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RESEARCH ARTICLE

On the *le*-semigroups whose semigroup of bi-ideal elements is a normal band

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Abstract. It is well known that the semigroup $\mathcal{B}(S)$ of all bi-ideal elements of an le-semigroup S is a band if and only if S is both regular and intra-regular. Here we show that $\mathcal{B}(S)$ is a band if and only if it is a normal band and give a complete characterization of the le-semigroups S for which the associated semigroup $\mathcal{B}(S)$ is in each of the seven nontrivial subvarieties of normal bands. We also show that the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of S forms a rectangular band and that $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(\mathcal{S})$.

1. Introduction

In the ideal theory of commutative rings, it was observed by W. Krull [15] that several results do not depend on the fact that the ideals are composed of elements. The same is true for the ideal theory of semigroups also. Consequently, these results can be formulated in a more general setting of lattice-ordered semigroups where an element represents an ideal of the ring or semigroup as an undivided entity. There are series of articles dealing with lattice-ordered semigroups, generalizing theorems from commutative ideal theory [1], [3], [5] and from the ideal theory of semigroups [12], [13], [21], [22]. Presently, lattice ordered semigroups are providing us a general setting not only for 'abstract ideal theory', but also

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for order-preserving transformations of a finite chain, power semigroups of an arbitrary semigroup, and for many other areas of algebra where the objects form similar kinds of lattice-ordered semigroups.

In the present paper we study le-semigroups globally; our aim here is to find out to what extent properties of the subsemigroup $\mathcal{B}(S)$ of all bi-ideal elements of an le-semigroup S affect the structure of the le-semigroup as a whole. In 1952, R.A. Good and D.R. Hughes [6] introduced the notion of bi-ideals of a semigroup; these have been generalized again and again to rings, semirings, ternary semirings, Γ-semigroups, etc [4], [8]–[11], [14], [23]. It has also been proved that this notion is very useful for characterizing different types of regularity of rings, semirings, and semigroups [2], [16]– [19]. In [12], N. Kehayopulu defined bi-ideal elements of an le-semigroup as a generalization of bi-ideals. Here we introduce the notion of minimal bi-ideal elements and show that the product of any two bi-ideal elements is a bi-ideal element, and that the product of any two minimal bi-ideal elements is a minimal bi-ideal element. Thus the set $\mathcal{B}(S)$ of all bi-ideal elements and the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements are subsemigroups of S. It is well known that S is both regular and intra-regular if and only if $b^2 = b$ for every bi-ideal element b of S, equivalently $\mathcal{B}(S)$ is a band. Here we show that $\mathcal{B}(S)$ is a locally testable semigroup and hence a normal band (since a band is locally testable if and only if it is a normal band) if S is both regular and intra-regular. The variety of normal bands has exactly eight subvarieties. Here we have characterized the le-semigroups S such that $\mathcal{B}(S)$ is in each of these subvarieties of normal bands.

This introduction is followed by preliminaries. In Section 3, we characterize the le-semigroups S such that $\mathcal{B}(S)$ is in each of the subvarieties of normal bands. In the last section, we show that the semigroup $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of S is a bi-ideal of the semigroup $\mathcal{B}(S)$ whereas the set $\mathcal{L}_m(S)$ of all minimal left ideal elements is a left ideal of the semigroup $\mathcal{L}(S)$ of all left ideal elements of S.

2. Preliminaries and foundations

An *le-semigroup* S is an algebra $(S, \cdot, \vee, \wedge, e)$ such that $(S, \cdot)$ is a semigroup, $(S, \vee, \wedge, e)$ is a lattice with a greatest element which is denoted by e, and for all $a, b, c \in S$,

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

For different examples and relevance, both classical and modern, of the le-semigroups we refer to [21]. Throughout the paper S will stand for an le-semigroup $(S, \cdot, \vee, \wedge, e)$.

The usual order relation $\leqslant$ on the set S is defined by: for $a, b \in S$

$$a \leqslant b \text{ if } a \vee b = b.$$

Since the multiplication is distributive over the lattice join, it follows that the order $\leqslant$ is compatible with the multiplication in S, that is, for all $a, b, c \in S$,

$$a \leqslant b \implies ac \leqslant bc \text{ and } ca \leqslant cb.$$

Let A be a nonempty subset of S. We denote $(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}$. A nonempty subset L is called a *left (right) ideal* of S if $SL \subseteq L$ $(LS \subseteq L)$ and $(L] \subseteq L$. A subset I is called an *ideal* if it is both a left and a right ideal of S. For $a \in S$, the left ideal generated by a is given by

$$(a]_l = \{x \in S \mid x \leqslant sa \text{ for some } s \in S \cup \{1\}\}.$$

An element $a \in S$ is called *regular* if $a \leqslant aea$; and *intra-regular* if $a \leqslant ea^2e$. If every element of S is regular (intra-regular) then the le-semigroup S is defined to be *regular (intra-regular)*. We also say that a is

(i) a *subsemigroup element* if $a^2 \leqslant a$;

(ii) a *left ideal element* if $ea \leqslant a$;

(iii) a *right ideal element* if $ae \leqslant a$;

(iv) a *bi-ideal element* if it is a subsemigroup element and $aea \leqslant a$.

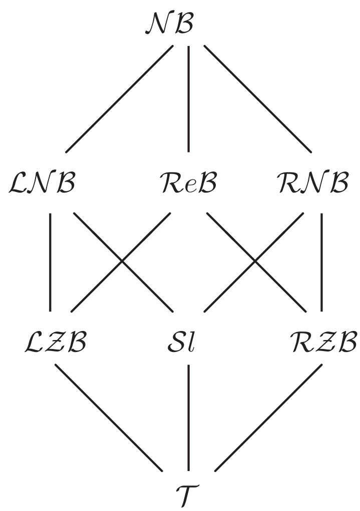
From the above definitions it is evident that every left and right ideal element is also a subsemigroup element. The definition of bi-ideal elements that we have given here is a little bit different from that of bi-ideal elements considered by Kehayopulu [12], Pasku and Petro [22]. According to these authors, a bi-ideal element b needs not satisfy $b^2 \leqslant b$, i.e. needs not be a subsemigroup element, and is actually an abstraction of the generalized bi-ideals (of a semigroup) and not of the bi-ideals.

Let $a \in S$. Then $b = a \vee a^2 \vee aea$ is the least bi-ideal element in S such that $a \leqslant b$. We call $a \vee a^2 \vee aea$ the *bi-ideal element generated by* a, and denote this by $\beta(a)$. Thus $a \in S$ is a bi-ideal element if and only if $\beta(a) = a$.

Now we recall some notions of semigroups (without order). A semigroup F is called *regular* if for every $a \in F$ there is $x \in F$ such that $a = axa$. By a *band* we mean a semigroup B such that $b^2 = b$ for all $b \in B$. A band S is *normal* if for all $a, b, c \in S$, $abca = acba$. A subsemigroup B of a semigroup F is called a *bi-ideal* of F if $BFB \subseteq B$.

In the diagram below, we use the following symbols to denote the different subvarieties of normal bands.

Normal band	$\mathcal{NB}$	$abcd = acbd$,
Rectangular band	$\mathcal{R}e\mathcal{B}$	$aba = a$,
Left normal band	$\mathcal{LNB}$	$abc = acb$,
Right normal band	$\mathcal{RNB}$	$abc = bac$,
Left zero band	$\mathcal{LZB}$	$ab = a$,
Right zero band	$\mathcal{RZB}$	$ab = b$,
Semilattice	$\mathcal{S}l$	$ab = ba$,
Trivial semigroup	$\mathcal{T}$	$a = b$.



A semigroup is called *locally finite* if every finitely generated subsemigroup is finite. A *locally testable semigroup* [24] is a semigroup which is locally finite and in which fSf is a semilattice for all idempotent $f \in S$. Nambooripad [20] proved that a regular semigroup S is locally testable if and only if fSf is a semilattice for all idempotent $f \in S$.

We refer the reader to [7] for the fundamentals of semigroup theory.

3. Subsemigroup of all bi-ideal elements

We denote the set of all left, right, and bi-ideal elements of S by $\mathcal{L}(S), \mathcal{R}(S)$, and $\mathcal{B}(S)$, respectively. Then $\mathcal{L}(S), \mathcal{R}(S)$, and $\mathcal{B}(S)$ are all nonempty, since e is a left ideal, a right ideal, and a bi-ideal element of S. Now for any two bi-ideal elements a and b of S, $(ab)^2 = (aba)b \leqslant ab$ and $abeab = (abea)b \leqslant ab$, since a is a bi-ideal element of S, which shows that the product of any two bi-ideal elements is a bi-ideal element. Thus $\mathcal{B}(S)$ is a subsemigroup of S. Similarly both $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are subsemigroups of S.

Now we show that the regularity of an *le*-semigroup is equivalent to the regularity of the semigroup $\mathcal{B}(S)$. This, we think, is well known. But as we have seen the sufficient part nowhere, for the sake of completeness, we include a proof.

Proposition 3.1. *Let S be an le-semigroup. Then S is regular if and only if the semigroup $\mathcal{B}(S)$ of all bi-ideal elements is regular.*

Proof. First assume that S is regular and that $b \in \mathcal{B}(S)$. Since b is a bi-ideal element, $beb \leqslant b$. On the other hand, $b \leqslant beb$ by the regularity of S. Thus we have $b = beb$ which shows that b is a regular element in $\mathcal{B}(S)$, since e is also a bi-ideal element of S.

Conversely, suppose that $\mathcal{B}(S)$ is a regular semigroup. Consider $a \in S$. Then $\beta(a) = a \vee a^2 \vee aea \in \mathcal{B}(S)$ and so there is $b \in \mathcal{B}(S)$ such that $a \vee a^2 \vee aea = (a \vee a^2 \vee aea)b(a \vee a^2 \vee aea) \leqslant (a \vee a^2 \vee aea)e(a \vee a^2 \vee aea) \leqslant aea$. This implies that $a \leqslant aea$. Thus S is a regular *le*-semigroup. $\square$

If S is a regular *le*-semigroup, then for every $a \in S$, $a \leqslant aea$ implies that $a^2 \leqslant aaea \leqslant aea$. Hence the bi-ideal element $\beta(a)$ generated by a reduces to the form $\beta(a) = aea$. Thus in a regular *le*-semigroup the notions of bi-ideal elements as we have defined and that defined by N. Kehayopulu [12] are the same. Therefore in a regular *le*-semigroup S, an element $b \in S$ is a bi-ideal element if and only if $b = ca$ for some right ideal element c and left ideal element a [12, Lemma 2]. This can be reframed as:

Theorem 3.2. *Let S be an le-semigroup. Then $\mathcal{R}(S)\mathcal{L}(S) \subseteq \mathcal{B}(S)$. If moreover, S is a regular le-semigroup, then $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

We also omit the proof of the following result, since this can be proved easily:

Proposition 3.3. *Let S be a regular le-semigroup. Then $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are bands.*

The following important result can be proved similarly to that in [13] for the quasi-ideal elements.

Theorem 3.4. *An le-semigroup S is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a band.*

Now we show that $\mathcal{B}(S)$ is in fact a normal band if S is both a regular and intra-regular *le*-semigroup.

Theorem 3.5. *Let S be an le-semigroup. Then S is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a normal band.*

Proof. Let $a, b, c \in \mathcal{B}(S)$. Then $(bab)(bcb) = ba(bbcb) \leqslant ba(beb) \leqslant bab$. Similarly, $(bab)(bcb) \leqslant bcb$. Thus $(bab)(bcb) \leqslant (bab) \wedge (bcb)$. Now let $u = (bab) \wedge (bcb)$. Then $u \leqslant bab$ and $u \leqslant bcb$. Since S is both regular and intra-regular, so $\mathcal{B}(S)$ is a band. Now $ueu = (bab \wedge bcb)e(bab \wedge bcb) = babebab \wedge babebcb \wedge bcbebab \wedge bcbebcb \leqslant bab \wedge babebcb \wedge bcbebab \wedge bcb \leqslant bab \wedge bcb = u$ shows that $u \in \mathcal{B}(S)$ which implies that $u = u^2 \leqslant (bab)(bcb)$. Thus $(bab) \wedge (bcb) \leqslant (bab)(bcb)$ and hence $(bab)(bcb) = (bab) \wedge (bcb)$.

Then $b\mathcal{B}(S)b = \{bab \mid a \in \mathcal{B}(S)\}$ is a semilattice for every $b \in \mathcal{B}(S)$. Thus $\mathcal{B}(S)$ is a locally testable semigroup. Since a locally testable semigroup is a band if and only if it is a normal band [24, Theorem 5], so $\mathcal{B}(S)$ is a normal band.

The converse follows from the Theorem 3.4. □

An ordered semigroup S is said to be *left (right) duo* if every left (right) ideal of S is a right (left) ideal of S; and S is said to be *duo* if S is both left and right duo.

Lemma 3.6. *An le-semigroup S is left duo if and only if $ae \leqslant ea$ for all $a \in S$.*

Proof. First assume that S is left duo and let $a \in S$. Then the left ideal $(a]_l = \{x \in S \mid x \leqslant sa$ for some $s \in S\}$ generated by a is a right ideal also. Then $ae \in (a]_l$ implies that there is some $s \in S$ such that $ae \leqslant sa$ and this implies that $ae \leqslant ea$.

Conversely let L be a left ideal of S and $a \in L$. Then for every $s \in S$, $as \leqslant ae \leqslant ea \in L$ implies that $as \in L$. Thus L is a right ideal of S and hence S is left duo. □

Immediately we have:

Proposition 3.7. *An le-semigroup S is duo if and only if $ae = ea$ for all $a \in S$.*

Let S be a regular left duo *le*-semigroup. Then for every $a \in S$, $a \leqslant aea \leqslant (ae)aea \leqslant ea^2ea$ shows that S is intra-regular. Hence $\mathcal{B}(S)$ is a band. In fact we have:

Theorem 3.8. *An le-semigroup S is regular left duo if and only if $\mathcal{B}(S)$ is a left normal band.*

Proof. First assume that S is regular left duo. Then $\mathcal{B}(S)$ is a band. Let $a, b, c \in \mathcal{B}(S)$. Then $abc = (abc)(abc) = aabcabc \leqslant a(ae)cabc \leqslant a(ea)cabc \leqslant acabc = aca(bc)(bc) \leqslant acab(cb)e \leqslant acabecb \leqslant acb$. Similarly $acb \leqslant abc$. Thus $abc = acb$ and hence $\mathcal{B}(S)$ is a left normal band.

Conversely, assume that $\mathcal{B}(S)$ is a left normal band. Then S is regular. Also for every $a \in S$, both ea and aea are bi-ideal elements of S, and hence $ae = (ae)(ae)(ae) = (aea)(ea)e = (aea)e(ea)$ [since $\mathcal{B}(S)$ is a normal band] $= (aeae^2)a \leqslant ea$ which shows that S is left duo. □

The left-right dual of this theorem is as follows:

Theorem 3.9. *An le-semigroup S is regular right duo if and only if $\mathcal{B}(S)$ is a right normal band.*

A band is a semilattice if and only if it is both a left and a right normal band. Hence it follows immediately that:

Theorem 3.10. *An le-semigroup S is regular duo if and only if $\mathcal{B}(S)$ is a semilattice.*

Theorem 3.11. *Let S be an le-semigroup. Then $\mathcal{B}(S)$ is a rectangular band if and only if S is regular and $eae = ebe$ for all $a, b \in S$.*

Proof. First assume that $\mathcal{B}(S)$ is a rectangular band and that $a, b \in S$. Since $\mathcal{B}(S)$ is a band, so S is regular and hence $\beta(a) = aea$ and $\beta(b) = beb$. Then $\beta(a) = \beta(a)\beta(b)\beta(a)$ implies that $a \leqslant aea = (aea)(beb)(aea) \leqslant ebe$. Then $eae \leqslant e^2be^2 \leqslant ebe$. Similarly $\beta(b) = \beta(b)\beta(a)\beta(b)$ implies that $ebe \leqslant eae$. Thus $eae = ebe$ for all $a, b \in S$.

Conversely let $a \in S$. Since S is regular, so $a \leqslant aea \leqslant aeaea \leqslant aea^2ea$, by the given condition. Thus $a \leqslant ea^2e$, and hence S is intra-regular. Therefore $\mathcal{B}(S)$ is a band, by Theorem 3.4. Now let a, b be two bi-ideal elements of S. Since a is a bi-ideal element and S is already known to be regular, then $aea = a$, and so $a = aea = aeaea = aeabaea = aba$; and hence $\mathcal{B}(S)$ is a rectangular band. □

Theorem 3.12. *Let S be an le-semigroup. Then $\mathcal{B}(S)$ is a left zero band if and only if S is regular and $ae \leqslant eb$ for all $a, b \in S$.*

Proof. First assume that $\mathcal{B}(S)$ is a left zero band and $a, b \in S$. Since $\mathcal{B}(S)$ is band, so S is regular and hence $\beta(ae) = ae^2ae$ and $\beta(b) = beb$. Then $\beta(ae) = \beta(ae)\beta(b)$ implies that $ae \leqslant ae^2ae = (ae^2ae)(beb) \leqslant eb$. Thus $ae \leqslant eb$ for all $a, b \in S$.

Conversely let $a \in S$. Since S is regular, so $a \leqslant aea \leqslant aeaea \leqslant ae^2a^2a$, by the given condition. Thus $a \leqslant ea^2e$, and hence S is intra-regular. Therefore $\mathcal{B}(S)$ is a band, by Theorem 3.4. Now let a, b be two bi-ideal elements of S. Since S is regular, $a = aea$, so that $ab \leqslant ae = ae(ae) \leqslant ae^2a \leqslant aea = a$ and $a = aeaea \leqslant ae(ae) \leqslant ae(eab) \leqslant (aea)b = ab$. Thus $a = ab$ and hence $\mathcal{B}(S)$ is a left zero band. □

The left-right dual of this theorem is as follows:

Theorem 3.13. *Let S be an le-semigroup. Then $\mathcal{B}(S)$ is a right zero band if and only if S is regular and $ea \leqslant be$ for all $a, b \in S$.*

4. Subsemigroup of all minimal bi-ideal elements

In this section we introduce minimal bi-ideal elements and minimal left ideal elements, and show that the set of all minimal bi-ideal elements of S is a subsemigroup of $\mathcal{B}(S)$.

Definition 4.1. Let S be an *le*-semigroup. A bi-ideal element b is said to be minimal if for every bi-ideal element a of S,

$$a \leqslant b \text{ implies that } a = b.$$

Minimal left (right) ideal elements are defined similarly.

We denote the set of all minimal bi-ideal, left ideal, and right ideal elements of S by $\mathcal{B}_m(S), \mathcal{L}_m(S)$, and $\mathcal{R}_m(S)$, respectively.

Now we show that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. For this consider $a, b \in \mathcal{B}_m(S)$. Then ab is a bi-ideal element. To check the minimality, let c be a bi-ideal element such that $c \leqslant ab$. Then ca and bc are bi-ideal elements such that $ca \leqslant aba \leqslant a$. Then by minimality of a we have $ca = a$. Similarly, $bc = b$. Then $ab = cabc \leqslant cec \leqslant c$ and hence $c = ab$. Thus $ab \in \mathcal{B}_m(S)$.

Similarly, it can be proved that both $\mathcal{L}_m(S)$ and $\mathcal{R}_m(S)$ are subsemigroups of $\mathcal{B}(S)$.

We also have:

Theorem 4.2. *If S is an le-semigroup then $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$.*

Proof. First consider $a \in \mathcal{R}_m(S)$ and $c \in \mathcal{L}_m(S)$, and denote $b = ac$. Then b is a bi-ideal element, by Theorem 3.2. To show the minimality of b, let $p \leqslant b$ be a bi-ideal element of S. Then pe is a right ideal element of S and $pe \leqslant be = ace \leqslant ae \leqslant a$ implies by the minimality of a as a right ideal element that $pe = a$. Similarly we have $ep = c$, since c is a minimal left ideal element. Then $p \leqslant b = ac = peep \leqslant p$ implies that $p = b$; and so b becomes a minimal bi-ideal element. Thus $\mathcal{R}_m(S)\mathcal{L}_m(S) \subseteq \mathcal{B}_m(S)$.

Now consider $b \in \mathcal{B}_m(S)$. Then be and eb are a right ideal element and a left ideal element, respectively. Let $a \leqslant be$ be a right ideal element of S. Then ab is a bi-ideal element of S such that $ab \leqslant beb \leqslant b$, and so $ab = b$, since b is a minimal bi-ideal element. Then $a \leqslant be = abe \leqslant ae \leqslant a$ implies that $a = be$. Thus be is a minimal right ideal element of S. Similarly eb is a minimal left ideal element of S. Then $beeb$ is a bi-ideal element, by Theorem 3.2. Now $beeb \leqslant b$ implies that $b = beeb$; and so $b \in \mathcal{R}_m(S)\mathcal{L}_m(S)$. Thus $\mathcal{B}_m(S) \subseteq \mathcal{R}_m(S)\mathcal{L}_m(S)$. Hence $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$. □

Theorem 4.3. *a) Let S be an le-semigroup such that the set $\mathcal{L}_m(S)$ of all minimal left ideal elements is non-empty. Then $\mathcal{L}_m(S)$ is a left ideal of the semigroup $\mathcal{L}(S)$. Moreover, $\mathcal{L}_m(S)$ is a right zero band.*
b) Let S be an le-semigroup such that the set $\mathcal{R}_m(S)$ of all minimal right ideal elements is non-empty. Then $\mathcal{R}_m(S)$ is a right ideal of the semigroup $\mathcal{R}(S)$. Moreover, $\mathcal{R}_m(S)$ is a left zero band.

Proof. a) Let $l \in \mathcal{L}(S)$ and $a \in \mathcal{L}_m(S)$. Then la is a left ideal element such that $la \leqslant ea \leqslant a$. This implies that $la = a$, since a is a minimal left ideal element. Hence $la \in \mathcal{L}_m(S)$ and so $\mathcal{L}(S)\mathcal{L}_m(S) \subseteq \mathcal{L}_m(S)$. Thus $\mathcal{L}_m(S)$ is a left ideal of $\mathcal{L}(S)$.

Now $la = a$ for every $l \in \mathcal{L}(S)$ and $a \in \mathcal{L}_m(S)$ implies that $ab = b$ for every $a, b \in \mathcal{L}_m(S)$; and hence $\mathcal{L}_m(S)$ is a right zero band.
b) Follows as the left-right dual of a). □

Now we characterize the semigroup $\mathcal{B}_m(S)$ of all minimal bi-ideal elements of S.

Theorem 4.4. *Let S be an le-semigroup such that the set $\mathcal{B}_m(S)$ of all minimal bi-ideal elements is non-empty. Then $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(S)$. Moreover, $\mathcal{B}_m(S)$ is a rectangular band.*

Proof. We have already shown that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. Now consider $a, c \in \mathcal{B}_m(S)$ and $b \in \mathcal{B}(S)$. Then abc is a bi-ideal element of S. To show the minimality of abc, let $d \leqslant abc$ be a bi-ideal element of S. Then da is a bi-ideal element of S and $da \leqslant abca \leqslant a$ implies that $a = da$, since a is a minimal bi-ideal element. Similarly minimality of c implies that $c = cd$. Then $abc = dabcd \leqslant d$ and so $d = abc$ which shows that abc is a minimal bi-ideal element of S. Thus $\mathcal{B}_m(S)\mathcal{B}(S)\mathcal{B}_m(S) \subseteq \mathcal{B}_m(S)$ and hence $\mathcal{B}_m(S)$ is a bi-ideal of $\mathcal{B}(S)$.

