4. *The restrictions of the mappings α and β on the* $\operatorname{Im} Q$ *and* $\operatorname{Im} L_S$ *define a projectivity between these lattices, which is a part of "triangular Galois theory":*

$$\operatorname{Im} Q \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \operatorname{Im} L_S \underset{L_S}{\overset{R_S}{\rightleftarrows}} \operatorname{Im} G = \operatorname{Im} R_S,$$

$$\operatorname{Im} Q \underset{Q}{\overset{G}{\rightleftarrows}} \operatorname{Im} G.$$

Symmetrically the other triangular Galois theory is constructed, which supplements the previous and in common the square is obtained, consisting of four pairs of co-ordinated mappings with the diagonal (G, Q): two projectivities and three dualities. Some known results of Zhou Zhengping (1983), J.J. Hutchinson (1987), S. Kyuno, M.-S.B. Smith (1989) are obtained as particular cases.

Compositions of dualities in a nondegenerated Morita context $(R, {}_RM_S, {}_SN_R, S)$ are studied *in the article* [45]. Mappings between the lattices of submodules $\mathbb{L}({}_RM)$, $\mathbb{L}(N_S)$, $\mathbb{L}({}_SS)$ and $\mathbb{L}(S_S)$, which are defined as annihilators are studied. In particular, the following mappings are considered:

$$\mathbb{L}({}_RM) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \mathbb{L}({}_SS) \underset{l}{\overset{r}{\rightleftarrows}} \mathbb{L}(S_S),$$

$$\mathbb{L}({}_RM) \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \mathbb{L}(S_S),$$

where

$$\alpha_M({}_RK) = \{s \in S \mid Ms \subseteq K\}, \quad \beta_M({}_SJ) = \{m \in M \mid [N, m] \subseteq J\};$$
$$r({}_SJ) = \{s \in S \mid Js = 0\}, \quad l(J_S) = \{s \in S \mid sJ = 0\};$$
$$G_M({}_RK) = \{s \in S \mid Ks = 0\}, \quad Q_M({}_SJ) = \{m \in M | mJ = 0\}.$$

The pairs (r, l) and (G_M, Q_M) form Galois connections, so they define the dualities:

$$\operatorname{Im} l \underset{l}{\overset{r}{\rightleftarrows}} \operatorname{Im} r, \qquad \operatorname{Im} Q_M \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \operatorname{Im} G_M.$$

To obtain their composition the equality Im $r =$ Im G_M is necessary. Some weak conditions under which this relation holds are indicated. Then the composition of these dualities generates a projectivity, which coincides with the pair (α_M, β_M):

$$\operatorname{Im} Q_M \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \operatorname{Im} G_M = \operatorname{Im} r \underset{r}{\overset{l}{\rightleftarrows}} \operatorname{Im} l,$$

$$\operatorname{Im} Q_M \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \operatorname{Im} l.$$

For a given context, four such triangles are obtained and under suitable conditions the compositions of dualities can be formed. If the context is *nondegenerated*, then all these conditions are satisfied and so the compositions of dualities can be considered.

Theorem 12.5. *If the Morita context* $(R, {}_RM_S, {}_SN_R, S)$ *is nondegenerated, then we obtain four co-ordinated dualities* (r, l), (G_M, Q_M), (r', l'), (G_N, Q_N) *and two projectivities* (α_M, β_M), (α_N, β_N), *which are compositions of corresponding dualities.*

To the same cycle of works can be attributed *the article* [50], in which the equivalence of special subcategories is proved for the Morita context $(R, {}_RV_S, {}_SW_R, S)$, using the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ of Theorem 6.1.

For a fixed pair (r, s) of corresponding torsions some modifications of the functors

$$R\text{-Mod} \underset{T^V = V \otimes_S \text{-}}{\overset{T^W = W \otimes_R \text{-}}{\rightleftarrows}} S\text{-Mod}$$

are defined such that they are of the form $\mathcal{T}_r \underset{\overline{T}^V}{\overset{\overline{T}^W}{\rightleftarrows}} \mathcal{F}_s$. The subcategories $\mathcal{A} = \mathcal{F}_r \cap \mathcal{F}_I \subseteq R$-Mod and $\mathcal{B} = \mathcal{F}_s \cap \mathcal{F}_J \subseteq S$-Mod are considered.

Theorem 12.6. *For every pair of corresponding torsions* (r, s) *the functors* $\overline{T}^W$ *and* $\overline{T}^V$ *define an equivalence between the subcategories of torsion free accessible modules:* $\mathcal{A} \approx \mathcal{B}$.

For the smallest pair (r_I, r_J) of corresponding torsions this result was proved by W.K. Nicholson, J.F. Watters (1988) and F.C. Iglesias, J.G. Torrecillas (1995).

The article [44] is a survey of some results on radicals in modules.

The book [43] is devoted to torsions in modules and contains some new results on the following subjects:

Chapter 1. Adjoint functors and radicals;

Chapter 2. Morita contexts and torsions;

Chapter 3. Principal functors and lattices of submodules.

13. Divisible and reduced modules ([41, 46])

In the paper [41] the investigations of torsions and accompanying constructions in Morita contexts are continued. For a torsion r of R-Mod with radical filter (Gabriel topology) $\mathcal{F}(r)$ the *right* R-module M_R is called *r-divisible* if $MK = M$ for every $K \in \mathcal{F}(r)$. The class $\mathcal{D}_r$ of all r-divisible right R-modules is a torsion class Mod-R. For every torsion r of R-Mod there exists the greatest torsion s of R-Mod such that $\mathcal{D}_r = \mathcal{D}_s$.

For the Morita context $(R, {}_RU_S, {}_SV_R, S)$ with the trace ideals I and J the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ (Theorem 6.1) of the lattices of torsions for *left* R-modules implies a close relation between the corresponding classes of divisible *right* R-modules.

Theorem 13.1. *For the given Morita context there exists an isotone bijection between the subcategories of divisible modules* $\mathcal{D}_r$ $(r \geqslant r_I)$ *of* Mod-R *and subcategories of divisible modules* $\mathcal{D}_{r'}$ $(r' \geqslant r_J)$ *of* Mod-S, *where* (r, r') *is a pair of corresponding torsions in the isomorphism* $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$.

A similar result is obtained for r-reduced modules.

In the article [46] the notions of r-divisible and r-reduced modules are generalized for idempotent radicals. If r is an idempotent radical of R-Mod, then the *right* R-module D_R is called *r-divisible* if $D \otimes {}_R X = 0$ for every ${}_R X \in \mathcal{T}_r$. A module Y_R is called *r-reduced* if it has no nontrivial r-divisible submodule. Let $\mathcal{D}_r$ ($\mathcal{R}_r$) be the class of all r-divisible (r-reduced) right R-modules. Then the pair $(\mathcal{D}_r, \mathcal{R}_r)$ defines a torsion theory in Mod-R (i.e. it determines an idempotent radical r^* of Mod-R).

The connections between the classes $\mathcal{T}_r$, $\mathcal{F}_r$ of R-Mod and corresponding classes $\mathcal{D}_r$, $\mathcal{R}_r$ of Mod-R are shown. In particular, is proved the following.

Theorem 13.2. $\mathcal{D}_r = H^{-1}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{T}_r)$, $\mathcal{D}_r = [H_{\mathbb{Q}/\mathbb{Z}}(\mathcal{T}_r)]^{\uparrow}$, $H^{-1}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{R}_r) = \mathcal{D}_r^{\downarrow}$.

The inverse transition from the class of modules $\mathcal{L} \subseteq$ Mod-R to the idempotent radical $r_{(\mathcal{L})}$ of R-Mod is defined by the rule: $\mathcal{T}_{r_{(\mathcal{L})}} = [H_{\mathbb{Q}/\mathbb{Z}}(\mathcal{L})]^{\uparrow}$. The class $\mathcal{T}_{r_{(\mathcal{L})}}$ (as well as $r_{(\mathcal{L})}$) is described by the relations $\mathcal{T}_{r_{(\mathcal{L})}} = \mathcal{L}^{\perp}$ and $\mathcal{T}_{r_{(\mathcal{L})}} = H^{-1}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{L}^{\downarrow})$. The mappings $r \mapsto \mathcal{D}_r$ and $\mathcal{L} \mapsto r_{(\mathcal{L})}$ define a Galois connection. Closed elements of this connection are characterized. Furthermore, is true the following theorem.

Theorem 13.3. $\mathcal{D}_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{D}_{r_\alpha}$.

Some curious applications of these results are indicated in the case of Morita contexts: for the corresponding torsions r and s the equivalences of accompanying subcategories are shown:

$$\mathbb{L}_r \approx \mathbb{L}_s, \qquad \mathbb{K}_r \approx \mathbb{K}_s,$$

where $\mathbb{L}_r$ is the class of r-torsion free r-injective modules, and $\mathbb{K}_r$ is the class of r-divisible r-flat modules.

These results are closely related to the investigations of B. Stenström (1975), B.J. Müller (1974), Zhou Zhengping (1991), T. Kato, T. Ohtake (1979).

14. Methods of construction of preradicals and their approximations ([47–49])

In the paper [47] the three-sided relation between the ideals of the ring R, classes of left R-modules and preradicals of R-Mod is studied.

Every ideal I of R defines the classes of modules $\mathcal{T}_I$, $\mathcal{A}(I)$, $\mathcal{F}_I$ and also the preradicals $r^{(I)}$, $r_{(J)}$, where:

$$\mathcal{T}_I = \{{}_RM \mid IM = M\}, \qquad \mathcal{A}(I) = \{{}_RM \mid IM = 0\},$$
$$\mathcal{F}_I = \{{}_RM \mid \mathrm{Im} = 0 \Rightarrow m = 0\};$$
$$r^{(I)}(M) = IM, \qquad r_{(I)}(M) = \{m \in M \mid \mathrm{Im} = 0\}.$$

Various restrictions on the ideal I imply some special properties of associated classes of modules and preradicals. In this way some bijections between ideals, classes of modules and preradicals of diverse types are obtained. This method is used in the five cases: 1) I is an arbitrary ideal; 2) $I = I^2$; 3) I is a “still” ideal, i.e. with the condition (a): $a \in Ia \; \forall\, a \in I$; 4) I is a left direct summand of R; 5) I is a ring direct summand of R. As an example we expose the case when $I = I^2$.

Theorem 14.1. *There exists a bijection between:*
1) *idempotent ideals of R (I);*
2) *cotorsions of R-Mod $(r^{(I)})$;*
3) *jansian torsions of R-Mod $(r_{(I)})$;*
4) *TTF-classes of R-Mod $(\mathcal{A}(I))$;*
5) *three-fold torsion theories of R $((\mathcal{T}_I, \mathcal{A}(I), \mathcal{F}_I))$;*
6) *radical filters of R closed under intersection $(\mathcal{E}_{r_{(I)}})$.*

These results generalize and supplement some known facts on the construction of preradicals by classes of modules and by ideals: L. Bican, T. Kepka, P. Nemec (1982), Y. Kurata (1972), G. Azumaya (1973), J.S. Golan (1986), J.P. Jans (1965), etc.

Diverse methods of “improvement” of preradicals are known, i.e. of the construction of nearest preradicals with the required properties. *In the paper* [48] these methods are systematized and supplemented, using various means: functors, classes of modules, ideals, filters, etc. For example, for every preradical r it is possible to show the upper and the lower approximations by torsions or by cotorsions. In particular, is true the following.

Theorem 14.2. *If r is an idempotent radical of R-Mod and $\mathcal{E} = \{I \in \mathbb{L}({}_RR) \mid R/I \in \mathcal{T}_r\}$, then the set of left ideals $\overline{\mathcal{E}} = \{K \in \mathbb{L}({}_RR) \mid (K : a) \in \mathcal{E} \; \forall\, a \in R\}$ is a radical filter and the corresponding torsion $\overline{r}$ is the greatest torsion contained in r.*

The work [49] is a text-book on the theory of modules and contains an introduction in this theory, using methods of the theory of categories.

The basic subjects: principal functors; main classes of modules (projective, injective, flat); generators and cogenerators of R-Mod; homological classification of rings.

15. Natural classes of modules ([51–54])

In connection with diverse problems of module theory a group of authors (J. Dauns, 1997, 1999; Y. Zhou, 1996; A.A. Garcia, H. Rincon, J.R. Montes, 2001, etc.) introduced and studied so-called *natural* (or saturated) classes of modules, i.e. classes closed under submodules, direct sums and injective envelopes. A series of articles on this subject was completed by the book: J. Dauns, Y. Zhou, "Classes of modules", Chapman and Hall, 2006.

In the article [51] it is proved that natural classes are closed (see Section 1), i.e. they possess the description by sets of left ideals of the ring R (Theorem 1.2). The inner characterization of natural sets of left ideals (i.e. of the form $\Gamma(\mathcal{K})$, where $\mathcal{K}$ is a natural class) is obtained. Using the mappings Γ and Δ, defined by the rules:

$$\Gamma(\mathcal{K}) = \{(0:m) \mid m \in M, M \in \mathcal{K}\},$$
$$\Delta(\mathcal{E}) = \{{}_RM \mid (a:m) \in \mathcal{E} \ \forall\, m \in M\}$$

is proved the next theorem.

Theorem 15.1. *The operators* Γ *and* Δ *define an isotone bijection between natural classes* $\mathcal{K}$ *of* R-Mod *and natural sets* $\mathcal{E}$ *of left ideals of* R*. In this case the following relations hold:*

$$\Gamma(\mathcal{K}^{\perp}) = \mathcal{E}, \qquad \Delta(\mathcal{E}^{\perp}) = \mathcal{K}^{\perp}.$$

In this way the operators Γ and Δ determine the isomorphism of boolean lattices, which consist in natural classes of R-modules and natural sets of left ideals of R. Moreover, in this case the operators of complementation are concordant with Γ and Δ.

Similar questions are discussed *in the paper* [52], where the lattice R-cl of all *closed* classes of R-Mod is studied $\big($i.e. the classes $\mathcal{K} \subseteq R$-Mod such that $\mathcal{K} = \Delta\Gamma(\mathcal{K})\big)$.

Theorem 15.2. *The lattice* R*-cl of closed classes of* R-Mod *is a frame.*

For every class $\mathcal{K} \in R$-cl it is proved that its pseudocomplement in R-cl coincides with the class

$$\mathcal{K}^{\perp} = \{{}_RM \mid M \text{ has no nontrivial submodules from } \mathcal{K}\}.$$

An interesting relation between closed and natural classes of modules is elucidated.

Theorem 15.3. *The skeleton of the lattice* R-cl *coincides with the lattice of natural classes* R-nat *of the category* R-Mod*, where*

$$\mathrm{Sk}(R\text{-cl}) = \{\mathcal{K}^{\perp} \mid \mathcal{K} \in R\text{-cl}\}.$$

Similar results for closed and natural sets of left ideals of R are obtained.

The investigations of natural classes of modules are continued *in the article* [53], where the relation between the torsion free classes and natural classes of R-Mod is studied. Every torsion free class (i.e. of the form $\mathcal{F}_r$ for a torsion r) is natural, so we have the inclusion $i\colon \mathbb{P} \to R$-nat, where $\mathbb{P}$ is the family of all torsion free classes of R-Mod. The inverse mapping $\phi\colon R\text{-nat} \to \mathbb{P}$ is defined, where $\phi(\mathcal{K})$ is the smallest torsion free class containing $\mathcal{K}$. Various forms of presentation of the class $\phi(\mathcal{K})$ are shown, in particular: $\phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}$, $\phi(\mathcal{K}) = \mathrm{Cog}(\mathcal{K})$.

For every set of natural classes $\{\mathcal{K}_\alpha \mid \alpha \in \mathfrak{A}\}$ of R-Mod the relation $\big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\big)^{\uparrow} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$ is proved, from which follows the next theorem.

Theorem 15.4. *The mapping* ϕ *preserves the join of natural classes of modules:*

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha\Big)^{\uparrow\downarrow} = \bigvee_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow}).$$

In the paper [54] some results on natural and conatural classes of R-Mod are translated in the terms of left ideals of R. If $\mathcal{K} \in R$-nat, then $\mathcal{E} = \Gamma(\mathcal{K}) = \{I \in \mathbb{L}({}_RR \mid R/I \in \mathcal{K}\}$ is the corresponding natural set of left ideals of R. Diverse descriptions of natural sets of left ideals are obtained. The lattice R-Nat of natural sets of left ideals of R coincides with the skeleton of the lattice R-Cl of closed sets. All these facts are dualized, proving similar results for conatural sets of left ideals of R.

16. Preradicals associated to principal functors ([55–57])

The next cycle of works is devoted to the investigation of preradicals accompanying principal functors of module categories, i.e. functors of the form $H = H^U = \mathrm{Hom}_R({}_RU, \text{-})$, $H' = H_U = \mathrm{Hom}_R(\text{-}\,, {}_RU)$ and $T = T^U = U \otimes {}_S\text{-}$, where ${}_RU_S$ is an arbitrary bimodule.

In the article [55] from this point of view the functor

$$H = \mathrm{Hom}_R({}_RU, \text{-}) \colon R\text{-Mod} \to \mathcal{A}b$$

is studied for an arbitrary module ${}_RU \in R$-Mod. Associated preradicals and their properties, as well as the conditions of coincidence of some preradicals are shown. The module ${}_RU$ defines the trace ideal $I = \sum\{\mathrm{Im}\, f \mid f \colon {}_RU \to {}_RR\}$, which in its turn determines the classes of modules ${}_I\mathcal{T}$, ${}_I\mathcal{F}$ and $\mathcal{A}(I)$. Properties of these classes lead to preradicals of diverse types: r^I, r_I, $r^{(I)}$, $r_{(I)}$, where:

$$\mathcal{R}(r^I) = {}_I\mathcal{T}, \qquad \mathcal{P}(r_I) = {}_I\mathcal{F}, \qquad \mathcal{P}(r^{(I)}) = \mathcal{A}(I), \qquad \mathcal{R}(r_{(I)}) = \mathcal{A}(I).$$

Numerous relations between the studied classes of modules (and so between the corresponding preradicals) are elucidated. Some simple conditions, under which the "near" preradicals coincide are found. In particular, the following conditions are equivalent:

$$1)\ r^U = r^I; \quad 2)\ \bar{r}^U = r^I; \quad 3)\ \bar{r}^U = r^{(I)}; \quad 4)\ r^U = r^{(I)}; \quad 5)\ IU = U.$$

In the paper [56] similar questions are studied for the functor

$$T = U \otimes {}_S\text{-} \colon S\text{-Mod} \to \mathcal{A}b,$$

where $U_S \in$ Mod-S. Classes of modules accompanying this functor are shown. All constructions indicated earlier for the functor H, have their analogues for the functor T. In particular, there exist the radical t_U in S-Mod and the nearest idempotent radical $\bar{t}_U \leqslant t_U$, where $t_U({}_SM) = \{m \in M \mid U \otimes {}_Sm = 0\}$ and $\mathcal{R}(\bar{t}_U) = \mathrm{Ker}\, T$. Conditions under which $t_U = \bar{t}_U$ or t_U is a torsion are shown.

Further, the ideal $J = (0 : U_S)$ of S and the corresponding classes of left S-modules: ${}_J\mathcal{T}$, ${}_J\mathcal{F}$, $\mathcal{A}(J)$ as well as the preradicals defined by them are considered. Properties and connections between these classes and preradicals are studied. Also the close relation with the case of functor H, which follows from the adjointness of the functors T and H, is mentioned.

The case of *contravariant* functor

$$H' = \mathrm{Hom}_R(\text{-}\,, {}_RU) \colon R\text{-Mod} \to \mathcal{A}b,$$

where ${}_RU \in R$-Mod, is studied *in the article* [57]. The module ${}_RU$ defines the radical r_U and the nearest idempotent radical $\bar{r}_U \leqslant r_U$. Some conditions are indicated when $\bar{r}_U = r_U$ or r_U is a torsion. Further, the ideal $I = (0 : {}_RU)$ of R, which defines preradicals r^I, r_I, $r^{(I)}$ and $r_{(I)}$ is considered. Numerous relations between the classes of modules and preradicals appearing in this situation are shown.

An important remark: there exists a rather full analogy between the situations for the functors H' and T. The reason of this fact is shown: the functor $T = U \otimes {}_S\text{-}$ determines the same classes of modules as the functor $H' = \text{Hom}_S(\text{-}, U^*)$, where ${}_SU^* = \text{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}U_S, \mathbb{Q}/\mathbb{Z})$ and $\mathbb{Q}/\mathbb{Z}$ is the injective cogenerator of the category of abelian groups $\mathcal{A}b$.