If $b \in \mathcal{B}_m(S)$, then b is a subsemigroup element of S and so $b^2 \leqslant b$. Now minimality of b implies that $b^2 = b$. Thus $\mathcal{B}_m(S)$ is a band. Let $a, b \in \mathcal{B}_m(S)$. Then aba is a bi-ideal element such that $aba \leqslant aea \leqslant a$ which implies that $aba = a$, since a is a minimal bi-ideal element of S. Thus $\mathcal{B}_m(S)$ is a rectangular band. □

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On strongly almost m-ω_1-$p^{\omega+n}$-projective abelian p-groups

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Abstract. For any non-negative integers m and n we define the class of *strongly almost* m-ω_1-$p^{\omega+n}$*-projective groups* which properly encompasses the classes of *strongly* m-ω_1-$p^{\omega+n}$*-projective groups* and *strongly almost* ω_1-$p^{\omega+n}$*-projective groups*, defined by the author in Demonstr. Math. (2014) and Hacettepe J. Math. Stat. (2015), respectively. Certain results about this new group class are proved as well as it is shown that it shares many analogous basic properties as those of the aforementioned two group classes.

1. Introduction and terminology

Let all groups considered in this paper be p-primary abelian, for some arbitrary fixed prime p. Besides, everywhere in the text, m and n are arbitrary integers greater than or equal to $\{0\}$. Our notions and notations are in the most part standard and follow those from the classical books [9], [10] and [12]. The not well-known of them will be explained below in detail.

A class of groups that plays a major role in torsion abelian group theory is the one consisting of all almost direct sums of cyclic groups, introduced in [11] as follows.

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The separable group G is called *an almost direct sum of cyclic groups* if there is a collection $\mathcal{C}$ consisting of nice subgroups of G, satisfying the following three conditions:

(1) $\{0\} \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed with respect to ascending unions, i.e., if $H_i \in \mathcal{C}$ with $H_i \subseteq H_j$ whenever $i \leqslant j$ $(i, j \in I)$ then $\cup_{i \in I} H_i \in \mathcal{C}$;

(3) If K is a countable subgroup of G, then there is $L \in \mathcal{C}$ (that is, a nice subgroup L of G) such that $K \subseteq L$ and L is countable.

Furthermore, an important class of p-torsion groups is the class of all almost $p^{\omega+n}$-projective groups, where $n \geqslant 0$ is an integer, defined in [1] and [2] like this: The group G is called *almost $p^{\omega+n}$-projective* if there exists a p^n-bounded subgroup $P \leqslant G$ such that G/P is an almost direct sum of cyclic groups (note that P is necessarily nice in G because the quotient G/P is separable). It is demonstrated there that this is tantamount to the fact that G is isomorphic to S/B, where S is an almost direct sum of cyclic groups and B is p^n-bounded.

Using the specific nature of countable subgroups, we generalized in [2] the last concept to the following: A group G is said to be *almost ω_1-$p^{\omega+n}$-projective* if there is a countable subgroup $C \leqslant G$ such that G/C is almost $p^{\omega+n}$-projective. Notice that such a subgroup C can be chosen to satisfy the inequalities $p^{\omega+n}G \subseteq C \subseteq p^{\omega}G$, and thus resultantly C is of necessity nice in G.

On the other vein, we showed in [2] also that almost ω_1-$p^{\omega+n}$-projective groups can be characterized in a different way as follows: The group G is almost ω_1-$p^{\omega+n}$-projective if there exists a p^n-bounded subgroup $H \leqslant G$ such that G/H is the sum of a countable group and an almost direct sum of cyclic groups. As observed, such a subgroup H need not always be nice in G, and so in [7] was given the following definition: A group G is called *strongly almost ω_1-$p^{\omega+n}$-projective* if there is a p^n-bounded nice subgroup $N \leqslant G$ with G/N a sum of a countable group and an almost direct sum of cyclic groups. Note that almost $p^{\omega+n}$-projective groups are obviously strongly almost ω_1-$p^{\omega+n}$-projective. Some principal results concerning certain generalizations of strongly almost ω_1-$p^{\omega+n}$-projective groups were established in [4], [5], [6] and [8], respectively.

On the other hand, in order to extend some classical sorts of groups, e.g. $p^{\omega+n}$-projective groups and ω_1-$p^{\omega+n}$-projective groups, in [3] were introduced a few classes of groups by using a single parameter m. So, the objective of the present article is to develop that idea to some new concepts which use the term "almost", and also to find suitable relationships between them and the mentioned above group classes.

Definition 1.1. The group G is said to be *almost m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded subgroup A of G such that G/A is strongly almost ω_1-$p^{\omega+n}$-projective.

In particular, if A is nice in G, then G is called *strongly almost m-ω_1-$p^{\omega+n}$-projective.*

If $m = 0$ we obtain strongly almost ω_1-$p^{\omega+n}$-projective groups, while we obtain strongly almost ω_1-$p^{\omega+m}$-projective groups when $n = 0$.

Definition 1.2. The group G is said to be *weakly almost m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded nice subgroup X of G such that G/X is almost ω_1-$p^{\omega+n}$-projective.

Substituting $m = 0$ we yield almost ω_1-$p^{\omega+n}$-projective groups, while if $n = 0$ we yield strongly almost ω_1-$p^{\omega+m}$-projective groups. In fact, the first fact is trivial, while for the second one we have the following arguments: in view of Lemma 2.16 of [1] an almost ω_1-p^{ω}-projective group is actually a sum of a countable group and an almost direct sum of cyclic groups. Hence the definition of a strongly almost ω_1-$p^{\omega+m}$-projective group is directly applicable, and we are set.

Definition 1.3. The group G is said to be *decomposably almost m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded subgroup S of G with the property that G/S is a sum of a countable group and an almost $p^{\omega+n}$-projective group.

In particular, if S is nice in G, then G is called *nice decomposably almost m-ω_1-$p^{\omega+n}$-projective.* In addition, if the sum above is direct, we shall say that G is *(nice) direct decomposably almost m-ω_1-$p^{\omega+n}$-projective.*

If $m = 0$ we identify the sums of countable groups and almost $p^{\omega+n}$-projective groups. If $n = 0$ we unify all almost ω_1-$p^{\omega+m}$-projective groups.

As for the second part, choosing $m = 0$ we will again obtain the sums of countable groups and almost $p^{\omega+n}$-projective groups, but choosing $n = 0$ we will obtain strongly almost ω_1-$p^{\omega+m}$-projective groups.

Definition 1.4. The group G is called *nicely almost m-$p^{\omega+n}$-projective* if there is a p^m-bounded nice subgroup Y of G such that G/Y is almost $p^{\omega+n}$-projective.

Putting $m = 0$ we get almost $p^{\omega+n}$-projective groups, and putting $n = 0$ we get almost $p^{\omega+m}$-projective groups. Likewise, nicely almost

m-$p^{\omega+n}$-projective groups are both nice decomposably almost m-ω_1-$p^{\omega+n}$-projective and almost $p^{\omega+m+n}$-projective. Actually, almost $p^{\omega+m+n}$-projective groups are groups for which there is (not necessarily nice) a p^m-bounded subgroup M and, respectively, a p^n-bounded subgroup N, such that G/M is almost $p^{\omega+n}$-projective, respectively, G/N is almost $p^{\omega+m}$-projective.

Generally, the following self containments are fulfilled (this manifestly visualizes some immediate relationships between the new group classes):

- {strongly almost ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {decomposably almost n-ω_1-$p^{\omega+n}$-projective groups}.
- {strongly almost m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {weakly almost m-ω_1-$p^{\omega+n}$-projective groups}.
- {nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {weakly almost m-ω_1-$p^{\omega+n}$-projective groups}.
- {nicely almost m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups}.

2. Some more relationships

In this section we will prove certain basic relation properties of the groups from the above definitions. Throughout the rest of the paper, we once again recollect that m and n are arbitrary fixed naturals or zero.

We start with the following:

Theorem 2.1. *For any group G there exists a p^m-bounded subgroup K such that G/K is almost ω_1-$p^{\omega+n}$-projective if and only if G is almost ω_1-$p^{\omega+m+n}$-projective.*

Proof. We shall first show that G is as in the necessity of the theorem $\iff$ there exists $C \leqslant G$ such that p^mC is countable with $p^mC \subseteq p^\omega G$ and G/C is almost $p^{\omega+n}$-projective $\iff$ there exists $L \leqslant G$ with $p^{m+n}L$ countable and G/L is an almost direct sum of cyclic groups.

Since the second equivalence follows directly by the methods used in the proof of Theorem 2.21 from [2] or by an immediate application of the corresponding definitions, we will be concentrated on the first one. In fact, if $p^{m+n}L$ is countable, then $L = R \oplus T$, where R is countable and $p^{m+n}T = \{0\}$. Thus

$$G/L = G/(R \oplus T) \cong [G/(R \oplus p^nT)]/[(R \oplus T)/(R \oplus p^nT)]$$

being an almost direct sum of cyclic groups implies that $G/(R \oplus p^n T)$ is almost $p^{\omega+n}$-projective with $C = R \oplus p^n T$, so $p^m C = p^m R$ is countable, as asked for.

"$\Rightarrow$". Suppose by assumption that there is a p^m-bounded subgroup $K \leqslant G$ such that G/K is almost ω_1-$p^{\omega+n}$-projective. Owing to Theorem 2.25 of [2], there exists a countable (nice) subgroup C/K of G/K such that $(G/K)/(C/K) \cong G/C$ is almost $p^{\omega+n}$-projective and $C/K \subseteq p^\omega(G/K) = [\cap_{i<\omega}(p^i G + K)]/K$. Therefore, $C \leqslant G$, $C = K + L$ for some countable $L \leqslant C$ and $C \subseteq \cap_{i<\omega}(p^i G + K)$. These conditions together imply that $p^m C \subseteq L$ is countable and $p^m C \subseteq \cap_{i<\omega} p^{i+m} G = p^\omega G$, as required.

"$\Leftarrow$". Write $C = X \oplus V$, where X is countable and V is p^m-bounded. Hence $G/C = G/(X \oplus V) \cong [G/V]/(X \oplus V)/V$ is almost $p^{\omega+n}$-projective, where $(X \oplus V)/V \cong X$ is countable. Thus, in accordance with [2], G/V is almost ω_1-$p^{\omega+n}$-projective, as desired. Moreover, $(X \oplus V)/V$ can be chosen so that

$$\begin{aligned} p^m[(X \oplus V)/V] &= (p^m X \oplus V)/V \\ &= (p^m C \oplus V)/V \subseteq (p^\omega G + V)/V \subseteq p^\omega(G/V). \end{aligned}$$

This proves the preliminary claim.

Now, we have all the information necessary to prove the full assertion. To that aim we just will show that G is almost ω_1-$p^{\omega+m+n}$-projective $\iff$ there is $S \leqslant G$ such that $p^{m+n} S$ is countable and G/S is an almost direct sum of cyclic groups, which is precisely the stated above equivalence (compare with points (1) and (4) in Theorem 2.21 from [2]).

Necessity. Appealing to [2], G is almost ω_1-$p^{\omega+m+n}$-projective if there is a countable subgroup K with G/K being almost $p^{\omega+m+n}$-projective. Thus, again in view of [2], there exists $S \leqslant G$ containing K such that G/S is an almost direct sum of cyclic groups and $p^{m+n} S \subseteq K$. The last yields that $p^{m+n} S$ is countable, as required.

Sufficiency. Suppose now that there exists $S \leqslant G$ such that $p^{m+n} S$ is countable and G/S is an almost direct sum of cyclic groups. Therefore, the quotient $G/S \cong (G/p^{m+n} S)/(S/p^{m+n} S)$ being an almost direct sum of cyclic groups implies with the aid of [2] that $G/p^{m+n} S$ is almost $p^{\omega+m+n}$-projective. And since $p^{m+n} S$ is countable, again the application of [2] leads to G is almost ω_1-$p^{\omega+m+n}$-projective, as desired. $\square$

Remark 1. Note that the condition $p^m C \subseteq p^\omega G$ stated in the proof of Theorem 2.1 was at all redundant and therefore not further used. One of the important consequences of Theorem 2.1 is that (weakly) almost

m-ω_1-$p^{\omega+n}$-projective groups are almost ω_1-$p^{\omega+m+n}$-projective. Likewise, the central role of Theorem 2.1 is to demonstrate unambiguously that the concepts in Definitions 1.1 and 1.2 are nontrivial.

Imitating Theorem 2.1, it is quite natural to ask whether or not strongly almost m-ω_1-$p^{\omega+n}$-projective groups are exactly the strongly almost ω_1-$p^{\omega+m+n}$-projective ones. Referring to the following statement, this seems to be true.

Proposition 2.2. *If G is a strongly almost m-ω_1-$p^{\omega+n}$-projective group, then G is strongly almost ω_1-$p^{\omega+m+n}$-projective.*

Proof. Assume that there exists a p^m-bounded nice subgroup T of G such that G/T is strongly almost ω_1-$p^{\omega+n}$-projective. Thus there is a nice subgroup A/T of G/T with the property that $p^n A \subseteq T$ and $(G/T)/(A/T) \cong G/A$ is the sum of a countable group and an almost direct sum of cyclic groups. Hence $p^{n+m}A = \{0\}$ and A is nice in G (cf. [9]), which conditions ensure that G is strongly almost ω_1-$p^{\omega+m+n}$-projective, as claimed. □

As noted above, a question of some majority is of whether or not the converse holds, that is, whether or not every strongly almost ω_1-$p^{\omega+m+n}$-projective group is strongly almost m-ω_1-$p^{\omega+n}$-projective.

An other question of some interest, which immediately arises, is also whether or not almost $p^{\omega+m+n}$-projective groups are strongly almost m-ω_1-$p^{\omega+n}$-projective (and, in particular, weakly almost m-ω_1-$p^{\omega+n}$-projective). This is inspired by the fact that, taking $m = 0$, almost $p^{\omega+n}$-projective groups are themselves strongly almost ω_1-$p^{\omega+n}$-projective (cf. [7]).

In this way, we have the following weaker relationship:

Proposition 2.3. *If G is an almost $p^{\omega+m+n}$-projective group, then G is a (direct) decomposably almost m-ω_1-$p^{\omega+n}$-projective group.*

Proof. Let $P \leqslant G$ such that G/P is an almost direct sum of cyclic groups and $p^{m+n}P = \{0\}$. Since $G/P \cong [G/p^nP]/[P/p^nP]$, we deduce that G/p^nP is almost $p^{\omega+n}$-projective and hence it is a sum of a countable group and an almost $p^{\omega+n}$-projective group. But $p^m(p^nP) = \{0\}$ and so G is decomposably almost m-ω_1-$p^{\omega+n}$-projective, as promised. □

Remark 2. The converse implication is, however, not true as simple examples show. Nevertheless, decomposably almost m-ω_1-$p^{\omega+n}$-projective groups are eventually intermediate situated between almost $p^{\omega+m+n}$-projective groups and almost m-ω_1-$p^{\omega+n}$-projective groups.

For separable groups (i.e., groups without elements of infinite height) all of the above notions are tantamount; we do not consider here concrete examples to show that these concepts are independent for lengths beyond ω, but we refer the interested reader to [4], [5] or [6] for more details when the group length is $> \omega$.

Theorem 2.4. *Suppose G is a group such that $p^\omega G = \{0\}$. Then all of the next points are equivalent:*

(a) *G is almost ω_1-$p^{\omega+m+n}$-projective;*

(b) *G is almost m-ω_1-$p^{\omega+n}$-projective;*

(c) *G is strongly almost m-ω_1-$p^{\omega+n}$-projective;*

(d) *G is weakly almost m-ω_1-$p^{\omega+n}$-projective;*

(e) *G is decomposably almost m-ω_1-$p^{\omega+n}$-projective;*

(f) *G is nice decomposably almost m-ω_1-$p^{\omega+n}$-projective;*

(g) *G is nicely almost m-$p^{\omega+n}$-projective;*

(h) *G is almost $p^{\omega+m+n}$-projective.*

Proof. Apparently, all of the points (b)-(h) imply (a) and, in virtue of [2], we obtain that point (a) holds provided (h) is fulfilled. Moreover, it is easy to see that clause (g) implies all other ones. So, what remains to show is the implication (h) $\Rightarrow$ (g). To this purpose, [2] helps us to write that G/Z is an almost direct sum of cyclic groups for some subgroup $Z \leqslant G$ which is bounded by p^{m+n}. Thus $(G/Z[p^m])/(Z/Z[p^m]) \cong G/Z$ being an almost direct sum of cyclic groups guarantees again by [2] that $G/Z[p^m]$ is almost $p^{\omega+n}$-projective since $Z/Z[p^m] \cong p^m Z$ is obviously bounded by p^n. But $Z[p^m] = Z \cap G[p^m]$ and both Z and $G[p^m]$ are nice in G because G/Z is p^ω-bounded and $G/G[p^m] \cong p^m G \subseteq G$ is p^ω-bounded too. So, resulting, $Z[p^m]$ must be nice in G (see, e.g., [9]), and since $Z[p^m]$ is p^m-bounded, we consequently get the desired fact that G is nicely almost m-$p^{\omega+n}$-projective. $\square$

We now proceed with two useful necessary and sufficient conditions which are needed for applicable purposes in the next section.

Proposition 2.5. *The group G is strongly almost m-ω_1-$p^{\omega+n}$-projective if and only if there exists a p^m-bounded nice subgroup T of G such that $G/(T + p^{\omega+n}G)$ is almost $p^{\omega+n}$-projective and $p^{\omega+n}(G/T)$ is countable.*

Proof. It follows directly from [2] because the isomorphism

$$[G/T]/p^{\omega+n}(G/T) \cong G/(T + p^{\omega+n}G)$$

is fulfilled. □

Proposition 2.6. *The group G is weakly almost m-ω_1-$p^{\omega+n}$-projective if and only if there exists a p^m-bounded nice subgroup X of G such that $G/(X+p^{\omega+n}G)$ is almost ω_1-$p^{\omega+n}$-projective and $p^{\omega+n}(G/X)$ is countable.*

Proof. It follows immediately from [2] since the isomorphism

$$[G/X]/p^{\omega+n}(G/X) \cong G/(X + p^{\omega+n}G)$$

holds. □

3. Ulm subgroups and Ulm factors

In [7] it was proved that if the group G is strongly almost ω_1-$p^{\omega+n}$-projective, then so is $G/p^\alpha G$ for any ordinal α. Here we will give a simpler proof to the same fact devoted to almost ω_1-$p^{\omega+n}$-projective groups (see Proposition 2.13 (b) from [2], too).

Proposition 3.1. *If G is an almost ω_1-$p^{\omega+n}$-projective group, then $G/p^\alpha G$ is an almost ω_1-$p^{\omega+n}$-projective group for every ordinal α.*

Proof. For finite ordinals α, the assertion is self-evident. So, we will assume that α is infinite. By virtue of Theorem 2.21 (2) in [2], let G/A be the sum of a countable group and an almost direct sum of cyclic groups for some $A \leqslant G$ with $p^n A = \{0\}$. Thus, by utilizing the methods in [1] and [2], we deduce that $p^\alpha(G/A)$, being contained in a countable summand of G/A, remains countable and $[G/A]/p^\alpha(G/A)$ is again a sum of a countable group and an almost direct sum of cyclic groups. If $T \subseteq p^\alpha(G/A)$, the same is still true for $(G/A)/T$; we specially take $T = (p^\alpha G + A)/A$.

But the following isomorphisms hold:

$$[G/A]/(p^\alpha G + A)/A \cong G/(p^\alpha G + A) \cong [G/p^\alpha G]/(p^\alpha G + A)/p^\alpha G.$$

Observing that $p^n((p^\alpha G + A)/p^\alpha G) = \{0\}$, we are finished. □

Remark 4. Reciprocally, we showed in Theorem 2.16 of [2] that a group G is almost ω_1-$p^{\omega+n}$-projective if and only if $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is almost ω_1-$p^{\omega+n}$-projective.

Our further work in this section will be focussed on the behavior of the new group classes about Ulm subgroups and Ulm factors. Our main results presented below settle this matter in some aspect.

The following claim on niceness is pivotal. Its proof, although not difficult, is rather technical, so that we leave it to the interested readers.

Lemma 3.2. *Suppose N is a nice subgroup of a group A and $M \subseteq p^\lambda A$ for some infinite ordinal λ where $p^\lambda A$ is bounded. Then $(N+M)/M$ is nice in A/M.*

Lemma 3.3. *Suppose that A is a group with a subgroup B such that A/B is bounded. Thenthe following are true:*

(a) *If N is nice in B, then N is nice in A.*

(b) *If M is nice in A, then $M \cap B$ is nice in B.*

Proof. Appealing to [9], note that a subgroup V of a group W is nice if, for any limit ordinal δ, the equality $\cap_{\alpha<\delta}(V+p^\alpha W) = V + p^\delta W$.

(a) Since $p^j A \subseteq B$ for some $j \in \mathbb{N}$ and hence $p^\omega A = p^\omega B$, it suffices to check the equality only for the ordinal ω. In fact,

$$\begin{aligned}\cap_{i<\omega}(N+p^iA) &= \cap_{j\leqslant i<\omega}(N+p^iA) \subseteq \cap_{k<\omega}(N+p^kB)\\ &= N+p^\omega B \subseteq N+p^\omega A,\end{aligned}$$

as required.

(b) We subsequently deduce that

$$\begin{aligned}\cap_{\alpha<\delta}(M\cap B+p^\alpha B) &\subseteq \cap_{\alpha<\delta}(M+p^\alpha A)\cap B = (M+p^\delta A)\cap B\\ &= (M+p^\delta B)\cap B = M\cap B+p^\delta B,\end{aligned}$$

as required, where the last equality follows by the modular law. □

We now proceed by proving with the next crucial statement, needed for our further application.

Proposition 3.4. *Let A be a group and $\lambda \geqslant \omega$ an ordinal.*

(i) *If A is strongly almost ω_1-$p^{\omega+n}$-projective and $Z \subseteq p^\lambda A$, where $p^\lambda A$ is bounded, then A/Z is strongly almost ω_1-$p^{\omega+n}$-projective.*

(ii) *If $X \subseteq p^{\omega+n}A$, $p^{\omega+n}A$ is countable and A/X is strongly almost ω_1-$p^{\omega+n}$-projective, then A is also strongly almost ω_1-$p^{\omega+n}$-projective.*

Proof. (i) Let Q be a nice subgroup of A with $p^nQ = \{0\}$ and suppose A/Q is the sum of a countable group and an almost direct sum of cyclic groups, say $A/Q = K + S$. It is easily seen that $Q' = (Q + Z)/Z$ is p^n-bounded and in accordance with Lemma 3.2 it is nice in $A' = A/Z$ as well. In addition, $A'/Q' \cong A/(Q + Z) \cong [A/Q]/[(Q + Z)/Q]$ and $(Q + Z)/Q \subseteq (Q + p^\lambda A)/Q = p^\lambda(A/Q)$. Since $K \cap S \subseteq S$ is countable, there exists a countable nice subgroup C of S such that $K \cap S \subseteq C$. Consequently, $(A/Q)/C = [(K+C)/C] \oplus [S/C]$. Since S/C is p^ω-bounded, we derive that

$$(p^\lambda(A/Q) + C)/C \subseteq p^\lambda((A/Q)/C) = p^\lambda((K + C)/C)$$

is countable, whence so is $p^\lambda(A/Q)$. Furthermore, in virtue of Lemma 2.16 from [1], we observe that A/Q is actually almost ω_1-p^ω-projective. Since $p^\lambda(A/Q)$ is countable, we obtain the same for $(Q + Z)/Q$ and thus in accordance with Theorem 2.23 of [2], we conclude that A'/Q' is also ω_1-p^ω-projective, as required.

(ii) With the aid of [7] we observe that the quotient

$$[A/X]/p^{\omega+n}(A/X) = [A/X]/[p^{\omega+n}A/X] \cong A/p^{\omega+n}A$$

is almost $p^{\omega+n}$-projective. We next again employ [7] to derive that A is strongly almost ω_1-$p^{\omega+n}$-projective, as asserted. □

The next statement is pivotal.

Lemma 3.5. *Suppose that A is a group with a subgroup B such that A/B is bounded. Then*

(i) *A is almost $p^{\omega+n}$-projective if and only if B is almost $p^{\omega+n}$-projective.*

(ii) *A is strongly almost ω_1-$p^{\omega+n}$-projective if and only if B is strongly almost ω_1-$p^{\omega+n}$-projective.*

(iii) *A is (strongly) almost m-ω_1-$p^{\omega+n}$-projective if and only if B (strongly) almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. (i) It is straightforward.

(ii) Since $p^tA \subseteq B$ for some $t \in \mathbb{N}$, we obtain that $p^\omega A = p^\omega B$ and thus $p^{\omega+n}A = p^{\omega+n}B$. Moreover, in virtue of (i), $B/p^{\omega+n}B = B/p^{\omega+n}A$ is almost $p^{\omega+n}$-projective uniquely when $A/p^{\omega+n}A$ is almost $p^{\omega+n}$-projective, because the factor-group $(A/p^{\omega+n}A)/(B/p^{\omega+n}A) \cong A/B$ remains bounded. We finally apply [7] to conclude the claim.

(iii) "$\Rightarrow$". Let A/H be strongly almost ω_1-$p^{\omega+n}$-projective for some $H \leqslant A[p^m]$ (which is nice in A). Since $[A/H]/[(B+H)/H] \cong A/(B+H)$ remains bounded as an epimorphic image of A/B, we deduce with the help of (ii) that $(B+H)/H \cong B/(B \cap H)$ is strongly almost ω_1-$p^{\omega+n}$-projective. In addition, $B \cap H \leqslant B[p^m]$ (which is nice in B), and we are finished.

"$\Leftarrow$". Let B/L be strongly ω_1-$p^{\omega+n}$-projective factor-group for some $L \leqslant B[p^m]$ (which is nice in B). Since $[A/L]/[B/L] \cong A/B$ is bounded, point (ii) is applicable to infer that A/L is strongly almost ω_1-$p^{\omega+n}$-projective. But $L \leqslant A[p^m]$ (which is nice in A), and we are done.

The niceness in both directions follows immediately from Lemma 3.3. □

We have now at our disposal all the ingredients needed to prove the following basic assertion on both Ulm subgroups and Ulm factors pertaining to the other remaining group classes.

Proposition 3.6. *If the group G is either*

(a) *strongly almost m-ω_1-$p^{\omega+n}$-projective or*

(b) *weakly almost m-ω_1-$p^{\omega+n}$-projective or*

(c) *nice direct decomposably almost m-ω_1-$p^{\omega+n}$-projective or*

(d) *nicely almost m-$p^{\omega+n}$-projective,*

then the same are both $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.

Proof. (a) Suppose that G/T is strongly almost ω_1-$p^{\omega+n}$-projective for some nice p^m-bounded subgroup T of G. Thus

$$p^\alpha G/(p^\alpha G \cap T) \cong (p^\alpha G + T)/T = p^\alpha(G/T)$$

is also strongly almost ω_1-$p^{\omega+n}$-projective in view of [7], with $p^\alpha G \cap T$ being p^m-bounded and nice in $p^\alpha G$ (cf. [9]). Hence $p^\alpha G$ is strongly almost m-ω_1-$p^{\omega+n}$-projective as well.

To show the second part, we consequently apply again [7] to infer that

$$\begin{aligned}(G/T)/p^\alpha(G/T) &= (G/T)/(p^\alpha G + T)/T \cong G/(p^\alpha G + T)\\ &\cong (G/p^\alpha G)/(p^\alpha G + T)/p^\alpha G\end{aligned}$$

is also strongly almost ω_1-$p^{\omega+n}$-projective. Moreover, it is plainly observed that $(p^\alpha G + T)/p^\alpha G$ is bounded by p^m because so is T, and that

$(p^\alpha G+T)/p^\alpha G$ is nice in $G/p^\alpha G$ since it is well known that $p^\alpha G+T$ is nice in G - see, for example, [9].

(b) Suppose G/X is almost ω_1-$p^{\omega+n}$-projective for some nice $X \leqslant G$ with $p^m X=\{0\}$. Observe that the following relations are valid:

$$p^\alpha G/(p^\alpha G\cap X)\cong(p^\alpha G+X)/X\subseteq G/X.$$

But a subgroup of an almost ω_1-$p^{\omega+n}$-projective group is again almost ω_1-$p^{\omega+n}$-projective (cf. [2]). Thus $p^\alpha G/(p^\alpha G\cap X)$ is almost ω_1-$p^{\omega+n}$-projective as well. Moreover, $p^\alpha G\cap X$ is obviously p^m-bounded and also, in accordance with [9], it is nice in $p^\alpha G$. So, $p^\alpha G$ is weakly almost m-ω_1-$p^{\omega+n}$-projective.

Furthermore,

$$\begin{aligned}(G/X)/p^\alpha(G/X)&=(G/X)/(p^\alpha G+X)/X\cong G/(p^\alpha G+X)\\&\cong(G/p^\alpha G)/(p^\alpha G+X)/p^\alpha G\end{aligned}$$

is almost ω_1-$p^{\omega+n}$-projective too, owing to Proposition 3.1.

Besides, it is obviously seen that

$$p^m((p^\alpha G+X)/p^\alpha G)=(p^{\alpha+m}G+p^\alpha G)/p^\alpha G=\{0\},$$

and in the case of niceness that $(p^\alpha G+X)/p^\alpha G$ is nice in $G/p^\alpha G$ because it is well known that $p^\alpha G+X$ is nice in G, see [9], for instance.

(c) Accordingly, write $G/H=B\oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective for some p^m-bounded nice subgroup H of G. But

$$p^\alpha G/(p^\alpha G\cap H)\cong(p^\alpha G+H)/H=p^\alpha(G/H)=p^\alpha B\oplus p^\alpha R,$$

where $p^\alpha B$ is obviously countable and $p^\alpha R$ is by [2] almost $p^{\omega+n}$-projective. Since $p^\alpha G\cap H$ is p^m-bounded and nice in $p^\alpha G$ (see [9]), we derive that $p^\alpha G$ is nice direct decomposably almost m-ω_1-$p^{\omega+m}$-projective, as stated.

Concerning the other part, the direct sum

$$\begin{aligned}(B/p^\alpha B)\oplus(R/p^\alpha R)&\cong[G/H]/p^\alpha(G/H)\\&\cong G/(p^\alpha G+H)\cong[G/p^\alpha G]/(p^\alpha G+H)/p^\alpha G\end{aligned}$$

is again a direct sum of a countable group and an almost $p^{\omega+n}$-projective group, because of the obvious facts that $B/p^\alpha B$ is countable and $R/p^\alpha R$ is almost $p^{\omega+n}$-projective, where the later one exploits [2]. In this vein, it is self-evident that $(p^\alpha G+H)/p^\alpha G$ is bounded by p^m and, in conjunction with [9], that $(p^\alpha G+H)/p^\alpha G$ is nice in $G/p^\alpha G$, as required.

(d) Given a p^m-bounded nice subgroup Y of G such that G/Y is almost $p^{\omega+n}$-projective. Hence, in view of [2], $p^\alpha G/(p^\alpha G \cap Y) \cong (p^\alpha G + Y)/Y \subseteq G/Y$ is almost $p^{\omega+n}$-projective as well, with $p^\alpha G \cap Y$ being p^m-bounded and nice in $p^\alpha G$ (cf. [9]).

On the other hand,

$$\begin{aligned}(G/p^\alpha G)/(Y + p^\alpha G)/p^\alpha G &\cong G/(Y + p^\alpha G) \\ &\cong (G/Y)/(Y + p^\alpha G)/Y = (G/Y)/p^\alpha(G/Y)\end{aligned}$$

is almost $p^{\omega+n}$-projective by exploiting [2]. Since $(Y + p^\alpha G)/p^\alpha G \cong Y/(Y \cap p^\alpha G)$ is p^m-bounded and nice in $G/p^\alpha G$ (see [9]), the assertion follows. $\square$

Under some extra restrictions on α, we can say even a little more:

Proposition 3.7. *If G is a nice direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group, then $G/p^{\alpha+m}G$ is nicely almost m-$p^{\omega+n}$-projective for every ordinal $\alpha \leqslant \omega + n$. In particular, $G/p^{\omega+m+n}G$ is nicely almost m-$p^{\omega+n}$-projective.*

Proof. By Definition 1.3, we write that $G/H = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective for some p^m-bounded nice subgroup H of G. An appeal to the proof of Proposition 3.6 (c) gives that

$$[G/H]/p^\alpha(G/H) \cong G/(p^\alpha G + H) \cong [G/p^{\alpha+m}G]/(p^\alpha G + H)/p^{\alpha+m}G.$$

is almost $p^{\omega+n}$-projective with

$$p^m((p^\alpha G + S)/p^{\alpha+m}G) = p^{\alpha+m}G/p^{\alpha+m}G = \{0\},$$

so that the claim follows. The final part is an immediate consequence by taking $\alpha = \omega + n$. $\square$

The following somewhat supplies Proposition 3.5 listed above.

Proposition 3.8. *If G is a direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group, then $p^\alpha G$ is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective for all ordinals α. In particular, if $\alpha \geqslant \omega$, then $p^\alpha G$ is almost ω_1-$p^{\omega+m}$-projective.*

In addition, if G is a nice direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group and $\alpha \geqslant \omega$, then $p^\alpha G$ is strongly almost ω_1-$p^{\omega+m}$-projective.

Proof. Using Definition 1.3, let $S \leqslant G[p^m]$ such that $G/S = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective. If $\alpha \geqslant \omega$, then one sees that $p^\alpha G/(p^\alpha G \cap S) \cong (p^\alpha G + S)/S \subseteq p^\alpha(G/S) = K \oplus P$ where K is countable and P is p^n-bounded. Hence $p^\alpha G/(p^\alpha G \cap S)$ is also such a direct sum of a countable group and a p^n-bounded group (which itself is a direct sum of cyclic groups) with p^m-bounded intersection $S \cap p^\alpha G$, so that $p^\alpha G$ is almost ω_1-$p^{\omega+m}$-projective.

If now $\alpha < \omega$ is finite, then in virtue of [2] the quotient

$$p^\alpha G/(p^\alpha G \cap S) \cong (p^\alpha G + S)/S = p^\alpha(G/S) = p^\alpha B \oplus p^\alpha R$$

is again a direct sum of the countable group $p^\alpha B$ and the almost $p^{\omega+n}$-projective group $p^\alpha R$, as needed. That is why, in both cases, $p^\alpha G$ is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective.

The final part follows easily since S being nice in G yields that $S \cap p^\alpha G$ is nice in $p^\alpha G$ (cf. [9]). $\square$

We now strengthen the idea in the proof of Proposition 3.1 by the following statement; however we cannot yet establish that, for all ordinals α, the Ulm factor $G/p^\alpha G$ possesses the direct decomposable almost m-ω_1-$p^{\omega+n}$-projective property provided that the same holds for G.

Proposition 3.9. *If G is a direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group, then $G/p^\alpha G$ is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective for all ordinals $\alpha \geqslant \omega + n$.*

Proof. Utilizing Definition 1.3, write that $G/S = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective for some p^m-bounded subgroup S of G.

Standardly, the following isomorphisms are true:

$$(G/p^\alpha G)/[(S + p^\alpha G)/p^\alpha G] \cong G/(S + p^\alpha G) \cong (G/S)/[(S + p^\alpha G)/S].$$

Moreover, $(S + p^\alpha G)/S \subseteq p^\alpha(G/S) = p^\alpha B$. Therefore, setting $T = (S + p^\alpha G)/S$, we deduce that

$$(G/S)/T = (B \oplus R)/T \cong (B/T) \oplus R$$

is again a direct sum of a countable group and an almost $p^{\omega+n}$-projective group. And since $p^m((p^\alpha G + S)/p^\alpha G) = \{0\}$, we are finished. $\square$

Remark 5. When $\alpha = \omega$, we know by [2] or by Theorem 2.4 that $G/p^{\omega}G$ must be almost $p^{\omega+m+n}$-projective and thus in virtue of Proposition 2.3 it is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective. However, the unsettled situation is when $\omega < \alpha < \omega + n$.

Now, we are ready to establish the following:

Theorem 3.10 (First Reduction Criterion). *The group G is (strongly) almost m-ω_1-$p^{\omega+n}$-projective if and only if the following two conditions are fulfilled:*

(1) *$p^{\omega+m+n}G$ is countable;*

(2) *$G/p^{\omega+m+n}G$ is (strongly) almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". As observed before, G is almost ω_1-$p^{\omega+m+n}$-projective, so point (1) follows automatically appealing to [2]. Concerning point (2), it follows immediately from Proposition 3.6(a).

"$\Leftarrow$". Assume now that clauses (1) and (2) are valid. For convenience put $k = m+n$. By definition, let $L/p^{\omega+k}G \leqslant G/p^{\omega+k}G$ be a p^m-bounded subgroup such that $(G/p^{\omega+k}G)/(L/p^{\omega+k}G) \cong G/L$ is strongly almost ω_1-$p^{\omega+n}$-projective. Thus $p^mL \subseteq p^{\omega+k}G$. Since G/L is $p^{\omega+k+m}$-bounded, we see that $p^{\omega+n}(G/L)$ is bounded (by p^{2m}), and applying Proposition 3.4 (i) to G/L, we deduce that

$$(G/L)/(p^{\omega+n}G + L)/L \cong G/(p^{\omega+n}G + L)$$

is strongly almost ω_1-$p^{\omega+n}$-projective, because $(p^{\omega+n}G+L)/L \subseteq p^{\omega+n}(G/L)$. Putting $M = p^{\omega+n}G + L$, it is obvious that $p^{\omega+n}G \subseteq M$ and $p^mM = p^{\omega+k}G$. That is why, G/M is strongly almost ω_1-$p^{\omega+n}$-projective with $M \leqslant G$ satisfying the above two relations.

Furthermore, supposing that Y is a maximal p^m-bounded summand of $p^{\omega+n}G$, so there is a direct decomposition $p^{\omega+n}G = X \oplus Y$ and, by what we have just shown above, the inclusions $X \subseteq p^{\omega+n}G \subseteq M$ are true. We can without loss of generality assume that X is countable because of the following reasons: Since $p^{\omega+k}G = p^mX$ is countable, it follows that $X = K \oplus Z$ where K is countable and Z is p^m-bounded. Therefore, $p^{\omega+n}G = K \oplus Z \oplus Y = K \oplus Y'$ where $Y' = Z \oplus Y$, as needed.

We next routinely verify that $X[p] = (p^{\omega+k}G)[p]$ and thus $Y \cap p^{\omega+k}G = \{0\}$. So, suppose H is a $p^{\omega+k}$-high subgroup of G such that $H \supseteq Y$. Now, $G[p] = (p^{\omega+k}G)[p] \oplus H[p] = X[p] \oplus H[p]$ together with H being pure in G (cf. [9]) readily force that $G[p^m] = X[p^m] \oplus H[p^m]$ whenever $m \geqslant 1$. In fact, given $g \in G$ with $p^mg \in p^{\omega+k}G$, we write $p^mg = p^ma$

where $a \in p^{\omega+n}G = X \oplus Y$. Then $p^m g = p^m x$ for some $x \in X$, whence $g \in x + G[p^m] \subseteq X + H[p^m]$, as required.

Besides, $X \cap H[p^m] \subseteq X \cap H = \{0\}$ and consequently $(G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G$ because $p^{\omega+k}G = p^m X \subseteq X$. Since $M/p^{\omega+k}G \subseteq (G/p^{\omega+k}G)[p^m]$, it follows that $M \subseteq X \oplus H[p^m]$ and hence

$$M = (X \oplus H[p^m]) \cap M = X + H[p^m] \cap M$$

by virtue of the modular law. Substituting $P = H[p^m] \cap M$, we derive that $p^m P = \{0\}$ and that $M = X + P$. In addition, $M = M + p^{\omega+n}G = P + p^{\omega+n}G$ and so $G/(p^{\omega+n}G + P) \cong (G/P)/(p^{\omega+n}G + P)/P$ is strongly almost ω_1-$p^{\omega+n}$-projective.

We now claim that $p^{\omega+n}(G/P)$ is countable. In fact, $p^{\omega+n}(G/M)$ is countable because G/M is strongly almost ω_1-$p^{\omega+n}$-projective (see [7]). But we subsequently have that

$$\begin{aligned} p^{\omega+n}(G/M) &= p^n(p^\omega(G/M)) = p^n(\cap_{i<\omega}(p^iG + M)/M) \\ &= p^n(\cap_{i<\omega}(p^iG + P)/M) \cong p^n(\cap_{i<\omega}[(p^iG + P)/P]/[M/P]) \\ &= p^n(p^\omega(G/P)/[M/P]) = [p^{\omega+n}(G/P) + (M/P)]/[M/P] \\ &= p^{\omega+n}(G/P)/[M/P] \end{aligned}$$

since $M/P = (p^{\omega+n}G + P)/P \subseteq p^{\omega+n}(G/P)$. Moreover,

$$\begin{aligned} M/P &= M/(M \cap H[p^m]) \cong (M + H[p^m])/H[p^m] \\ &= (X + H[p^m])/H[p^m] \cong X/(X \cap H[p^m]) \cong X \end{aligned}$$

is countable. Finally, $p^{\omega+n}(G/P)$ is countable as well, as claimed.

Also, because $(p^{\omega+n}G + P)/P \leqslant p^{\omega+n}(G/P)$, Proposition 3.4 (ii) applied to G/P shows that G/P is strongly almost ω_1-$p^{\omega+n}$-projective with $p^m P = \{0\}$, as required.

As for the "niceness" property, it can be established as Theorem 3.12 quoted below. □

Now, with Proposition 2.5 at hand, we deduce the following consequence.

Corollary 3.11. *Suppose that $p^\lambda G$ is countable for some ordinal $\lambda \geqslant \omega$. Then the group G is (strongly) almost m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We henceforth have all the information to prove our next basic result.

Theorem 3.12 (Second Reduction Criterion)**.** *The group G is weakly almost m-ω_1-$p^{\omega+n}$-projective if and only if*

(1) *$p^{\omega+m+n}G$ is countable;*

(2) *$G/p^{\omega+m+n}G$ is weakly almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". It follows directly from [2] together with Proposition 3.6 (b).

"$\Leftarrow$". For our convenience, set $k = m+n$. By definition, let $T/p^{\omega+k}G \leqslant G/p^{\omega+k}G$ be a p^m-bounded nice subgroup such that

$$(G/p^{\omega+k}G)/(T/p^{\omega+k}G) \cong G/T$$

is almost ω_1-$p^{\omega+n}$-projective. Thus T is nice in G (see, e.g., [9]), and $p^mT \subseteq p^{\omega+k}G$. Applying Proposition 3.1 or Proposition 2.13 (b) in [2],

$$G/(T + p^{\omega+n}G) \cong [G/T]/(T + p^{\omega+n}G)/T = [G/T]/p^{\omega+n}(G/T)$$

is also almost ω_1-$p^{\omega+n}$-projective. Putting $T' = T + p^{\omega+n}G$, we see that G/T' is almost ω_1-$p^{\omega+n}$-projective and that $T' \supseteq p^{\omega+n}G$ remains nice in G and $p^mT' = p^mT + p^{\omega+k}G = p^{\omega+k}G$. So, replacing hereafter T' with T, we may without loss of generality assume that $p^{\omega+n}G \leqslant T$.

Suppose now Y is a maximal p^m-bounded summand of $p^{\omega+n}G$; so there exists a direct decomposition $p^{\omega+n}G = X \oplus Y$ and thus the inclusions $X \subseteq p^{\omega+n}G \subseteq T$ hold. We may also assume with no harm of generality that X is countable; in fact, $p^{\omega+k}G = p^mX$ is countable and therefore we can decompose $X = K \oplus Z$, where K is countable and Z is p^m-bounded (whence Z is a p^m-bounded summand of $p^{\omega+n}G$ and so $Z \subseteq Y$). Consequently, it is readily checked that $p^{\omega+n}G = K \oplus Y$ with countable summand K, as wanted.

Next, a straightforward check shows that $X[p] = (p^{\omega+k}G)[p] = (p^mX)[p]$ and thus $Y \cap p^{\omega+k}G = \{0\}$ because

$$(Y \cap p^{\omega+k}G)[p] = Y \cap (p^{\omega+k}G)[p] = Y \cap X[p] = \{0\}.$$

Let us now H be a $p^{\omega+k}$-high subgroup of G containing Y (thus H is maximal with respect to $H \cap p^{\omega+k}G = \{0\}$ with $H \supseteq Y$). We now assert that

$$(G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G.$$

In fact, as noted above, $X[p] = (p^{\omega+k}G)[p]$ and thereby $X \cap H = \{0\}$ because

$$(X \cap H)[p] = X[p] \cap H = (p^{\omega+k}G)[p] \cap H = \{0\}.$$

Since $G[p] = (p^{\omega+k}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and H is pure in G (see [9]), it plainly follows that $G[p^m] = X[p^m] \oplus H[p^m]$. To prove this, given $v \in G$ with $p^m v \in p^{\omega+k}G$, it suffices to show that $v \in X \oplus H[p^m]$. In fact, $p^m v = p^m d$ where $d \in p^{\omega+n}G = X \oplus Y$. Then $p^m d = p^m x$ for some $x \in X$ and so $p^m v = p^m x$. Therefore,

$$v \in x + G[p^m] = x + X[p^m] + H[p^m] \subseteq X + H[p^m],$$

as required. So, the assertion is sustained.

Furthermore, by what we have obtained above,

$$T/p^{\omega+k}G \subseteq (G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G$$

implies that $T \subseteq X \oplus H[p^m]$; note also that $X \subseteq T$. Put $L = T \cap H[p^m] \subseteq H$, so that it is clear that $L \cap p^{\omega+k}G = \{0\}$. Moreover, the modular law ensures that

$$T = (X \oplus H[p^m]) \cap T = X \oplus (T \cap H[p^m]) = X \oplus L.$$

We consequently conclude that $T = p^{\omega+n}G + T = p^{\omega+n}G + L$ and $G/T = G/(p^{\omega+n}G + L)$ is almost ω_1-$p^{\omega+n}$-projective. Observe also that L is p^m-bounded, and that L is nice in G. The first fact is trivial, as for the second one $L \cap p^{\omega+k}G = \{0\}$ easily forces that $L \cap p^{\omega+n}G$ is nice in $p^{\omega+n}G$ and thus it is nice in G. On the other hand, as noticed above, $p^{\omega+n}G + L = T$ is also nice in G. According to [9], these two conditions together imply that L is nice in G, as expected.