17. Preradicals and new operations in the lattices of submodules ([58–61])

The works [58–61] are related with some new operations in the lattices of submodules, defined by standard preradicals. We remind that standard preradicals α_N^M and ω_N^M are determined by a pair $N \subseteq M$, where $N \in \mathbb{L}({}_RM)$, as follows:

$$\alpha_N^M(X) = \sum_{f:\ M \to X} f(N), \qquad \omega_N^M(X) = \bigcap_{f:\ X \to M} f^{-1}(N), \qquad X \in R\text{-Mod}.$$

Denote by $\mathbb{L}^{\text{ch}}({}_RM)$ the lattice of *characteristic* (fully invariant) submodules of M (i.e. such $N \subseteq M$ that $f(N) \subseteq N$ for every $f\colon M \to M$). *In the article* [58] the relations between the lattice $\mathbb{L}^{\text{ch}}({}_RM)$ and some sublattices of the lattice R-pr of all preradicals of R-Mod are studied. The mappings

$$\alpha^M : \mathbb{L}^{\text{ch}}({}_RM) \to R\text{-pr}, \qquad \omega^M : \mathbb{L}^{\text{ch}}({}_RM) \to R\text{-pr},$$

which transfer N into α_N^M and N into ω_N^M, are injective, so we obtain the isomorphisms:

$$\mathbb{L}^{\text{ch}}({}_RM) \cong \text{Im}\,\alpha^M \subseteq R\text{-pr}, \qquad \mathbb{L}^{\text{ch}}({}_RM) \cong \text{Im}\,\omega^M \subseteq R\text{-pr}.$$

In other form the relation between $\mathbb{L}^{\text{ch}}({}_RM)$ and R-pr can be expressed defining on R-pr the equivalence relation:

$$r \cong_M s \Leftrightarrow r(M) = s(M).$$

Then R-pr is divided into the classes of equivalence, which have the form $[\alpha_N^M, \omega_N^M]$ and the isomorphism between $\mathbb{L}^{\text{ch}}({}_RM)$ and $R\text{-pr}/\cong_M$ holds.

Using the product and coproduct of preradicals and the standard preradicals α_N^M and ω_N^M, for $K, N \in \mathbb{L}^{\text{ch}}({}_RM)$ in the lattice $\mathbb{L}^{\text{ch}}({}_RM)$ four operations are defined:

1) *α-product* $K \cdot N$ is $\alpha_K^M \alpha_N^M(M) = \sum_{f:\ M \to N} f(K)$;

2) *ω-product* $K \odot N$ is $\omega_K^M \omega_N^M(M) = \bigcap_{f:\ N \to M} f^{-1}(K)$;

3) *α-coproduct* $N : K$ is $(\alpha_N^M : \alpha_K^M)(M)$;

4) *ω-coproduct* $N \odot K$ is $(\omega_N^M : \omega_K^M)(M)$.

Basic properties of these operations are shown, in particular the distributivity of diverse types. For example, is true the following theorem.

Theorem 17.1. *The following relations hold:*

$$(K_1 + K_2) \cdot N = (K_1 \cdot N) + (K_2 \cdot N);$$
$$(K_1 \cap K_2) \odot N = (K_1 \odot N) \cap (K_2 \odot N);$$
$$N : (K_1 + K_2) = (N : K_1) + (N : K_2);$$
$$N \odot (K_1 \cap K_2) = (N \odot K_1) \cap (N \odot K_2).$$

If ${}_RM = {}_RR$ (i.e. $\mathbb{L}^{\mathrm{ch}}({}_RM)$ is the lattice of ideals of R), then two operations coincide with multiplication and addition of ideals.

The foregoing ideas are developed *in the paper* [59], where the standard preradicals α_N^M and ω_N^M are used to define four operations in the lattice $\mathbb{L}({}_RM)$ of *all* submodules of M. Namely, *α-product* of $K, N \in \mathbb{L}({}_RM)$ is

$$K \cdot N = \alpha_K^M(N) = \sum_{f:\ M \to N} f(K),$$

and *ω-product* is

$$K \odot N = \omega_K^M(N) = \bigcap_{f:\ N \to M} f^{-1}(K).$$

A series of properties of these operations is shown (as associativity and distributivity of diverse types).

In a dual form the other two operations are defined, using the coproduct of preradicals. Namely, *α-coproduct* $N : K$ is defined by the relation $(N : K)/N = \alpha_K^M(M/N)$, and the *$\omega$-coproduct* $N \odot K$ is defined similarly: $(N \odot K)/N = \omega_K^M(M/N)$. Various forms of presentation of these operations, as well as a series of their properties are exposed. In the case $M =_R R$ we have:

$$I \cdot J = IJ, \qquad I \odot J = \bigcap_{f:\ {}_RI \to {}_RR} f^{-1}(I),$$
$$I : J = JR + I, \qquad I \odot J = \big(J : (0 : I)_r\big)_l.$$

A natural continuation of the previous investigations is the article [60], in which the *inverse operations* with respect to α-product and ω-coproduct are introduced and studied. Namely, for $N, K \in \mathbb{L}({}_RM)$ the *left quotient relative to α-product* is defined as

$$N \mathbin{/_{\cdot}} K = \sum \{L_\alpha \subseteq M \mid L_\alpha \cdot K \subseteq N\},$$

and the *right quotient relative to ω-coproduct* is defined by the rule:

$$N \mathbin{{}_{\odot}\backslash} K = \bigcap \{L_\alpha \subseteq M \mid N \odot L_\alpha \supseteq K\}.$$

The distributivity of the operations of α-product and ω-coproduct ensures the existence of these quotients for all submodules.

Various possibilities of representation of these quotients, as well as basic properties of the considered operations are shown. For example, is true the next theorem.

Theorem 17.2. *The following relations hold:*

$$\Big(\bigcap_{\alpha \in \mathfrak{A}} N_\alpha\Big) \mathbin{/_{\cdot}} K = \bigcap_{\alpha \in \mathfrak{A}} (N_\alpha \mathbin{/_{\cdot}} K), \quad N \mathbin{{}_{\odot}\backslash} \Big(\sum_{\alpha \in \mathfrak{A}} K_\alpha\Big) = \sum_{\alpha \in \mathfrak{A}} (N \mathbin{{}_{\odot}\backslash} K_\alpha).$$

In the particular case when $M =_R R$ we have:

$$I \mathbin{/_{\cdot}} J = (I : J)_l = \{a \in R \mid aJ \subseteq I\},$$
$$I \mathbin{{}_{\odot}\backslash} J = J(0 : I)_r, \text{ where } (0 : I)_r = \{b \in R \mid Ib = 0\}.$$

In the article [61] the inverse operations for ω-product and α-coproduct are introduced and investigated. In contrast to the previous cases, these operations are *partial*.

The *left quotient of N by K relative to ω-product $N \mathbin{/_{\odot}} K$* is defined as the smallest submodule $L \subseteq M$ with the property $L \odot K \supseteq N$. Similarly, the *right quotient $N \mathbin{{}_{:}\backslash} K$ relative to α-coproduct* is the greatest submodule $L \subseteq M$ such that $N : L \subseteq K$.

The existence of the quotients $N \mathbin{/_{\odot}} K$ and $N \mathbin{{}_{:}\backslash} K$ is equivalent to the relation $N \subseteq K$. Diverse forms of representation of these quotients are shown. Also properties of these operations and relations with the lattice operations of $\mathbb{L}({}_RM)$ are obtained.

Theorem 17.3. *The following relations hold:*

$$\Big(\sum_{\alpha \in \mathfrak{A}} N_\alpha\Big) \mathbin{/_{\odot}} K = \sum_{\alpha \in \mathfrak{A}} (N_\alpha \mathbin{/_{\odot}} K), \quad N_\alpha \subseteq K, \ \alpha \in \mathfrak{A};$$
$$N \mathbin{{}_{:}\backslash} \Big(\bigcap_{\alpha \in \mathfrak{A}} K_\alpha\Big) = \bigcap_{\alpha \in \mathfrak{A}} (N \mathbin{{}_{:}\backslash} K_\alpha), \quad N \subseteq K_\alpha, \ \alpha \in \mathfrak{A}.$$

18. Closure operators and preradicals ([62–66])

The cycle of works [62–66] is devoted to closure operators in module categories and their relations with preradicals of these categories. Basic types of closure operators of R-Mod, their properties and connections, as well as operations with closure operators are investigated. The question of the interrelations between closure operators and preradicals of R-Mod has a special interest.

Earlier (in the works [7, 9] and [20]) the fact that every idempotent radical of R-Mod defines a special closure operator of this category was remarked and used. The notion of *radical closure* of R-Mod was defined and studied.

A more general notion of closure operator of R-Mod is studied in works of D. Dikranjan, E. Giuli, W. Tholen, etc. A result of these investigations is the book: D. Dikranjan, W. Tholen "Categorical structures of closure operators", Kluwer Acad. Publ., 1995.

In the article [62] the main types of closure operators (weakly hereditary and idempotent) are described by dense and closed submodules. Let $\mathbb{CO}$ be the class of all closure operators of R-Mod. For an operator $C \in \mathbb{CO}$ a submodule $N \in \mathbb{L}({}_RM)$ is called *C-dense* (*C-closed*) in M if $C_M(N) = M$ $(C_M(N) = M)$. We denote:

$$\mathcal{F}_1^C(M) = \{N \subseteq M \mid C_M(N) = M\},$$
$$\mathcal{F}_2^C(M) = \{N \subseteq M \mid C_M(N) = N\}.$$

In this way every operator $C \in \mathbb{CO}$ defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, distinguishing in each $M \in R$-Mod the sets of C-dense or C-closed submodules. Basic properties of these functions are shown. Also, the possibilities of restoration of C by the functions $\mathcal{F}_1^C$ or $\mathcal{F}_2^C$ are indicated.

For an abstract function $\mathcal{F}$ the definitions of a function *of type* $\mathcal{F}_1$ and a function *of type* $\mathcal{F}_2$ are given, using properties of the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$.

Theorem 18.1. *There exists an isotone bijection between weakly hereditary closure operators of $C \in \mathbb{CO}$ and abstract functions of type $\mathcal{F}_1$. So weakly hereditary closure operators are described by dense submodules.*

Dual result for idempotent closure operators is obtained.

Theorem 18.2. *There exists an antiisotone bijection between idempotent closure operators of $\mathbb{CO}$ and abstract functions of the type $\mathcal{F}_2$, so idempotent closure operators are described by closed submodules.*

Further the case, when an operator $C \in \mathbb{CO}$ is simultaneously weakly hereditary and idempotent, is considered. Then the operator C can be reestablished both by $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$. Properties of the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ are indicated in this case and analogues of previous theorems are proved. Namely, bijections are obtained between weakly hereditary idempotent closure operators and functions of type $\mathcal{F}_1$ or of type $\mathcal{F}_2$ with the property of transitivity:

(∗∗) If $N \subseteq P \subseteq M$, $N \in \mathcal{F}(P)$ and $P \in \mathcal{F}(M)$, then $N \in \mathcal{F}(M)$.

On the basis of previous results *in the paper [63]* the characterizations by the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ are obtained for some other important classes of closure operators: hereditary, weakly hereditary maximal, hereditary maximal, minimal and cohereditary. In each of these cases conditions on the functions $\mathcal{F}_1^C$ or $\mathcal{F}_2^C$, which are necessary and sufficient for the restoration of the operator C are indicated. As an example, we consider the case of *hereditary* closure operators, i.e. such $C \in \mathbb{CO}$ that in the situation $L \subseteq N \subseteq M$ the relation $C_N(L) = C_M(L) \cap N$ holds. Such an operator is weakly hereditary, so it is described by $\mathcal{F}_1^C$ (Theorem 18.1). For an abstract function $\mathcal{F}$ the following condition is considered:

(∗∗∗) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(P)$.

Theorem 18.3. *There exists an isotone bijection between hereditary closure operators $C \in \mathbb{CO}$ and abstract functions of type $\mathcal{F}_1$ with the condition* (∗∗∗).

A similar method is used for the characterization of the rest of named types of closure operators.

The article [64] is a continuation of these researches and contains the study of the basic four operations in the class $\mathbb{CO}$ of closure operators of R-Mod: meet ($\wedge$), join ($\vee$), multiplication ($\cdot$), comultiplication ($\#$). Principal properties of these operations and relations between them are shown. In particular, some properties of distributivity are proved.

Theorem 18.4. *The following relations hold:*

$$(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha)\cdot D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D), \qquad (\bigvee_{\alpha\in\mathfrak{A}} C_\alpha)\cdot D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D);$$

$$(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha)\,\#\, D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \,\#\, D), \qquad (\bigvee_{\alpha\in\mathfrak{A}} C_\alpha)\,\#\, D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \,\#\, D).$$

Some other distributivity relations are true under additional conditions. For example, is proved the following theorem.

Theorem 18.5. a) *If the operator* $C \in \mathbb{CO}$ *is* hereditary, *then*

$$C \# (\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha) = \bigwedge_{\alpha \in \mathfrak{A}} (C \# D_\alpha)$$

for arbitrary $D_\alpha \in \mathbb{CO}$, $\alpha \in \mathfrak{A}$.

b) *If the operator* $C \in \mathbb{CO}$ *is* minimal, *then*

$$C \cdot (\bigvee_{\alpha \in \mathfrak{A}} D_\alpha) = \bigvee_{\alpha \in \mathfrak{A}} (C \cdot D_\alpha)$$

for arbitrary $D_\alpha \in \mathbb{CO}$, $\alpha \in \mathfrak{A}$.

The question on the preservation of properties of closure operators by the indicated operations in $\mathbb{CO}$ is studied. Some results on this subject are:

1) if C_α, $\alpha \in \mathfrak{A}$, are weakly hereditary, then $\bigvee_{\alpha \in \mathfrak{A}} C_\alpha$ is weakly hereditary;
2) if C_α, $\alpha \in \mathfrak{A}$, are maximal (minimal), then $\bigvee_{\alpha \in \mathfrak{A}} C_\alpha$ is maximal (minimal);
3) if C_α, $\alpha \in \mathfrak{A}$, are hereditary, then $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is hereditary;
4) if C_α, $\alpha \in \mathfrak{A}$, are maximal, then $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is maximal.

Some relations between closure operators and preradicals of R-Mod are studied *in the article* [66]. Three mappings are defined between the classes $\mathbb{CO}$ of closure operators and $\mathbb{PR}$ of preradicals of R-Mod:

$$\Phi \colon \mathbb{CO} \to \mathbb{PR}, \qquad \Psi_1 \colon \mathbb{PR} \to \mathbb{CO}, \qquad \Psi_2 \colon \mathbb{PR} \to \mathbb{CO},$$

where

$$\begin{aligned} \Phi(C) &= r_C, & r_C(M) &= C_M(0); \\ \Psi_1(r) &= C^r, & [(C^r)_M(N)]/N &= r(M/N); \\ \Psi_2(r) &= C_r, & (C_r)_M(N) &= N + r(M). \end{aligned}$$

Denote by $\mathbb{M}\mathrm{ax}(\mathbb{CO})$ ($\mathbb{M}\mathrm{in}(\mathbb{CO})$) the class of all maximal (minimal) closure operators of R-Mod. The pair of mappings (Φ, Ψ_1) determines an isomorphism $\mathbb{M}\mathrm{ax}(\mathbb{CO}) \cong \mathbb{PR}$, while the pair (Φ, Ψ_2) defines an isomorphism $\mathbb{M}\mathrm{in}(\mathbb{CO}) \cong \mathbb{PR}$. Using these isomorphisms, in continuation some bijections are obtained between preradicals of diverse types (idempotent, radical, hereditary etc.) and closure operators with special properties. In particular, is proved the following.

Theorem 18.6. 1) *The pair of mappings* (Φ, Ψ_1) *determines isotone bijections between:*

- *idempotent preradicals of* $\mathbb{PR}$ *and maximal weakly hereditary closure operators of* $\mathbb{CO}$*;*
- *pretorsions of* $\mathbb{PR}$ *and maximal hereditary closure operators of* $\mathbb{CO}$*.*

2) *The pair of mappings* (Φ, Ψ_2) *defines isotone bijections between:*

- *idempotent preradicals of* $\mathbb{PR}$ *and minimal weakly hereditary closure operators of* $\mathbb{CO}$*;*
- *pretorsions of* $\mathbb{PR}$ *and minimal hereditary closure operators of* $\mathbb{CO}$*;*
- *cotorsions of* $\mathbb{PR}$ *and weakly hereditary and cohereditary closure operators of* $\mathbb{CO}$*.*

The influence of the mappings Φ, Ψ_1 and Ψ_2 on operations in the classes $\mathbb{CO}$ and $\mathbb{PR}$ is studied *in the paper* [65]. In particular, it is proved that the mapping Φ preserves the meets and joins:

$$\Phi(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha) = \bigwedge_{\alpha\in\mathfrak{A}} [\Phi(C_\alpha)], \qquad \Phi(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha) = \bigvee_{\alpha\in\mathfrak{A}} [\Phi(C_\alpha)].$$

Furthermore, the mapping Φ transforms the coproducts of $\mathbb{CO}$ into the products of $\mathbb{PR}$: $\Phi(C \# D) = \Phi(C) \cdot \Phi(D)$.

The mapping Ψ_1 preserves the meets and joins:

$$\Psi_1(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha) = \bigwedge_{\alpha\in\mathfrak{A}} [\Psi_1(r_\alpha)], \qquad \Psi(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha) = \bigvee_{\alpha\in\mathfrak{A}} [\Psi_1(r_\alpha)].$$

Moreover, Ψ_1 transforms the products of $\mathbb{PR}$ into the coproducts of $\mathbb{CO}$: $\Psi_1(r \cdot s) = \Psi_1(r) \# \Psi_1(s)$, and the coproducts of $\mathbb{PR}$ into the products of $\mathbb{CO}$: $\Psi_1(r : s) = \Psi_1(r) \cdot \Psi_1(s)$.

Some similar results are obtained for the mapping Ψ_2.

On the whole the works of this cycle show the expediency and usefulness of the combined investigations of preradicals and closure operators of R-Mod.

19. Instead of conclusion

The present work reflects the aspiration to embrace by one survey the majority of results of author on radicals and torsions in modules. The requirement of conciseness made impossible the minimal full exposition of results, and so we limit ourselves by formulation of questions and of small part of results. Furthermore, it is impossible to give the definitions and preliminary facts, which strongly restrict the possibilities to show the "entourage" in which the exposed results were obtained. The references in the main text are reduced to the minimum, otherwise the bibliography sharply increases.

The effort to overcome these difficulties led us to the presented form of the review, which can give a general idea of the direction of investigations and of the type of results.

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The groups whose cyclic subgroups are either ascendant or almost self-normalizing

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Abstract. The main result of this paper shows a description of locally finite groups, whose cyclic subgroups are either almost self-normalizing or ascendant. Also, we obtained some natural corollaries of the above situation.

Introduction

The subgroups of a group G are connected with some natural families of subgroups. One of them is the following. Let H be a subgroup of a group G. We construct an ascending series

$$\langle 1\rangle = H_0 \leqslant H_1 \leqslant \dots H_\alpha \leqslant H_{\alpha+1} \leqslant \dots H_\gamma \leqslant G,$$

where $H_1 = H$, $H_2 = N_G(H_1) = N_G(H)$, $H_{\alpha+1} = N_G(H_\alpha)$ for every ordinal $\alpha < \gamma$, $H_\lambda = \bigcup_{\mu<\lambda} H_\mu$ for every limit ordinal $\lambda < \gamma$, and $N_G(H_\gamma) = H_\gamma$. This chain is called the *upper normalized chain* of H in G. Here the two natural types of subgroups appear. If $H_\gamma = G$, then a subgroup H is called *ascendant* in G. If $H_\gamma = H$ (that is $N_G(H) = H$), then a subgroup H is called *self-normalizing* in G. Thus, every subgroup of a group is naturally connected with the two types of subgroups: an ascendant and a self-normalizing subgroups. The presence of a large family of ascendant subgroups has a strong influence on the group structure. For example, if

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every subgroup of a group G is ascendant, then G is locally nilpotent [16]. Moreover, if every cyclic subgroup of a group G is ascendant, then G is locally nilpotent [6, Theorem 2]. More precisely, the subgroup $\mathbf{Gru}(G)$ of an arbitrary group G, generated by all ascendant cyclic subgroups of G, is locally nilpotent. This subgroup is called the *Gruenberg radical* of G. Every finitely generated subgroup of $\mathbf{Gru}(G)$ is ascendant in G and nilpotent [6, Theorem 2]. A group G is said to be a *Gruenberg group*, if $G = \mathbf{Gru}(G)$.

L.A. Kurdachenko and H. Smith [13] have considered the groups, whose subgroups are either subnormal or self-normalizing. A natural generalization of this paper was an article [12]. In [12] L.A. Kurdachenko et al. considered the groups, whose finitely generated subgroups are either ascendant or self-normalizing. From their results it follows that locally finite groups, whose cyclic subgroups are either ascendant or self-normalizing, have the same structure. Here we discuss a more general situation.

We remark that the groups, in which some family of subgroups divides into two types of subgroups, which often have the opposite properties, considered by other authors (see, for example, [15], [17]).

Let H be a subgroup of a group G. Then H is called *almost self-normalizing* in G, if H has finite index in $N_G(H)$.

In this paper we consider the groups whose cyclic subgroups are either almost self-normalizing or ascendant. The main result is the following

Theorem A. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Suppose that $G \neq \mathbf{Gru}(G)$. Then the following assertions hold:*

(i) *a factor-group $G/\mathbf{Gru}(G)$ is finite;*
(ii) *$G = Q \leftthreetimes R$, where Q is a normal Sylow σ'-subgroup of G, R is a Sylow σ-subgroup of G, $\sigma = \Pi(G/\mathbf{Gru}(G))$;*
(iii) *R is a Chernikov subgroup;*
(iv) *$\mathbf{Gru}(G) = C_R(Q) \times Q$;*
(v) *if $g \notin \mathbf{Gru}(G)$, then $C_G(g)$ is finite;*
(vi) *$\mathbf{Gru}(G)$ is nilpotent-by-finite.*

We obtained the following additional information about the structure of a factor-group $G/\mathbf{Gru}(G)$.