What remains to illustrate is that $p^{\omega+n}(G/L)$ is countable. Indeed, we have $p^{\omega+n}(G/L) = (p^{\omega+n}G + L)/L = T/L$. Also,

$$\begin{aligned} T/L &= T/(T \cap H[p^m]) \cong (T + H[p^m])/H[p^m] \\ &= (p^{\omega+n}G + H[p^m])/H[p^m] \cong p^{\omega+n}G/(p^{\omega+n}G \cap H[p^m]). \end{aligned}$$

But as obtained above, $p^{\omega+n}G = X \oplus Y$ and since $Y \subseteq H$, we have with the aid of the modular law that $p^{\omega+n}G \cap H = (X \oplus Y) \cap H = (X \cap H) \oplus Y = Y$, whence $p^{\omega+n}G \cap H[p^m] = Y[p^m]$. We therefore establish that

$$T/L \cong (X \oplus Y)/Y[p^m] \cong X \oplus (Y/Y[p^m]) \cong X \oplus p^m Y = X.$$

Since X is shown above to be countable, so does $T/L = p^{\omega+n}(G/L)$. We finally apply Proposition 2.6 to get the desired claim. □

Mimicking the method demonstrated above, with Proposition 2.6 in hand we can state:

Corollary 3.13. *Let $\lambda \geqslant \omega$ be an ordinal such that $p^\lambda G$ is countable. Then the group G is weakly almost m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We now ready to establish our next reduction theorem.

Theorem 3.14 (Third Reduction Criterion)**.** *The group G is nicely almost m-$p^{\omega+n}$-projective if and only if*

(1) *$p^{\omega+m+n}G$ is countable;*

(2) *$G/p^{\omega+m+n}G$ is nicely almost m-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". Clause (1) follows immediately as above.

As for clause (2), it follows directly by Proposition 3.6 (d).

"$\Leftarrow$". Assume that (1) and (2) are fulfilled, so that let there exist a nice p^m-bounded subgroup $A/p^{\omega+m+n}G$ of $G/p^{\omega+m+n}G$ with $A \leqslant G$ such that G/A is almost $p^{\omega+n}$-projective. Thus, as we have seen before, $p^m A \subseteq p^{\omega+k}G$ for $k = m + n$, and A is nice in G. Imitating the same technique as in Theorems 3.10 and 3.12, we can find a p^m-bounded nice subgroup N of G such that G/N is almost $p^{\omega+n}$-projective, and so we complete the arguments. $\square$

Same as above, we derive:

Corollary 3.15. *Let $\lambda \geqslant \omega$ be an ordinal for which $p^\lambda G$ is countable. Then the group G is nicely almost m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We will be now concentrated on nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups, which are somewhat difficult to handle. So, we will restrict our attention on the ideal case $n = 1$ by showing that the investigation of nice decomposably almost m-ω_1-$p^{\omega+1}$-projective groups can be reduced to these of length not exceeding $\omega + m + 1$. Specifically, the following holds:

Theorem 3.16 (Fourth Reduction Criterion)**.** *The group G is nice direct decomposably almost m-ω_1-$p^{\omega+1}$-projective if and only if*

(1) *$p^{\omega+m+1}G$ is countable;*

(2) *$G/p^{\omega+m+1}G$ is nice direct decomposably almost m-ω_1-$p^{\omega+m+1}$-projective.*

Proof. The "and only if" part follows directly as above in a combination with Proposition 3.6 (c), respectively.

Concerning the "if" part, we set for simpleness $k = m + 1$. Using the corresponding definition, suppose $T/p^{\omega+k}G \leqslant G/p^{\omega+k}G$ is a p^m-bounded nice subgroup such that $[G/p^{\omega+k}G]/[T/p^{\omega+k}G] \cong G/T$ is a direct sum of a countable group and an almost $p^{\omega+1}$-projective group. Hence T is nice in G (see, e.g., [9]), and $p^mT \subseteq p^{\omega+k}G$. Also, it is routinely checked that $[G/T]/p^{\omega+1}(G/T) \cong G/(T + p^{\omega+1}G)$ is almost $p^{\omega+1}$-projective. Henceforth, the proof goes on imitating the same scheme of proof as that in Theorems 3.10 and 3.12 to infer the wanted statement. $\square$

Remark 6. As observed in Proposition 3.6 (c), the necessity in Theorem 3.16 is valid for any natural n. However, the sufficiency probably fails for each other $n > 1$.

4. Open questions

We close the work with certain challenging problems which are worthwhile for a further study.

Problem 1. Is it true that weakly almost n-ω_1-$p^{\omega+m}$-projective groups are almost m-ω_1-$p^{\omega+n}$-projective?

Problem 2. Are (strongly) almost m-ω_1-$p^{\omega+n}$-projective groups strongly almost ω_1-$p^{\omega+m+n}$-projective?

Problem 3. Does it follow that nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups are strongly almost m-ω_1-$p^{\omega+n}$-projective?

Correction. In the proof of Theorem 2.23 from [2], on lines 4 and 6 the phrase *"almost $p^{\omega+n}$-projective"* should be stated as *"almost ω_1-$p^{\omega+n}$-projective"*. The omission "ω_1" was involuntarily.

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On solvable Z_3-graded alternative algebras

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Abstract. Let $A = A_0 \oplus A_1 \oplus A_2$ be an alternative Z_3-graded algebra. The main result of the paper is the following: if A_0 is solvable and the characteristic of the ground field not equal 2,3 and 5, then A is solvable.

1. Introduction

Let R be an algebra over a field F. Let G be a finite group of automorphisms of R, and $R^G = \{x \in R | \phi(x) = x \text{ for all } \phi \in G\}$ be a fixed points subalgebra of R.

For Lie algebras there is a classical Higman result: if a Lie algebra L has an automorphism ϕ of simple order p without fixed points ($L^\phi = 0$), then L is nilpotent [1]. Moreover, nil index $h(p)$ in this case depends only on the order p. The explicit estimate of the function $h(p)$ was found in the paper of Kreknin and Kostrikin [2]. At the same time Kreknin proved that a Lie ring with a regular automorphism of an arbitrary finite order is solvable [3]. It is also worth mentioning here a result of Makarenko [4] who proved that if a Lie algebra L admits an automorphism of a prime order p with a finite-dimensional fixed-point subalgebra of dimension t, then L has a nilpotent ideal of nilpotency class bounded in terms of p and of codimension bounded in terms of t and p.

If R is an associative algebra with a finite group of automorphisms G then classical Bergman-Isaacs theorem says that if the subalgebra of fixed

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points R^G is nilpotent and R has no $|G|$-torsion, then R is nilpotent [5]. Kharchenko proved that under the same conditions, if R^G is a PI-ring, then R is a PI-ring [6]. For Jordan algebras the analogue of Kharchenko's result was proved by Semenov [7].

The Bergman-Isaacs theorem was partially generalized by Martindale and Montgomery to the case when G is a finite group of so called Jordan automorphisms, that is a linear automorphisms that are automorphisms of the adjoint Jordan algebra $R^{(+)}$ (note that in this case R^G is not a subalgebra in R, but a subalgebra in $R^{(+)}$) [8].

Note, that in general for Jordan algebras Bergman-Isaacs theorem is false - there is an example of a solvable non-nilpotent Jordan algebra J with an automorphism of second order ϕ such that the ring of invariants J^ϕ is nilpotent. However, Zhelyabin in [11] proved, that if a Jordan algebra J over a field of characteristic not equal 2,3 admits an automorphism of second order ϕ such that the algebra of invariants J^ϕ is solvable, then J is solvable.

For alternative algebras in [12] it was proved that if A is an alternative algebra over a field of characteristic not equal 2 with an automorphism g of second order then the solvability of the algebra of fixed points A^g implies the solvability of A. On the other hand, if the characteristic of the ground field is zero and G is a finite group of automorphisms of an alternative algebra A, then again the solvability of the algebra of fixed points A^G implies the solvability of A [7]. At the same time it is not known if the similar result is true in positive characteristic.

In this work we study a special case of the problem for alternative algebras: we consider a Z_3-graded alternative algebra $A = A_0 \oplus A_1 \oplus A_2$ and prove, that if the characteristic of the ground field not equal 2,3 and 5 and A_0 is solvable, then A is solvable.

As a consequence we obtain the following result: if A is an alternative algebra with an automorphism ϕ of order $2^k 3^l$, then under the same conditions on the characteristic of the ground field, the solvability of the subalgebra of fixed points A^ϕ implies the solvability of A.

2. Definitions and preliminary results

Let F be a field of characteristic not equal 2,3,5, A be an algebra over F. If $x, y, z \in A$ then $(x, y, z) = (xy)z - x(yz)$ is the associator of elements x, y, z, $x \circ y = xy + xy$ is a Jordan product of elements x and y and $[x, y] = xy - yx$ is a commutator of the elements x and y.

Definition. An algebra A is called Z_3-graduated if A is a direct sum of subspaces $A_i,\ i \in Z_3 : A = A_0 \oplus A_1 \oplus A_2$ and $A_iA_j \subseteq A_{i+j}$.

If A is a Z_3-graded algebra, then for every $i \in Z_3$ and $x \in A$ by x_i we will denote the projection of the element x to the subspace A_i and if $M \subset A$ then $M_i = \{x_i |\ x \in M\}$. An ideal I of A is called *homogeneous* if $I_j \subset I, \quad j = 0,1,2$. If I is a homogeneous ideal of A, then the factor-algebra A/I is also a Z_3-graded algebra.

If ϕ is an automorphism of the algebra A, then by A^{ϕ} we denote the subalgebra of fixed points of ϕ, that is $A^g = \{x \in A|\ \phi(x) = x\}$.

Define subsets A^i, $A^{<i>}$ and $A^{(i)}$ as:

$$A^2 = A^{<2>} = A^{(1)} = AA, \quad A^n = \sum_{i=1}^{n-1} A^i A^{n-i}, \quad A^{<n>} = A^{<n-1>}A$$

$$A^{(1)} = A^2, \quad A^{(i)} = A^{(i-1)}A^{(i-1)}.$$

Definition. An algebra A is called *nilpotent* if $A^i = 0$ for some i. An algebra is called *solvable*, if $A^{(i)} = 0$ for some i.

It is clear that $A^{(i)} \subset A^{2^i}$, so every nilpotent algebra is solvable. If A is an associative algebra then the inverse is also true: every solvable associative algebra is nilpotent. But in general, a solvable algebra is not necessary nilpotent. An example of an alternative solvable non-nilpotent algebra was constructed by Dorofeev [15](can also be found in [13]).

Definition. An algebra A is called alternative, if for all $x, y \in A$:

$$(x, x, y) = (y, x, x) = 0. \tag{1}$$

Let A be an alternative algebra. We will need the following identities on A (that are the linearizations of the well-known Moufang identities):

$$(x_1, x_2y, z) + (x_2, x_1y, z) = (x_1, y, z)x_2 + (x_2, y, z)x_1. \tag{2}$$

$$(x_1, yx_2, z) + (x_2, yx_1, z) = x_1(x_2, y, z) + x_2(x_1, y, z). \tag{3}$$

$$(x_1 \circ x_2, y, z) = (x_1, x_2y + yx_2, z) + (x_2, x_1y + yx_1, z). \tag{4}$$

$$(x_1 \circ x_2, y, z) = (x_1, y, z) \circ x_2 + (x_2, y, z) \circ x_1. \tag{5}$$

Also, in A the following equalities hold([13]):

$$2[(a, b, c), d] = ([a, b], c, d) + ([b, c], a, d) + ([c, a], b, d), \tag{6}$$

$$(dx, y, z) + (d, x, [y, z]) = d(x, y, z) + (d, y, z)x. \tag{7}$$

Let $D(A)$ be the associator ideal of A, that is an ideal generated by all associators (x, y, z), $x, y, z \in A$. In [13] it was shown that

$$D(A) = (A, A, A) + (A, A, A)A = (A, A, A) + A(A, A, A), \tag{8}$$

where $(A, A, A) = \{\sum_i (x_i, y_i, z_i) |\ \ x_i, y_i, z_i \in A\}$.

Let $J_2(A) = \{\sum_i \alpha_i a_i^2 \,|\, \alpha_i \in F, a \in A\}$ and $J_6(A) = \{\sum_i \alpha_i a_i^6 \,|\, \alpha_i \in F, a \in A\}$. Suppose A is an alternative algebra, then if $\mathrm{char}(F) \neq 2$ then $J_2(A)$ is an ideal of A and if $\mathrm{char}(F) \neq 2, 3, 5$ then $J_6(A)$ is also an ideal in A(see, for example, [13]).

3. Properties of Z_3-graded alternative algebras

In this section we will get some technical results that we will need. Throughout this section $A = A_0 \oplus A_1 \oplus A_2$ is an arbitrary alternative Z_3-graded algebra.

Lemma 1.

1)
$$(A_0^2, A_1, A_2) \subset A_0^2 \tag{9}$$

2) *For every $x \in A_1$, $y \in A_2$, $a_1, a_2 \in A_0$:*

$$(x(a_1a_2))y = x((a_1a_2)y) + a', \quad (y(a_1a_2))x = y((a_1a_2)x) + a'' \tag{10}$$

for some $a', a'' \in A_0^2$.

3)
$$(A_0A_1)(A_0^2A_2) \subset A_0^2, \quad (A_0A_2)(A_0^2A_1) \subset A_0^2 \tag{11}$$
$$(A_1A_0^2)(A_2A_0) \subset A_0^2, (A_2A_0^2)(A_1A_0) \subset A_0^2. \tag{12}$$

Proof. Let $x \in A_1$, $y \in A_2$ and $a_1, a_2 \in A_0$. Then using (7) we get:

$$(a_1a_2, x, y) = -(a_1, a_2, [x, y]) + a_1(a_2, x, y) + (a_1, x, y)a_2 \subset A_0^2.$$

And (9) is proved. It is easy to see that (10) follows from (9).

Let us prove (11) and (12). It is easy to see that they are similar and it is enough to prove one of these inclusions. Using (7) and (9) we compute:

$$\begin{aligned}(A_0A_1)(A_0^2A_2) &\subset A_0(A_1(A_0^2A^2)) + (A_0, A_1, A_0^2A_2)\\ &\subset A_0^2 + (A_0, A_1, A_0^2)A_2 + A_0^2(A_0, A_1, A_2) + (A_1, A_0^2, A_2)\\ &\subset A_0^2 + ((A_0^2)A_1)A_2 \subset A_0^2.\end{aligned}$$

□

Remark. From (10) it is follows that $A_0^2 + (A_1 A_0^2)A_2 = A_0^2 + A_1(A_0^2 A_2)$ and $A_0^2 + (A_2 A_0^2)A_1 = A_0^2 + A_2(A_0^2 A_1)$. This allows us to omit brackets in such a sentences without ambiguity.

Lemma 2.

$$(D(A))_1 \subseteq A_0A_1 + (A_1, A_2, A_1) + (A_2, A_0, A_2). \tag{13}$$

$$(D(A))_2 \subseteq A_0A_2 + (A_2, A_1, A_2) + (A_1, A_0, A_1). \tag{14}$$

Proof. It is enough to prove one of these equations. Let us prove (13). Using (8) we have:

$$\begin{aligned}(D(A))_1 \subseteq\ & (A_1, A_1, A_1)A_1 + (A_0, A_0, A_0)A_1 + (A_2, A_2, A_2)A_1 \\ & + (A_1, A_0, A_0)A_0 + (A_1, A_2, A_1)A_0 + (A_2, A_2, A_0)A_0 \\ & + (A_1, A_1, A_0)A_2 + (A_2, A_2, A_1)A_2 + (A_0, A_2, A_0)A_2 \\ & + (A_0, A_0, A_1) + (A_1, A_2, A_1) + (A_2, A_0, A_2).\end{aligned}$$

Using (6) we get:

$$\begin{aligned}(A_1, A_0, A_0)A_0 &\subseteq A_0A_1 + [A_0, (A_1, A_0, A_0)] \\ &\subseteq A_0A_1 + (A_0, A_1, A_0) \subseteq A_0A_1.\end{aligned}$$

Similarly, we obtain that

$$(A_1, A_2, A_1)A_0 + (A_2, A_2, A_0)A_0 \subseteq A_0A_1 + (A_2, A_0, A_2).$$

By (2) we compute:

$$\begin{aligned}(A_1, A_1, A_0)A_2 &\subseteq (A_2, A_1, A_0)A_1 + (A_1, A_0, A_0) + (A_1, A_2, A_1) \\ &\subseteq A_0A_1 + (A_1, A_2, A_1), \\ (A_2, A_2, A_1)A_2 &\subseteq A_0A_1 + (A_2, A_0, A_2) + (A_1, A_2, A_1).\end{aligned}$$

And, finally, using (1) we obtain the following inclusion:

$$(A_0, A_2, A_0)A_2 \subseteq A_0A_1 + (A_2, A_0, A_2).$$

Summing up the obtained inclusions we finally have that:

$$(D(A))_1 \subseteq A_0A_1 + (A_1, A_2, A_1) + (A_2, A_0, A_2). \qquad \square$$

Lemma 3.

1)

$$(A_0^2A_1, A_0, A_2) \subset A_0^2, \quad (A_0^2A_2, A_0, A_1) \subset A_0^2. \tag{15}$$

2)

$$((A_1, A_2, A_1) + (A_2, A_0, A_2))A_0^2(A_1 \circ A_1) \subset A_0^2. \tag{16}$$

$$((A_2, A_1, A_2) + (A_1, A_0, A_1))A_0^2(A_2 \circ A_2) \subset A_0^2. \tag{17}$$

3) *For all $n \geqslant 2$:*

$$(A_1A_0^{<n>})(A_0A_2) \subset A_1(A_0^{<n+1>})A_2 + A_0^2, \tag{18}$$

$$(A_2A_0^{<n>})(A_0A_1) \subset A_2(A_0^{<n+1>})A_1 + A_0^2. \tag{19}$$

Proof. It is easy to see that it is enough to prove only one inclusion in every statement. We will prove the first inclusion in all cases.

By (7) we have:

$$\begin{aligned}(A_0^2A_1, A_0, A_2) &\subset A_0^2(A_1, A_0, A_2) + (A_0^2, A_0, A_2)A_1 + (A_0^2, A_1, A_2)\\ &\subset A_0^2 + (A_0^2A_2)A_1 \subset A_0^2.\end{aligned}$$

And (15) is proved.

Using (6), (9) and (15) we compute:

$$\begin{aligned}&((A_1, A_2, A_1) + (A_2, A_0, A_2))A_0^2(A_1 \circ A_1)\\ &\quad\subset A_0^2 + [(A_1, A_2, A_1) + (A_2, A_0, A_2), A_0^2(A_1 \circ A_1)]\\ &\quad\subset A_0^2 + (A_0^2(A_1 \circ A_1), A_0, A_1) + (A_0^2(A_1 \circ A_1), A_2, A_2)\\ &\quad\subset A_0^2 + (A_0^2(A_1 \circ A_1), A_2, A_2).\end{aligned}$$

Using (2) and (4) we have:

$$\begin{aligned}(A_0^2(A_1 \circ A_1), A_2, A_2) &\subset (A_1, A_0^2, A_2) + A_0^2 + (A_1 \circ A_1, A_0^2, A_2)A_2\\ &\subset A_0^2 + (A_1, A_0^2, A_0)A_2 \subset A_0^2.\end{aligned}$$

Thus, $((A_1, A_2, A_1) + (A_2, A_0, A_2))A_0^2(A_1 \circ A_1) \subset A_0^2$.

Let us prove (18). Using (9) and (15) we get:

$$\begin{aligned}(A_1A_0^{<n>})(A_0A_2) &\subset ((A_1A_0^{<n>})A_0)A_2 + (A_1A_0^{<n>}, A_0, A_2)\\ &\subset A_0^2 + (A_1A_0^{<n+1>})A_2 + (A_1, A_0^{<n>}, A_0)A_2\\ &\subset A_0^2 + A_1(A_0^{n+1}A_2).\end{aligned}$$

□

Lemma 4.

1) *Let* char$(F) \neq 2$. *Then A is solvable if and only if $J_2(A)$ is solvable.*
2) *Let* char$(F) \neq 2,3,5$. *Then A is solvable if and only if $J_6(A)$ is solvable.*

Proof. The proof is similar for both cases. Let us prove 2.

If A is solvable then clearly $J_6(A)$ is solvable.

Suppose $J_6(A)$ is solvable. Consider the factor algebra $\overline{A} = A/J_6(A)$. Then for every $\overline{x}$ in $\overline{A}$: $\overline{x}^6 = 0$, that is $\overline{A}$ is a nil algebra of nil-index 6. Since the characteristic of the ground field F not equal 2,3 or 5, then by Zhevlakov's theorem $\overline{A}$ is solvable ([14], the proof can also be found in [13]). Thus, A is solvable. $\square$

Lemma 5. *Let A be a Z_3-graded alternative algebra over a field F of characteristic not equal 2,3,5. Then we have the following inclusions:*

1) $(J_2(A))_1 \subset A_0 \circ A_1 + A_2 \circ A_2$, $(J_2(A))_2 \subset A_0 \circ A_2 + A_1 \circ A_1$.
2) $(J_6(A))_0 \subset A_0^2 + A_1 A_0^2 A_2 + A_2 A_0^2 A_1$.

Proof. The first assertion is obvious.

Let us prove 2. We will use the following notation: if $u, v \in A$ then $u \equiv v$ means that $u - v \in A_0^2 + A_1 A_0^2 A_2 + A_2 A_0^2 A_1$

Let $x \in A_1, y \in A_2, a \in A_0$. It is sufficient to prove that $((x+y+a)^6)_0 \equiv 0$.

First we will proof the following inclusion:

$$x(y,x,a)x^2 + x^2(y,x,a)x \in A_0^2 \tag{20}$$

Indeed, using (5) and (2) we have

$$\begin{aligned} A_0^2 \ni 2(xy,x^3,a) &= (xy,x^2,a) \circ x + (xy,x,a) \circ x^2 \\ &= x(xy,x,a)x + (xy,x,a)x^2 + (xy,x,a) \circ x^2 \\ &= x(y,x,a)x^2 + 2(y,x,a)x^3 + x^2(y,x,a)x. \end{aligned}$$

Thus, $x(y,x,a)x^2 + x^2(y,x,a)x \in A_0^2$. Similarly, one can prove the following inclusion:

$$y(x,y,a)y^2 + y^2(x,y,a)y \in A_0^2. \tag{21}$$

Consider $p = (x+y+a)^3$. Then we have:

$$\begin{aligned} p_0 &= x^3 + y^3 + a^3 + (x \circ a)y + (y \circ a)x + (x \circ y)a, \\ p_1 &= x^2 y + (x \circ y)x + y^2 a + (a \circ y)y + a^2 x + (a \circ x)a, \\ p_2 &= y^2 x + (y \circ x)y + x^2 a + (a \circ x)x + a^2 y + (a \circ y)a. \end{aligned}$$

Since $((x+y+a)^6)_0 = p_0^2 + p_1 \circ p_2$, then it is enough to proof that:

$$(x^2y + (x\circ y)x) \circ (y^2x + (y\circ x)y) \equiv 0, \tag{22}$$

$$(y^2a + (a\circ y)y) \circ (y^2x + (y\circ x)y) \equiv 0, \tag{23}$$

$$(a^2x + (a\circ x)a) \circ (y^2x + (y\circ x)y) \equiv 0, \tag{24}$$

$$(x^2y + (x\circ y)x) \circ (x^2a + (a\circ x)x) \equiv 0, \tag{25}$$

$$(y^2a + (a\circ y)y) \circ (x^2a + (a\circ x)x) \equiv 0, \tag{26}$$

$$(a^2x + (a\circ x)a) \circ (x^2a + (a\circ x)x) \equiv 0, \tag{27}$$

$$(x^2y + (x\circ y)x) \circ (a^2y + (a\circ y)a) \equiv 0, \tag{28}$$

$$(y^2a + (a\circ y)y) \circ (a^2y + (a\circ y)a) \equiv 0, \tag{29}$$

$$(a^2x + (a\circ x)a) \circ (a^2y + (a\circ y)a) \equiv 0. \tag{30}$$

The equivalences (22),(27) and (29) are obvious. Let us prove (23). We have:

$$\begin{aligned}
&(y^2a)(y^2x) + (y^2x)(y^2a)\\
&\quad = (y^2ay)(yx) - (y^2a, y, yx) + ((y^2x)y^2)a - (y^2x, y^2, a)\\
&\quad \equiv -(a,y,x)y^3 - y(y^2x,y,a) - (y^2x,y,a)y \equiv y(y,x,a)y^2,\\
&(y^2a)((y\circ x)y) + (y\circ x)y)(y^2a)\\
&\quad = ((y^2a)(y\circ x)))y + (y^2a, y\circ x, y) + (y\circ x)(y^3a) + (y\circ x, y, y^2a)\\
&\quad \equiv (y^2(a(y\circ x)))y + (y^2, a, y\circ x)y + (y^2a, y\circ x, y) + (y\circ x, y, y^2a)\\
&\quad \equiv y(y,a,x)y^2 + 2(y\circ x, y, a)y^2 = 3y(y,a,x)y^2,\\
&((a\circ y)y)(y^2x) + (y^2x)((ay+ya)y)\\
&\quad = ((a\circ y)y^2)(yx) + ((a\circ y)y, y, yx) + y((yx)(ay^2)) + (y, yx, ay^2)\\
&\qquad + (y^2xy)(ay) - (y^2x, y, ay)\\
&\quad \equiv y((yx)a)y^2 - y(yx, a, y^2) + y^2(y,x,a)y + ((y^2xy)a)y\\
&\qquad - (y^2xy, a, y) - y(x,y,a)y^2\\
&\quad \equiv y^2(y,x,a)y + y^2((xy)a)y + (y^2, xy, a)y - y(x,a,y)y^2 - y(x,y,a)y^2\\
&\equiv y^2(y,x,a)y,\\
&((a\circ y)y)((y\circ x)y)) + ((y\circ x)y))((a\circ y)y)\\
&\quad = (ay^2)((y\circ x)y) + (yay)(yxy + xy^2) + (y\circ x)(y(a\circ y)y)\\
&\qquad + (y\circ x, y, (a\circ y)y)
\end{aligned}$$

$$
\begin{aligned}
&\equiv a(y^2((y\circ x)y)) + (a, y^2, (y\circ x)y) + (yay^2)(xy) - (yay, y, xy)\\
&\quad + ((yay)x)y^2 - (yay, x, y^2) + y^2(x, y, a)y\\
&\equiv 3y^2(x, y, a)y + y((ay)x)y^2 + (y, ay, x)y^2\\
&\equiv 3y^2(x, y, a)y + y(a(yx))y^2 + y(a, y, x)y^2 + y(y, a, x)y^2\\
&\equiv 3y^2(x, y, a)y.
\end{aligned}
$$

Summing up the obtained equations we have:

$$
\begin{aligned}
&(y^2a + (a\circ y)y)\circ(y^2x + (y\circ x)y)\\
&\qquad \equiv y(y, x, a)y^2 + 3y(y, a, x)y^2 + y^2(y, x, a)y + 3y^2(x, y, a)y \equiv 0.
\end{aligned}
$$

That proves (23). Using similar arguments one can obtain (25).

Let us prove (24):

$$
\begin{aligned}
&(a^2x)\circ(y^2x + (y\circ x)y)\\
&\qquad \equiv a^2(xy^2x + (y\circ x)y) + (a^2, x, y^2x + (y\circ x)y) \equiv 0,\\
&((a\circ x)a)\circ(y^2x + (y\circ x)y)\\
&\qquad = (xa^2)\circ(y^2x + (y\circ x)y) + (axa)(y^2x + (y\circ x)y)\\
&\qquad\quad + (y^2x + (y\circ x)y)(axa)\\
&\qquad \equiv a((xa)(y^2x + (y\circ x)y)) + (a, xa, (y^2x + (y\circ x)y))\\
&\qquad\quad + ((y^2x + (y\circ x)y)(ax))a - ((y^2x + (y\circ x)y), ax, a)\\
&\qquad \equiv a(a, x, (y^2x + (y\circ x)y)) - ((y^2x + (y\circ x)y), x, a)a \equiv 0.
\end{aligned}
$$

Thus, $(a^2x)\circ(y^2x + (y\circ x)y) + ((a\circ x)a)\circ(y^2x + (y\circ x)y) \equiv 0$ and (24) is proved. Similarly, one can prove (28).

Consider (26). We have:

$$
\begin{aligned}
(y^2a)\circ(x^2a) &= ((y^2a)x^2)a - (y^2a, x^2, a) + ((x^2a)y^2)a - (x^2a, y^2, a)\\
&\equiv -a(y^2a, x^2, a) - a(x^2, y^2, a) \equiv 0.
\end{aligned}
$$

Similarly, $(y^2a)\circ(ax^2) + (ay^2)\circ(ax^2) + (ay^2)\circ(x^2a) \equiv 0$. Further, we compute:

$$
\begin{aligned}
(y^2a)(xax) + (ay^2)(xax) &= y^2(axax) + (y^2, a, xax) + a(y^2(xax)) + (a, y^2, xax)\\
&\equiv y(((y(ax))a)x) - y(y(ax), a, x) \equiv -y(y(ax), a, x).
\end{aligned}
$$

Using (7) and (9) we get:

$$
\begin{aligned}
-y(y(ax), a, x) &= (y(ax), a, y)x - (y(ax), a, yx) - ([y(ax), a], y, x)\\
&\equiv (y(ax), a, y)x = ((ax, a, y)y)x = (((x, a, y)a)y)x \equiv 0.
\end{aligned}
$$

Using similar computations one can prove that $(xax)(y^2a)+(xax)(ay^2)\equiv 0$. Finally,

$$\begin{aligned}
&(yay)(xax) + (xax)(yay)\\
&\quad = ((yay)x)(ax) - (yay, x, ax) + x((ax)(yay)) + (x, ax, yay)\\
&\quad \equiv ((ya)(yx))(ax) + (ya, y, x)(ax) + x(((ax)y)(ay)) - x(ax, y, ay)\\
&\quad \equiv (y(a(yx)))(ax) + (y, a, yx)(ax) + (ya, y, x)(ax) + x((((ax)y)a)y)\\
&\qquad - x((ax)y, a, y) - x(ax, y, ay)\\
&\quad \equiv ((y, a, x)y)(ax) + ((a, y, x)y)(ax) - x(y(ax, a, y))\\
&\qquad - x(y(ax, y, a)) = 0.
\end{aligned}$$

And (26) is proved. The equality (30) can be proved in a similar way. □

4. The main part

Recall that if A is an algebra, then by $D(A)$ we denote the ideal generated by associators. Define subalgebras K_i and T_i as

$$K_1 := J_2(A),\ T_1 := D(K_1),\quad K_i := J_2(T_{i-1}),\ T_i := D(K_i).$$

It is easy to see that:

$$A \supseteq K_1 \supseteq T_1 \supseteq K_2 \supseteq ... \supseteq K_i \supseteq T_i \supseteq ...$$

Lemma 6. *If for some $i \geqslant 1$ T_i or K_i is solvable, then A is solvable.*

Proof. By lemma 4 A is solvable if and only if $J_2(A) = K_1$ is solvable. Since $D(K_1)$ is a homogeneous ideal, then $K_1/D(K_1)$ - is an associative Z_3-graded algebra with a solvable even part. Thus, by Bergman-Isaacs theorem $K_1/D(K_1)$ is nilpotent and if $D(K_1)$ is solvable, then A is solvable.

Similar arguments show that K_i and T_i are solvable if and only if T_{i-1} is solvable. □

Lemma 7. *Let A be a Z_3-graded algebra and $A_0 = 0$. If* $\operatorname{char} F \neq 2, 3$, *then A is solvable.*

Proof. Consider $J_3(A) = \{\sum\limits_i x_i^3 | x_i \in A\}$. Using similar arguments as in lemma 4 we get, that A is solvable if and only if $J_3(A)$ is solvable. For all $x \in A_1$ and $y \in A_2$ we have:

$$(x + y)^3 = x^3 + y^3 + x^2y + yx^2 + xyx + y^2x + xy^2 + yxy.$$

But $x^3 \in A_0$, $y^3 \in A_0$ and $xy \in A_0$. Thus $(x+y)^3 = 0$ and $J_3(A) = 0$. $\square$

Theorem 1. *Let A be a Z_3-graded alternative algebra over a field F. If A_0 is solvable and* $\operatorname{char} F \neq 2, 3, 5$, *then A is solvable.*

Proof. Let $A_0^{(m)} = 0$ and $n = 2^m$. Consider T_n and define $I = J_6(T_n)$. By lemmas 4 and 6 it is enough to prove that I is solvable. By lemma 5 we have $I_0 \subset A_0^2 + (T_n)_2(A_0^2)(T_n)_1 + (T_n)_1(A_0^2)(T_n)_2$.

Our aim now is to prove that

$$(T_n)_1(A_0^2)(T_n)_2 \subset A_0^2 + (T_{n-1})_1 A_0^{<3>}(T_{n-1})_2. \tag{31}$$

Indeed, since $T_n \subset K_n = J_2(T_{n-1})$ then by lemma 5:

$$(T_n)_2 \subset A_0 \circ (T_{n-1})_2 + (T_{n-1})_1 \circ (T_{n-1})_1.$$

By (12) and (18) we have that

$$(T_n)_1 A_0^2 (A_0 \circ (T_{n-1})_2) \subset (T_{n-1})_1 A_0^{<3>}(T_{n-1})_2 + A_0^2.$$

Using inclusion (13) we get:

$$\begin{aligned}(T_n)_1 &= (D(K_n))_1 \\ &\subseteq A_0(K_n)_1 + ((K_n)_1, (K_n)_2, (K_n)_1) + ((K_n)_2, A_0, (K_n)_2).\end{aligned}$$

And now it is left to use inclusions (11) and (16) to prove (31). Similar reasons shows us that $(T_n)_2(A_0^2)(T_n)_1 \subset A_0^2 + (T_{n-1})_2 A_0^{<3>}(T_{n-1})_1$ and we may conclude that

$$I_0 \subset A_0^2 + (T_{n-1})_1 A_0^{<3>}(T_{n-1})_2 + (T_{n-1})_2 A_0^{<3>}(T_{n-1})_1.$$

Now we can continue to use similar arguments and get that

$$I_0 \subset A_0^2 + (T_{n-2})_1 A_0^{<4>}(T_{n-2})^2 + (T_{n-2})_2 A_0^{<4>}(T_{n-2})_1.$$

And finally, we will get that

$$I_0 \subset A_0^2 + A_1 A_0^{<n>} A_2 + A_2 A_0^{<n>} A_1. \tag{32}$$

Let us prove that $A_1 A_0^{<n>} A_2 \subset A_0^2$. For this we will prove that for all $k \geqslant 2$:

$$A_1(A_0^k, A_0, A_0)A_2 \subset A_0^2. \tag{33}$$

Indeed, using (2) and (9) we have:

$$A_1(A_0^k, A_0, A_0)A_2 \subset A_1((A_2, A_0, A_0)A_0^k) + A_1(A_0^k, A_2, A_0) \subset A_0^2.$$

Moreover, from (18) and (33) we see that

$$A_1(((...((A_0^2, A_0, A_0)A_0)A_0)...A_0)A_2 \subset A_0^2. \tag{34}$$

Now we can use (34) to obtain the following inclusions:

$$\begin{aligned} A_1A_0^{<n>}A_2 &\subset A_1(((A_0^2A_0^2)A_0)...A_0)A_2 + A_0^2 \\ &\subset A_1(((A_0^2A_0^2)A_0^2)...A_0^2)A_2 + A_0^2 \\ &\subset A_1(((A_0^{(2)}A_0^{(2)})...)A_0^{(2)})A_2 + A_0^2 \\ &\subset ... \subset A_1A_0^{(m)}A_2 + A_0^2 = A_0^2. \end{aligned}$$

Similarly, $A_2A_0^{<n>}A_1 \subset A_0^2$. Thus, $I_0 \subset A_0^2 = A_0^{(1)}$.

Now we can start from the beginning with the ideal I and construct an ideal I' such that I (and, thus, A) is solvable if and only if I' is solvable and $I'_0 \subset I_0^2 \subset A_0^{(2)}$. Repeating this construction, in the end we will construct an sublagebra $\widetilde{I}$ such that A is solvable if and only if $\widetilde{I}$ is solvable and $\widetilde{I}_0 \subset A_0^{(m)} = 0$. But by lemma 7 $\widetilde{I}$ is solvable, so A is also solvable.

Corollary 1. *Let A be an alternative algebra with an automorphism ϕ of order 3. If* $\operatorname{char} F \neq 2, 3, 5$ *and the subalgebra A^ϕ of fixed points with respect to ϕ is solvable, then A is solvable.*

Proof. If the ground field F is algebraically closed, then we can consider subspaces $A_\xi = \{x \in A| \ \ \phi(x) = \xi x\}$ and $A_{\xi^2} = \{x \in A| \ \ \phi(x) = \xi^2 x\}$, where ξ is a primitive cube root of unity. It is easy to see that $A = A_\xi \oplus A_{\xi^2} \oplus A^\phi$ and A is a Z_3-graded algebra. Since A^ϕ is solvable, then by theorem 1 A is solvable.

If F is not algebraically closed we can consider it's algebraic closure $\overline{F}$ and an algebra $\overline{A} = A \otimes_F \overline{F}$. Then $\overline{A}$ is an alternative algebra over $\overline{F}$ and A is solvable if and only if $\overline{A}$ is solvable. We can define an automorphisms $\overline{\phi}$ on $\overline{A}$ by putting: $\overline{\phi}(a \otimes \alpha) = \phi(a) \otimes \alpha$ for all $a \in A, \alpha \in \overline{F}$. Then $\overline{\phi}$ is an automorphism of order 3 and the subalgebra of fixed points $\overline{A}^{\overline{\phi}} = A^\phi \otimes \overline{F}$ is solvable. Thus, $\overline{A}$ is solvable and, finally, A is solvable. $\square$

Corollary 2. *Let $A = \sum\limits_{i=0}^{n-1} A_i$ be a Z_n-graded alternative algebra, where $n = 2^k 3^l$ and $k + l \geqslant 1$. If* $\operatorname{char} F \neq 2, 3, 5$ *and the subalgebra A_0 is solvable, then A is solvable.*

Proof. If $k = 0$ then by corollary 1 A is solvable. Suppose $k \geqslant 1$. We will use an induction on l. If $l = 0$ then the result follows from the paper of Smirnov [12]. Let $l \geqslant 1$. Then we can consider subspaces $\widehat{A}_0 = \sum_i A_{3i}$, $\widehat{A}_1 = \sum_i A_{1+3i}$, $\widehat{A}_2 = \sum_i A_{2+3i}$. Then $A = \widehat{A}_0 \oplus \widehat{A}_1 \oplus \widehat{A}_2$ - is a Z_3-gradation of A. By theorem 1 A is solvable if and only if $\widehat{A}_0$ is solvable. On the other hand it is easy to see that $\widehat{A}_0$ is a $Z_{n'}$-graded algebra, where $n' = 2^k 3^{l-1}$ and $(\widehat{A}_0)_0 = A_0$ is solvable. Now we may use the induction and get that $\widehat{A}_0$ is solvable. Hence, A is solvable. $\square$

Corollary 3. *Let A be an alternative algebra with an automorphism ϕ of order $2^k 3^l$. If $\operatorname{char} F \neq 2, 3, 5$ and the subalgebra A^ϕ of fixed points with respect to ϕ is solvable, then A is solvable.*

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Vector bundles on projective varieties and representations of quivers*

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Abstract. We present equivalences between certain categories of vector bundles on projective varieties, namely *cokernel bundles*, *Steiner bundles*, *syzygy bundles*, and *monads*, and full subcategories of representations of certain quivers. As an application, we provide decomposability criteria for such bundles.

1. Introduction

Vector bundles over algebraic varieties play a central role in algebraic geometry, and many interesting problems are still open. In particular, constructing indecomposable vector bundles on a variety X with rank smaller than the $\dim X$ is not an easy task for certain choices of X, especially for projective spaces.

Monads are one of the most important tools for constructing such bundles; indeed, the majority of examples of low rank bundles on projective spaces, namely the Horrocks–Mumford bundle of rank 2 on $\mathbb{P}^4$, Horrocks'

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parent bundle of rank 3 on $\mathbb{P}^5$, and the rank $2k$ instanton bundles on $\mathbb{P}^{2k+1}$, are obtained as cohomologies of certain monads.

The goal of this paper is to show that the theory of representations of quivers might also be an interesting tool for the construction of useful monads and cokernel bundles on projective varieties. More precisely, we present equivalences between certain categories of vector bundles on projective varieties and full subcategories of representations of certain quivers. In this way, we translate the problems of constructing indecomposable vector bundles on $\mathbb{P}^n$ with low rank into a (possibly still very hard) problem of linear algebra. As an application of these results, we give decomposability criteria for cokernel bundles, syzygy bundles and monads.

Let us now present more precisely the results proved here, starting with *cokernel bundles*, a class a vector bundles introduced by Brambilla in [1]. Let X be a nonsingular projective variety of dimension n, and let $\mathcal{E}$ and $\mathcal{F}$ be simple vector bundles on X such that

(i) $\mathrm{Hom}(\mathcal{F},\mathcal{E}) = \mathrm{Ext}^1(\mathcal{F},\mathcal{E}) = 0$;

(ii) $\mathcal{E}^\vee \otimes \mathcal{F}$ is globally generated;

(iii) $\dim \mathrm{Hom}(\mathcal{E},\mathcal{F}) \leqslant 3$.

A *cokernel bundle of type* $(\mathcal{E},\mathcal{F})$ on X is a vector bundle $\mathcal{C}$ with a resolution of the form

$$0 \longrightarrow \mathcal{E}^a \xrightarrow{\alpha} \mathcal{F}^b \longrightarrow \mathcal{C} \longrightarrow 0\ .$$

We prove (cf. Thm 3.5 below):

Theorem 1.1. *The category of cokernel bundles of type* $(\mathcal{E},\mathcal{F})$ *is equivalent to a full subcategory of the category of representation of the Kronecker quiver with* $w = \dim \mathrm{Hom}(\mathcal{E},\mathcal{F})$ *arrows:*

$$\bullet \ \underset{w}{\overset{1}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}}\ \bullet$$

As application of this equivalence, we obtain new proofs of simplicity and exceptionality criteria for cokernel bundles that were originally established by Brambilla in [1, Thm. 4.3] (cf. Thm 3.8 below) and Soares in [10, Theorem 2.2.7] (cf. Cor 3.13 below).

Next, we consider 1^{st}-syzygy bundles on projective spaces; recall that *syzygy bundles* are those given as kernel of surjective morphisms of the form

$$\mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_m)^{a_m} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^c .$$

Let $\mathcal{G} := \ker \alpha$; we refer to [2] as a general reference on syzygy bundles.

The case $m = 1$ can be regarded as a cokernel bundle; for the remainder of the paper, we focus on the case $m = 2$, though it is not hard to generalize our results for $m > 2$ (see Remark 4.6 below). More precisely, we prove the following result, including a new decomposability criterion for syzygy bundles.

Theorem 1.2. *For any fixed integers $d_1 > d_2 > 0$, there is a faithful functor from the category of representations of the quiver*

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "w_1"'] & \bullet & \bullet \arrow[l, shift left=2, "1"'] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2, "w_2"] \end{tikzcd}$$

to the category of syzygy bundles given by sequences of the form

$$0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{a_2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^c \to 0 \tag{1}$$

where $w_j = h^0(\mathcal{O}_{\mathbb{P}^n}(d_j))$, $j = 1, 2$. Moreover, if $a_1^2 + a_2^2 + c^2 - w_1 a_1 c - w_2 a_2 c > 1$, then $\mathcal{G}$ is decomposable.

Finally, we consider the relation between monads and representations of quivers. Recall that a *monad* on a nonsingular projective variety X is a complex of locally free sheaves of the form

$$M^\bullet : \ \mathcal{A}^a \longrightarrow \mathcal{B}^b \longrightarrow \mathcal{C}^c \tag{2}$$

whose only nontrivial cohomology is the middle one, which we assume, in this paper, to also be a locally free sheaf. We prove:

Theorem 1.3. *If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are simple vector bundles, then the category of monads of the form (2) is equivalent to a full subcategory of the category of representations of the quiver*

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "m"'] & \bullet \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "n"'] & \bullet \end{tikzcd}$$

where $m = \dim \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $n = \dim \mathrm{Hom}(\mathcal{B}, \mathcal{C})$. In addition, if $a^2 + b^2 + c^2 - mab - nbc > 1$ then the cohomology sheaf of (2) is decomposable.

This generalizes the results of [4] (in particular, [4, Thm 1.1]) concerning linear monads on $\mathbb{P}^n$, i.e. when $\mathcal{A} = \mathcal{O}_{\mathbb{P}^n}(-1)$, $\mathcal{B} = \mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{C} = \mathcal{O}_{\mathbb{P}^n}(1)$.

Furthermore, if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are elements of distinct blocks of an n-block collection generating the bounded derived category $D^b(X)$ of coherent sheaves of $\mathcal{O}_X$-modules, then we also prove that the cohomology sheaf $\mathcal{E}$ of (2) is decomposable, if and only if the corresponding quiver representation is decomposable, cf. Theorem 5.5.

Notation. Throughout this paper, κ denotes an algebraically closed field with characteristic zero, and X is always a nonsingular projective variety over κ of dimension n.

2. Preliminary definitions and results

In this section we revise some key definitions and results on the theory of representations of quivers and on the derived category of coherent sheaves that will be relevant in the following sections.

2.1. Representations of quivers

We begin by revising some basic facts about representations of quivers. Recall that a *quiver* Q consists on a pair (Q_0, Q_1) of sets where Q_0 is the set of vertices and Q_1 is the set of arrows and a pair of maps $t, h : Q_1 \to Q_0$ the tail and head maps. An example is the *Kronecker quiver*, denoted K_w, which consists of 2 vertices and w arrows.

$$\bullet \; \begin{array}{c} \xrightarrow{1} \\ \vdots \\ \xrightarrow[w]{} \end{array} \; \bullet \tag{3}$$

A *representation* $R = (\{V_i\}, \{A_a\})$ of Q consists of a collection of finite dimensional κ-vector spaces $\{V_i; i \in Q_0\}$ together with a collection of linear maps $\{A_a : V_{t(a)} \to V_{h(a)}; a \in Q_1\}$. A *morphism* f between two representations $R_1 = (\{V_i\}, \{A_a\})$ and $R_2 = (\{W_i\}, \{B_a\})$ is a collection of linear maps $\{f_i\}$ such that for each $a \in Q_1$ the diagram bellow is commutative

$$\begin{tikzcd} V_{t(a)} \arrow[r, "A_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\ W_{t(a)} \arrow[r, "B_a"'] & W_{h(a)} \end{tikzcd}$$

With these definitions, representations of Q form an abelian category hereby denoted by $\mathfrak{R}(Q)$.

Given a representation $R \in \mathfrak{R}(Q)$, we associate a vector $\mathbf{v} \in \mathbb{Z}^{Q_0}$ called *dimension vector*, whose entries are $\mathbf{v}_i = \dim V_i$.