Corollary A1. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Let $G \neq \mathbf{Gru}(G)$, $F = G/\mathbf{Gru}(G)$ and $\sigma = \Pi(F)$. Suppose that the Sylow σ'-subgroup of G is infinite. Then*

(i) *if $p \in \sigma$ and $p \neq 2$, then Sylow p-subgroup of F is cyclic;*
(ii) *Sylow 2-subgroup of F is cyclic or a generalized quaternion group;*
(iii) *every subgroup of order pq of F, $p, q \in \sigma$, is cyclic.*

Let G be a Chernikov group and D be the divisible part of G. Put $\mathbf{Sp}(G) = \Pi(D)$.

Corollary A2. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Let $G \neq \mathbf{Gru}(G)$, $F = G/\mathbf{Gru}(G)$ and $\sigma = \Pi(F)$. Suppose that the Sylow σ'-subgroup of G is finite and $\mathbf{Sp}(G) = \{p\}$ for some prime $p \in \sigma$. Then*
(i) *if q is a prime and $q \notin \{2, p\}$, then Sylow q-subgroup of F is cyclic;*
(ii) *if $p \neq 2$, then Sylow 2-subgroup of F is cyclic or generalized quaternion group.*

Corollary A3. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Let $G \neq \mathbf{Gru}(G)$, $F = G/\mathbf{Gru}(G)$ and $\sigma = \Pi(F)$. Suppose that the Sylow σ'-subgroup of G is finite and $|\mathbf{Sp}(G)| \geqslant 2$. Then*
(i) *if $q \in \sigma$ is a prime and $q \neq 2$, then Sylow q-subgroup of F is cyclic;*
(ii) *if $2 \in \sigma$, then Sylow 2-subgroup of F is cyclic or generalized quaternion group.*

An important special case of the ascendant subgroups are the subnormal subgroups. A subnormal subgroup is exactly an ascendant subgroup having finite upper normalized chain. From Theorem A we can obtain the description of locally finite groups whose cyclic subgroups are either almost self-normalizing or subnormal.

The subgroup $\mathbf{B}(G)$, generated by all cyclic subnormal subgroups of G, is called the *Baer radical* of G. Every finitely generated subgroup of $\mathbf{B}(G)$ is subnormal in G and nilpotent (see, for example, [14, Theorem 2.5.1]), so that a subgroup $\mathbf{B}(G)$ is locally nilpotent. A group G is said to be a *Baer group*, if $G = \mathbf{B}(G)$.

Let G be a group and A be an abelian normal subgroup of G. Then A is said to be *G-quasifinite* if A is infinite, but every proper G-invariant subgroup of A is finite.

Theorem B. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or subnormal. Suppose that $G \neq \mathbf{B}(G)$. Then the following assertions hold:*
(i) *a factor-group $G/\mathbf{B}(G)$ is finite;*

(ii) *$G = Q \leftthreetimes R$, where Q is a normal Sylow σ'-subgroup of G, R is a Sylow σ-subgroup of G, $\sigma = \Pi(G/\mathbf{B}(G))$;*
(iii) *R is a Chernikov subgroup;*
(iv) *$\mathbf{B}(G) = C_R(Q) \times Q$;*
(v) *if $g \notin \mathbf{B}(G)$, then $C_G(g)$ is finite;*
(vi) *$\mathbf{B}(G)$ includes a finite G-invariant σ-subgroup K such that*

$$\mathbf{B}(G)/K = QK/K \times U_1/K \times \ldots \times U_k/K,$$

where U_j/K is a G-quasifinite divisible Chernikov p_j-subgroup, $p_j \in \sigma$, $1 \leqslant j \leqslant k$;
(vii) *$\mathbf{B}(G)$ is nilpotent.*

Trivially, every normal subgroup is a special case of subnormal subgroup, and we come to

Corollary B1. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or normal. Suppose that G is a not Dedekind group. Then the following assertions hold:*
(i) *a factor-group $G/\mathbf{B}(G)$ is finite cyclic;*
(ii) *if $g \notin \mathbf{B}(G)$, then $C_G(g)$ is finite;*
(iii) *every subgroup of $\mathbf{B}(G)$ is G-invariant, in particular, $\mathbf{B}(G)$ is a Dedekind group;*
(iv) *Sylow 2-subgroup of $\mathbf{B}(G)$ is Chernikov, moreover, if this Sylow 2-subgroup is infinite, then $\mathbf{B}(G)$ is abelian and $G/\mathbf{B}(G)$ has order 2.*

1. Preliminaries and lemmas

Lemma 1. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). If H is a subgroup of G, then every cyclic subgroup of H is either almost self-normalizing or ascendant (respectively, subnormal).*

Proof. Let C be a cyclic subgroup of H and suppose that C is not ascendant (respectively, subnormal) in H. Then C can not be ascendant (respectively, subnormal) in G. It follows that the index $|N_G(C) : C|$ is finite. An inclusion $N_H(C) \leqslant N_G(C)$ shows that the index $|N_H(C) : C|$ is finite. □

Lemma 2. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). If A is an infinite periodic abelian subgroup of G, then the Gruenberg radical (respectively, Baer radical) of G includes A.*

Proof. Indeed, for each element $x \in A$ we have $A \leqslant C_G(x)$, which follows that the index $|N_G(\langle x\rangle) : \langle x\rangle|$ is infinite. Thus, $x \in \mathbf{Gru}(G)$ (respectively, $x \in \mathbf{B}(G)$). Hence $A \leqslant \mathbf{Gru}(G)$ (respectively, $A \leqslant \mathbf{B}(G)$). □

Lemma 3. *Let L be a locally nilpotent periodic subgroup of G. If L is not Chernikov, then the centralizer of every element of L is infinite.*

Proof. Suppose first that the set $\Pi(L)$ is infinite. Let g be an arbitrary element of L and $|g| = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s}$, where $p_1, \ldots, p_s$ are primes, $p_j \neq p_m$ whenever $j \neq m$. Since $\Pi(L)$ is infinite, the set $\pi = \Pi(L) \setminus \{p_1, \ldots, p_s\}$ is infinite. Then the Sylow π-subgroup L_π of L is infinite. The fact, that L is locally nilpotent, implies the inclusion $L_\pi \leqslant C_G(g)$, which follows that $C_G(g)$ is infinite.

Suppose now that the set $\Pi(L)$ is finite. Since L is not Chernikov, there exists a prime p such that the Sylow p-subgroup P of L is not Chernikov. Let x be an arbitrary element of L. If x is a p'-element, then $P \leqslant C_G(x)$, which follows again that $C_G(x)$ is infinite. Assume that $x \in P$. Since P is not Chernikov, P includes an $\langle x\rangle$-invariant abelian subgroup A, which is not Chernikov [21]. Then its lower layer $H = \Omega_1(A)$ is an infinite elementary abelian subgroup. Clearly H is an $\langle x\rangle$-invariant subgroup. Let $1 \neq b_1 \in H$. Put $K_1 = \langle b_1\rangle^{\langle x\rangle}$, then K_1 is a finite $\langle x\rangle$-invariant subgroup. Since H is elementary abelian, H includes a subgroup B_1 such that $H = K_1 \times B_1$. We note that the index $|H : B_1|$ is finite. Then the index $|H : B_1^y|$ is also finite for every element $y \in \langle x\rangle$. Since an element x has finite order, a family $\{B_1^y | y \in \langle x\rangle\}$ is finite. Then the intersection $C_1 = \bigcap_{y \in \langle x\rangle} B_1^y$ has finite index in H. In particular, C_1 is infinite. By such choice C_1 is $\langle x\rangle$-invariant and $K_1 \cap C_1 = \langle 1\rangle$. Let $1 \neq b_2 \in C_1$ and $K_2 = \langle b_2\rangle^{\langle x\rangle}$, then K_2 also is a finite $\langle x\rangle$-invariant subgroup such that $K_1 \cap K_2 = \langle 1\rangle$. Since H is elementary abelian, H includes a subgroup B_2 such that $H = K_1K_2 \times B_2$. Using the similar arguments and ordinary induction, we construct the family $\{K_n | n \in \mathbb{N}\}$ of finite $\langle x\rangle$-invariant subgroups such that $K_1 \cdot \ldots \cdot K_m \cap K_{m+1} = \langle 1\rangle$ for every $m \in \mathbb{N}$. It follows that $\langle K_n | n \in \mathbb{N}\rangle = \mathbf{Dr}_{n\in\mathbb{N}} K_n$.

Since $\langle x, K_n\rangle$ is a finite p-subgroup, it is nilpotent. Since K_n is its normal subgroup, $K_n \cap \zeta(\langle x, K_n\rangle) \neq \langle 1\rangle$. Let $1 \neq z_n \in K_n \cap \zeta(\langle x, K_n\rangle)$ and put $Z = \langle z_n | n \in \mathbb{N}\rangle$. An equality $\langle K_n | n \in \mathbb{N}\rangle = \mathbf{Dr}_{n\in\mathbb{N}} K_n$ implies that Z is an infinite elementary abelian subgroup. By its choice $Z \leqslant C_G(\langle x\rangle)$. It follows that $C_G(x)$ is infinite. □

Corollary 1. *Let L be an infinite periodic nilpotent subgroup of G. Then the centralizer of every element of L is infinite.*

Proof. If L is a not Chernikov subgroup, then result follows from Lemma 3. Therefore, suppose that L is a Chernikov subgroup. Since L is nilpotent, every subgroup of L is subnormal. In particular, L is a Baer group. Then L is central-by-finite [8, Corollary 1 to Lemma 4]. In particular, $C_L(x)$ is infinite for each element $x \in L$. □

Corollary 2. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). Suppose that L is a locally nilpotent periodic subgroup of G. If L is not Chernikov, then the Gruenberg radical (respectively, Baer radical) of G includes L.*

Proof. In fact, by Lemma 3 $C_G(g)$ is infinite for each element $g \in L$, which follows that the index $|N_G(\langle g\rangle) : \langle g\rangle|$ is infinite and hence $g \in \mathbf{Gru}(G)$ (respectively, $g \in \mathbf{B}(G)$). □

Corollary 3. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). Suppose that p is a prime such that the Sylow p-subgroup P of G is not Chernikov. Then the Gruenberg radical (respectively, Baer radical) of G includes P. In particular, P is normal in G.*

Corollary 4. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). Suppose that L is an infinite periodic nilpotent subgroup of G. Then the Gruenberg radical (respectively, Baer radical) of G includes L.*

Proof. In fact, by Corollary 1 $C_G(g)$ is infinite for each element $g \in L$, which follows that the index $|N_G(\langle g\rangle) : \langle g\rangle|$ is infinite and hence $g \in \mathbf{Gru}(G)$ (respectively, $g \in \mathbf{B}(G)$). □

Let G be a Chernikov group. Denote by $\mathbf{D}(G)$ the maximal normal divisible abelian subgroup of G. A subgroup $\mathbf{D}(G)$ is called a *divisible part* of G.

Lemma 4. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Then Sylow p-subgroup of $G/\mathbf{Gru}(G)$ is finite for every prime p. Moreover, for every $p \in \Pi(G/\mathbf{Gru}(G))$ every Sylow p-subgroup of G is Chernikov.*

Proof. Put $B = \mathbf{Gru}(G)$. Let $p \in \Pi(G/B)$ and let P/B be the Sylow p-subgroup of G/B. Let $B \neq xB \in P/B$. Without loss of generality we can assume that x is an p-element. Let B_p be the Sylow p-subgroup of B. Since B is locally nilpotent, B_p is G-invariant. Then a product $\langle x\rangle B_p$ is a p-subgroup. Suppose that B_p is not Chernikov. Let C be the Sylow p-subgroup of G, including $\langle x\rangle B_p$. Then C is a not Chernikov subgroup, and Corollary 3 proves that $C \leqslant B$. In this case $x \in B$, what contradicts to the choice of element x. This contradiction shows that B_p is a Chernikov subgroup.

Suppose now that P/B is not Chernikov. Then P/B includes an infinite elementary abelian subgroup A/B [2, Theorem 8]. Without loss of generality we can assume that A/B is countable. Then A/B has an ascending series of finite subgroups

$$A_1/B \leqslant A_2/B \leqslant \ldots \leqslant A_n/B \leqslant \ldots$$

such that $A/B = \bigcup_{n\in\mathbb{N}} A_n/B$. Since A_1/B is finite, A_1 includes a finite subgroup K_1 such that $A_1 = K_1B$. Choose in K_1 Sylow p-subgroup S_1. Since K_1 is finite, $S_1(B \cap K_1)/(B \cap K_1)$ is a Sylow p-subgroup of $K_1/(B \cap K_1)$. On the other hand, $K_1/(B \cap K_1) \cong K_1B/B$ is a p-group. It follows that $S_1(B \cap K_1)/(B \cap K_1) = K_1/(B \cap K_1)$, or $S_1(B \cap K_1) = K_1$. In turn out it follows that $A_1 = S_1B$. Choose in A_2 a finite subgroup K_2 such that $K_1 \leqslant K_2$ and $A_2 = K_2B$. Let S_2 be the Sylow p-subgroup of K_2, including S_1. Using the above arguments, we can prove that $A_2 = S_2B$. Using similarly arguments and ordinary induction, we construct an ascending series

$$S_1 \leqslant S_2 \leqslant \ldots \leqslant S_n \leqslant \ldots$$

of finite p-subgroups such that $A = \Big(\bigcup_{n\in\mathbb{N}} S_n\Big)B$. Put $S = \bigcup_{n\in\mathbb{N}} S_n$, then S is a p-subgroup and isomorphism $S/(S \cap B) \cong SB/B = A/B$ shows that S is not Chernikov. Since B_p is a normal p-subgroup, then SB_p is a p-subgroup. Let D be the Sylow p-subgroup of G, including SB_p. The fact that S is not Chernikov, implies that D is not Chernikov. Corollary 3 proves that $D \leqslant B$. In this case $S \leqslant B$ and therefore $SB = A \leqslant B$, what contradicts to the choice of A. This contradiction shows that P/B is a Chernikov subgroup.

Let Q be the Sylow p-subgroup of B, then $B = B_p \times Q$. Since B_p and P/B are Chernikov, P/Q likewise is a Chernikov group. In particular, it is countable. Then P includes a p-subgroup R such that $P = QR$ (see, for example, [3, Theorem 2.4.5]). Denote by W the divisible part of R. Since W is abelian and infinite, Lemma 2 shows that $W \leqslant B$. An inclusion $Q \leqslant B$ implies that $WQ \leqslant B$. In turn out, it follows that P/B is finite. □

Let G be a group. Recall that a subgroup H of a group G is called *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element g of G.

Lemma 5. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Then the factor-group $G/\mathbf{Gru}(G)$ is finite. Moreover, if $\pi = \Pi(G) \setminus \Pi(G/\mathbf{Gru}(G))$ then $G = Q \leftthreetimes R$, where Q is a Sylow π-subgroup of G and R is a Chernikov subgroup.*

Proof. Put $B = \mathbf{Gru}(G)$. Suppose that G/B is infinite. Then G/B includes an infinite abelian subgroup A/B [10]. Since the Sylow p-subgroups of G/B are finite for each prime p by Lemma 4, the set $\Pi(A/B)$ is infinite. Let $p \in \Pi(A/B)$ and let P/B be the Sylow p-subgroup of A/B. By this choice P/B is non-identity, i.e. B does not include P. Lemma 4 shows that P/B is finite. Being almost locally nilpotent, P has a Carter subgroup C, that is maximal locally nilpotent self-normalizing subgroup. We remark also that all Carter subgroup of P are conjugate and abnormal [20, Theorem 2.1 and Corollary 2.2]. Since C is abnormal in P, CB is also abnormal. Then CB/B is abnormal in P/B. On the other hand, P/B is abelian, which follows that $CB/B = P/B$.

Let a be an arbitrary element of A. Then C^a is maximal locally nilpotent self-normalizing subgroup of P. As we noted above, the subgroups C and C^a are conjugate in P, that is there exists an element x of P such that $C^a = C^x$. It follows that $ax^{-1} \in N_A(C)$, which follows the equality $A = PN_A(C)$. Take into account an equality $P = CB$, we obtain that $A = BN_A(C)$. If we suppose that a subgroup C is not Chernikov, then the fact that C is locally nilpotent together with Corollary 1 imply the inclusion $C \leqslant B$. But this contradicts to the choice of P. This contradiction shows that C is a Chernikov subgroup. The isomorphism

$$A/B \cong BN_A(C)/B \cong N_A(C)/(B \cap N_A(C))$$

shows that $N_A(C)/(B \cap N_A(C))$ has infinite set $\Pi(N_A(C)/(B \cap N_A(C)))$ and is abelian. Since C is abnormal in P, $N_P(C) = C$. It follows that

$$B \cap N_A(C) = B \cap P \cap N_A(C) = B \cap N_P(C) = B \cap C.$$

Together with $C/(B \cap C) \cong CB/B = P/B$ and the fact, that P/B is finite, it follows that $N_A(C)/C$ is an abelian group with infinite set

$\Pi(N_A(C)/C)$. Put $N_A(C) = K$. Since $K/C_K(C)$ is a Chernikov group (see, for example, [3, Theorem 1.5.16]), $C_K(C)C/C$ is an abelian group with infinite set $\Pi(C_K(C)C/C)$. Let S/C be the Sylow σ'-subgroup of $C_K(C)C/C$, where $\sigma = \Pi(C)$. Clearly S/C is countable, so that $S = C \leftthreetimes U$ (see, for example, [3, Theorem 2.4.5]), where U is a Sylow σ'-subgroup of S. An inclusion $C \leqslant C_K(C)$ implies that $S = C \times U$. The fact that U is an infinite abelian subgroup implies that a subgroup $C_S(y)$ is infinite for every element $y \in C$. In turn out, it implies that a cyclic subgroup $\langle y \rangle$ is ascendant in G. Hence $y \in B$. In other words, $C \leqslant B$. But in this case the equality $P = CB$ implies that $P = B$, which contradicts to the choice of P. This contradiction proves that G/B is finite.

In particular, a set π is finite. Let Q be a Sylow π'-subgroup of G. The choice of π shows that $Q \leqslant B$, so that Q is normal in G. Using Lemma 4 we obtain that for each prime $p \in \pi$ every Sylow p-subgroup of G is Chernikov. It follows that B/Q is countable. Take in account that G/B is finite, we obtain that G/Q is countable. Then $G = Q \leftthreetimes R$, where R is a Sylow π-subgroup of G (see, for example, [3, Theorem 2.4.5]). Clearly R is a Chernikov subgroup. □

Lemma 6. *Let G be a group and P be a normal Chernikov divisible p-subgroup of G such that $G/C_G(P)$ is finite. Then P includes a finite G-invariant subgroup F such that $P/F = U_1/F \times \ldots \times U_k/F$, where U_j/F is a G-quasifinite subgroup, $1 \leqslant j \leqslant k$.*

Proof. Let

$$\mathfrak{Q} = \{Q \mid Q \text{ is an infinite } G\text{-invariant subgroup of } P\}.$$

Since P is Chernikov, it satisfies the minimal condition on subgroups, and therefore $\mathfrak{Q}$ has a minimal element, say V_1. Clearly V_1 is G-quasifinite. Then V_1 must be divisible and hence V_1 has a direct complement in P (see, for example, [4, Theorem 21.2]). It follows that P includes a G-invariant subgroup R_1 such that $P = V_1R_1$ and the intersection $V_1 \cap R_1$ is finite (see, for example, [11, Corollary 5.11]). Put $F_1 = V_1 \cap R_1$, then $P/F_1 = V_1/F_1 \times R_1/F_1$. Clearly V_1/F_1 is G-quasifinite. Now we choose in R_1/F_1 a G-invariant G-quasifinite subgroup V_2/F_1. Using the above arguments, we have found a G-invariant subgroup R_2/F_1 such that $R_1/F_1 = (V_2/F_1)(R_2/F_1)$ and the intersection $V_2/F_1 \cap R_2/F_1 = F_2/F_1$ is finite. By such choice we have

$$P/F_2 = (V_1F_2/F_2) \times V_2/F_2 \times R_2/F_2.$$

The both factors V_1F_2/F_2 and V_2/F_2 are G-quasifinite. Using the similar arguments, after finitely many steps we prove the required result. □

Lemma 7. *Let G be a locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Suppose that $G = Q \leftthreetimes F$, where $Q = \mathbf{Gru}(G)$, $\Pi(Q) \cap \Pi(F) = \varnothing$ and F is a finite subgroup. If Q is infinite, then*

(i) *Q is nilpotent-by-finite;*
(ii) *$C_F(Q) = \langle 1 \rangle$;*
(iii) *if $g \notin Q$, then $C_G(g)$ is finite;*
(iv) *if $p \in \Pi(F)$ and $p \neq 2$, then Sylow p-subgroup of F is cyclic;*
(v) *Sylow 2-subgroup of F is cyclic or generalized quaternion group;*
(vi) *every subgroup of order pq of F, $p, q \in \Pi(F)$, is cyclic.*

Proof. Choose an arbitrary element $g \notin Q$ and suppose that $C_G(g)$ is infinite. Since g has finite order, the index $|C_G(g) : \langle g \rangle|$ is infinite. In turn out, it implies $\langle g \rangle$ has infinite index in its normalizer, which show that $\langle g \rangle$ is ascendant in G. This contradiction shows that $C_G(g)$ is finite.

Let $y \in C_F(Q)$, then $Q \leqslant C_G(y)$, in particular, $C_G(y)$ is infinite. As we have seen above in this case $y \in \mathbf{Gru}(G) = Q$, i.e. $y \in Q \cap F = \langle 1 \rangle$.

Let $p \in \Pi(F)$ and g be an element of F, having order p. By above proved $C_Q(g)$ is finite. Then Q is nilpotent-by-finite [19, Theorem 1.2].