The *Euler form* on $\mathbb{Z}^{Q_0}$ is a bilinear form associated to Q, given by

$$< \mathbf{v}, \mathbf{w} >= \sum_{i \in Q_0} \mathbf{v}_i \mathbf{w}_i - \sum_{a \in Q_1} \mathbf{v}_{t(a)} \mathbf{w}_{h(a)}.$$

The *Tits form* is the corresponding quadratic form, given by

$$q(\mathbf{v}) =< \mathbf{v}, \mathbf{v} > .$$

For instance, the Tits form of the Kronecker quiver with w arrows is given by

$$q_w(\mathbf{v}) = a^2 + b^2 - wab, \quad \mathbf{v} = (a, b) \in \mathbb{Z}^2. \tag{4}$$

Definition 2.1. A vector $\mathbf{v} \in \mathbb{Z}^{Q_0}$ is a root if there is an indecomposable representation R of Q with dimension vector $\mathbf{v}$. Moreover, $\mathbf{v}$ is a Schur root if there is a representation R of Q with dimension vector $\mathbf{v}$ satisfying $\mathrm{Hom}(R, R) = \kappa$.

Clearly, every Schur root is a root; note also that the condition $\mathrm{Hom}(R, R) = \kappa$ is an open condition in the affine space

$$\oplus_{a \in Q_1} \mathrm{Hom}(\kappa^{\mathbf{v}_{t(a)}}, \kappa^{\mathbf{v}_{h(a)}})$$

of all representations with fixed dimension vector $\mathbf{v}$. Thus if $\mathbf{v}$ is a Schur root, then $\mathrm{Hom}(R, R) = \kappa$ for a generic representation with dimension vector $\mathbf{v}$. In particular, if $\mathbf{v}$ is a Schur root, then generic representation with dimension vector $\mathbf{v}$ is indecomposable. A reference for generic representations and Schur roots is [8]. For more information about roots and root systems, we refer to [5].

The following two facts will be very relevant in what follows. The first one follows from Kac's theory of infinite root systems [5].

Proposition 2.2. *Let Q be a quiver with Tits form q. If $\mathbf{v}$ is a dimension vector satisfying $q(\mathbf{v}) > 1$, then every representation with dimension vector $\mathbf{v}$ is decomposable.*

The second fact follows from [5, Prop 1.6] and [9, Thm 4.1].

Proposition 2.3. *Let Q be the Kronecker quiver with $w \geqslant 3$, and let $\mathbf{v} \in \mathbb{Z}^2$ be a dimension vector. If $q_w(\mathbf{v}) \leqslant 1$, then $\mathbf{v}$ is a Schur root.*

2.2. Derived categories

In [7], Miró-Roig and Soares gave a cohomological characterisation of Steiner bundles and later Marques and Soares [6], gave a cohomological characterisation of a class of bundles given as cohomology of monads. Both results will be relevant for us, so we review them here.

Let $D^b(X)$ be the bounded derived category of the abelian category of coherent sheaves of $\mathcal{O}_X$-modules. An *exceptional collection* is an ordered collection $(\mathcal{F}_0, \cdots, \mathcal{F}_m)$ of objects of $D^b(X)$ such that

$$\mathrm{Hom}^0_{D^b(X)}(\mathcal{F}_i, \mathcal{F}_i) \simeq \kappa, \;\; \mathrm{Ext}^p(\mathcal{F}_i, \mathcal{F}_i) = 0, \;\; \text{for all } \; p \geqslant 1,$$

$$\mathrm{Ext}^p(\mathcal{F}_i, \mathcal{F}_j) = 0 \;\text{ for all } i > j, \text{ and } p \geqslant 0.$$

In addition, if

$$\mathrm{Ext}^p(\mathcal{F}_i, \mathcal{F}_j) = 0 \;\text{ for }\; i \leqslant j \;\text{ and }\; p \neq 0\,,$$

then $(\mathcal{F}_0, \ldots, \mathcal{F}_m)$ is called a *strongly exceptional collection*. It is a *full (strongly) exceptional collection* if it generates $D^b(X)$.

An exceptional collection $(\mathcal{F}_0, \cdots, \mathcal{F}_m)$ is called a *block* if

$$\mathrm{Ext}^p(\mathcal{F}_j, \mathcal{F}_i) = 0 \;\; \forall \;\; p \geqslant 0 \;\text{ and }\; i \neq j.$$

An *m-block collection* of type $(t_0, \ldots, t_m)$ is an exceptional collection $\boldsymbol{B} = (\boldsymbol{\mathcal{F}}_0, \ldots, \boldsymbol{\mathcal{F}}_m)$ where each $\boldsymbol{\mathcal{F}}_i = (\mathcal{F}^i_1, \ldots, \mathcal{F}^i_{t_i})$ is a block.

Definition 2.4. Let $\boldsymbol{B} = (\boldsymbol{\mathcal{F}}_0, \ldots, \boldsymbol{\mathcal{F}}_m)$ be an m-block collection of type $(t_0, \ldots, t_m)$. The left dual m-block collection of $\mathbf{B}$ is the m-block collection ${}^\vee\mathbf{B}$ of type $(u_0, \ldots, u_m)$ with $u_i = u_{m-i}$

$${}^\vee\mathbf{B} = (\boldsymbol{\mathcal{H}}_0, \ldots, \boldsymbol{\mathcal{H}}_m) = (\mathcal{H}^0_1, \ldots, \mathcal{H}^0_{u_0}, \ldots, \mathcal{H}^m_1, \ldots, \mathcal{H}^m_{u_m})$$

where

$$\mathrm{Hom}^k_{D^b(X)}(\mathcal{H}^i_j, \mathcal{F}^l_p) = 0$$

for all indices, with the only exception

$$\mathrm{Ext}^i(\mathcal{H}^i_j, \mathcal{F}^{m-i}_j) \simeq \kappa.$$

These conditions uniquely determine ${}^\vee\mathbf{B}$.

We are now able to define Steiner bundles in the sense of [7] and state their cohomological characterisation.

Definition 2.5. A vector bundle $\mathcal{S}$ on X is a Steiner bundle of type $(\mathcal{F}_0, \mathcal{F}_1)$ if it is given by a short exact sequence of the form

$$0 \longrightarrow \mathcal{F}_0^a \xrightarrow{\alpha} \mathcal{F}_1^b \longrightarrow \mathcal{S} \longrightarrow 0$$

such that $a, b \geqslant 1$ and $(\mathcal{F}_0, \mathcal{F}_1)$ is an ordered pair of vector bundles on X satisfying

(i) $(\mathcal{F}_0, \mathcal{F}_1)$ is strongly exceptional;

(ii) $\mathcal{F}_0^\vee \otimes \mathcal{F}_1$ is globally generated.

The cohomological characterisation is the following, cf. [7, Thm 2.4].

Theorem 2.6. *Let X be a smooth projective variety of dimension n with an n-block collection $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \ldots, \boldsymbol{\mathcal{F}}_n)$, $\boldsymbol{\mathcal{F}}_i = (\mathcal{F}_1^i, \ldots, \mathcal{F}_{t_i}^i)$ of locally free sheaves which generate $D^b(X)$, and let ${}^\vee\mathbf{B}$ be its left dual basis. Let $\mathcal{F}_{i_0}^i \in \boldsymbol{\mathcal{F}}_i$ and $\mathcal{F}_{j_0}^j \in \boldsymbol{\mathcal{F}}_j$, where $0 \leqslant i < j \leqslant n$ and $1 \leqslant i_0 \leqslant a_i$, $1 \leqslant j_0 \leqslant a_j$, and let $\mathcal{S}$ be a locally free sheaf on X. Then $\mathcal{S}$ is a Steiner bundle of type $(\mathcal{F}_{i_0}^i, \mathcal{F}_{j_0}^j)$ given by the short exact sequence*

$$0 \longrightarrow (\mathcal{F}_{i_0}^i)^a \longrightarrow (\mathcal{F}_{j_0}^j)^b \longrightarrow \mathcal{S} \longrightarrow 0$$

if and only if $(\mathcal{F}_{i_0}^i)^\vee \otimes \mathcal{F}_{j_0}^j$ is globally generated and all $\operatorname{Ext}^l(\mathcal{H}_p^m, \mathcal{S})$ vanish, with the only exceptions of

$$\dim \operatorname{Ext}^{n-i-1}(\mathcal{H}_{i_0}^{n-i}, \mathcal{S}) = a \quad \text{and} \quad \dim \operatorname{Ext}^{n-j}(\mathcal{H}_{j_0}^{n-j}, \mathcal{S}) = b. \tag{5}$$

Now we turn our attention to the cohomological characterisation for the bundles obtained as cohomology of monads, due to Marques and Soares in [6].

Definition 2.7. A monad $M^\bullet$ on a smooth projective variety X is a complex of locally free coherent sheaves on X

$$M^\bullet : \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$$

such that α is injective, β is surjective; the coherent sheaf $\mathcal{E} = \ker \beta / \operatorname{im} \alpha$ is called the cohomology of $M^\bullet$.

The following two definitions are important for the main result we would like to present.

Definition 2.8. Let $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \cdots, \boldsymbol{\mathcal{F}}_m)$, $\boldsymbol{\mathcal{F}}_i = (\mathcal{F}_1^i, \cdots, \mathcal{F}_{t_i}^i)$, be an m-block collection. A coherent sheaf $\mathcal{E}$ on X has *natural cohomology with respect to* $\mathbf{B}$ if for each $0 \leqslant p \leqslant m$ and $1 \leqslant j \leqslant t_p$ there is at most one $q \geqslant 0$ such that $\mathrm{Ext}^q(\mathcal{F}_j^p, \mathcal{E}) \neq 0$.

Definition 2.9. Let X be a smooth projective variety with an m-block collection $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \cdots, \boldsymbol{\mathcal{F}}_m)$, $\boldsymbol{\mathcal{F}}_i = (\mathcal{F}_1^i, \cdots, \mathcal{F}_{t_i}^i)$ of coherent sheaves on X. A Beilinson monad for $\mathcal{E}$ is a bounded complex $G^\bullet$ in $D^b(X)$ whose terms are finite direct sums of elements of $\mathbf{B}$ and whose cohomology is $\mathcal{E}$, that is,

$$\bigoplus_{i \in \mathbb{Z}} H^i(G^\bullet) = H^0(G^\bullet) = \mathcal{E}.$$

The next result tell us when a coherent sheaf $\mathcal{E}$ on X is isomorphic to a Beilinson monad $G^\bullet$, see [6, Cor 1.7].

Lemma 2.10. *Let X be a smooth projective variety of dimension n with an n-block collection $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \cdots, \boldsymbol{\mathcal{F}}_n)$ generating $D^b(X)$. Let ${}^\vee\mathbf{B} = (\boldsymbol{\mathcal{H}}_0, \cdots, \boldsymbol{\mathcal{H}}_n)$ with $\boldsymbol{\mathcal{H}}_i = (\mathcal{H}_1^i, \cdots, \mathcal{H}_{u_i}^i)$, be its left dual n-block collection. Then each coherent sheaf $\mathcal{E}$ on X is isomorphic to a Beilinson monad $G^\bullet$ with each G^r given by*

$$G^r = \bigoplus_{p,q} \mathrm{Ext}^{n-q+r}(\mathcal{H}_p^{n-q}, \mathcal{E}) \otimes \mathcal{F}_p^q.$$

The cohomological characterisation for monads is the following, cf. [6, Thm 2.2].

Theorem 2.11. *Let X be a nonsingular projective variety of dimension n, and let $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \cdots, \boldsymbol{\mathcal{F}}_n)$, where $\boldsymbol{\mathcal{F}}_i = (\mathcal{F}_1^i, \cdots, \mathcal{F}_{t_i}^i)$, be an n-block collection of coherent sheaves on X generating $D^b(X)$. Let ${}^\vee\mathbf{B}$ be its left dual n-block collection, and let $\mathcal{F}_{i_0}^i$, $\mathcal{F}_{j_0}^j$, and $\mathcal{F}_{k_0}^k$ be elements of the blocks $\boldsymbol{\mathcal{F}}_i, \boldsymbol{\mathcal{F}}_j$ and $\boldsymbol{\mathcal{F}}_k$, respectively, with $0 \leqslant i < j < k \leqslant n$.*

A torsion-free sheaf $\mathcal{E}$ on X is the cohomology sheaf of a monad of the form

$$M^\bullet : (\mathcal{F}_{i_0}^i)^a \longrightarrow (\mathcal{F}_{j_0}^j)^b \longrightarrow (\mathcal{F}_{k_0}^k)^c \tag{6}$$

for some $b \geqslant 1$ and $a, c \geqslant 0$ if and only if $\mathcal{E}$ has:

(1) *rank* $b \cdot \mathrm{rk}(\mathcal{F}_{j_0}^j) - a \cdot \mathrm{rk}(\mathcal{F}_{i_0}^i) - c \cdot \mathrm{rk}(\mathcal{F}_{k_0}^k)$;

(2) *Chern polynomial* $c_t(\mathcal{E}) = c_t(F_{j_0}^j)^b c_t(F_{i_0}^i)^{-a} c_t(F_{k_0}^k)^{-c}$;

(3) *natural cohomology with respect to* ${}^\vee\mathbf{B}$.

Remark 2.12. The original statement of [6, Thm 2.2] requires $a, b, c \geqslant 1$. However, following the same steps of the proof of [6, Thm 2.2], one can prove that the result also holds for $a, c \geqslant 0$; in other words, one can allow for degenerate monads.

This result will be very useful in the last section of this paper, in which we study the decomposability of sheaves given by the cohomology of monads of the above form.

3. Cokernel and Steiner bundles

In this section we explain the relation between cokernel and Steiner bundles and representations of the Kronecker quiver.

3.1. Cokernel bundles

Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on a nonsingular projective variety X of dimension $n \geqslant 2$, satisfying the following conditions:

(1) $\mathcal{E}$ and $\mathcal{F}$ are simple, that is, $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathrm{Hom}(\mathcal{F}, \mathcal{F}) = \kappa$;

(2) $\mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$;

(3) $\mathrm{Ext}^1(\mathcal{F}, \mathcal{E}) = 0$;

(4) the sheaf $\mathcal{E}^\vee \otimes \mathcal{F}$ is globally generated;

(5) $W = \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ has dimension $w \geqslant 3$.

The next definition is due to Brambilla [1].

Definition 3.1. A *cokernel bundle of type* $(\mathcal{E}, \mathcal{F})$ on $\mathbb{P}^n$ is a vector bundle $\mathcal{C}$ with resolution of the form

$$0 \longrightarrow \mathcal{E}^a \xrightarrow{\alpha} \mathcal{F}^b \longrightarrow \mathcal{C} \longrightarrow 0 \tag{7}$$

where $\mathcal{E}, \mathcal{F}$ satisfy the conditions (1) through (5) above, $a \geqslant 0$ and $b \cdot \mathrm{rk}(\mathcal{F}) - a \cdot \mathrm{rk}(\mathcal{E}) \geqslant n$.

Cokernel bundles of type $(\mathcal{E}, \mathcal{F})$ form a full subcategory of the category of coherent sheaves on X; this category will be denoted by $\mathfrak{C}_X(\mathcal{E}, \mathcal{F})$.

Let us now see how cokernel bundles are related to quivers. Fix a basis $\boldsymbol{\sigma} = \{\sigma_1, \cdots, \sigma_w\}$ of $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$.

Definition 3.2. A representation $R = (\{\kappa^a, \kappa^b\}, \{A_i\}_{i=1}^w)$ of K_w is $(\mathcal{E}, \mathcal{F}, \boldsymbol{\sigma})$-*globally injective* when the map

$$\alpha(P) := \sum_{i=1}^{w} A_i \otimes \sigma_i(P) \ : \ \kappa^a \otimes \mathcal{E}_P \to \kappa^b \otimes \mathcal{F}_P$$

is injective for every $P \in X$; here, $\mathcal{E}_P$ and $\mathcal{F}_P$ denote the fibers of $\mathcal{E}$ and $\mathcal{F}$ over the point P, respectively.

$(\mathcal{E}, \mathcal{F}, \boldsymbol{\sigma})$-globally injective representations of K_w form a full subcategory of the category of representations of K_w; we denote it by $\mathfrak{R}(K_w)^{gi}$. From now on, since $(\mathcal{E}, \mathcal{F}, \boldsymbol{\sigma})$ are fixed, we will just refer to globally injective representations. It is a simple exercise to establish the following properties of $\mathfrak{R}(K_w)^{gi}$.

Lemma 3.3. *The category $\mathfrak{R}(K_w)^{gi}$ is closed under sub-objects, i.e. every subrepresentation R' of a representation R in $\mathfrak{R}(K_w)^{gi}$ is also in $\mathfrak{R}(K_w)^{gi}$.*

Lemma 3.4. *The category $\mathfrak{R}(K_w)^{gi}$ is closed under extensions and under direct summands, that is, respectively:*

(i) *if $R_1, R_2 \in \mathfrak{R}(K_w)^{gi}$ and*

$$0 \longrightarrow R_1 \longrightarrow R \longrightarrow R_2 \longrightarrow 0$$

is a short exact sequence in $\mathfrak{R}(K_w)^{gi}$, then $R \in \mathfrak{R}(K_w)^{gi}$;

(ii) *if $R \in \mathfrak{R}(K_w)^{gi}$ with $R \simeq R_1 \oplus R_2$, then $R_i \in \mathfrak{R}(K_w)^{gi}$, $i = 1, 2$.*

Our next result relates the category of globally injective representations of K_w to the category of cokernel bundles.

Theorem 3.5. *For every choice of basis $\boldsymbol{\sigma}$ of $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, there is an equivalence between $\mathfrak{R}(K_w)^{gi}$, the category of $(\mathcal{E}, \mathcal{F}, \boldsymbol{\sigma})$-globally injective representations of K_w, and $\mathfrak{C}_X(\mathcal{E}, \mathcal{F})$, the category of cokernel bundles of type $(\mathcal{E}, \mathcal{F})$.*

Proof. Given a basis $\boldsymbol{\sigma}$ of $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, we construct a functor

$$\mathbf{L}_{\boldsymbol{\sigma}} : \mathfrak{R}(K_w)^{gi} \to \mathfrak{C}_X(\mathcal{E}, \mathcal{F})$$

and show that it is essentially surjective and fully faithful.

Let $R = (\{\kappa^a, \kappa^b\}, \{A_i\}_{i=1}^w)$ be a globally injective representation of K_w. Define a map $\alpha : \mathcal{E}^a \to \mathcal{F}^b$ given by

$$\alpha = A_1 \otimes \sigma_1 + \cdots + A_w \otimes \sigma_w.$$

Since R is globally injective, we have that $\dim\operatorname{coker}\alpha(P) = b\cdot\operatorname{rk}(\mathcal{F}) - a\cdot\operatorname{rk}(\mathcal{E})$ for each $P\in X$. Therefore α is injective as a map of sheaves, and $\mathcal{C} := \operatorname{coker}\alpha$ is a cokernel bundle.

Now given two globally injective representations

$$R_1 = (\{\kappa^a,\kappa^b\},\{A_i\}_{i=1}^w) \ \text{ and } \ R_2 = (\{\kappa^c,\kappa^d\},\{B_i\}_{i=1}^w),$$

and a morphism $f=(f_1,f_2)$ between them, let $\mathbf{L}_{\boldsymbol{\sigma}}(R_1)=\mathcal{C}_1, \mathbf{L}_{\boldsymbol{\sigma}}(R_2)=\mathcal{C}_2$ be the cokernel bundles and α_1, α_2 the maps associated to R_1 and R_2, respectively. We want to define a morphism $\mathbf{L}_{\boldsymbol{\sigma}}(f):\mathcal{C}_1\to\mathcal{C}_2$.

Since we have $f_1:\kappa^a\to\kappa^c, f_2:\kappa^b\to\kappa^d$, we have maps $f_1^{'} = f_1\otimes\mathbb{1}_{\mathcal{E}}\in \operatorname{Hom}(\mathcal{E}^a,\mathcal{E}^c)$ and $f_2^{'} = f_2\otimes\mathbb{1}_{\mathcal{F}}\in\operatorname{Hom}(\mathcal{F}^b,\mathcal{F}^d)$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^a \arrow[r, "\alpha_1"] \arrow[d, "f_1^{'}"'] & \mathcal{F}^b \arrow[r, "\pi_1"] \arrow[d, "f_2^{'}"'] & \mathcal{C}_1 \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathcal{E}^c \arrow[r, "\alpha_2"] & \mathcal{F}^d \arrow[r, "\pi_2"] & \mathcal{C}_2 \arrow[r] & 0 \end{tikzcd} \tag{8}$$

where π_1,π_2 are the projections. Applying the left exact contravariant functor $\operatorname{Hom}(-,\mathcal{C}_2)$ to the upper sequence on (8) we find a map $\phi\in \operatorname{Hom}(\mathcal{C}_1,\mathcal{C}_2)$ and we define $\mathbf{L}_{\boldsymbol{\sigma}}(f):=\phi$.

Now given $\mathcal{C}$ an object of $\mathfrak{C}_X(\mathcal{E},\mathcal{F})$ we take $\alpha=\sum_{i=1}^w A_i\otimes\sigma_i$, with $A_i\in\operatorname{Hom}(\kappa^a,\kappa^b), i=1,\cdots,w$. Hence $R=(\{\kappa^a,\kappa^b\},\{A_i\}_{i=1}^w)$ is a globally injective representation of $\mathfrak{R}(K_w)$ such that $\mathbf{L}_{\boldsymbol{\sigma}}(R)=\mathcal{C}$. Therefore $\mathbf{L}_{\boldsymbol{\sigma}}$ is essentially surjective.

Finally, we need to prove that $\mathbf{L}_{\boldsymbol{\sigma}}$ is fully faithful. To check that it is full, given $\phi\in\operatorname{Hom}(\mathbf{L}_{\boldsymbol{\sigma}}(R_1),\mathbf{L}_{\boldsymbol{\sigma}}(R_2))$ we want $f=(f_1,f_2)\in\operatorname{Hom}(R_1,R_2)$ such that $\mathbf{L}_{\boldsymbol{\sigma}}(f)=\phi$. Let $\tilde{\phi}=\phi\pi_1\in\operatorname{Hom}(\mathcal{F}^b,\mathcal{C}_2)$. Let us apply the left exact covariant functor $\operatorname{Hom}(\mathcal{F}^b,-)$ to the lower sequence in diagram (9) below:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^a \arrow[r, "\alpha_1"] \arrow[d, dashed, "f_1^{'}"'] & \mathcal{F}^b \arrow[r, "\pi_1"] \arrow[d, dashed, "f_2^{'}"'] & \mathcal{C}_1 \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathcal{E}^c \arrow[r, "\alpha_2"] & \mathcal{F}^d \arrow[r, "\pi_2"] & \mathcal{C}_2 \arrow[r] & 0 \end{tikzcd} \tag{9}$$

we conclude that

$$\rho_2:\operatorname{Hom}(\mathcal{F}^b,\mathcal{F}^d)\to\operatorname{Hom}(\mathcal{F}^b,\mathcal{C}_2) \tag{10}$$

is an isomorphism since $\operatorname{Hom}(\mathcal{F}^b,\mathcal{E}^c)=\operatorname{Ext}^1(\mathcal{F}^b,\mathcal{E}^c)=0$. It follows that there is a morphism $f_2^{'}\in\operatorname{Hom}(\mathcal{F}^b,\mathcal{F}^d)$ such that

$$\rho_2(f_2^{'}) = \pi_2 f_2^{'} = \phi\pi_1$$

with $f_2^{'} = f_2 \otimes \mathbb{1}_{\mathcal{F}}$ and $f_2 \in \mathrm{Hom}(\kappa^b, \kappa^d)$.

Consider $\tilde{\tilde{\phi}} = f_2^{'}\alpha_1 \in \mathrm{Hom}(\mathcal{E}^a, \mathcal{F}^d)$. Applying the left exact covariant functor $\mathrm{Hom}(\mathcal{E}^a, -)$ to the lower sequence on (9) we get

$$0 \longrightarrow \mathrm{Hom}(\mathcal{E}^a, \mathcal{E}^c) \xrightarrow{\gamma_1} \mathrm{Hom}(\mathcal{E}^a, \mathcal{F}^d) \xrightarrow{\gamma_2} \mathrm{Hom}(\mathcal{E}^a, \mathcal{C}_2) \longrightarrow \cdots$$

Once we have an exact sequence,

$$\gamma_2(f_2^{'}\alpha_1) = \pi_2 f_2^{'}\alpha_1 = \phi\pi_1\alpha_1 = 0$$

then $f_2^{'}\alpha_1 \in \ker\gamma_2 = \mathrm{im}\,\gamma_1$, and there is a map $f_1^{'} \in \mathrm{Hom}(\mathcal{E}^a, \mathcal{E}^c)$ such that $\gamma_1(f_1^{'}) = \alpha_2 f_1^{'} = f_2^{'}\alpha_1$ and $f_1^{'} = f_1 \otimes \mathbb{1}_{\mathcal{E}}$ with $f_1 \in \mathrm{Hom}(\kappa^a, \kappa^c)$.

Since $\alpha_1 = \sum_{i=1}^{w} A_i \otimes \sigma_i$, $\alpha_2 = \sum_{i=1}^{w} B_i \otimes \sigma_i$, $\alpha_2 f_1^{'} = f_2^{'}\alpha_1$, and $\boldsymbol{\sigma}$ is a basis then $f_2 A_i = B_i f_1, i = 1, \cdots, w$, thus

$$f = (f_1, f_2) \in \mathrm{Hom}_{\mathfrak{R}(K_w)^{gi}}(R_1, R_2).$$

Now we need to prove that $\mathbf{L}_{\boldsymbol{\sigma}}(f) = \phi$. Suppose $\mathbf{L}_{\boldsymbol{\sigma}}(f) = \overline{\phi}$ such that $\overline{\phi}\pi_1 = \pi_2 f_2^{'} = \phi\pi_1$. Then $(\overline{\phi} - \phi)\pi_1 = 0$ and $\mathcal{C}_1 = \mathrm{im}\,\pi_1 \subset \ker(\overline{\phi} - \phi)$ therefore $\overline{\phi} = \phi$.

Finally, we show that $\mathbf{L}_{\boldsymbol{\sigma}} : \mathrm{Hom}(R_1, R_2) \to \mathrm{Hom}(\mathbf{L}_{\boldsymbol{\sigma}}(R_1), \mathbf{L}_{\boldsymbol{\sigma}}(R_2))$ is injective. Let $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathrm{Hom}(R_1, R_2)$ be morphisms such that $\mathbf{L}_{\boldsymbol{\sigma}}(f) = \phi_1 = \phi_2 = \mathbf{L}_{\boldsymbol{\sigma}}(g)$, that is, $\phi_1 - \phi_2 = 0$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^a \arrow[r, "\alpha_1"] \arrow[d, "f_1^{'}-g_1^{'}"'] & \mathcal{F}^b \arrow[r, "\pi_1"] \arrow[d, "f_2^{'}-g_2^{'}"'] & \mathcal{C}_1 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathcal{E}^c \arrow[r, "\alpha_2"] & \mathcal{F}^d \arrow[r, "\pi_2"] & \mathcal{C}_2 \arrow[r] & 0
\end{tikzcd} \tag{11}$$

Given $\phi_1 - \phi_2 = 0 \in \mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_2)$, doing the same construction as before,

$$0\pi_1 = 0 \in \mathrm{Hom}(\mathcal{F}^b, \mathcal{C}_2) \simeq \mathrm{Hom}(\mathcal{F}^b, \mathcal{F}^d)$$

with isomorphism given by ρ_2 in (10). Since

$$\rho_2(f_2^{'} - g_2^{'}) = \pi_2 \circ (f_2^{'} - g_2^{'}) = 0$$

then $f_2^{'} - g_2^{'} = 0$ and so $f_2^{'} = g_2^{'}$. Similarly, $0\alpha_1 = 0 \in \mathrm{Hom}(\mathcal{E}^a, \mathcal{F}^d)$ and

$$\gamma_1(f_1^{'} - g_1^{'}) = \alpha_2(f_1^{'} - g_1^{'}) = 0\alpha_1 = 0.$$

Since γ_1 injective, $f_1^{'} - g_1^{'} = 0$, then $f_1^{'} = g_1^{'}$. Therefore $\mathbf{L}_{\boldsymbol{\sigma}}$ is faithful. $\square$

Remark 3.6. Note that the functor $\mathbf{L}_{\boldsymbol{\sigma}}$ depends on the choice of the basis $\boldsymbol{\sigma}$. However let $\boldsymbol{\sigma}'$ be another basis for $\mathrm{Hom}(\mathcal{E},\mathcal{F})$. Let $\mathbf{L}_{\boldsymbol{\sigma}'}$ be the equivalence between the category of $(\mathcal{E},\mathcal{F},\boldsymbol{\sigma}')$-globally injective representations of K_w and the cokernel bundles on $\mathbb{P}^n$. Then if $\mathbf{G}$ is the inverse functor of $\mathbf{L}_{\boldsymbol{\sigma}'}$ we have that the functor $\mathbf{G}\circ\mathbf{L}_{\boldsymbol{\sigma}'}$ gives an equivalence between the categories $(\mathcal{E},\mathcal{F},\boldsymbol{\sigma})$- and $(\mathcal{E},\mathcal{F},\boldsymbol{\sigma}')$-globally injective representations of K_w.

Lemma 3.7. *For any choice of basis $\boldsymbol{\sigma}$, the functor $\mathbf{L}_{\boldsymbol{\sigma}} : \mathfrak{R}(K_w)^{gi} \to \mathfrak{C}_X(\mathcal{E},\mathcal{F})$ defined above is additive and exact. In particular, if $R \simeq R_1 \oplus R_2$ is a globally injective representation, then $\mathbf{L}_{\boldsymbol{\sigma}}(R) \simeq \mathbf{L}_{\boldsymbol{\sigma}}(R_1) \oplus \mathbf{L}_{\boldsymbol{\sigma}}(R_2)$.*

Proof. Checking the additivity of $\mathbf{L}_{\boldsymbol{\sigma}}$ is a simple exercise. We show its exactness in detail.

Let us prove that $\mathbf{L}_{\boldsymbol{\sigma}}$ preserves exact sequences. Let $R_1 = (\{\kappa^{a_1},\kappa^{b_1}\}, \{A_i\}_{i=1}^w)$, $R_2 = (\{\kappa^{a_2},\kappa^{b_2}\},\{B_i\}_{i=1}^w)$ and $R_3 = (\{\kappa^{a_3},\kappa^{b_3}\},\{C_i\}_{i=1}^w)$ be globally injective representations of K_w and let $f : R_1 \to R_2$ and $g : R_2 \to R_3$ be morphisms such that the sequence

$$0 \longrightarrow R_1 \xrightarrow{f} R_2 \xrightarrow{g} R_3 \longrightarrow 0$$

is exact. We want to prove that

$$0 \longrightarrow \mathcal{C}_1 \xrightarrow{\varphi} \mathcal{C}_2 \xrightarrow{\psi} \mathcal{C}_3 \longrightarrow 0$$

is also exact, where $\mathcal{C}_i = \mathbf{L}_{\boldsymbol{\sigma}}(R_i), i = 1,2,3$ and $\varphi = \mathbf{L}_{\boldsymbol{\sigma}}(f), \psi = \mathbf{L}_{\boldsymbol{\sigma}}(g)$. From the exact sequence of representations we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d, dashed] & \\
0 \arrow[r] & \mathcal{E}^{a_1} \arrow[r, "\alpha_1"] \arrow[d, "\mathbb{1}_{\mathcal{E}}\otimes f_1"] & \mathcal{F}^{b_1} \arrow[r, "\pi_1"] \arrow[d, "\mathbb{1}_{\mathcal{F}}\otimes f_2"] & \mathcal{C}_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathcal{E}^{a_2} \arrow[r, "\alpha_2"] \arrow[d, "\mathbb{1}_{\mathcal{E}}\otimes g_1"] & \mathcal{F}^{b_2} \arrow[r, "\pi_2"] \arrow[d, "\mathbb{1}_{\mathcal{F}}\otimes g_2"] & \mathcal{C}_2 \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \mathcal{E}^{a_3} \arrow[r, "\alpha_3"] \arrow[d] & \mathcal{F}^{b_3} \arrow[r, "\pi_3"] \arrow[d] & \mathcal{C}_3 \arrow[r] \arrow[d, dashed] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We need to show that φ is injective and ψ is surjective.

- ψ is surjective:

  It follows from the fact that $\pi_3(\mathbb{1}_{\mathcal{F}} \otimes g_2)$ is surjective.

- φ is injective.

  Let us suppose $\varphi(s) = 0, s \in \mathcal{C}_1$. Then $s = \pi_1(v), v \in \mathcal{F}^{b_1}$ and

$$0 = \varphi\pi_1(v) = \pi_2(\mathbb{1}_{\mathcal{F}} \otimes f_2)(v).$$

  Since $\ker \pi_2 = \mathrm{im}\alpha_2$, there is $u \in \mathcal{E}^{a_2}$ such that

$$(\mathbb{1}_{\mathcal{F}} \otimes f_2)(v) = \alpha_2(u) \tag{12}$$

Note that

$$\alpha_3(\mathbb{1}_{\mathcal{E}} \otimes g_1)(u) = (\mathbb{1}_{\mathcal{F}} \otimes g_2)(\alpha_2)(u) = (\mathbb{1}_{\mathcal{F}} \otimes g_2)(\mathbb{1}_{\mathcal{F}} \otimes f_2)(v) = 0$$

and since α_3 is injective, $(\mathbb{1}_{\mathcal{E}} \otimes g_1)(u) = 0$ so $u = (\mathbb{1}_{\mathcal{E}} \otimes f_1)(u')$ with $u' \in \mathcal{E}^{a_1}$. We have

$$\alpha_2(u) = \alpha_2(\mathbb{1}_{\mathcal{E}} \otimes f_1)(u') = (\mathbb{1}_{\mathcal{F}} \otimes f_2)\alpha_1(u').$$

From (12) we have $(\mathbb{1}_{\mathcal{F}} \otimes f_2)(v) = (\mathbb{1}_{\mathcal{F}} \otimes f_2)(\alpha_1(u'))$. Since $(\mathbb{1}_{\mathcal{F}} \otimes f_2)$ is injective, it follows that $v = \alpha_1(u')$ therefore

$$s = \pi_1(v) = \pi_1\alpha_1(u') = 0.$$

Now suppose $R \simeq R_1 \oplus R_2$. Let us prove that $\mathbf{L}_{\boldsymbol{\sigma}}(R_1 \oplus R_2) \simeq \mathbf{L}_{\boldsymbol{\sigma}}(R_1) \oplus \mathbf{L}_{\boldsymbol{\sigma}}(R_2)$. We have the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & R_1 \arrow[r, "i_{R_1}"] & R_1 \oplus R_2 \arrow[r, shift left, "\pi_{R_2}"] & R_2 \arrow[l, shift left, "i_{R_2}"] \arrow[r] & 0 \end{tikzcd}$$

where i_{R_j} is the inclusion and π_{R_j} the projection, $j = 1, 2$. Since the sequence above is split, $\pi_{R_2} \circ i_{R_2} = \mathbb{1}_{R_2}$. Now since $\mathbf{L}_{\boldsymbol{\sigma}}$ is an exact functor, we have

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{L}_{\boldsymbol{\sigma}}(R_1) \arrow[r, "\mathbf{L}_{\boldsymbol{\sigma}}(i_{R_1})"] & \mathbf{L}_{\boldsymbol{\sigma}}(R_1 \oplus R_2) \arrow[r, shift left, "\mathbf{L}_{\boldsymbol{\sigma}}(\pi_{R_2})"] & \mathbf{L}_{\boldsymbol{\sigma}}(R_2) \arrow[l, shift left, "\mathbf{L}_{\boldsymbol{\sigma}}(i_{R_2})"] \arrow[r] & 0 \end{tikzcd} \tag{13}$$

Then

$$\mathbf{L}_{\boldsymbol{\sigma}}(\pi_{R_2} \circ i_{R_2}) = \mathbf{L}_{\boldsymbol{\sigma}}(\pi_{R_2}) \circ \mathbf{L}_{\boldsymbol{\sigma}}(i_{R_2}) = \mathbf{L}_{\boldsymbol{\sigma}}(\mathbb{1}_{R_2}) = \mathbb{1}_{\mathbf{L}_{\boldsymbol{\sigma}}(R_2)}$$

therefore the sequence (13) is split. Hence $\mathbf{L}_{\boldsymbol{\sigma}}(R_1 \oplus R_2) \simeq \mathbf{L}_{\boldsymbol{\sigma}}(R_1) \oplus \mathbf{L}_{\boldsymbol{\sigma}}(R_2)$. $\square$

As an application of the previous results, we give a new, functorial proof for a result due to Brambilla, cf. [1, Thm 4.3].

Theorem 3.8. *Let $\mathcal{C}$ be a cokernel bundle of type $(\mathcal{E},\mathcal{F})$, given by the resolution*

$$0 \longrightarrow \mathcal{E}^a \xrightarrow{\alpha} \mathcal{F}^b \longrightarrow \mathcal{C} \longrightarrow 0\ , \tag{14}$$

and let $w = \dim \operatorname{Hom}(\mathcal{E},\mathcal{F})$.

(i) *If $\mathcal{C}$ is simple, then $a^2+b^2-wab \leqslant 1$.*

(ii) *If $a^2+b^2-wab \leqslant 1$, then there exists a non-empty open subset $U \subset \operatorname{Hom}(\mathcal{E}^a,\mathcal{F}^b)$ such that for every $\alpha \in U$ the corresponding cokernel bundle is simple.*

Proof. To prove (i), let $\mathcal{C}$ be a cokernel bundle given by resolution (14) and suppose $\mathcal{C}$ is simple. By Theorem 3.5 there is a globally injective representation R of K_w such that $\mathcal{C} = \mathbf{L}_{\boldsymbol{\sigma}}(R)$. Since $\mathbf{L}_{\boldsymbol{\sigma}}$ is full, we have that $\kappa = \operatorname{Hom}(\mathcal{C},\mathcal{C}) \simeq \operatorname{Hom}(R,R)$, thus R is simple and therefore, by Proposition 2.2, $q_w(a,b) = a^2+b^2-wab \leqslant 1$.

For the second claim, note that if $q_w(a,b) \leqslant 1$, there is a generic representation R with dimension vector (a,b) such that R is Schur, by Proposition 2.2. Then there is a non-empty open subset

$$U \subset \operatorname{Hom}(\kappa^a,\kappa^b)\otimes\kappa^w \simeq \operatorname{Hom}(\mathcal{E}^a,\mathcal{F}^b)$$

such that every $R \in U$ is simple. Since $\operatorname{Hom}(\mathcal{C},\mathcal{C}) \simeq \operatorname{Hom}(R,R) = \kappa$, it follows that $\mathcal{C}$ is simple. $\square$

The previous Theorem implies that if $a^2+b^2-wab > 1$ then $\mathcal{C}$ is not simple. However, more is true, and it is not difficult to establish the following stronger statement.

Proposition 3.9. *Under the same conditions as in Theorem 3.8, if $a^2+b^2-wab > 1$, then $\mathcal{C}$ is decomposable.*

Under more restrictive conditions, Brambilla proved in [1, Thm 6.3] that if $\mathcal{C}$ is a *generic* cokernel bundle such that $a^2+b^2-wab > 1$, then $\mathcal{C} \simeq \mathcal{C}_k^n \oplus \mathcal{C}_{k+1}^m$, where $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ are *Fibonacci bundles*, $n,m \in \mathbb{N}$ (we refer to [1] for the definition of Fibonacci bundles).

Proof. Let $\mathcal{C}$ be any cokernel bundle given by the exact sequence (14), such that $a^2 + b^2 - wab > 1$. Then there is a globally injective representation R of K_w, such that $\mathcal{C} = \mathbf{L}_{\boldsymbol{\sigma}}(R)$ with dimension vector (a, b) satisfying and $q_w(a, b) = a^2 + b^2 - wab > 1$. By Lemma 2.2, R is decomposable. Then by Lemma 3.7, $\mathcal{C}$ is also decomposable. □

Next, recall that a vector bundle $\mathcal{E}$ on X is *exceptional* if it is simple and $\mathrm{Ext}^p(\mathcal{E}, \mathcal{E}) = 0$ for $p \geqslant 1$.

Proposition 3.10. *Under the same conditions as in Theorem 3.8, if $\mathcal{C}$ is exceptional, then $a^2 + b^2 - wab = 1$.*

Proof. Since the functor $\mathbf{L}_{\boldsymbol{\sigma}}$ is exact, we have an isomorphism

$$\mathrm{Ext}^1(R, R) \simeq \mathrm{Ext}^1(\mathbf{L}_{\boldsymbol{\sigma}}(R), \mathbf{L}_{\boldsymbol{\sigma}}(R)).$$

Now we know from [8] that

$$q_w(a, b) = \dim \mathrm{Hom}(R, R) - \dim \mathrm{Ext}^1(R, R) \tag{15}$$

hence if $\mathcal{C}$ is an exceptional cokernel bundle, then $q_w(a, b) = a^2 + b^2 - wab = 1$. □

However, the converse of the Proposition 3.10 is not true. For instance, consider the generic cokernel bundle given by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(4)^{35} \longrightarrow \mathcal{C} \longrightarrow 0 .$$

We have $q_{35}(1, 35) = 1$, but from the long exact sequence of cohomologies, $\mathrm{Ext}^2(\mathcal{C}, \mathcal{C}) \simeq \kappa^{35}$ hence $\mathcal{C}$ is not exceptional. In order to establish the converse statement, we need stronger assumption, provided by Steiner bundles.

3.2. Steiner bundles

Note that the Steiner bundles of type $(\mathcal{E}, \mathcal{F})$ satisfying $\dim \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \geqslant 3$, are a particular case of cokernel bundles, therefore all results in the previous section also hold for such Steiner bundles. Furthermore, the additional hypotheses satisfied by the sheaves $\mathcal{E}$ and $\mathcal{F}$ allow to establish the converse of Lemma 3.7 and Proposition 3.10.

Let us first consider the converse of Lemma 3.7; more precisely, we prove the following statement.

Theorem 3.11. *Let X be a nonsingular projective variety of dimension n, and let $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \cdots, \boldsymbol{\mathcal{F}}_n)$ be an n-block collection generating $D^b(X)$. A Steiner bundle of type $(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0})$ such that $w = \dim \mathrm{Hom}(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0}) \geqslant 3$ is decomposable if and only if, for any choice of basis γ for $\mathrm{Hom}(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0})$, the corresponding $(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0}, \gamma)$-globally injective representation of K_w is also decomposable.*

The theorem follows easily from Lemma 3.7 and the following claim. Let $\mathfrak{S}_{\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0}}(X)$ denote the category of Steiner bundles of type $(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0})$ over X.

Proposition 3.12. *The category $\mathfrak{S}_{\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0}}(X)$ is closed under direct summands.*

Proof. Let ${}^{\vee}\mathbf{B} = (\boldsymbol{\mathcal{H}}_0, \cdots, \boldsymbol{\mathcal{H}}_n)$ where $\boldsymbol{\mathcal{H}}_i = (\mathcal{H}^i_1, \cdots, \mathcal{H}^i_{u_i})$, be the n-block collection which is left dual to $\mathbf{B}$, and let $\mathcal{S}$ be a Steiner bundle of type $(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0})$ given by the short exact sequence

$$0 \longrightarrow (\mathcal{F}^i_{i_0})^a \longrightarrow (\mathcal{F}^j_{j_0})^b \longrightarrow \mathcal{S} \longrightarrow 0\ ,$$

where $\mathcal{F}^i_{i_0}$ and $\mathcal{F}^j_{j_0}$ are elements of blocks $\boldsymbol{\mathcal{F}}_i$ and $\boldsymbol{\mathcal{F}}_j$ respectively, $0 \leqslant i < j \leqslant n$.

If $\mathcal{S} \simeq \mathcal{S}_1 \oplus \mathcal{S}_2$, $0 \neq \mathcal{S}_i \subsetneq \mathcal{S}$, $i = 1, 2$, then we have that

$$\mathrm{Ext}^p(\mathcal{H}^m_q, \mathcal{S}) \simeq \mathrm{Ext}^p(\mathcal{H}^m_q, \mathcal{S}_1) \oplus (\mathcal{H}^m_q, \mathcal{S}_2).$$

It follows that $\mathrm{Ext}^p(\mathcal{H}^m_q, \mathcal{S}_l)$, $l = 1, 2$, vanish except for

$$\mathrm{Ext}^{n-i-1}(\mathcal{H}^{n-i}_{i_0}, \mathcal{S}_l) = a_l,\ a_l \geqslant 0,\ l = 1, 2$$

and

$$\mathrm{Ext}^{n-j}(\mathcal{H}^{n-j}_{j_0}, \mathcal{S}_l) = b_l,\ b_l \geqslant 0,\ l = 1, 2$$

with $a_1 + a_2 = a$ and $b_1 + b_2 = b$. Then from the cohomological characterisation, Theorem 2.11, one of the following possibilities must hold.

1) For $a_l \neq 0$ and $b_l \neq 0$, $l = 1, 2$, the bundles $\mathcal{S}_l$ are Steiner bundles given by

$$0 \longrightarrow (\mathcal{F}^i_{i_0})^{a_l} \longrightarrow (\mathcal{F}^j_{j_0})^{b_l} \longrightarrow \mathcal{S}_l \longrightarrow 0\ .$$

2) For $a_1, b_1, b_2 \neq 0$ and $a_2 = 0$, we have

$$0 \longrightarrow (\mathcal{F}^i_{i_0})^{a_1} \longrightarrow (\mathcal{F}^j_{j_0})^{b_1} \longrightarrow \mathcal{S}_1 \longrightarrow 0 \text{ and } \mathcal{S}_2 \simeq (\mathcal{F}^j_{j_0})^{b_2}.$$

3) For $a_1 = 0$ and $b_1, a_2, b_2 \neq 0$, we have

$$\mathcal{S}_1 \simeq (\mathcal{F}^i_{i_0})^{a_1} \text{ and } 0 \longrightarrow (\mathcal{F}^i_{i_0})^{a_2} \longrightarrow (\mathcal{F}^j_{j_0})^{b_2} \longrightarrow \mathcal{S}_2 \longrightarrow 0 .$$

□

To complete this section, we consider the converse of Proposition 3.10.

Proposition 3.13. *Let $\mathcal{S}$ be a Steiner bundle of type $(\mathcal{E}, \mathcal{F})$ with $w = \dim \operatorname{Hom}(\mathcal{E}, \mathcal{F}) \geqslant 3$, given by the short exact sequence:*

$$0 \longrightarrow \mathcal{E}^a \xrightarrow{\alpha} \mathcal{F}^b \longrightarrow \mathcal{S} \longrightarrow 0 . \tag{16}$$

(i) *If $\mathcal{S}$ is exceptional then $a^2 + b^2 - wab = 1$.*

(ii) *If $a^2 + b^2 - wab = 1$ then there is a non-empty open subset $U \subset \operatorname{Hom}(\mathcal{E}^a, \mathcal{F}^b)$ such that for every $\alpha \in U$ the corresponding bundle $\mathcal{S}$ is exceptional.*

Proof. The first claim is just Proposition 3.10. For the second statement, we first show that if $\mathcal{S}_1, \mathcal{S}_2$ are Steiner bundles of type $(\mathcal{E}, \mathcal{F})$, then $\operatorname{Ext}^p(\mathcal{S}_1, \mathcal{S}_2) = 0$ for $p \geqslant 2$.