Suppose first that Q is not Chernikov. Then Q includes an F-invariant abelian subgroup A, which is not Chernikov [21]. Let y be an arbitrary element of F. The equality $\Pi(Q) \cap \Pi(F) = \varnothing$ implies that $A = C_A(y) \times [A, y]$ [1, Proposition 2.12]. By above proved $C_A(y)$ is finite, so that $[A, y]$ has finite index in A. It is valid for every element $y \in F$. Therefore, the finiteness of F implies that a subgroup $C = \bigcap_{y \in F} [A, y]$ has finite index in A. By its choice $C_C(y) = \langle 1 \rangle$ for every element $y \in F$. Put $E = \bigcap_{y \in F} C^y$, then E has finite index in A (in particular, E is infinite), E is F-invariant and $C_E(y) = \langle 1 \rangle$ for every element $y \in F$. Since F is finite, we can choose in E minimal F-invariant subgroup V. Then $C_V(y) = \langle 1 \rangle$ for every element $y \in F$. Therefore, F satisfies the conditions (iv)-(vi) by [9, Satz V.8.15].

Suppose now that Q is a Chernikov subgroup. Denote by D the divisible part of Q. The equality $\Pi(Q) \cap \Pi(F) = \varnothing$ implies again that $D = C_D(y) \times [D, y]$ [1, Proposition 2.12] for every element $y \in F$. If we suppose that $C_D(y) \neq \langle 1 \rangle$, then $C_D(y)$ must be infinite, and we obtain a contradiction with condition (iii). This contradiction shows that $C_D(y) = \langle 1 \rangle$ for every element $y \in F$. Again choose in D a minimal

F-invariant subgroup W. Then $C_W(y) = \langle 1 \rangle$ for every element $y \in F$. Therefore, F satisfies the conditions (iv)-(vi) by [9, Satz V.8.15]. □

Lemma 8. *Let G be a Chernikov group and P be a divisible part of G. Suppose that P is a p-subgroup and $C_G(g)$ is finite for each p'-element g. Then the following assertions hold:*

(i) *if q is a prime and $q \notin \{2, p\}$, then Sylow q-subgroup of $G/\mathbf{Gru}(G)$ is cyclic;*

(ii) *if $p \neq 2$, then Sylow 2-subgroup of $G/\mathbf{Gru}(G)$ is cyclic or generalized quaternion group.*

Proof. Put $C = O_p(G)$. Our conditions yields that $\mathbf{Gru}(G) = C$. Let q be a prime, $q \neq p$, and Q/C be a Sylow q-subgroup of G/C. Since C is a p-subgroup, $Q = C \leftthreetimes R$, where R is a Sylow q-subgroup of Q (see, for example, [3, Theorem 2.4.5]). Choose in R an arbitrary abelian subgroup A. Let

$$\mathfrak{S} = \{S |\ S \text{ is an infinite } A\text{-invariant subgroup of } P\}.$$

Since P is Chernikov, it satisfies the minimal condition on subgroups, and therefore $\mathfrak{S}$ has a minimal element, say V. Clearly V is A-quasifinite. In particular, it follows that V is a divisible abelian subgroup. Since A is a p'-subgroup, $V = C_V(y) \times [V, y]$ for each element $y \in A$ [1, Proposition 2.12]. The fact, that A is abelian, implies that the subgroups $C_A(y)$ and $[V, y]$ are A-invariant. Therefore, if we assume that $C_A(y) \neq \langle 1 \rangle$, then $C_A(y)$ must be infinite, and we obtain a contradiction. This contradiction shows that $C_A(y) = \langle 1 \rangle$ for each element $y \in A$. Using Lemma 3.1 of paper [7] we obtain that a subgroup A is cyclic. In particular, $\zeta(R)$ is cyclic. Let $\langle d \rangle = \Omega_1(\zeta(R))$. Suppose that x is an element of R, having order q. If $\langle d, x \rangle \neq \langle d \rangle$, then $\langle d, x \rangle$ is an elementary abelian subgroup of order q^2. But in this case it is not cyclic, and we obtain a contradiction with above proved. This contradiction shows that $\langle d, x \rangle = \langle d \rangle$. In other words, R has only one subgroup of order q. Then R (and hence Q/C) is cyclic, whenever $q \neq 2$, and Q/C is cyclic or generalized quaternion group, whenever $q = 2$ (see, for example, [9, Satz III.8.2]). □

Lemma 9. *Let G be an infinite periodic group whose cyclic subgroups are either almost self-normalizing or ascendant. If $g \notin \mathbf{Gru}(G)$, then $C_G(g)$ is finite.*

Proof. Choose an arbitrary element $g \notin \mathbf{Gru}(G)$ and suppose that $C_G(g)$ is infinite. Since g has finite order, the index $|C_G(g) : \langle g \rangle|$ is infinite. In turn out, it implies that $\langle g\rangle$ has infinite index in its normalizer, which show that $\langle g\rangle$ is ascendant in G. This contradiction shows that $C_G(g)$ is finite. □

Lemma 10. *Let G be an infinite periodic group whose cyclic subgroups are either almost self-normalizing or ascendant. If K is a finite normal subgroup of G, then every cyclic subgroup of G/K is either almost self-normalizing or ascendant. More precisely, for each element $g \notin \mathbf{Gru}(G)$ the centralizer $C_{G/K}(gK)$ is finite.*

Proof. At once we note that $K \leqslant \mathbf{Gru}(G)$. In fact, since K is finite, every element g of K has only finitely many conjugates in G. Then $C_G(g)$ has finite index in G, in particular, $C_G(g)$ is infinite. As we have seen above, in this case the index $|N_G(\langle g\rangle) : \langle g\rangle|$ is infinite, which show that $\langle g\rangle$ is ascendant in G.

Let g be an arbitrary element of $G \setminus \mathbf{Gru}(G)$ and suppose that the index $|N_{G/K}(\langle gK\rangle) : \langle gK\rangle|$ is infinite. Put $V/K = N_{G/K}(\langle gK\rangle)$ and $X = \langle g, K\rangle$. By its choice X is a normal subgroup of infinite subgroup V. Since g has finite order, a subgroup X is finite. It follows that $C_V(X)$ has finite index in V, in particular, it is infinite. An inclusion $C_V(X) \leqslant C_V(g)$ shows that $C_G(g)$ is infinite and we obtain a contradiction. This contradiction shows that a cyclic subgroup $\langle gK\rangle$ has finite index in its normalizer. It follows that $N_{G/K}(\langle gK\rangle)$ is finite and hence $C_{G/K}(gK)$ is finite.

If $g \in \mathbf{Gru}(G)$, then a cyclic subgroup $\langle g\rangle$ is ascendant in G. Therefore, $\langle gK\rangle$ is ascendant in G/K. In other words, every cyclic subgroup of G/K is either almost self-normalizing or ascendant. □

Now we can describe the general structure of locally finite groups, whose cyclic subgroups are either almost self-normalizing or ascendant.

2. The proofs of the main results

Proof of Theorem A. Put $B = \mathbf{Gru}(G)$ and let Q be a Sylow π-subgroup of G, $\pi = \Pi(G) \setminus \sigma$. Lemma 5 shows that G/B is finite and a group G has a semidirect decomposition: $G = Q \leftthreetimes R$, where R is a Chernikov Sylow σ-subgroup of G.

(v) follows from Lemma 9.

Let $y \in C_R(Q)$, then $Q \leqslant C_G(y)$, in particular, $C_G(y)$ is infinite. As we have seen above in this case $y \in \mathbf{Gru}(G)$.

Suppose that Q is infinite and consider a factor-group G/C, where $C = C_R(Q)$ is a Sylow σ-subgroup of B. We have $G/C = QC/C \leftthreetimes R/C$.

Every cyclic subgroup of QC/C is ascendant in G/C. Let $zC \notin QC/C$ and suppose that $C_{G/C}(zC)$ is infinite. Then $C_{QC/C}(zC) = Z/C$ is infinite. We have $Z = D \times C$, where D is a Sylow π-subgroup of Z. The fact, that Z/C is infinite, implies that D is infinite. For every element $d \in D$ we have $[z, d] \in C$. On the other hand, $D \leqslant Q$ and Q is a normal subgroup of G, therefore $[z, d] \in Q$, that is $[z, d] \in C \cap Q = \langle 1 \rangle$. This shows that $D \leqslant C_G(z)$, in particular $C_G(z)$ is infinite. However it contradicts to (v). This contradiction shows that $C_{G/C}(zC)$ is finite for every element $zC \notin QC/C$. It is not hard to prove that in this case a subgroup $\langle zC \rangle$ has finite index in its normalizer. In other words, every cyclic subgroup of G/C is ascendant or almost self-normalizing. Furthermore, $QC/C = \mathbf{Gru}(G/C)$ and we can use Lemma 7. By Lemma 7 QC/C and hence Q is nilpotent-by-finite.

If Q is finite, then $\mathbf{Gru}(G)$ is a Chernikov subgroup, in particular, it is abelian-by-finite. □

Now we can obtain some additional information about the structure of factor-group $G/\mathbf{Gru}(G)$.

Proof of Corollary A1. Put $B = \mathbf{Gru}(G)$, then $B = P \times R$, where P is a Sylow p-subgroup of B and R is a Sylow p'-subgroup of B. We have $G = Q \leftthreetimes R$, where R is a Chernikov Sylow π-subgroup of G, $\pi = \Pi(G) \setminus \sigma$. Consider a factor-group G/C, where $C = C_R(Q) = R \cap B$ is a Sylow σ-subgroup of B. We have $G/C = QC/C \leftthreetimes R/C$. As in a proof of Theorem A we can show that every cyclic subgroup of QC/C is ascendant in G/C and every cyclic subgroup of G/C, which does not lie in QC/C, is almost self-normalizing. Furthermore, $QC/C = \mathbf{Gru}(G/C)$, hence we can apply Lemma 7 and prove this result. □

Proof of Corollary A2. Put $B = \mathbf{Gru}(G)$. Then $B = P \times R$, where P is a Sylow p-subgroup of B and R is a Sylow p'-subgroup of B. By our conditions R is finite. Consider a factor-group G/R. Suppose that the Sylow p'-subgroup S/R of $\mathbf{Gru}(G/R)$ is non-identity. Let $R \neq xR \in S/R$. Since Sylow p-subgroup of $\mathbf{Gru}(G/R)$ includes BR/R, $x \notin B$. Lemma 10 shows that $C_{G/R}(xR)$ is finite. On the other hand, since $\mathbf{Gru}(G/R)$ is locally nilpotent, $PR/R \leqslant C_{G/R}(xR)$ and we obtain a contradiction. This contradiction shows that $\mathbf{Gru}(G/R)$ is a p-subgroup. If $xR \notin \mathbf{Gru}(G/R)$ and xR is a p'-element, then $x \notin B$ and $C_G(x)$ is finite by Lemma 9. Using Lemma 10 we obtain that $C_{G/R}(xR)$ is finite. The application of Lemma 8 yields that Sylow q-subgroup of $(G/R)/\mathbf{Gru}(G/R)$ is cyclic whenever $q \notin \{2, p\}$, and Sylow 2-subgroup of $(G/R)/\mathbf{Gru}(G/R)$ is cyclic or generalized quaternion group, if $p \neq 2$. Remain to note that the Sylow q-subgroups of F are isomorphic to Sylow q-subgroups of $(G/R)/\mathbf{Gru}(G/R)$, because $\mathbf{Gru}(G/R)$ is a p-subgroup. □

Proof of Corollary A3. Let $p, r \in \mathbf{Sp}(G)$, $B = \mathbf{Gru}(G)$. Then $B = P \times R \times S$, where P is a Sylow p-subgroup of B, R is a Sylow r-subgroup of B, S is a Sylow $\{p, r\}'$-subgroup of B. Consider a factor-group G/D, where $D = R \times S$. Let $D \neq xD$ and assume that xD is a p'-element. In this case $xD \notin PD/D = B/D$. Let $Z/D = C_{B/D}(xD)$. We have $Z = C \times D$, where C is a Sylow p-subgroup of Z. By choice of C we have $[x, C] \leqslant D$. On the other hand, $C \leqslant P$ and since P is G-invariant, then $[x, C] \leqslant P$. Thus $[x, C] \leqslant D \cap P = \langle 1 \rangle$, which shows that $C_{B/D}(xD) = C_B(x)D/D$. The fact that $x \notin B$ together with Lemma 9 shows that $C_B(x)$ is finite, so that $C_{B/D}(xD)$ is finite. Since G/B is finite, $C_{G/D}(xD)$ is likewise finite.

If we suppose now that $\mathbf{Gru}(G/D)$ is a not p-subgroup, then it contains some p'-element yD. Since $\mathbf{Gru}(G/D)$ is locally nilpotent, $B/D \leqslant C_{G/D}(xD)$, in particular, $C_{G/D}(xD)$ is infinite, which contradicts to above proved. This contradiction proves that $\mathbf{Gru}(G/D)$ is a p-subgroup.

The application of Lemma 8 yields that the Sylow q-subgroup of the factor-group $(G/D)/\mathbf{Gru}(G/D)$ is cyclic whenever $q \notin \{2, p\}$, and Sylow 2-subgroup of $(G/D)/\mathbf{Gru}(G/D)$ is cyclic or generalized quaternion group, if $p \neq 2$. Since $\mathbf{Gru}(G/D)$ is a p-subgroup, the Sylow q-subgroups of F are isomorphic to Sylow q-subgroups of $(G/D)/\mathbf{Gru}(G/D)$.

Consider now a factor-group G/PS. Using the above arguments, we obtain that the Sylow p-subgroup of F is cyclic whenever $p \neq 2$, and Sylow p-subgroup of F is cyclic or generalized quaternion group, if $p = 2$. □

Proof of Theorem B. Put $B = \mathbf{B}(G)$. Repeating almost word to word the proof of Theorem A, we will prove assertions (i)-(v). If Q is infinite, then using arguments of a proof of Lemma 7, we obtain that Q is nilpotent-by-finite. Let W be a nilpotent normal subgroup of Q, having finite index. Then $Q = WH$ for some finite subgroup H. An inclusion $H \leqslant \mathbf{B}(G)$ implies that H is subnormal in G. Then WH is a nilpotent subgroup [8, Lemma 4]. A subgroup $R \cap B$ is Chernikov, and being a Baer group, it is central-by-finite [8, Corollary 1 to Lemma 4]. In particular, $R \cap B$ is nilpotent. Moreover, let D be a divisible part of $R \cap B$. Then $D \leqslant \zeta(R \cap B)$. Let T be a finite subgroup such that $R \cap B = TD$. Clearly T is normal in $R \cap B$, and hence in B. Since G/B is finite, $T^G = U$ is also finite. Then $(R \cap B)/U = UD/U$ is divisible. Therefore, using Lemma 6 we obtain (vi). □

Proof of Corollary B1. Let L be a locally nilpotent radical of G. Since $\mathbf{B}(G) \leqslant L$, G/L is finite by Theorem B. In particular, L is infinite.

Suppose first that L is not Chernikov. Then $C_L(g)$ is infinite by Lemma 3. It follows that every cyclic subgroup of L is G-invariant. Then every subgroup of L is G-invariant. In particular, $L = \mathbf{B}(G)$. Furthermore, L is a Dedekind group. Then either L is abelian, or $L = Q \times E \times R$, where Q is a quaternion group, E is elementary abelian 2-subgroup and R is an abelian $2'$-subgroup (see, for example, [14, Theorem 6.1.1]). We note here, that E must be finite. In fact, every non-identity cyclic subgroup of E is G-invariant, and being a subgroup of order 2, lies in the center of G. Hence if we suppose that E is infinite, then $\zeta(G)$ is infinite. But in this case $C_G(g)$ is infinite for each element $g \in G$, so that every cyclic subgroup of G is normal in G, and G must be Dedekind.

Let H be an infinite subgroup of L. If $x \in C_G(H)$, then $H \leqslant C_G(x)$, so that $C_G(x)$ is infinite and $x \in \mathbf{B}(G)$. Hence $C_G(H) \leqslant L$, in particular, $C_G(L) \leqslant L$. The fact, that every subgroup of L is G-invariant, implies that $G/C_G(L)$ is abelian (see, for example, [18, Theorem 1.5.1]).

If Sylow 2-subgroup D of L is infinite, then by above proved L is abelian. If we suppose that $\Omega_1(D)$ is infinite, then using the above arguments, we obtain that G is a Dedekind group. This implies that $\Omega_1(D)$ is finite. Then D is a Chernikov group. Being infinite, D includes a quasicyclic 2-subgroup W. As we have seen above, W is G-invariant and $C_G(W) \leqslant L$. Furthermore, $G/C_G(W)$ is isomorphic to a periodic subgroup of $Aut(W)$. We recall that $Aut(W)$ is isomorphic to the multiplicative group of a ring of integer 2-adic numbers (see, for example, [5, Section 113, Example 3]). Recall also, that a periodic subgroup of the multiplicative group of a ring of integer 2-adic numbers has order 2 (see, for example, [5, Section 128, Example 2]). Thus in this case the factor-group G/L has order 2.

Suppose now that there exists an odd prime p such that Sylow p-subgroup P of L is infinite. By above proved P is abelian. Assume that $\Omega_1(P)$ is infinite. Then again we have an inclusion $C_G(\Omega_1(P)) \leqslant L$. Since every subgroup of $\Omega_1(P)$ is G-invariant, $G/C_G(\Omega_1(P))$ is a cyclic group, whose order divides $p-1$. Hence G/L is a cyclic group, whose order divides $p-1$. Suppose now that $\Omega_1(P)$ is finite. Then P is a Chernikov group. In this case the orders of elements of P are not bounded. Since every subgroup of P is G-invariant, G/P is isomorphic to a periodic subgroup of the multiplicative group of a ring of integer p-adic numbers (see, for example, [18, Theorem 1.5.6]). We recall that a periodic subgroup of the multiplicative group of a ring of integer p-adic numbers is cyclic and its order divides $p-1$ (see, for example, [5, Section 128, Example 2]). Thus in this case the factor-group G/L is cyclic and its order divides $p-1$.

Suppose now that the Sylow p-subgroups of L are finite for all primes p. Since L is not Chernikov subgroup, $\Pi(L)$ is infinite. Let $\sigma = \Pi(G/L)$, then σ is finite by Theorem B. It follows that Sylow σ-subgroup K of L is finite. Theorem B shows that $C_G(g)$ is finite for each $g \notin L$. Lemma 10 implies that $C_{G/K}(gK)$ is finite for each $g \notin L$. Obviously L/K is normal Sylow σ'-subgroup of G/K, so that $G/K = L/K \leftthreetimes S/K$, where S/K is a finite Sylow σ-subgroup of G/K. Since $\Pi(L/K)$ is infinite, we can find in L/K a finite σ'-subgroup R/K such that $R/K \cap C_{G/K}(gK) = \langle 1 \rangle$ for every element $gK \in S/K$ (recall that every subgroup of L/K is G-invariant). Taking into account the fact, that S/K is abelian ($S/K \cong G/L$) and Satz V.8.15 of a book [9], we obtain that every Sylow subgroup of S/K is cyclic, and therefore S/K is cyclic.

Consider now the case when L is a Chernikov subgroup. By Theorem B the Baer radical $\mathbf{B}(G)$ is nilpotent. Corollary 1 shows that every element of $\mathbf{B}(G)$ has infinite centralizers. Then every cyclic subgroup of $\mathbf{B}(G)$ is G-invariant, and therefore every subgroup of $\mathbf{B}(G)$ is G-invariant. As above we can shows that $\mathbf{B}(G)$ includes a centralizer of each its infinite subgroup. Since $G/\mathbf{B}(G)$ is finite, $\mathbf{B}(G)$ is infinite. Being Chernikov, $\mathbf{B}(G)$ includes a quasicyclic p-subgroup for some prime p. Using the above arguments, we obtain that $G/\mathbf{B}(G)$ is cyclic and its order divides $p-1$. Furthermore, if $p=2$, then $\mathbf{B}(G)$ is abelian and $G/\mathbf{B}(G)$ has order 2. □

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Volume **21** (2016). Number 1, pp. 128–143 RESEARCH ARTICLE

Co-intersection graph of submodules of a module

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Abstract. Let M be a unitary left R-module where R is a ring with identity. The co-intersection graph of proper submodules of M, denoted by $\Omega(M)$, is an undirected simple graph whose the vertex set $V(\Omega)$ is a set of all non-trivial submodules of M and there is an edge between two distinct vertices N and K if and only if $N + K \neq M$. In this paper we investigate connections between the graph-theoretic properties of $\Omega(M)$ and some algebraic properties of modules . We characterize all of modules for which the co-intersection graph of submodules is connected. Also the diameter and the girth of $\Omega(M)$ are determined. We study the clique number and the chromatic number of $\Omega(M)$.

1. Introduction

The investigation of the interplay between the algebraic structures-theoretic properties and the graph-theoretic properties has been studied by several authors. As a pioneer, J. Bosak [4] in 1964 defined the graph of semigroups. Inspired by his work, B. Csakany and G. Pollak [7] in 1969, studied the graph of subgroups of a finite group. The Intersection graphs of finite abelian groups studied by B. Zelinka [11] in 1975. Recently, in 2009, the intersection graph of ideals of a ring, was considered by I. Chakrabarty et. al. in [5]. In 2012, on a graph of ideals researched

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by A. Amini et. al. in [2] and Also, intersection graph of submodules of a module introduced by S. Akbari et. al. in [1]. Motivated by previous studies on the intersection graph of algebraic structures, in this paper we define the *co-intersection graph* of submodules of a module. Our main goal is to study the connection between the algebraic properties of a module and the graph theoretic properties of the graph associated to it.

Throughout this paper R is a ring with identity and M is a unitary left R-module. We mean from a non-trivial submodule of M is a nonzero proper left submodule of M.

The co-intersection graph of an R-module M, denoted by $\Omega(M)$, is defined the undirected simple graph with the vertices set $V(\Omega)$ whose vertices are in one to one correspondence with all non-trivial submodules of M and two *distinct* vertices are adjacent if and only if the sum of the corresponding submodules of M is not-equal M.

A submodule N of an R-module M is called *superfluous* or*small* in M (we write $N \ll M$), if for every submodule $X \subseteq M$, the equality $N + X = M$ implies $X = M$, i.e., a submodule N of M is called small in M, if $N + L \neq M$ for every proper submodule L of M. The radical of R-module M written $\mathrm{Rad}(M)$, is sum of all small submodules of M.

A non-zero R-module M is called *hollow*, if every proper submodule of M is small in M.

A non-zero R-module M is called *local*, if has a largest submodule, i.e., a proper submodule which contains all other proper submodules.

An R-module M is said to be A-*projective* if for every epimorphism $g : A \to B$ and homomorphism $f : M \to B$, there exists a homomorphism $h : M \to A$, such that $gh = f$. A module P is projective if P is A-projective for every R-module A. If P is P-projective, then P is also called self-(or quasi-)projective.

A non-zero R-module M is said to be *simple*, if it has no non-trivial submodule. A nonzero R-module M is called *indecomposable*, if it is not a direct sum of two non-zero submodules. For an R-module M, the length of M is the length of composition series of M, denoted by $l_R(M)$.

An R-module M has finite length if $l_R(M) < \infty$, i.e., M is Noetherian and Artinian. The ring of all endomorphisms of an R-module M is denoted by $\mathrm{End}_R(M)$.

Let Ω=($V(\Omega)$,$E(\Omega)$) be a graph with vertex set $V(\Omega)$ and edge set $E(\Omega)$ where an edge is an unordered pair of distinct vertices of Ω. Graph Ω is finite, if $\mathrm{Card}(V(\Omega)) < \infty$, otherwise Ω is infinite. A subgraph of a graph Ω is a graph Γ such that $V(\Gamma) \subseteq V(\Omega)$ and $E(\Gamma) \subseteq E(\Omega)$. By order of Ω, we mean the number of vertices of Ω and we denoted it by $|\Omega|$. If X and Y are two adjacent vertices of Ω, then we write $X \leftrightarrow Y$.

The *degree* of a vertex v in a graph Ω, denoted by $\deg(v)$, is the number of edges incident with v. A vertex v is called *isolated* if $\deg(v) = 0$. Let U and V be two distinct vertices of Ω. An U, V-path is a path with starting vertex U and ending vertex V. For distinct vertices U and V, $d(U, V)$ is the least length of an U, V-path. If Ω has no such a path, then $d(U, V) = \infty$. The *diameter* of Ω, denoted by $\mathrm{diam}(\Omega)$ is the supremum of the set $\{d(U, V)\colon U$ and V are distinct vertices of $\Omega\}$.

A *cycle* in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. We mean of (X, Y, Z) is a cycle of length 3. The *girth* of a graph is the length of its shortest cycle. A graph with no-cycle has infinite girth.

By a *null graph*, we mean a graph with no edges. A graph is said to be *connected* if there is a path between every pair of vertices of the graph.

A *tree* is a connected graph which does not contain a cycle.

A *star graph* is a tree consisting of one vertex adjacent to all the others.

A *complete* graph is a graph in which every pair of distinct vertices are adjacent. The complete graph with n distinct vertices, denoted by K_n.

By a *clique* in a graph Ω, we mean a complete subgraph of Ω and the number of vertices in a largest clique of Ω, is called the clique number of Ω and is denoted by $\omega(\Omega)$.

An *independent set* in a graph is a set of pairwise non-adjacent vertices. An *independence number* of Ω, written $\alpha(\Omega)$, is the maximum size of an independent set.

For a graph Ω, let $\chi(\Omega)$, denote the *chromatic number* of Ω, i.e., the minimum number of colors which can be assigned to the vertices of Ω such that every two adjacent vertices have different colors.

2. Connectivity, diameter and girth of $\Omega(M)$

In this section, we characterize all modules for which the co-intersection graph of submodules is not connected. Also the diameter and the girth of $\Omega(M)$ are determined. Finally we study some modules whose co-intersection graphs are complete.

Theorem 2.1. *Let M be an R-module. Then the graph $\Omega(M)$ is not connected if and only if M is a direct sum of two simple R-modules.*

Proof. Assume that $\Omega(M)$ is not connected. Suppose that Ω_1 and Ω_2 are two components of $\Omega(M)$. Let X and Y be two submodules of M such that $X \in \Omega_1$ and $Y \in \Omega_2$. Since there is no X,Y-path, then $M = X + Y$. Now, if $X \cap Y \neq (0)$, then by

$$X \cap Y + X = X \neq M \quad \text{and} \quad X \cap Y + Y = Y \neq M$$

implies that there is a X,Y-path by $X \cap Y$, to form $X \leftrightarrow X \cap Y \leftrightarrow Y$, a contradiction. Hence, $X \cap Y = (0)$ and $M = X \oplus Y$. Now, we show that X and Y are minimal submodules of M. To see this, let Z be a submodule of M such that $(0) \neq Z \subseteq X$ then $Z + X = X \neq M$. Hence Z and X are adjacent vertices, which implies that $Z \in \Omega_1$. Hence there is no Z,Y-path and by arguing as above, we have $M = Z + Y$, since Z and Y are not adjacent vertices. But since

$$X = X \cap M = X \cap (Z + Y) = Z + X \cap Y = Z$$

by Modularity condition, X is a minimal submodule of M.

A similar argument shows that Y is also a minimal submodule of M and in fact every non-trivial submodule of M is a minimal submodule, which yields that every non-trivial submodule is also maximal. But, minimality of X and Y implies that, they are simple R-modules and since $M = X \oplus Y$, we are done.

Conversely, suppose that $\Omega(M)$ is connected. Let $M = X \oplus Y$, where X and Y are simple R-modules. Let $M_1 = X \times \{0\}$ and $M_2 = \{0\} \times Y$. Then M_1 and M_2 are minimal submodules of M. Moreover, M_1 and M_2 are simple R-modules. But, $M = M_1 \oplus M_2$ and $M_1 \cong M/M_2$ and $M_2 \cong M/M_1$. Consequently, M_1 and M_2 are maximal submodules of M. Therefore, M_1 and M_2 are two maximal and minimal submodules of M. We show that M_1 is an isolated vertex in $\Omega(M)$. To see this, let N be a vertex in $\Omega(M)$, with $N + M_1 \neq M$. Then, maximality of M_1 implies that $N + M_1 = M_1$, and hence $N \subseteq M_1$. Then, minimality of M_1 implies that $M_1 = N$. Hence, M_1 is an isolated vertex in $\Omega(M)$. Thus, $\Omega(M)$ is not connected, a contradiction. This completes the proof. □

Example 2.2. Let $\mathbb{Z}_{pq}$ be a $\mathbb{Z}$-module, such that p and q are two distinct prime numbers. Then $\Omega(\mathbb{Z}_{pq})$ is not connected. Because, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are simple $\mathbb{Z}$-modules and by Theorem 2.1, $\Omega(\mathbb{Z}_p \oplus \mathbb{Z}_q)$ is not connected. Since $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, $\Omega(\mathbb{Z}_{pq})$ is not connected. But, we consider $\mathbb{Z}_{p_1p_2p_3}$ as $\mathbb{Z}$-module, such that p_i is a prime number, for $i = 1, 2, 3$. We know $M_1 = p_1\mathbb{Z}_{p_1p_2p_3}$, $M_2 = p_2\mathbb{Z}_{p_1p_2p_3}$ and $M_3 = p_3\mathbb{Z}_{p_1p_2p_3}$ are the only maximal submodules of $\mathbb{Z}_{p_1p_2p_3}$. Also, $K = p_1p_2\mathbb{Z}_{p_1p_2p_3}$, $L = p_1p_3\mathbb{Z}_{p_1p_2p_3}$ and $N = p_2p_3\mathbb{Z}_{p_1p_2p_3}$ are the other submodules of $\mathbb{Z}_{p_1p_2p_3}$. Hence, $\Omega(\mathbb{Z}_{p_1p_2p_3})$ is connected (see Fig. 1).

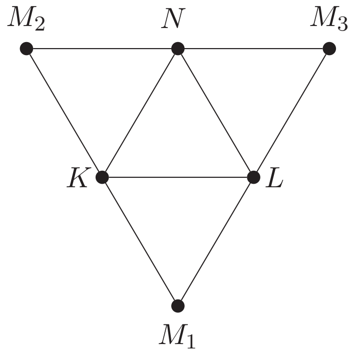

FIGURE 1. $\Omega(\mathbb{Z}_{p_1p_2p_3})$.

Corollary 2.3. *Let M be an R-module. If $\Omega(M)$ is connected, then the following hold:*

(1) *every pair of maximal submodules of M, have non-trivial intersection, and there exists a path between them;*
(2) *every pair of minimal submodules of M, have non-trivial sum, and there is an edge between them.*

Proof. (1) Let M_1 and M_2 be two maximal submodules of M. Clearly, $M_1 \cap M_2 \neq M$. Let $M_1 \cap M_2 = (0)$. Since $M = M_1 + M_2$, $M = M_1 \oplus M_2$. So $M/M_1 \cong M_2$ and $M/M_2 \cong M_1$, hence M_1 and M_2 are two simple R-modules. Now, by Theorem 2.1, $\Omega(M)$ is not connected, which is a contradiction by hypothesis. Hence $M_1 \cap M_2 \neq (0)$, and there exists a path to form $M_1 \leftrightarrow M_1 \cap M_2 \leftrightarrow M_2$ between them.

(2) Let M_1 and M_2 be two minimal submodules of M such that $M = M_1 + M_2$. If $M_1 \cap M_2 = (0)$, then $M = M_1 \oplus M_2$, such that M_1 and M_2 are two simple R-modules, then by Theorem 2.1, $\Omega(M)$ is not connected, which is a contradiction by hypothesis. Also if $M_1 \cap M_2 \neq (0)$, since $(0) \subsetneqq M_1 \cap M_2 \subseteq M_i \subsetneqq M$, for $i = 1, 2$, by minimality of M_1 and M_2 implies that $M_1 \cap M_2 = M_1 = M_2$, which is a contradiction by hypothesis $M_1 \neq M_2$. Therefore, $M \neq M_1 + M_2$, and there is an edge between them. □

Corollary 2.4. *Let M be an R-module. If $|\Omega(M)| \geqslant 2$, and $\Omega(M)$ is not connected, then the following hold:*
(1) *$\Omega(M)$ is a null graph;*
(2) *$l_R(M) = 2$.*

Proof. (1) Suppose that $\Omega(M)$ is not connected, then by Theorem 2.1, $M = M_1 \oplus M_2$, such that M_1 and M_2 are two simple R-modules. So any non-trivial submodule of M is simple. In fact any non-trivial submodule of M is minimal and consequently a maximal submodule. Hence for each two distinct non-trivial submodules K and L of M, we have $M = K + L$, thus there is no edge between two distinct vertices K and L of the graph $\Omega(M)$. Therefore, $\Omega(M)$ is a null graph.

(2) It is clear by Theorem 2.1. □

Theorem 2.5. *Let M be an R-module. If $\Omega(M)$ is connected, then $diam(\Omega(M)) \leqslant 3$.*

Proof. Let A and B be two non-trivial distinct submodules of M. If $A + B \neq M$ then A and B are adjacent vertices of $\Omega(M)$, so $d(A, B) = 1$. Suppose that $A + B = M$. If $A \cap B \neq (0)$, then there exists a path $A \leftrightarrow A \cap B \leftrightarrow B$ of length 2, so $d(A, B) = 2$. Now, if $A \cap B = (0)$, then $M = A \oplus B$, and since $\Omega(M)$ is connected, by Corollary 2.3(1), implies that at least one of A and B should be non-maximal. Assume that B is not maximal. Hence there exists a submodule X of M such that $B \subsetneqq X \subsetneqq M$, and $B + X = X \neq M$. Now, if $A + X \neq M$, then there exists a path $A \leftrightarrow X \leftrightarrow B$ of length 2, then $d(A, B) = 2$. But, if $A + X = M$, then by Modularity condition, $X = X \cap (A \oplus B) = (X \cap A) \oplus B$. Now, if $X \cap A = (0)$, then $X = B$, a contradiction with existence X. Also, if $X \cap A \neq (0)$, then there exists a path $A \leftrightarrow X \cap A \leftrightarrow X \leftrightarrow B$ of length 3, so $d(A, B) \leqslant 3$. Therefore, $\mathrm{diam}(\Omega(M)) \leqslant 3$. □

Remark 2.6. Let R be an integral domain. Then $\Omega(R)$ is a connected graph with $\mathrm{diam}(\Omega(R)) = 2$.

Proof. Suppose that I and J are two ideals of integral domain R. Now, if $I + J \neq R$, then I and J are adjacent vertices, then $d(I, J) = 1$. But, if $I + J = R$, there exist two possible cases $I \cap J = (0)$ or $I \cap J \neq (0)$. The first case implies that $R = I \oplus J$, then there is idempotent e in R, such that $I = Re$ and $J = R(1 - e)$. Since integral domain R has no zero divisor, then $e = 0$ or $e = 1$, thus $I = (0)$ and $J = R$ or $I = R$ and $J = (0)$, this is a contradiction. In second case, since $IJ = I \cap J \neq (0)$

and $I + IJ = I \neq R$, $J + IJ = J \neq R$, then there exists a path to form $I \leftrightarrow IJ \leftrightarrow J$, then $d(I, J) = 2$. Consequently, $\Omega(R)$ is a connected graph and $\operatorname{diam}(\Omega(R)) = 2$. □

Theorem 2.7. *Let M be an R-module, and $\Omega(M)$ a graph, which contains a cycle. Then girth$(\Omega(M)) = 3$.*

Proof. On the contrary, assume that $\operatorname{girth}(\Omega(M)) \geqslant 4$. This implies that every pair of distinct non-trivial submodules M_1 and M_2 of M with $M_1 + M_2 \neq M$ should be comparable. Because, if X and Y are two distinct non-trivial submodules of M with $X + Y \neq M$ such that $X \nsubseteq Y$ and $Y \nsubseteq X$, then $X \subsetneqq X + Y$ and $Y \subsetneqq X + Y$. As $X + Y + X = X + Y \neq M$ and $Y + X + Y = X + Y \neq M$, hence $\Omega(M)$ has a cycle to form $(X, X + Y, Y)$ of length 3, a contradiction. Now, since $\operatorname{girth}(\Omega(M)) \geqslant 4$, $\Omega(M)$ contains a path of length 3, say $A \leftrightarrow B \leftrightarrow C \leftrightarrow D$. Since every two submodules in this path are comparable and every chain of non-trivial submodules of length 2 induces a cycle of length 3 in $\Omega(M)$, the only two possible cases are $A \subseteq B$, $C \subseteq B$ or $B \subseteq A$, $B \subseteq C$, $D \subseteq C$. The first case yields $A + B = B \neq M$, $C + B = B \neq M$, $A + C \subseteq B \neq M$, then (A, B, C) is a cycle of length 3 in $\Omega(M)$, a contradiction. In the second case, we have $B + A = A \neq M$, $B + C = C \neq M$, $B + D \subseteq C \neq M$ and $C + D = C \neq M$, then (B, C, D) is a cycle of length 3 in $\Omega(M)$, which again this is a contradiction. Consequently, $\operatorname{girth}(\Omega(M)) = 3$, and the proof is complete. □

Example 2.8. Since $\mathbb{Z}$ is an integral domain, then by Remark 2.6, $\Omega(\mathbb{Z})$ is a connected graph and contains a cycle $(2\mathbb{Z},\ 4\mathbb{Z},\ 6\mathbb{Z})$, then by Theorem 2.7, $\operatorname{girth}(\Omega(\mathbb{Z})) = 3$.

Theorem 2.9. *Let M be a Noetherian R-module. Then, $\Omega(M)$ is complete if and only if M contains a unique maximal submodule.*

Proof. Suppose that M is a Noetherian R-module, then M has at least one maximal submodule. Moreover every nonzero submodule of M contained in a maximal submodule. Therefore, if M possesses a unique maximal submodule, say U, then U contains every nonzero submodule of M. Assume that K and L are two distinct vertices of $\Omega(M)$. Then $K \subseteq U$ and $L \subseteq U$, hence $K + L \subseteq U \neq M$. Therefore, $\Omega(M)$ is complete.

Conversely, suppose that $\Omega(M)$ is complete. Let X and Y be two distinct maximal submodules of M. Then $X + Y \neq M$, since $X \subseteq X + Y$ and $Y \subseteq X + Y$, by maximality of X and Y, we have $X + Y = X = Y$, a contradiction. Consequently, M contains a unique maximal submodule, and the proof is complete. □

Theorem 2.10. *Let M be an Artinian R-module. Then $\Omega(M)$ is connected if and only if M contains a unique minimal submodule.*

Proof. Suppose that M is an Artinian R-module, then M has at least one minimal submodule. Moreover, every nonzero submodule of M contains a minimal submodule. Therefore, if M possesses a unique minimal submodule, say L, then L contained in every nonzero submodule of M. Assume that A and B are two distinct vertices of $\Omega(M)$. Then $L \subseteq A$ and $L \subseteq B$, hence $L + A = A \neq M$ and $L + B = B \neq M$. Then there is A, B-path, to form $A \leftrightarrow L \leftrightarrow B$. Therefore, $\Omega(M)$ is connected.

Conversely, suppose that $\Omega(M)$ is connected. Let N_1 and N_2 be two distinct minimal submodules of M. Since $(0) \subseteq N_1 \cap N_2 \subseteq N_i \subsetneqq M$, for $i = 1, 2$, by minimality of N_1 and N_2, if $N_1 \cap N_2 \neq (0)$, then $N_1 \cap N_2 = N_1 = N_2$, a contradiction. If $N_1 \cap N_2 = (0)$, then the only two possible cases are $N_1 + N_2 = M$ or $N_1 + N_2 \neq M$. If $N_1 + N_2 = M$, then $M = N_1 \oplus N_2$ such that N_1 and N_2 are two simple R-modules. Then by Theorem 2.1, $\Omega(M)$ is not connected, a contradiction. But, if $N_1 + N_2 \neq M$, $N_1 = N_1/(N_1 \cap N_2) \cong (N_1 + N_2)/N_2$ and N_1 is simple, then N_2 is maximal submodule of M. Also, similarly N_1 is maximal submodule of M. Since, $(0) \subsetneqq N_i \subseteq N_1 + N_2 \subsetneqq M$, for $i = 1, 2$, by maximality of N_1 and N_2, we have $N_1 + N_2 = N_1 = N_2$, which again this is a contradiction. Consequently, M contains a unique minimal submodule, and the proof is complete. □

Proposition 2.11. *Let M be an R-module, with the graph $\Omega(M)$. Then M is a hollow if and only if $\Omega(M)$ is a complete graph.*

Proof. Suppose that K_1 and K_2 are two distinct vertices of $\Omega(M)$. Since M is a hollow R-module, then $K_1 \ll M$ and $K_2 \ll M$. Then by [3, Proposition 5.17(2)] $K_1 + K_2 \ll M$. Thus, $K_1 + K_2 \neq M$. Therefore, $\Omega(M)$ is a complete graph.

Conversely, assume that $\Omega(M)$ is a complete graph. Let N is a non-trivial submodule of M. Since $\Omega(M)$ is complete, N is adjacent to every other vertex of $\Omega(M)$. Then $N + X \neq M$, for every proper submodule X of M, thus $N \ll M$. Hence, M is a hollow R-module. □

Corollary 2.12. *Let M be an R-module and N be a non-trivial submodule of M. If $|\Omega(M)| = n$, then N is a non-trivial small submodule of M if and only if* $\deg(N) = n - 1$, $n \in \mathbb{N}$.

Proof. It is clear. □

Example 2.13. We consider $\mathbb{Z}_{12}$ as $\mathbb{Z}_{12}$- module. The non-trivial submodules of $\mathbb{Z}_{12}$ are $M_1 = \{0, 6\}$, $M_2 = \{0, 4, 8\}$, $M_3 = \{0, 3, 6, 9\}$, $M_4 = \{0, 2, 4, 6, 8, 10\}$ such that $M_1 = \{0, 6\}$ is the only non-trivial small submodule of $\mathbb{Z}_{12}$ and $|\Omega(\mathbb{Z}_{12})| = 4$. Then, by Corollary 2.12, $\deg(M_1) = 3$ (see Fig. 2).

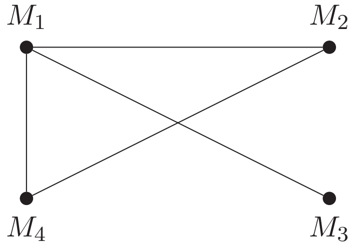

FIGURE 2. $\Omega(\mathbb{Z}_{12})$.

Example 2.14. For every prime number p and for all $n \in \mathbb{N}$ with $n \geqslant 2$, the co-intersection graph of $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, is a complete graph. Because, $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is local, then it is hollow. Hence, by Proposition 2.11, $\Omega(\mathbb{Z}_{p^n})$ is complete. Also, since the number of non-trivial submodules of $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is equal $n-1$. Therefore, $\Omega(\mathbb{Z}_{p^n})$ is a complete graph with $n-1$ vertices, i.e., $\Omega(\mathbb{Z}_{p^n})$=K_{n-1} (see Fig. 3 for $p = 2$ and $n = 5$).

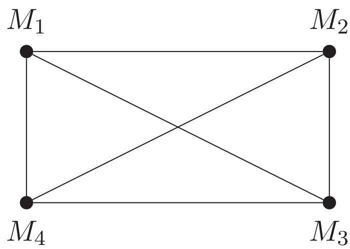

FIGURE 3. $\Omega(\mathbb{Z}_{32})$.

Example 2.15. For every prime number p, the co-intersection graph of $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, is a complete graph. Because, by [10, 41.23, Exercise (6)], for every prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is hollow. Therefore, by Proposition 2.11, $\Omega(\mathbb{Z}_{p^\infty})$ is complete.

Corollary 2.16. *Let M be an R-module. Then $\Omega(M)$ is complete, if one of the following holds:*

(1) *if M is an indecomposable R-module, such that every pair of non-trivial submodules of M, have zero intersection;*
(2) *if M is a local R-module;*
(3) *if M is a self-(or quasi-) projective R-module and $\mathrm{End}_R(M)$ is a local ring.*

Proof. (1) It is clear by definition.

(2) Since local R-modules are hollow, it follows from Proposition 2.11.

(3) Since, M is a self- (or quasi-) projective R-module and $\mathrm{End}_R(M)$ is a local ring, M is hollow by [9, Proposition 2.6]. Then it follows from Proposition 2.11. □

3. Clique number, chromatic number and some finiteness conditions

Let M be an R-module. In this section, we obtain some results on the clique and the chromatic number of $\Omega(M)$. We also study the condition under which the chromatic number of $\Omega(M)$ is finite. Finally, it is proved that $\chi(\Omega(M))$ is finite, provided $\omega(\Omega(M))$ is finite.

Lemma 3.1. *Let M be an R-module and $\omega(\Omega(M)) < \infty$. Then the following hold:*

(1) $l_R(M) < \infty$;
(2) $\omega(\Omega(M)) = 1$ *if and only if either* $|\Omega(M)| = 1$ *or* $|\Omega(M)| \geqslant 2$ *and M is a direct sum of two simple R-modules(i.e., $\Omega(M)$ is null);*
(3) *if* $\omega(\Omega(M)) > 1$, *then the number of minimal submodules of M is finite.*

Proof. (1) Let $M_0 \subset M_1 \subset \cdots \subset M_i \subset M_{i+1} \subset \ldots$, be an infinite strictly increasing sequence of submodules of M. For $i < j$, $M_i + M_j = M_j \neq M$, so similarly for infinite strictly decreasing sequence of submodules of M. Hence, any infinite strictly increasing or decreasing sequence of submodules of M induces a clique in $\Omega(M)$ which contradicts the finiteness $\omega(\Omega)$. This implies that for infinite strictly (increasing and decreasing) sequence of submodules of M, $M_n = M_{n+i}$ for $i = 1, 2, 3, \ldots$. Thus, M should be Noetherian and Artinian. Therefore, $l_R(M) < \infty$.

(2) Suppose that $\omega(\Omega) = 1$ and $|\Omega(M)| \geqslant 2$. This implies that $\Omega(M)$ is not connected. Hence, by Theorem 2.1, M is a direct sum of two simple R-modules.

Conversely, it is clear by Theorem 2.1.

(3) Since $\omega(\Omega) > 1$, by Part (2), M is not a direct sum of two simple R-modules. Then, by Theorem 2.1, $\Omega(M)$ is not connected. Therefore, by Corollary 2.3(2), every pair of minimal submodules of M, have non-trivial sum. Suppose that $\Omega^{\star}(M)$ is a subgraph of $\Omega(M)$ with the vertex set $V^{\star}=\{L \leqslant M | L \text{ is } minimal \text{ submodule } of M\}$. Then $\Omega^{\star}(M)$ is a clique in M, and $\text{Card}(V^{\star}) = \omega(\Omega^{\star}(M)) \leqslant \omega(\Omega(M)) < \infty$. Hence, then the number of minimal submodules of M is finite. □

Remark 3.2. Let M be an R-module with the length $l_R(M)$ and N be a submodule of M and $\triangle(\Omega) = \max\{\deg(v_i) | v_i \in V(\Omega)\}$, then:

(1) Clearly, $\omega(\Omega(N)) \leqslant \omega(\Omega(M))$ and $\omega(\Omega(M/N)) \leqslant \omega(\Omega(M))$. Hence, $\omega(\Omega(M)) < \infty$, implies that $\omega(\Omega(N)) < \infty$ and $\omega(\Omega(M/N)) < \infty$.

(2) Clearly, $l_R(M) \leqslant \omega(\Omega(M)) + 1$. Also if $\Omega(M)$ is a connected graph, then $\omega(\Omega(M)) \leqslant \chi(\Omega(M)) \leqslant \triangle(\Omega) + 1$ by Theorem 10.3(1) of [6, p. 289]. Hence, $\triangle(\Omega) < \infty$, implies that $\chi(\Omega(M)) < \infty$, $\omega(\Omega(M)) < \infty$, and $l_R(M) < \infty$.

Theorem 3.3. *Let M be an R-module and $|\Omega(M)| \geqslant 2$. Then the following conditions are equivalent:*

(1) $\Omega(M)$ *is a star graph;*
(2) $\Omega(M)$ *is a tree;*
(3) $\chi(\Omega(M) = 2$*;*
(4) $l_R(M) = 3$*, M has a unique minimal submodule L such that every non-trivial submodule contains L is maximal submodule of M.*

Proof. (1) ⇒ (2) ⇒ (3) It follows from definitions.

(3) ⇒ (4), Let $\chi(\Omega(M) = 2$. Then $\Omega(M)$ is not null and by Corollary 2.4(1), $\Omega(M)$ is connected. By Remark 3.2(2), $\omega(\Omega(M)) \leqslant \chi(\Omega(M))$, hence $\omega(\Omega(M)) < \infty$ and by Lemma 3.1(1), $l_R(M) < \infty$. Then M is Artinian. Hence M contains a minimal submodule L. We show that L is unique. Let there exist two minimal submodules L_1 and L_2 of M. Then by Corollary 2.3(2), $L_1 + L_2 \neq M$. Since $(L_1 + L_2) + L_1 = L_1 + L_2 \neq M$ and $(L_1 + L_2) + L_2 = L_1 + L_2 \neq M$, then $(L_1, L_1 + L_2, L_2)$ is a cycle of length 3 in $\Omega(M)$, which contradicts $\chi(\Omega(M) = 2$. Hence, L is a unique minimal submodule of M. Suppose that L contained in every non-trivial submodule of M. If K is a non-trivial submodule of M such that $L \subsetneqq K$, we show that K is a maximal submodule of M. Let $L \subsetneqq K \subsetneqq X \subsetneqq M$, since $L + K = K$, $K + X = X$ and $L + X = X$, (L, K, X) is a cycle of

length 3, which is a contradiction. Consequently, K is a maximal submodule contains L, and $(0) \subsetneqq L \subsetneqq K \subsetneqq M$, is a composition series of M with length 3. Therefore, $l_R(M) = 3$.

(4) $\Rightarrow$ (1) Suppose that $l_R(M) = 3$ and M has a unique minimal submodule L, such that every non-trivial submodule L_i, $(i \in I)$ of M contains L, is a maximal submodule of M. Then, $(0) \subsetneqq L \subsetneqq L_i \subsetneqq M$, for all $i \in I$, are composition series of M with length 3, such that $L_i + L = L_i \neq M$ and $L_i + L_j = M$ for $i \neq j$, Therefore, $\Omega(M)$ is a star graph. The proof is complete. □

Lemma 3.4. *Let M be an R-module and N a vertex of the graph $\Omega(M)$. If $\deg(N) < \infty$, then $l_R(M) < \infty$.*

Proof. Suppose that N contains an infinite strictly increasing sequence of submodules $N_0 \subset N_1 \subset N_2 \subset \cdots$. Then $N_i + N = N \neq M$, for all $i \in I$, which contradicts $\deg(N) < \infty$. Similarly, if N contains an infinite strictly decreasing sequence of submodules, which again yields a contradiction. Also assume that M/N contains an infinite strictly increasing sequence of submodules $M_0/N \subset M_1/N \subset M_2/N \subset \cdots$. Since $N \subset M_0 \subset M_1 \subset M_2 \subset \cdots$. Then $M_i + N = M_i \neq M$, for all $i \in I$, a contradiction. Similarly, if M/N contains an infinite strictly decreasing sequence of submodules, which again yields a contradiction. Hence, N and M/N can not contain an infinite strictly increasing or decreasing sequence of submodules. Thus, they are Noetherian R-module as well as Artinian R-module. Hence, M is Noetherian R-module as well as Artinian R-module. Therefore, $l_R(M) < \infty$. □

Lemma 3.5. *Let M be an R-module and N a minimal submodule of M. Assume that L is a non-trivial submodule of M such that $L + N = M$. Then, L is a maximal submodule of M.*

Proof. Let U be a submodule of M such that $(0) \neq L \subseteq U \subsetneqq M$. Then $L + U = U$ and $(0) \subseteq L \cap N \subseteq U \cap N \subseteq N$. Since N is a minimal submodule of M, $L \cap N = N$ or $U \cap N = (0)$. If $L \cap N = N$ then $N \subseteq L$ thus $N + U \subseteq L + U = U$ and $M = L + N \subseteq N + U$, implies that $M = U$, which is a contradiction. Hence $U \cap N = (0)$ and $U = U \cap (L + N) = L + U \cap N = L$ by Modularity condition. Therefore, L is a maximal submodule of M. □

Theorem 3.6. *Let M be an R-module with the graph $\Omega(M)$ and N is a minimal submodule of M, such that $\deg(N) < \infty$. If $\Omega(M)$ is connected, then the following hold:*

(1) *the number of minimal submodules of M is finite;*
(2) $\chi(\Omega(M)) < \infty$*;*
(3) *if* $\mathrm{Rad}(M) \neq (0)$*, then* $\Omega(M)$ *has a vertex which is adjacent to every other vertex.*

Proof. (1) Let $\Sigma = \{K \leqslant M|\ \mathrm{K}\ \textit{be}\ \mathrm{a}\ \textit{minimal}\ \mathrm{submodule}\ \textit{of}\, M\}$. Clearly, $\Sigma \neq \varnothing$. Since $\Omega(M)$ is connected, then by Corollary 2.3(2), for all $K \in \Sigma$, $K + N \neq M$, for N and every $K \in \Sigma$ are minimal submodules of M and adjacent vertices of $\Omega(M)$ with $\deg(N) < \infty$. Hence, $\mathrm{Card}(\Sigma) < \infty$, thus the number of minimal submodules of M is finite.

(2) Let $\{U_i\}_{i\in I}$ be the family of non-trivial submodules which are not adjacent to N. Thus, $U_i + N = M$, for all $i \in I$. Hence by Lemma 3.5, U_i is maximal submodule of M, for all $i \in I$. Since $U_i + U_j = M$, for $i \neq j$, distinct vertices U_i and U_j are not two adjacent vertices of $\Omega(M)$. Hence, one can color all $\{U_i\}_{i\in I}$ by a color, and other vertices, which are a finite number of adjacent vertices N, by a new color to obtain a proper vertex coloring of $\Omega(M)$. Therefore, $\chi(\Omega(M)) < \infty$.

(3) In order to establish this part, consider $\mathrm{Rad}(M)$. Since N is a vertex of $\Omega(M)$ and $\deg(N) < \infty$, by Lemma 3.4, $l_R(M) < \infty$ and thus M is Noetherian. Then by [3, Proposition 10.9], M is finitely generated and by [3, Theorem 10.4(1)], $\mathrm{Rad}(M) \ll M$. Now, we know that $\mathrm{Rad}(M) \neq M$, otherwise, $M = (0)$, which is a contradiction. Hence, $\mathrm{Rad}(M)$ is a non-trivial submodule of M and for each non-trivial submodule K of M, we have $K + \mathrm{Rad}(M) \neq M$. Consequently, $\Omega(M)$ has the vertex $\mathrm{Rad}(M)$, which is adjacent to every other vertex. □

Corollary 3.7. *Let M be an R-module with the graph $\Omega(M)$. Then the following hold:*
(1) *if M has no maximal or no minimal submodule, then $\Omega(M)$ is infinite;*
(2) *if M contains a minimal submodule and every minimal submodule of M has finite degree, then $\Omega(M)$ is either null or finite.*

Proof. (1) If M has no maximal submodule, since $(0) \subsetneqq M$ and (0) is not maximal, there exists a submodule M_0 of M, such that $(0) \subsetneqq M_0 \subsetneqq M$, and M_0 is not maximal, there exists a submodule M_1 of M , such that $(0) \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M$. Consequently, there exists $(0) \subsetneqq M_1 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M$, and for $i < j$, $M_i + M_j = M_j \neq M$. Thus M contains an infinite strictly increasing sequence of submodules. Therefore, $\Omega(M)$ is infinite. If M has no minimal submodule, since $M \supsetneqq (0)$ and M is not minimal, there exists a submodule N_0, such that $M \supsetneqq N_0 \supsetneqq (0)$, and N_0 is not minimal, there exists a submodule N_1, such that $M \supsetneqq N_0 \supsetneqq N_1 \supsetneqq (0)$. Consequently, there exists $M \supsetneqq N_0 \supsetneqq N_1 \supsetneqq \cdots \supsetneqq (0)$, and for $i < j$, $N_i + N_j = N_i \neq M$. Thus M contains an infinite strictly decreasing sequence of submodules. Therefore, $\Omega(M)$ is infinite.

(2) Suppose that $\Omega(M)$ is not null and by contrary assume that $\Omega(M)$ is infinite. Since $\Omega(M)$ is not null, by Corollary 2.4(1), $\Omega(M)$ is connected and since every minimal submodule of M has finite degree, by Lemma 3.4, $l_R(M) < \infty$. Hence, M is Artinian and by Theorem 3.6(1), the number of minimal submodules of M is finite. Since $\Omega(M)$ is infinite, and $V(\Omega(M)) = \{N_i | i \in I\}$, there exists a minimal submodule N which $N \subseteq N_i$, for each $i \in I$, then $N + N_i = N_i \neq M$, for each $i \in I$. This contradicts $\deg(N) < \infty$. Hence, $\Omega(M)$ is a finite graph. □

Theorem 3.8. *Let M be an R-module such that $\Omega(M)$ is infinite and $\omega(\Omega(M)) < \infty$. Then the following hold:*

(1) *the number of maximal submodules of M is infinite;*
(2) *the number of non-maximal submodules of M is finite;*
(3) $\chi(\Omega(M)) < \infty$*;*
(4) $\alpha(\Omega(M)) = \infty$*.*

Proof. (1) On the contrary, assume that the number of maximal submodules of M is finite. Since $\Omega(M)$ is infinite, $\Omega(M)$ has an infinite clique which contradicts the finiteness of $\omega(\Omega(M))$.

(2) Suppose that $\omega(\Omega(M)) < \infty$, then by Lemma 3.1(1), $l_R(M) < \infty$. Also for each $U \leqslant M$, $l_R(M/U) \leqslant l_R(M)$, $l_R(M/U) < \infty$. We claim that the number of non-maximal submodules of M is finite. To see this, assume that

$$T_n = \{X \lneqq M | l_R(M/X) = n\} \quad \text{and} \quad n_0 = \max\{n | \operatorname{Card}(T_n) = \infty\}.$$

Since $T_1 = \{X \lneqq M | l_R(M/X) = 1\}$, then M/X is a simple R-module, thus X is a maximal submodule of M. Hence, $T_1 = \{X \lneqq M | X \leqslant^{\max} M\}$. By Part(1), T_1 is infinite, then there exists n_0 and $n_0 \geqslant 1$. Since $l_R(M/X) < l_R(M)$ and by Remark 3.2(2), $l_R(M) \leqslant \omega(\Omega(M))+1$. Clearly, $1 \leqslant n_0 \leqslant \omega(\Omega(M))$. However, since $l_R(M) < \infty$, Theorem 5 of [8, p. 19], implies that every proper submodule of length n_0 is contained in a submodule of length $n_0 + 1$. Moreover, by the definition of n_0, the number of submodules of length $n_0 + 1$ is finite. Hence there exists a submodule N of M such that $l_R(M/N) = n_0 + 1$ and N contains an infinite number of submodules $\{N_i\}_{i \in I}$ of M, where $l_R(M/N_i)) = n_0$, for all $i \in I$. Now,

$\omega(\Omega(M)) < \infty$ implies that, there exist submodules K and L of M, with $K, L \subseteq N$ and $l_R(M/K) = l_R(M/L) = n_0$, such that $K + L = M$. Since $K \cap L \subseteq N$ and $M/(K \cap L) \cong M/K \oplus M/L$,

$$\begin{aligned} n_0 + 1 = l_R(M/N) &\geqslant l_R(M/(K \cap L)) = l_R(M/K \oplus M/L) \\ &= l_R(M/K) + l_R(M/L) = 2n_0. \end{aligned}$$

Then $n_0 = 1$. Thus, only T_1 is infinite. Consequently, the number of non-maximal submodules of M is finite.

(3) In order to establish this Part, if $\omega(\Omega(M)) = 1$, there is nothing to prove. Let $\omega(\Omega(M)) > 1$. Since, the sum of two distinct maximal submodules is equal M, they are not two adjacent vertices of $\Omega(M)$. Now, by Part (1), the number of maximal submodules of M is infinite. Hence, one can color all maximal submodules by a color, and other vertices, which are finite number, by a new color, to obtain a proper vertex coloring of $\Omega(M)$. Therefore, $\chi(\Omega(M)) < \infty$.

(4) Suppose that $S = \{N \leqslant M | N \leqslant^{\max} M\}$. Since each two elements of S are not two adjacent vertices of $\Omega(M)$, then S is an independent set of the graph $\Omega(M)$. By Part (1), $\text{Card}(S) = \infty$. Hence, $\alpha(\Omega(M)) = \infty$. □

4. Conclusions and future work

In this work we investigated many fundamental properties of the graph $\Omega(M)$ such as connectivity, the diameter, the girth, the clique number, the chromatic number and obtain some interesting results with finiteness conditions on them. However, in future work, shall search the supplement of this graph and research on deeper properties of them.

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RESEARCH ARTICLE

Parafunctions of triangular matrices and m-ary partitions of numbers

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Abstract. Using the machinavy of paradeterminants and parapermanents developed in [2] we get new relations for some number-theoretical functions natural argument that were studied in [3].

Introduction

Partition polynomials together with corresponding linear recurrent equations appear in different areas of mathematics. Therefore, it is important to develop the general theory of partition polynomials, which would unify the results obtained in this area of mathematics. One of these general approaches to studying partition polynomials and its corresponding linear recurrent equations is their study with the help of triangular matrices (tables) machinery [1, 2].

The present paper continues the study of properties and interrelations of three number-theoretical functions of a natural argument, which was started in [3]. These functions are the functions $b_m(n), \xi_m(n)$, ([3], p.68-69.) respectively generalizing the number $p(n)$ of unordered partitions of a positive integer n into positive integer summands and the sum of divisors of a positive integer $\sigma(n)$, as well as the function $d_m(n)$, which for $m = 2$ equals $(-1)^{t(n)}$, where $t(n)$ is the n-th term of the Prouhet-Thue-Morse

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sequence [4]. In [3] the authors apply the methods of combinatorial analysis (generatrix method) and linear algebra. As for us, in order to study these functions, we use the general theory of partition polynomials developed with the help of triangular matrix calculus machinery developed by the first autor. At that, we received new relations between these functions and all the proofs are considerably simplified. As the result we get a seweral new relations between functions $b_m(n), \xi_m(n)$ and $d_m(n)$ and express them via paradeterminants and parapermanents.

1. Preliminaries

This section includes some necessary notions and their properties, which will be needed in the next section.

1.1. Some notions and results concerning triangular matrices

In this section we provide basic notions and results about paradeterminants and parapermanents that will be used for the proving of the main results of the paper.

Let K be some number field.

Definition 1 ([1, 2])**.** A triangular table

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1}$$

of numbers from a number field K is called *a triangular matrix*, an element a_{11} is an upper element of this triangular matrix, and a number n is its order.

Definition 2 ([1, 2])**.** If A is the triangular matrix (1), then its paradeterminant and parapermanent are the following numbers, respectively:

$$\operatorname{ddet}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\}, \tag{2}$$

$$\operatorname{pper}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\},$$

where the summation is over the set of natural solutions of the equality $p_1 + \ldots + p_r = n$ and

$$b_{ij} = \{a_{ij}\} = \prod_{k=j}^{i} a_{ik}, \quad 1 \leqslant j \leqslant i \leqslant n.$$

For a parapermanent and paradeterminant of a matrix we will use notations shown in (15) and (16) respectively.

Definition 3 ([1,2]). To each element a_{ij} of the given triangular matrix (1) we correspond a triangular matrix with this element in the bottom left corner, which we call *a corner* of the given triangular matrix and denote by $R_{ij}(A)$.

It is obvious that the corner $R_{ij}(A)$ is a triangular matrix of order $(i - j + 1)$. The corner $R_{ij}(A)$ includes only those elements a_{rs} of the triangular matrix (1), the indices of which satisfy the relations $j \leqslant s \leqslant r \leqslant i$.

Sometimes we extend the range of indeces in (1) from 1,…,n to 0,1,…,n+1 and agree that

$$\begin{aligned} \operatorname{ddet}(R_{01}(A)) &= \operatorname{ddet}(R_{n,n+1}(A)) = \operatorname{pper}(R_{01}(A)) \\ &= \operatorname{pper}(R_{n,n+1}(A)) = 1. \end{aligned} \tag{3}$$

When finding values of the paradeterminant and the parapermanent of triangular matrices, it is convenient to use *algebraic complements.*

Definition 4 ([1,2]). *Algebraic complements* D_{ij}, P_{ij} to a factorial product $\{a_{ij}\}$ of a key element a_{ij} of the matrix (1) are, respectively, numbers

$$D_{ij} = (-1)^{i+j} \cdot \operatorname{ddet}(R_{j-1,1}) \cdot \operatorname{ddet}(R_{n,i+1}), \tag{4}$$

$$P_{ij} = \operatorname{pper}(R_{j-1,1}) \cdot \operatorname{pper}(R_{n,i+1}), \tag{5}$$

where $R_{j-1,1}$ and $R_{n,i+1}$ are corners of the triangular matrix (1).

Theorem 1 ([1,2]). *If A is the triangular matrix (1), then the parafunctions of this matrix can be decomposed by the elements of the last row. With that, the following equalities hold:*

$$\operatorname{ddet}(A) = \sum_{s=1}^{n} \{a_{ns}\} D_{ns} = \sum_{s=1}^{n} (-1)^{n+s} \{a_{ns}\} \cdot \operatorname{ddet}(R_{s-1,1}), \tag{6}$$

$$\operatorname{pper}(A) = \sum_{s=1}^{n} \{a_{ns}\} P_{ns} = \sum_{s=1}^{n} \{a_{ns}\} \cdot \operatorname{pper}(R_{s-1,1}), \tag{7}$$

where

$$b_{ij} = \{a_{ij}\} = \prod_{k=j}^{i} a_{ik}, \quad 1 \leqslant j \leqslant i \leqslant n.$$

Theorem 2 (Relation between parapermanent and paradeterminant [1, 2])**.** *If A is the triangular matrix* (1)*, then the following relation holds*

$$\operatorname{pper}(a_{ij})_{1\leqslant j\leqslant i\leqslant n} = \operatorname{ddet}\left((-1)^{\delta_{ij}+1} a_{ij}\right)_{1\leqslant j\leqslant i\leqslant n}. \tag{8}$$

Corollary 1. *For any triangular matrix $(b_{ij})_{1\leqslant j\leqslant i\leqslant n}$, the following equality holds*

$$\operatorname{ddet}(b_{ij})_{1\leqslant j\leqslant i\leqslant n} = \operatorname{pper}((-1)^{\delta_{ij}+1} b_{ij})_{1\leqslant j\leqslant i\leqslant n}.$$

Theorem 3 ([5])**.** *The following is true*

$$\begin{vmatrix} a_{11} & a_1 & 0 & \dots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_2 & \dots & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 & 0 & 0 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \dots & a_{n-2,n-2} & a_{n-2} & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1} \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix}$$

$$= \left\langle \begin{matrix} a_{11} & & & & & & \\ a_1 \frac{a_{21}}{a_{22}} & a_{22} & & & & & \\ a_1 \frac{a_{31}}{a_{32}} & a_2 \frac{a_{32}}{a_{33}} & a_{33} & & & & \\ \vdots & \dots & \dots & \ddots & & & \\ a_1 \frac{a_{n-2,1}}{a_{n-2,2}} & a_2 \frac{a_{n-2,2}}{a_{n-2,3}} & a_3 \frac{a_{n-2,3}}{a_{n-2,4}} & \dots & a_{n-2,n-2} & & \\ a_1 \frac{a_{n-1,1}}{a_{n-1,2}} & a_2 \frac{a_{n-1,2}}{a_{n-1,3}} & a_3 \frac{a_{n-1,3}}{a_{n-1,4}} & \dots & a_{n-2} \frac{a_{n-1,n-2}}{a_{n-1,n-1}} & a_{n-1,n-1} & \\ a_1 \frac{a_{n1}}{a_{n2}} & a_2 \frac{a_{n2}}{a_{n3}} & a_3 \frac{a_{n3}}{a_{n4}} & \dots & a_{n-2} \frac{a_{n,n-2}}{a_{n,n-1}} & a_{n-1} \frac{a_{n,n-1}}{a_{nn}} & a_{nn} \end{matrix} \right\rangle \tag{9}$$

1.2. Some data from the general theory of partition polynomials

We will need also the following three results.

Theorem 4 ([2], Theorem 2.5.3). *The following three equalities are equipotent:*

$$A_n = \left\langle \begin{array}{ccccc} x(1) & & & & \\ \frac{x(2)}{x(1)} & x(1) & & & \\ \vdots & \dots & \ddots & & \\ \cdot\frac{x(n-1)}{x(n-2)} & \frac{x(n-2)}{x(n-3)} & \dots & x(1) & \\ \frac{x(n)}{x(n-1)} & \frac{x(n-1)}{x(n-2)} & \dots & \frac{x(2)}{x(1)} & x(1) \end{array} \right\rangle,$$

$$A_n = x_1 A_{n-1} - x_2 A_{n-2} + \ldots + (-1)^{n-1} x_n A_0, \quad A_0 = 1,$$

$$A_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-k} \frac{k!}{\lambda_1! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} \cdot \ldots x_n^{\lambda_n}, \; k = \lambda_1 + \ldots + \lambda_n.$$

Theorem 5 ([7]). *Let polynomials* $y_n(x_1, x_2, \ldots, x_n), n = 0, 1, \ldots,$ *satisfy the recurrence relation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + \ldots + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1} a_n x_n y_0, \quad (10)$$

where $y_0 = 1$. *Then the relations*

$$y_n = \text{ddet} \begin{pmatrix} a_1 x_1 & & & \\ a_2 \frac{x_2}{x_1} & x_1 & & \\ \vdots & \dots & \ddots & \\ a_n \frac{x_n}{x_{n-1}} & \dots & \frac{x_2}{x_1} & x_1 \end{pmatrix}, \quad (11)$$

$$y_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-k} \Big(\sum_{i=1}^{n} \lambda_i a_i \Big) \frac{(k-1)!}{\lambda_1! \lambda_2! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n}, \quad (12)$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$, *hold.*

Theorem 6 ([2], Theorem 3.6.1). *The following formulae of inversion of partition polynomials written as parafunctions of triangular matrices are valid:*

1) $$b_i = \left\langle \tau_{sr} \frac{a_{s-r+1}}{a_{s-r}} \right\rangle_{1 \leqslant r \leqslant s \leqslant i}, \tag{13}$$

$$a_i = \left\langle \tau^{-1}_{s,s-r+1} \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leqslant r \leqslant s \leqslant i}, \quad i = 1, 2, \ldots; \tag{14}$$

2) $$b_i = \left[\tau_{sr} \frac{a_{s-r+1}}{a_{s-r}} \right]_{1 \leqslant r \leqslant s \leqslant i},$$

$$a_i = (-1)^{i-1} \left\langle \tau^{-1}_{s,s-r+1} \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leqslant r \leqslant s \leqslant i}, \quad i = 1, 2, \ldots,$$

where a_i, b_i are arbitrary real variables, τ_{rs} are rational numbers.

2. Parafunctions of triangular matrices and *m*-ary partitions of numbers

Our first theorem show how functions $b_m(n)$, $\xi_m(n)$, $d_m(n)$, studied in [3] can be expressed via paradeterminant and parapermanent.

Theorem 7. *The following equalities hold:*

$$b_m(n) = \left[\begin{array}{ccccc} \xi_m(1) & & & & \\ \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{2}\xi_m(1) & & & \\ \vdots & \ldots & \ddots & & \\ \frac{\xi_m(n-1)}{\xi_m(n-2)} & \frac{\xi_m(n-2)}{\xi_m(n-3)} & \ldots & \frac{1}{n-1}\xi_m(1) & \\ \frac{\xi_m(n)}{\xi_m(n-1)} & \frac{\xi_m(n-1)}{\xi_m(n-2)} & \ldots & \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{n}\xi_m(1) \end{array} \right], \tag{15}$$

$$d_m(n) = (-1)^n \left\langle \begin{array}{ccccc} \xi_m(1) & & & & \\ \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{2}\xi_m(1) & & & \\ \vdots & \ldots & \ddots & & \\ \frac{\xi_m(n-1)}{\xi_m(n-2)} & \frac{\xi_m(n-2)}{\xi_m(n-3)} & \ldots & \frac{1}{n-1}\xi_m(1) & \\ \frac{\xi_m(n)}{\xi_m(n-1)} & \frac{\xi_m(n-1)}{\xi_m(n-2)} & \ldots & \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{n}\xi_m(1) \end{array} \right\rangle, \tag{16}$$

$$\xi_m(n) = (-1)^{n-1} \left\langle \begin{array}{ccccc} b_m(1) & & & & \\ 2 \cdot \frac{b_m(2)}{b_m(1)} & b_m(1) & & & \\ \vdots & \ldots & \ddots & & \\ (n-1) \cdot \frac{b_m(n-1)}{b_m(n-2)} & \frac{b_m(n-2)}{b_m(n-3)} & \ldots & b_m(1) & \\ n \cdot \frac{b_m(n)}{b_m(n-1)} & \frac{b_m(n-1)}{b_m(n-2)} & \ldots & \frac{b_m(2)}{b_m(1)} & b_m(1) \end{array} \right\rangle, \tag{17}$$

$$\xi_m(n) = (-1)^n \left\langle \begin{array}{ccccc} d_m(1) & & & & \\ 2\cdot\frac{d_m(2)}{d_m(1)} & d_m(1) & & & \\ \vdots & \dots & \ddots & & \\ (n-1)\cdot\frac{d_m(n-1)}{d_m(n-2)} & \frac{d_m(n-2)}{d_m(n-3)} & \dots & d_m(1) & \\ n\cdot\frac{d_m(n)}{d_m(n-1)} & \frac{d_m(n-1)}{d_m(n-2)} & \dots & \frac{d_m(2)}{d_m(1)} & d_m(1) \end{array} \right\rangle, \tag{18}$$

$$b_m(n) = (-1)^n \left\langle \begin{array}{ccccc} d_m(1) & & & & \\ \frac{d_m(2)}{d_m(1)} & d_m(1) & & & \\ \vdots & \dots & \ddots & & \\ \frac{d_m(n-1)}{d_m(n-2)} & \frac{d_m(n-2)}{d_m(n-3)} & \dots & d_m(1) & \\ \frac{d_m(n)}{d_m(n-1)} & \frac{d_m(n-1)}{d_m(n-2)} & \dots & \frac{d_m(2)}{d_m(1)} & d_m(1) \end{array} \right\rangle, \tag{19}$$

$$d_m(n) = (-1)^n \left\langle \begin{array}{ccccc} b_m(1) & & & & \\ \frac{b_m(2)}{b_m(1)} & b_m(1) & & & \\ \vdots & \dots & \ddots & & \\ \cdot\frac{b_m(n-1)}{b_m(n-2)} & \frac{b_m(n-2)}{b_m(n-3)} & \dots & b_m(1) & \\ \frac{b_m(n)}{b_m(n-1)} & \frac{b_m(n-1)}{b_m(n-2)} & \dots & \frac{b_m(2)}{b_m(1)} & b_m(1) \end{array} \right\rangle. \tag{20}$$

Proof. Relations (15), (16) follows from recurrent relations of Theorem 3 (from [3], p. 70). Indeed, each of these equalities is a result of expansion of the paradeterminants on the right side of (15) or (16) by elements of the last raw. Relations (17), (18) can be obtained by inversion of (15), (16) using Theorem 6; (19), (20) follows directly from Theorem 2 in [3], p. 69, and the above Theorem 3 on the relation between paradeterminants and determinants. □

The following theorem gives recurrent relations between functions $b_m(n), \xi_m(n), d_m(n)$.

Theorem 8. *The following equalities hold:*

$$\xi_m(n) = -\Big(b_m(1)\xi_m(n-1) + b_m(2)\xi_m(n-2) + \ldots + b_m(n-1)\xi_m(1) - nb_m(n)\xi_m(0)\Big), \tag{21}$$

$$\xi_m(n) = -\Big(d_m(1)\xi_m(n-1) + d_m(2)\xi_m(n-2) + \ldots + d_m(n-1)\xi_m(1) + nd_m(n)\xi_m(0)\Big), \tag{22}$$

$$b_m(n) = -\Big(d_m(1)b_m(n-1) + d_m(2)b_m(n-2) + \ldots + d_m(n-1)b_m(1) + d_m(n)b_m(0)\Big), \tag{23}$$

$$d_m(n) = -\Big(b_m(1)d_m(n-1) + b_m(2)d_m(n-2) + \ldots + b_m(n-1)d_m(1) + b_m(n)d_m(0)\Big), \tag{24}$$

where $b_m(0) = d_m(0) = \xi_m(0) = 1$.

Proof. To prove (21) multiply both sides of (17) by $(-1)^{n-1}$ and expand paradeterminant on the right side of the obtained equality by elements of the last row. As the result, we get

$$\begin{aligned}(-1)^{n-1}\xi_m(n) = b_m(1)(-1)^{n-2}\xi_m(n-1) - b_m(2)(-1)^{n-3}\xi_m(n-2)\\ + \ldots + (-1)^{n-2}b_m(n-1)(-1)^0\xi_m(n-(n-1))\\ + (-1)^{n-1}b_m(n)(-1)^{-1}\xi_m(n-n)\end{aligned}$$

and hence (21). Similarly, one can prove the relation (22) using (18). Relations (23), (24) can be obtained from (19) and (20) respectively. Let us prove, for example, (23). Multiply both sides of (19) by $(-1)^n$ and expand paradeterminant on the right side of obtained equality by elements of the last row. Then

$$\begin{aligned}(-1)^n b_m(n) = d_m(1)(-1)^{n-1}b_m(n-1) - d_m(2)(-1)^{n-2}b_m(n-2)\\ + \ldots + (-1)^{n-2}d_m(n-1)(-1)^{1)}b_m(n-(n-1))\\ + (-1)^{n-1}d_m(n)(-1)^0 b_m(n-n),\end{aligned}$$

and the required relation follow immediately. □

In the next theorem, we describe partition polynomials as defined in [6] presenting m-ary numbers $b_m(n), \xi_m(n), d_m(n)$.

Theorem 9. *The following equalities hold:*

$$d_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{\xi_m^{\lambda_1}(1)\cdot\ldots\cdot\xi_m^{\lambda_n}}{\lambda_1!\cdot\ldots\cdot\lambda_n! 1^{\lambda_1}\cdot\ldots\cdot n^{\lambda_n}}, \tag{25}$$

$$\xi_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{k-1} \frac{n(k-1)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!}\cdot b_m^{\lambda_1}(1)\cdot\ldots\cdot b_m^{\lambda_n}(n), \tag{26}$$

$$\xi_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{n(k-1)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!}\cdot d_m^{\lambda_1}(1)\cdot\ldots\cdot d_m^{\lambda_n}(n), \tag{27}$$

$$b_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{k!}{\lambda_1! \cdot \ldots \cdot \lambda_n!} \cdot d_m^{\lambda_1}(1) \cdot \ldots \cdot d_m^{\lambda_n}(n), \quad (28)$$

$$d_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{k!}{\lambda_1! \cdot \ldots \cdot \lambda_n!} \cdot b_m^{\lambda_1}(1) \cdot \ldots \cdot b_m^{\lambda_n}(n), \quad (29)$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof. Partition polynomial corresponding to parapermanent (15), were described by Kachi and Tzermias [3, Theorem 1, p. 68]. Paradeterminant of the same matrix corresponds to the partition polynomial that differs from the previous one only by sign $(-1)^{n-k}$. Thus (25) holds. The relations for partition polynomials (26), (27) and (28), (29) follow directly from theorems 5 and 4 respectively. □

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On nilpotent Lie algebras of derivations with large center

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Abstract. Let $\mathbb{K}$ be a field of characteristic zero and A an associative commutative $\mathbb{K}$-algebra that is an integral domain. Denote by R the quotient field of A and by $W(A) = R\operatorname{Der} A$ the Lie algebra of derivations on R that are products of elements of R and derivations on A. Nilpotent Lie subalgebras of the Lie algebra $W(A)$ of rank n over R with the center of rank $n-1$ are studied. It is proved that such a Lie algebra L is isomorphic to a subalgebra of the Lie algebra $u_n(F)$ of triangular polynomial derivations where F is the field of constants for L.

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and A be an associative commutative algebra over $\mathbb{K}$ with identity, without zero divisors. A $\mathbb{K}$-linear mapping $D : A \longrightarrow A$ is called $\mathbb{K}$-*derivation* of A if D satisfies the Leibniz's rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. The set $\operatorname{Der} A$ of all $\mathbb{K}$-derivations on A forms a Lie algebra over $\mathbb{K}$ with respect to the operation $[D_1, D_2] = D_1D_2 - D_2D_1,\ D_1, D_2 \in \operatorname{Der} A$. Denote by R the quotient field of the integral domain A. Each derivation D of A is uniquely extended to a derivation of R by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$. Denote by $\operatorname{Der} R$ the Lie algebra (over $\mathbb{K}$) of all $\mathbb{K}$-derivations on R.

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Define the mapping $rD : R \longrightarrow R$ by $(rD)(s) = r \cdot D(s)$ for all $r, s \in R$. It is easy to see that rD is a derivation of R. The R-linear hull of the set $\{rD | r \in R, D \in \operatorname{Der} A\}$ forms the vector space $R \operatorname{Der} A$ over R, which is a Lie subalgebra of $\operatorname{Der} R$. Observe that $R \operatorname{Der} A$ is a Lie algebra over $\mathbb{K}$ but not always over R, and $\operatorname{Der} A$ is embedded in a natural way into $R \operatorname{Der} A$. Many authors study the Lie algebra of derivations $\operatorname{Der} A$ and its subalgebras, see [2–7].

This paper deals with nilpotent Lie subalgebras of the Lie algebra $R \operatorname{Der} A$. Let L be a Lie subalgebra of $R \operatorname{Der} A$. The subfield $F = F(L)$ of R consisting of all $a \in R$ such that $D(a) = 0$ for all $D \in L$ is called *the field of constants for* L. Let us denote by RL the R-linear hull of L and, analogously, by FL the linear hull of L over its field of constants $F = F(L)$. The *rank of* L *over* R is defined as the dimension of the vector space RL over R, i.e. $\operatorname{rank}_R L = \dim_R RL$.

The main results of the paper:

- (Theorem 1) If L is a nilpotent Lie subalgebra of the Lie algebra $R \operatorname{Der} A$ of rank n over R such that its center $Z(L)$ is of rank $n-1$ over R and F is the field of constants for L, then the Lie algebra FL is contained in the Lie subalgebra of $R \operatorname{Der} A$ of the form

$$F\left\langle D_1, aD_1, \frac{a^2}{2!}D_1, \ldots, \frac{a^s}{s!}D_1, D_2, aD_2, \ldots, \frac{a^s}{s!}D_2, \ldots, \right.$$
$$\left. D_{n-1}, \ldots, \frac{a^s}{s!}D_{n-1}, D_n \right\rangle,$$

  where $D_1, D_2, \ldots, D_n \in FL$ are such that $[D_i, D_j] = 0$, $i, j = 1, \ldots, n$, and $a \in R$ is such that $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$ and $D_n(a) = 1$.
- (Theorem 2) The Lie algebra FL is isomorphic to some subalgebra of the Lie algebra $u_n(F)$ of triangular polynomial derivations.

Recall that the Lie algebra $u_n(\mathbb{K})$ of triangular polynomial derivations consists of all derivations of the form

$$D = f_1(x_2, \ldots, x_n)\frac{\partial}{\partial x_1} + f_2(x_3, \ldots, x_n)\frac{\partial}{\partial x_2} + \cdots$$
$$+ f_{n-1}(x_n)\frac{\partial}{\partial x_{n-1}} + f_n\frac{\partial}{\partial x_n},$$

where $f_i \in \mathbb{K}[x_{i+1}, \ldots, x_n], i = 1, \ldots, n-1$, and $f_n \in \mathbb{K}$. It is a Lie subalgebra of the Lie algebra $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations on the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$. Such subalgebras are studied in [2, 3]. As Lie algebras, they are locally nilpotent but not nilpotent.

We use the standard notations. The Lie algebra $R\operatorname{Der} A$ is denoted by $W(A)$, as in [7]. The linear hull of elements $D_1, D_2, \dots, D_n$ over the field $\mathbb{K}$ is denoted by $\mathbb{K}\langle D_1, D_2, \dots, D_n\rangle$. If L is a Lie subalgebra of a Lie algebra M, then the normalizer of L in M is the set $N_M(L) = \{x \in M \mid [x, L] \subseteq L\}$. Obviously, $N_M(L)$ is the largest subalgebra of M in which L is an ideal.

1. Nilpotent Lie subalgebras of $R\operatorname{Der} A$ with the center of large rank

We use Lemmas 1-5 proved in [7].

Lemma 1 ([7, Lemma 1]). *Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then*
(a) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.
(b) *If $a, b \in \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2$, then $[aD_1, bD_2] = ab[D_1, D_2]$.*

Lemma 2 ([7, Lemma 2]). *Let L be a nonzero Lie subalgebra of the Lie algebra $W(A)$, and F be the field of constants for L. Then FL is a Lie algebra over F, and if L is abelian, nilpotent or solvable, then the Lie algebra FL has the same property.*

Lemma 3 ([7, Theorem 1]). *Let L be a nilpotent Lie subalgebra of finite rank over R of the Lie algebra $W(A)$, and F be the field of constants for L. Then FL is finite dimensional over F.*

Lemma 4 ([7, Lemma 4]). *Let L be a Lie subalgebra of the Lie algebra $W(A)$, and I be an ideal of L. Then the vector space $RI \cap L$ over $\mathbb{K}$ is an ideal of L.*

Lemma 5 ([7, Lemma 5]). *Let L be a nilpotent Lie subalgebra of rank $n > 0$ over R of the Lie algebra $W(A)$. Then:*
(a) *L contains a series of ideals*

$$0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = L$$

such that $\operatorname{rank}_R I_k = k$ for each $k = 1, \dots, n$.
(b) *L possesses a basis $D_1, \dots, D_n$ over R such that $I_k = (RD_1 + \cdots + RD_k) \cap L$, $k = 1, \dots, n$, and $[L, D_k] \subset I_{k-1}$.*
(c) $\dim_F FL/FI_{n-1} = 1$.

Lemma 6. *Let L be a nilpotent Lie subalgebra of the Lie algebra $W(A)$, and F be the field of constants for L. If L is of rank $n > 0$ over R and its center $Z(L)$ is of rank $n - 1$ over R, then L contains an abelian ideal I such that $\dim_F FL/FI = 1$.*

Proof. Since the center $Z(L)$ is of rank $n-1$ over R, we can take linearly independent over R elements $D_1, D_2, \dots, D_{n-1} \in Z(L)$. Let us consider

$$I = RZ(L) \cap L = (RD_1 + \cdots + RD_{n-1}) \cap L.$$

In view of Lemma 4, I is an ideal of the Lie algebra L. Let us show that I is an abelian ideal.

We first show that for an arbitrary element $D = r_1D_1 + r_2D_2 + \cdots + r_{n-1}D_{n-1} \in I$, its coefficients $r_1, r_2, \dots, r_{n-1} \in \bigcap_{i=1}^{n-1} \operatorname{Ker} D_i$. For each $D_i \in Z(L)$, $i = 1, \dots, n-1$, let us consider

$$[D_i, D] = [D_i, r_1D_1 + r_2D_2 + \cdots + r_{n-1}D_{n-1}] = \sum_{j=1}^{n-1} [D_i, r_jD_j].$$

By Lemma 1, $[D_i, r_jD_j] = r_j[D_i, D_j] + D_i(r_j)D_j = D_i(r_j)D_j$. Since $D_i \in Z(L)$, we get

$$[D_i, D] = \sum_{j=1}^{n-1} D_i(r_j)D_j = 0.$$

This implies that $D_i(r_1) = D_i(r_2) = \cdots = D_i(r_{n-1}) = 0$ because $D_1, D_2, \dots, D_{n-1} \in L$ are linearly independent over R. Therefore, $r_j \in \bigcap_{i=1}^{n-1} \operatorname{Ker} D_i$ for $j = 1, \dots, n-1$.

Now we take arbitrary $D, D' \in I$ and show that $[D, D'] = 0$. Let $D = a_1D_1 + a_2D_2 + \cdots + a_{n-1}D_{n-1}$ and $D' = b_1D_1 + b_2D_2 + \cdots + b_{n-1}D_{n-1}$. Then

$$[D, D'] = \sum_{i,j=1}^{n-1} (a_ib_j[D_i, D_j] + a_iD_i(b_j)D_j - b_jD_j(a_i)D_i) = 0$$

since $a_i, b_j \in \bigcap_{i=1}^{n-1} \operatorname{Ker} D_i$ for all $i, j = 1, \dots, n-1$, and I is an abelian ideal.

It is easy to see that FI is an abelian ideal of the Lie algebra FL over F and $\dim_F FL/FI = 1$ in view of Lemma 5(c). □

Remark 1. It follows from the proof of Lemma 6 that for an arbitrary $D = a_1D_1 + a_2D_2 + \cdots + a_{n-1}D_{n-1} \in FI$, the inclusions $a_i \in \bigcap_{k=1}^{n-1} \operatorname{Ker} D_k$ hold for $i = 1, \dots, n-1$.

Lemma 7. *Let L be a Lie subalgebra of rank n over R of the Lie algebra $W(A)$, $\{D_1, D_2, \ldots, D_n\}$ be a basis of L over R, and F be the field of constants for L. Let there exists $a \in R$ such that $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$ and $D_n(a) = 1$. Then if $b \in R$ satisfies the conditions $D_1(b) = D_2(b) = \cdots = D_{n-1}(b) = 0$ and $D_n(b) \in F\langle 1, a, \ldots, \frac{a^s}{s!}\rangle$ for some integer $s \geqslant 0$, then $b \in F\langle 1, a, \ldots, \frac{a^s}{s!}, \frac{a^{s+1}}{(s+1)!}\rangle$.*

Proof. Since $D_n(b) \in F\langle 1, a, \ldots, \frac{a^s}{s!}\rangle$, the equality $D_n(b) = \sum_{i=0}^{s} \beta_i \frac{a^i}{i!}$ holds for some $\beta_i \in F$, $i = 0, \ldots, s$. Take an element $c = \sum_{i=0}^{s} \beta_i \frac{a^{i+1}}{(i+1)!}$ from R. It is easy to check that $D_1(c) = D_2(c) = \cdots = D_{n-1}(c) = 0$, because $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$ by the conditions of the lemma. Since $D_n(a) = 1$, the equality $D_n(c) = \sum_{i=0}^{s} \beta_i \frac{a^i}{i!} = D_n(b)$ holds, and so $D_k(b - c) = 0$ for all $k = 1, \ldots, n$. This implies that $b - c \in F$, hence for some $\gamma \in F$, we obtain

$$b = \gamma + c = \gamma + \sum_{i=0}^{s} \beta_i \frac{a^{i+1}}{(i+1)!}.$$

Thus,

$$b \in F\left\langle 1, a, \ldots, \frac{a^s}{s!}, \frac{a^{s+1}}{(s+1)!} \right\rangle,$$

and the proof is complete. □

Theorem 1. *Let L be a nilpotent Lie subalgebra of the Lie algebra $W(A)$, and let $F = F(L)$ be the field of constants for L. If L is of rank n and its center $Z(L)$ is of rank $n-1$ over R, then there exist $D_1, D_2, \ldots, D_n \in FL$, $a \in R$, and an integer $s \geqslant 0$ such that FL is contained in the Lie subalgebra of $W(A)$ of the form*

$$F\left\langle D_1, aD_1, \frac{a^2}{2!}D_1, \ldots, \frac{a^s}{s!}D_1, D_2, aD_2, \ldots, \frac{a^s}{s!}D_2, \ldots, D_{n-1}, \ldots, \frac{a^s}{s!}D_{n-1}, D_n \right\rangle,$$

where $[D_i, D_j] = 0$ for $i, j = 1, \ldots, n$, $D_n(a) = 1$, and $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$.

Proof. It is easy to see that the vector space over F of the form

$$F\left\langle D_1, aD_1, \frac{a^2}{2!}D_1, \dots, \frac{a^s}{s!}D_1, D_2, aD_2, \dots, \frac{a^s}{s!}D_2, \dots, \right.$$
$$\left. D_{n-1}, \dots, \frac{a^s}{s!}D_{n-1}, D_n\right\rangle$$

is a Lie algebra over F. We denote it by $\widetilde{L}$.

By Lemma 6, the Lie algebra L contains an abelian ideal I such that FI is of codimension 1 in FL over F. The ideal I contains an R-basis $\{D_1, D_2, \dots, D_{n-1}\}$ of the center $Z(L)$. Let us take an arbitrary element $D_n \in L$ that is not in $Z(L)$. Then $\{D_1, D_2, \dots, D_{n-1}, D_n\}$ is an R-basis of L and $FL = FI + FD_n$, where FI is an abelian ideal of FL.

Let us consider the action of the inner derivation $\operatorname{ad} D_n$ on the vector space FI. It is easy to see that $\dim_F \operatorname{Ker}(\operatorname{ad} D_n) = n - 1$. Indeed, let

$$D = r_1D_1 + r_2D_2 + \cdots + r_{n-1}D_{n-1} \in \operatorname{Ker}(\operatorname{ad} D_n).$$

Then

$$[D_n, D] = \sum_{i=1}^{n-1}[D_n, r_iD_i] = \sum_{i=1}^{n-1} D_n(r_i)D_i = 0$$

whence $D_n(r_i) = 0$ for all $i = 1, \dots, n-1$.

By Remark 1, $r_1, r_2, \dots, r_{n-1} \in F$. Thus, $\operatorname{Ker}(\operatorname{ad} D_n) = F\langle D_1, D_2, \dots, D_{n-1}\rangle$ and $\dim_F \operatorname{Ker}(\operatorname{ad} D_n) = n - 1$.

The Jordan matrix of the nilpotent operator $\operatorname{ad} D_n$ over F has $n - 1$ Jordan blocks. Denote by $J_1, J_2, \dots, J_{n-1}$ the corresponding Jordan chains. Without loss of generality, we may take $D_1 \in J_1$, $D_2 \in J_2$, …, $D_{n-1} \in J_{n-1}$ to be the first elements in the corresponding Jordan bases.

If $\dim_F F\langle J_1\rangle = \dim_F F\langle J_2\rangle = \cdots = \dim_F F\langle J_{n-1}\rangle = 1$, then $FL = F\langle D_1, D_2, \dots, D_n\rangle$ and FL is an abelian Lie algebra. It is the algebra from the conditions of the theorem if $s = 0$.

Let

$$\dim_F F\langle J_1\rangle \geqslant \dim_F F\langle J_2\rangle \geqslant \cdots \geqslant \dim_F F\langle J_{n-1}\rangle$$

and $\dim_F F\langle J_1\rangle = s + 1$, $s \geqslant 1$. Write the elements of the basis J_1 as follows:

$$J_1 = \Big\{D_1, \sum_{i=1}^{n-1} a_{1i}D_i, \sum_{i=1}^{n-1} a_{2i}D_i, \dots, \sum_{i=1}^{n-1} a_{si}D_i\Big\}.$$

By the definition of a Jordan basis,

$$D_1 = [D_n, \sum_{i=1}^{n-1} a_{1i}D_i] = \sum_{i=1}^{n-1} D_n(a_{1i})D_i$$

whence $D_n(a_{11}) = 1$ and $D_n(a_{1i}) = 0$ for all $i \neq 1$.

By Remark 1, $\sum\limits_{i=1}^{n-1} a_{1i} D_i \in FI$ implies $a_{1i} \in \bigcap\limits_{k=1}^{n-1} \operatorname{Ker} D_k,\ i = 1, \ldots, n-1$, and thus $a_{12}, a_{13}, \ldots, a_{1,n-1} \in F,$ and $a_{11} \notin F$. Let us write $a_{11} = a$. Then $a_{11}, a_{12}, \ldots, a_{1,n-1} \in F\langle 1, a\rangle$.

We shall show that $a_{21}, a_{22}, \ldots, a_{2,n-1} \in F\langle 1, a, \frac{a^2}{2!}\rangle$. By the definition of a Jordan basis,

$$[D_n, \sum_{i=1}^{n-1} a_{2i} D_i] = \sum_{i=1}^{n-1} D_n(a_{2i}) D_i = \sum_{i=1}^{n-1} a_{1i} D_i$$

whence $D_n(a_{2i}) = a_{1i} \in F\langle 1, a\rangle$ for $i = 1, \ldots, n-1$. Then, by Lemma 7, $a_{2i} \in \bigcap\limits_{k=1}^{n-1} \operatorname{Ker} D_k$ implies $a_{2i} \in F\langle 1, a, \frac{a^2}{2!}\rangle$, $i = 1, \ldots, n-1$. Assume that $a_{mi} \in F\langle 1, a, \ldots, \frac{a^m}{m!}\rangle$ for all $m = 1, \ldots, s-1$ and $i = 1, \ldots, n-1$. Then

$$[D_n, \sum_{i=1}^{n-1} a_{m+1,i} D_i] = \sum_{i=1}^{n-1} a_{mi} D_i$$

whence $D_n(a_{m+1,i}) = a_{mi}$ for $i = 1, \ldots, n-1$. The coefficients $a_{m+1,i}$ satisfy the conditions of Lemma 7, so that $a_{m+1,i} \in F\langle 1, a, \ldots, \frac{a^{m+1}}{(m+1)!}\rangle$. Reasoning by induction, we get that all coefficients a_{ji}, $i = 1, \ldots, n-1$, $j = 1, \ldots, s$, of the elements from the basis J_1 belong to $F\langle 1, a, \ldots, \frac{a^s}{s!}\rangle$, and thus $F\langle J_1\rangle \subseteq \widetilde{L}$.

Consider the basis

$$J_2 = \Big\{D_2, \sum_{i=1}^{n-1} b_{1i} D_i, \sum_{i=1}^{n-1} b_{2i} D_i, \ldots, \sum_{i=1}^{n-1} b_{ti} D_i\Big\},$$

where $1 \leqslant t+1 \leqslant s$ and $\dim_F F\langle J_2\rangle = t+1$. By the definition of a Jordan basis, $[D_n, \sum\limits_{i=1}^{n-1} b_{1i} D_i] = \sum\limits_{i=1}^{n-1} D_n(b_{1i}) D_i = D_2$, and thus $D_n(b_{12}) = 1$ and $D_n(b_{1i}) = 0$ for all $i \neq 2$. Set $b_{12} = b \notin F$ and consider $D_n(b-a) = 0$. It follows from Remark 1 that $a, b \in \bigcap\limits_{i=1}^{n-1} \operatorname{Ker} D_i$, so $b - a = \delta \in F$. The latter means that $b \in F\langle 1, a\rangle$. Moreover, $b_{1i} \in F$ for $i \neq 2$ in view of Remark 1. Thus, $b_{11}, b_{12}, \ldots, b_{1,n-1} \in F\langle 1, a\rangle$. Reasoning as for J_1 and using Lemma 7, one can show that $b_{2i} \in F\langle 1, a, \frac{a^2}{2!}\rangle$ and prove by induction that $b_{ji} \in F\langle 1, a, \ldots, \frac{a^t}{t!}\rangle$ for all $j = 1, \ldots, t$ and $i = 1, \ldots, n-1$. Since $t \leqslant s$, we have $F\langle J_2\rangle \subseteq \widetilde{L}$.

In the same way, one can show that the subspaces $F\langle J_3\rangle,\ F\langle J_4\rangle,\dots,$ $F\langle J_{n-1}\rangle$ lie in $\widetilde{L}$. Therefore, the Lie algebra FL is contained in the Lie subalgebra $\widetilde{L}$ of $W(A)$. □

Theorem 2. *Let L be a nilpotent Lie subalgebra of the Lie algebra $W(A)$, and let $F = F(L)$ be its field of constants. If L is of rank $n \geqslant 3$ and its center $Z(L)$ is of rank $n-1$ over R, then the Lie algebra FL over F is isomorphic to a finite dimensional subalgebra of the Lie algebra $u_n(F)$ of triangular polynomial derivations.*

Proof. By Theorem 1, the Lie algebra FL is contained in the Lie subalgebra $\widetilde{L}$ of $W(A)$, which is of the form $F\langle D_1, aD_1, \frac{a^2}{2!}D_1,\dots,\frac{a^s}{s!}D_1, D_2,$ $aD_2,\dots,\frac{a^s}{s!}D_2,\dots,D_{n-1},\dots,\frac{a^s}{s!}D_{n-1}, D_n\rangle$, where $[D_i, D_j] = 0$ for $i, j = 1,\dots,n$, $D_n(a) = 1$ and $D_1(a) = D_2(a) = \cdots = D_{n-1}(a) = 0$. The Lie algebra $\widetilde{L}$ is isomorphic (as a Lie algebra over F) to the subalgebra

$$F\left\langle \frac{\partial}{\partial x_1}, x_n\frac{\partial}{\partial x_1}, \frac{x_n^2}{2!}\frac{\partial}{\partial x_1},\dots,\frac{x_n^s}{s!}\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-1}}, x_n\frac{\partial}{\partial x_{n-1}},\dots,\right.$$
$$\left.\frac{x_n^s}{s!}\frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_n}\right\rangle$$

of the Lie algebra $u_n(F)$ of triangular polynomial derivations over F. □

2. Example of a maximal nilpotent Lie subalgebra of the Lie algebra $\widetilde{W}_n(\mathbb{K})$

Lemma 8 ([8, Lemma 4])**.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. For a rational function $\phi \in \mathbb{K}(t)$, write $\phi' = \frac{d\phi}{dt}$. If $\phi \in \mathbb{K}(t)\setminus\mathbb{K}$, then does not exist a function $\psi \in \mathbb{K}(t)$ such that $\psi' = \frac{\phi'}{\phi}$.*

Let us denote by $\mathbb{K}[X] = \mathbb{K}[x_1, x_2,\dots,x_n]$ the polynomial algebra, by $\mathbb{K}(X) = \mathbb{K}(x_1, x_2,\dots,x_n)$ the field of rational functions in n variables over $\mathbb{K}$, and by $\widetilde{W}_n(\mathbb{K})$ the Lie algebra of derivations on the field $\mathbb{K}(X)$. We think that the first part of the following statement is known.

Proposition 1. *The subalgebra $L = \mathbb{K}\langle x_1\frac{\partial}{\partial x_1}, x_2\frac{\partial}{\partial x_2},\dots,x_n\frac{\partial}{\partial x_n}\rangle$ of the Lie subalgebra of $\widetilde{W}_n(\mathbb{K})$ is isomorphic to a Lie subalgebra of the Lie algebra $u_n(\mathbb{K})$ of triangular polynomial derivations, but is not conjugated with any Lie subalgebra of this Lie algebra by an automorphism of the Lie algebra $\widetilde{W}_n(\mathbb{K})$.*

Proof. Let us show that L is a maximal nilpotent Lie subalgebra of $\widetilde{W}_n(\mathbb{K})$. Obviously, L is abelian, and so it is nilpotent. Let us show that L coincides with its normalizer in $\widetilde{W}_n(\mathbb{K})$, which will imply that L is maximal nilpotent (in view of the well-known fact from the theory of Lie algebras that a proper Lie subalgebra of a nilpotent Lie algebra does not coincide with its normalizer, see [1, p.58]).

Let D be an arbitrary element of the normalizer $N = N_{\widetilde{W}_n(\mathbb{K})}(L)$. Then $[D, x_i\frac{\partial}{\partial x_i}] \in L$ for each $i = 1, \ldots, n$. D can be uniquely written as $D = \sum\limits_{j=1}^{n} f_j \frac{\partial}{\partial x_j}$, where $f_1, \ldots, f_n \in \mathbb{K}(X)$. Using the following equations

$$\Big[x_i\frac{\partial}{\partial x_i}, \sum_{j=1}^{n} f_j\frac{\partial}{\partial x_j}\Big] = \sum_{j=1}^{n}\Big[x_i\frac{\partial}{\partial x_i}, f_j\frac{\partial}{\partial x_j}\Big] = $$
$$= \sum_{\substack{j=1\\ j\neq i}}^{n} x_i\frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial x_j} + \Big(x_i\frac{\partial f_i}{\partial x_i} - f_i\Big)\frac{\partial}{\partial x_i},$$

we obtain that

$$x_i\frac{\partial f_j}{\partial x_i} = \alpha_j x_j, i \neq j, \text{ and } x_i\frac{\partial f_i}{\partial x_i} - f_i = \alpha_i x_i \tag{1}$$

for $\alpha_i, \alpha_j \in \mathbb{K}, i, j = 1, \ldots, n$. We rewrite the first equation in (1) in the form $\frac{\partial f_j}{\partial x_i} = \frac{\alpha_j x_j}{x_i}$ and consider f_j as a rational function in x_i over the field $\mathbb{K}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. By Lemma 8 with $\phi = \phi(x_i) = x_i$, we have $\alpha_j = 0$. Thus, $\frac{\partial f_j}{\partial x_i} = 0$ for all $i \neq j$. This means that $f_j \in \mathbb{K}(x_j)$ for each $j = 1, \ldots, n$.

Write $f_i = \frac{u_i}{v_i}$, where $u_i, v_i \in \mathbb{K}[x_i]$ are relatively prime and $v_i \neq 0$. Then the second equation in (1) is rewritten as

$$x_i\frac{u_i'v_i - u_iv_i' - \alpha_i v_i^2}{v_i^2} = \frac{u_i}{v_i},$$

where $'$ denotes the derivative with respect to the variable x_i. But then $x_i(u_i'v_i - u_iv_i' - \alpha_i v_i^2)v_i = u_iv_i^2$, whence we have that the polynomial v_i must divide v_i'. It is possible only if $v_i \in \mathbb{K}^*$, i.e. f_i is a polynomial in x_i with coefficients in $\mathbb{K}$. Since $x_i(f_i' - \alpha_i) = f_i$, we have that f_i is a polynomial of degree 1. It is easy to see that $f_i = \gamma_i x_i$ with $\gamma_i \in \mathbb{K}$ for all $i = 1, \ldots, n$. Thus $D \in L$, that is, $L = N$ and L is a maximal nilpotent Lie subalgebra of $\widetilde{W}_n(\mathbb{K})$.

If L is conjugated by an automorphism of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ with some Lie subalgebra of $u_n(\mathbb{K})$, then L is contained in a nilpotent Lie subalgebra of $u_n(\mathbb{K})$. Therefore, L is not coincide with its normalizer in $\widetilde{W}_n(\mathbb{K})$, which contradicts the fact proved above. However, the subalgebra $\mathbb{K}\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\rangle$ of the Lie algebra $u_n(\mathbb{K})$ is obviously isomorphic to L. $\square$

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