Indeed, suppose $\mathcal{S}_i$, $i = 1, 2$, are given by short exact sequences

$$0 \longrightarrow \mathcal{E}^{a_i} \longrightarrow \mathcal{F}^{b_i} \longrightarrow \mathcal{S}_i \longrightarrow 0 \tag{17}$$

Applying the functor $\operatorname{Hom}(-, \mathcal{F})$ to the sequence (17) for $i = 1$, we have $\operatorname{Ext}^p(\mathcal{S}_1, \mathcal{F}) = 0$, $p \geqslant 2$. Applying $\operatorname{Hom}(-, \mathcal{E})$ to the same sequence, we obtain $\operatorname{Ext}^q(\mathcal{S}_1, \mathcal{E}) = 0, q \geqslant 0$. Finally applying the functor $\operatorname{Hom}(\mathcal{S}_1, -)$ to the sequence (17), $i = 2$, we conclude that $\operatorname{Ext}^j(\mathcal{S}_1, \mathcal{S}_2) = 0$ for $j \geqslant 2$.

Now to prove the second claim, start by supposing that $q_w(a, b) = a^2 + b^2 - wab = 1$. By Theorem 3.8 item (ii) there exists a non-empty open subset $U \subset \operatorname{Hom}(\mathcal{E}^a, \mathcal{F}^b)$ such that for every $\alpha \in U$ the associated bundle $\mathcal{S}$ is simple. From (15) we see that $\operatorname{Ext}^1(\mathcal{S}, \mathcal{S}) = 0$. Finally, from the considerations above, we have $\operatorname{Ext}^p(\mathcal{S}, \mathcal{S}) = 0$ for $p \geqslant 2$. Hence $\mathcal{S}$ is exceptional. □

Remark 3.14. Soares also proved in [10, Thm 2.2.7], using a different method, that a generic Steiner bundle of type $(\mathcal{E}, \mathcal{F})$ given by the short exact sequence (16) is exceptional if and only if $a^2 + b^2 - wab = 1$.

4. Syzygy bundles and quivers

In this section we relate a different class of vector bundles, the syzygy bundles, with representations of quivers. A locally free sheaf $\mathcal{G}$ given by the short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_m)^{a_m} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{c} \longrightarrow 0 \tag{18}$$

is called a *syzygy bundle*. Here, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a surjective map of sheaves on $\mathbb{P}^n$ given by

$$\alpha(f_1, f_2, \ldots, f_m) = \sum_{i=1}^{m} \alpha_i f_i$$

where $f_1, \ldots, f_m$ are homogeneous polynomials of degree $d_1, \ldots, d_m$ in $\kappa[X_0, \ldots, X_n]$ and d_i are distinct positive integers. Let us assume $0 \leqslant d_m < \cdots < d_1$.

Note that for $m = 1$, the dual bundle $\mathcal{G}^*$ is a cokernel bundle. However, the same is not true for $m > 1$, since the bundle $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(d_1)^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_m)^{a_m}$ is not simple.

To relate syzygy bundles with representations of quivers, we restrict ourselves, for the sake of simplicity, to the case $m = 2$. The results for the general case are the same, but the notation becomes more complicated. Thus we set $m = 2$, and consider exact sequences of the form

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b} \xrightarrow{\alpha_1, \alpha_2} \mathcal{O}_{\mathbb{P}^n}^{c} \longrightarrow 0 \tag{19}$$

with $d_1 > d_2$. We denote by $\mathfrak{S}yz(d_1, d_2)$ the category of syzygy bundles given by short exact sequences as in (19) above.

Fix, for $i = 1, 2$, a basis $\boldsymbol{\sigma}_i = \{f_1^i, \ldots, f_{w_i}^i\}$ of $H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))$, where $w_i = \binom{n+d_i}{d_i}$. Consider the quiver below, which will be denoted by A_{w_1,w_2}:

$$\bullet \;\underset{w_1}{\overset{1}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}}\; \bullet \;\underset{w_2}{\overset{1}{\substack{\longleftarrow \\ \vdots \\ \longleftarrow}}}\; \bullet \tag{20}$$

If (a, b, c) is a dimension vector of this quiver, its Tits form is given by

$$q_{w_1,w_2}(a, b, c) = a^2 + b^2 + c^2 - w_1 ab - w_2 bc. \tag{21}$$

Let $R = (\{\kappa^a, \kappa^b, \kappa^c\}, \{A_i\}_1^{w_1}, \{B_j\}_1^{w_2})$ be a representation of A_{w_1,w_2}, where each A_i is a $c \times a$ matrix, and each B_j is a $c \times b$ matrix with entries in κ. We define

$$\alpha_1 = \sum_{i=1}^{w_1} A_i \otimes f_i^1 \text{ and } \alpha_2 = \sum_{j=1}^{w_2} B_j \otimes f_j^2,$$

so that we have a map

$$(\alpha_1, \alpha_2) : \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \to \mathcal{O}_{\mathbb{P}^n}^c. \tag{22}$$

Definition 4.1. A representation R of A_{w_1,w_2} is $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$-*globally surjective* if the map (α_1, α_2) is surjective.

Denote by $\mathfrak{R}(A_{w_1,w_2})^{gs}$ the category of $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$-globally surjective representations of A_{w_1,w_2}. We will now build a functor $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}$ between the $\mathfrak{R}(A_{w_1,w_2})^{gs}$ and the category of syzygy bundles $\mathfrak{S}yz(d_1, d_2)$.

First, let $R = (\{\kappa^a, \kappa^b, \kappa^c\}, \{A_i\}_{i=1}^{w_1}, \{B_j\}_{i=1}^{w_2})$ be a globally surjective representation of A_{w_1,w_2}. We define the sheaf

$$\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R) := \ker(\alpha_1, \alpha_2),$$

where (α_1, α_2) is the map defined above in (22). Note that, since R is globally surjective, $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R)$ is a vector bundle, and it is given by the exact sequence (19).

Now let $\{g_1, g_2, h\}$ be a morphism between the globally surjective representations

$$R = (\{\kappa^a, \kappa^b, \kappa^c\}, \{A_i\}_{i=1}^{w_1}, \{B_j\}_{i=1}^{w_2}) \text{ and}$$

$$R' = (\{\kappa^{a'}, \kappa^{b'}, \kappa^{c'}\}, \{A'_i\}_{i=1}^{w_1}, \{B'_j\}_{i=1}^{w_2}).$$

The following diagram commutes for $i = 1, \ldots, w_1$ and $j = 1, \ldots, w_2$.

$$\begin{tikzcd}
\kappa^a \arrow[rr, bend left, "g_1"] \arrow[dr, "A_i"] & & \kappa^{a'} \arrow[dr, "A'_i"] & \\
& \kappa^c \arrow[rr, "h"] & & \kappa^{c'} \\
\kappa^b \arrow[ur, "B_i"'] \arrow[rr, bend right, "g_2"'] & & \kappa^{b'} \arrow[ur, "B'_j"'] &
\end{tikzcd} \tag{23}$$

It induces the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r, "i_1"] \arrow[d, dashed, "\phi"] & \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \arrow[r, "\alpha_1, \alpha_2"] \arrow[d, "M"] & \mathcal{O}_{\mathbb{P}^n}^c \arrow[r] \arrow[d, "h\otimes \mathbb{1}_{\mathcal{O}_{\mathbb{P}^n}}"] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r, "i_2"] & \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a'} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b'} \arrow[r, "\alpha_1', \alpha_2'"] & \mathcal{O}_{\mathbb{P}^n}^{c'} \arrow[r] & 0 \end{tikzcd} \tag{24}$$

where

$$M = \begin{pmatrix} g_1 \otimes \mathbb{1}_{\mathcal{O}_{\mathbb{P}^n}(-d_1)} & 0 \\ 0 & g_2 \otimes \mathbb{1}_{\mathcal{O}_{\mathbb{P}^n}(-d_2)} \end{pmatrix}$$

The commutativity of (23) implies the commutativity of the right square in (24). We then have an induced morphism $\phi : \mathcal{G} = \mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R) \to \mathcal{G}' = \mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R')$, which we define to be $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(g_1, g_2, h)$.

Lemma 4.2. *The functor $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}$ is faithful and essentially surjective.*

Proof. We prove that $\mathrm{Hom}(R, R') \to \mathrm{Hom}(\mathbf{G}(R), \mathbf{G}(R'))$ is injective. Let $\{g_1, g_2, h\}$ be a morphism between R and R' such that $\mathbf{G}(\{g_1, g_2, h\}) = 0$, that is, $\phi = 0$. Since the diagram (24) commutes if $\phi = 0$ then $g_1 = g_2 = h = 0$, hence $\mathbf{G}$ is faithful.

Let $\mathcal{G}$ be a syzygy bundle with resolution

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \xrightarrow{\alpha_1, \alpha_2} \mathcal{O}_{\mathbb{P}^n}^c \longrightarrow 0$$

Then the maps α_1 and α_2 are given by

$$\alpha_1 = \sum_{i=1}^{w_1} A_i \otimes f_i^1 \text{ and } \alpha_2 = \sum_{j=1}^{w_2} B_j \otimes f_j^2$$

with $A_i \in \mathrm{Hom}(k^a, k^c)$ and $B_j \in \mathrm{Hom}(k^b, k^c)$. Therefore

$$R = (\{k^a, k^b, k^c\}, \{A_i\}_1^{w_1}, \{B_j\}_1^{w_2})$$

is a globally surjective representation of (20) such that $\mathbf{G}(R) = \mathcal{G}$. □

Remark 4.3. Note that $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}$ is not full, since not every

$$M \in \mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b, \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a'} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b'}\right)$$

is necessarily diagonal. It follows that the categories $\mathfrak{R}(A_{w_1,w_2})^{gs}$ and $\mathfrak{S}yz(d_1, d_2)$ are not, in general, equivalent.

This completes the proof of the first part of Theorem 1.2. To establish its second part, we first need the following two lemmas.

Lemma 4.4. *The category of globally surjective representations of A_{w_1,w_2} is closed under quotients, and hence closed under direct summands.*

Proof. Let $R = (\{\kappa^a, \kappa^b, \kappa^c\}, \{A_i\}_{i=1}^{w_1}, \{B_j\}_{j=1}^{w_2})$ be a $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$-globally surjective representation of A_{w_1,w_2} and $R' = (\{\kappa^{a'}, \kappa^{b'}, \kappa^{c'}\}, \{A'_i\}_{i=1}^{w_1}, \{B'_j\}_{j=1}^{w_2})$ be a subrepresentation of R. We want to prove that the quotient representation

$$R/R' = (\{\kappa^a/\kappa^{a'}, \kappa^b/\kappa^{b'}, \kappa^c/\kappa^{c'}\}, \{C_i\}_{i=1}^{w_1}, \{D_j\}_{j=1}^{w_2}),$$

where C_i and D_j are the maps induced by A_i and B_j respectively, is also globally surjective. We have the diagram

$$\begin{tikzcd}[row sep=large]
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\kappa^{a'} \arrow[d, "l_1"'] \arrow[r, shift left=2, "A'_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "A'_{w_1}"'] & \kappa^{c'} \arrow[d, "l_3"'] & \kappa^{b'} \arrow[d, "l_2"'] \arrow[l, shift right=2, "B'_1"'] \arrow[l, phantom, "\vdots"] \arrow[l, shift left=2, "B'_{w_2}"] \\
\kappa^{a} \arrow[d, "p_1"'] \arrow[r, shift left=2, "A_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "A_{w_1}"'] & \kappa^{c} \arrow[d, "p_3"'] & \kappa^{b} \arrow[d, "p_2"'] \arrow[l, shift right=2, "B_1"'] \arrow[l, phantom, "\vdots"] \arrow[l, shift left=2, "B_{w_2}"] \\
\kappa^a/\kappa^{a'} \arrow[d] \arrow[r, shift left=2, "C_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "C_{w_1}"'] & \kappa^c/\kappa^{c'} \arrow[d] & \kappa^b/\kappa^{b'} \arrow[d] \arrow[l, shift right=2, "D_1"'] \arrow[l, phantom, "\vdots"] \arrow[l, shift left=2, "D_{w_2}"] \\
0 & 0 & 0
\end{tikzcd}$$

where l_i are the inclusions and p_i the projections $i = 1, 2, 3$. Now consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \arrow[r, "(\alpha_1,\alpha_2)"] \arrow[d, "M"] & \mathcal{O}_{\mathbb{P}^n}^c \arrow[d, "p_3 \otimes \mathbb{1}_{\mathcal{O}_{\mathbb{P}^n}}"] \\
\mathcal{O}_{\mathbb{P}^n}(-d_1)^{(a-a')} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{(b-b')} \arrow[r, "(\gamma_1,\gamma_2)"] & \mathcal{O}_{\mathbb{P}^n}^{(c-c')}
\end{tikzcd}$$

where

$$M = \begin{pmatrix} p_1 \otimes \mathbb{1}_{\mathcal{O}_{\mathbb{P}^n}(-d_1)} & 0 \\ 0 & p_2 \otimes \mathbb{1}_{\mathcal{O}_{\mathbb{P}^n}(-d_2)} \end{pmatrix}$$

and $\gamma_1 = \sum_{i=1}^{w_1} C_i \otimes f_i^1$, $\gamma_2 = \sum_{j=1}^{w_2} D_j \otimes f_j^2$.

Since p_i is surjective and (α_1, α_2) is surjective for every $P \in \mathbb{P}^n$, we have that the map (γ_1, γ_2) is also surjective for every point $P \in \mathbb{P}^n$, hence the quotient representation R/R' is globally surjective. □

Lemma 4.5. *Let R be a decomposable globally surjective representation of A_{w_1,w_2}. Then $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R)$ is also decomposable.*

Proof. Let $R \simeq R_1 \oplus R_2$ be a decomposable globally surjective representation. From Lemma 4.4 we have that R_1 and R_2 are globally surjective. Let $\mathcal{G}_i = \mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R_i)$, $i = 1, 2$ be given by the short exact sequence

$$0 \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_i} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b_i} \xrightarrow{\alpha^i} \mathcal{O}_{\mathbb{P}^n}^{c_i} \longrightarrow 0$$

where $\alpha_i = (\alpha_1^i, \alpha_2^i)$, $i = 1, 2$. Since

$$\begin{aligned} \mathcal{G} = \mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R) = \ker(\alpha_1 \oplus \alpha_2) &\simeq \ker \alpha_1 \oplus \ker \alpha_2 = \\ &= \mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R_1) \oplus \mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R_2), \end{aligned}$$

it follows that $\mathcal{G}$ is decomposable. □

We are finally in position to complete the proof of Theorem 1.2. Indeed, fix bases $\boldsymbol{\sigma}_j$ for $H^0(\mathcal{O}_{\mathbb{P}^n}(d_j))$, $j = 1, 2$. For every syzygy bundle $\mathcal{G}$ given by a short exact sequence of the form (19), one can find a $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$-globally surjective representation R of A_{w_1,w_2} with dimension vector (a, b, c) with $\mathbf{G}_{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2}(R) = \mathcal{G}$. If $q_{w_1,w_2}(a, b, c) > 1$, then R is decomposable, by Lemma 2.2, and it must decompose as a sum of $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$-globally surjective representations by Lemma 4.4. Therefore Lemma 4.5 implies that $\mathcal{G}$ is also decomposable.

Remark 4.6. All the results can be generalized for syzygy bundles with $m \geqslant 2$. To build the associated quiver, we add a vertex to the quiver with $w_i = \dim H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))$ arrows from this vertex to the vertex associated to $\mathcal{O}_{\mathbb{P}^n}^{\oplus c}$, for each term $\mathcal{O}_{\mathbb{P}^n}(-d_i)^{\oplus a_i}$.

5. Monads and representations of quivers

Recall that a *monad* $M^\bullet$ on a projective variety X is a complex of locally free sheaves

$$M^\bullet : \ \mathcal{A}^{\oplus a} \xrightarrow{\alpha} \mathcal{B}^{\oplus b} \xrightarrow{\beta} \mathcal{C}^{\oplus c} \tag{25}$$

where α is injective and β is surjective. The coherent sheaf $\mathcal{E} := \ker \beta / \mathrm{im}\ \alpha$ is called the *cohomology of* $M^\bullet$; note that $\mathcal{E}$ is locally free if and only if the map α_P on the fibers is injective for every point $P \in X$.

Now let $m = \dim \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $n = \dim \mathrm{Hom}(\mathcal{B}, \mathcal{C})$. We also assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are simple vector bundles, and that the cohomology sheaf $\mathcal{E}$ is locally free. We will denote the category of such monads by $\mathfrak{M}_{\mathcal{A},\mathcal{B},\mathcal{C}}$, regarding it as a full subcategory of the category of complexes of coherent sheaves on X.

Next, consider the quiver $K_{m,n}$ given by the graph

$$\begin{tikzcd} \bullet \arrow[r, shift left=3, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "m"'] & \bullet \arrow[r, shift left=3, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "n"'] & \bullet \end{tikzcd}$$

The category of representations of $K_{m,n}$ is denoted by $\mathfrak{R}(K_{m,n})$. Note that its Tits form is given by

$$q_{m,n}(a, b, c) = a^2 + b^2 + c^2 - mab - nbc. \tag{26}$$

5.1. Proof of Theorem 1.3

We begin by describing a functor from $\mathfrak{M}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ to $\mathfrak{R}(K_{m,n})$ in a manner similar to what was done in the previous sections. Choose bases $\boldsymbol{\gamma} = \{\gamma_1, \cdots, \gamma_m\}$ of $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $\boldsymbol{\sigma} = \{\sigma_1, \cdots, \sigma_n\}$ of $\mathrm{Hom}(\mathcal{B}, \mathcal{C})$. We can write

$$\alpha = \sum_{i=1}^{m} A_i \otimes \gamma_i \quad \text{and} \quad \beta = \sum_{j=1}^{n} B_j \otimes \sigma_j$$

where each A_i is a $b \times a$ matrix with entries in κ, and each B_j is a $c \times b$ matrix with entries in κ.

Now let

$$\mathbf{G}_{\boldsymbol{\gamma},\boldsymbol{\sigma}} : \mathfrak{M}_{\mathcal{A},\mathcal{B},\mathcal{C}} \to \mathfrak{R}(K_{m,n}) \tag{27}$$

be the functor that to each monad $M^\bullet$ as in (25) with maps α and β, associates the representation $R = (\{\kappa^a, \kappa^b, \kappa^c\}, \{A_i\}_{i=1}^m, \{B_j\}_{j=1}^m)$. Let $\varphi_\bullet = (f, g, h)$ be a morphism between the monads $M_1^\bullet$ and $M_2^\bullet$ below

$$\begin{tikzcd}
\mathcal{A}^{a_1} \arrow[r, "\alpha_1"] \arrow[d, "f"'] & \mathcal{B}^{b_1} \arrow[r, "\beta_1"] \arrow[d, "g"'] & \mathcal{C}^{c_1} \arrow[d, "h"'] \\
\mathcal{A}^{a_2} \arrow[r, "\alpha_2"'] & \mathcal{B}^{b_2} \arrow[r, "\beta_2"'] & \mathcal{C}^{c_2}
\end{tikzcd}$$

Since $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are simple, it follows that

$$(f, g, h) = (A \otimes \mathbb{1}_{\mathcal{A}}, B \otimes \mathbb{1}_{\mathcal{B}}, C \otimes \mathbb{1}_{\mathcal{C}})$$

where A, B and C are, respectively, $a_2 \times a_1$, $b_2 \times b_1$ and $c_2 \times c_1$ matrices with entries in κ. If

$$\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet) = (\{\kappa^{a_1}, \kappa^{b_1}, \kappa^{c_1}\}, \{A_i^1\}_{i=1}^m, \{B_j^1\}_{j=1}^n)$$

$$\text{and} \quad \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_2^\bullet) = (\{\kappa^{a_2}, \kappa^{b_2}, \kappa^{c_2}\}, \{A_i^2\}_{i=1}^m, \{B_j^2\}_{j=1}^n),$$

we then have

$$\begin{tikzcd}
\kappa^{a_1} \arrow[r, shift left=3, "A_1^1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "A_m^1"'] \arrow[d, "A"'] & \kappa^{b_1} \arrow[r, shift left=3, "B_1^1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "B_n^1"'] \arrow[d, "B"'] & \kappa^{c_1} \arrow[d, "C"'] \\
\kappa^{a_2} \arrow[r, shift left=3, "A_1^2"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "A_m^2"'] & \kappa^{b_2} \arrow[r, shift left=3, "B_1^2"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "B_n^2"'] & \kappa^{c_2}
\end{tikzcd} \tag{28}$$

$$BA_i^1 = A_i^2 A \ \text{ and } \ CB_j^1 = B_j^2 B \ \text{ for } \ i = 1, \cdots, m \ \text{ and } \ j = 1, \cdots, n.$$

Hence the matrices A,B and C define a morphism between the representations. From the construction of the functor we see that $\mathbf{G}_{\gamma,\boldsymbol{\sigma}}$: $\mathrm{Hom}(M_1^\bullet, M_2^\bullet) \to \mathrm{Hom}(\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet), \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_2^\bullet))$ is an isomorphism, thus we have the following result, which corresponds to the first part of Theorem 1.3.

Proposition 5.1. *The category $\mathfrak{M}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ is equivalent to a full subcategory of $\mathfrak{R}(K_{m,n})$.*

Let us further characterise the subcategory of $\mathfrak{R}(K_{m,n})$ obtained in this way. The monad conditions imply that $\alpha(P)$ is injective and $\beta(P)$ is surjective for every $P \in X$. Therefore we say that a representation $R = (\{\kappa^a, \kappa^b, \kappa^c\}, \{A_i\}_{i=1}^m, \{B_j\}_{j=1}^n)$ is $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$-globally injective and surjective if $\alpha(P) = \sum_{i=1}^m A_i \otimes \gamma_i(P)$ is injective and $\beta(P) = \sum_{j=1}^n B_j \otimes \sigma_j(P)$ is surjective, for every $P \in X$. In addition, the matrices A_i and B_j must satisfy quadratic equations imposed by the condition $\beta\alpha = 0$:

$$\sum_{1 \leqslant i \leqslant j \leqslant m} (B_i A_j + B_j A_i)(\sigma_i \gamma_j) = 0;$$

note that the precise relation depends on the choice of bases $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}$. We denote by $\mathfrak{G}_{m,n}^{\mathrm{gis}}$ the full subcategory of $\mathfrak{R}(K_{m,n})$ consisting of the objects satisfying the conditions above.

In order to prove the second part of Theorem 1.3, our first goal is to prove that $\mathfrak{G}_{m,n}^{\mathrm{gis}}$ is closed under direct summands.

Lemma 5.2. *The category $\mathfrak{G}_{m,n}^{\mathrm{gis}}$ is closed under direct summands.*

Proof. It is a general fact that if S is a subrepresentation of a quiver representation R which satisfies the given relations, then S also satisfies the same relations.

Moreover, every subrepresentation of a $\boldsymbol{\gamma}$-globally injective representation will also be $\boldsymbol{\gamma}$-globally injective (cf. Lemma 3.3 above), while any quotient representation of a $\boldsymbol{\sigma}$-globally surjective representation will also be $\boldsymbol{\sigma}$-globally surjective (cf. Lemma 4.4 above). $\square$

Next, the previous lemma allows us to relate the decomposability of the monad with the decomposability of the associated quiver representation.

Proposition 5.3. *A monad $M^\bullet$ is decomposable if and only if the associated quiver representation $\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M^\bullet)$ is decomposable. In addition, if $\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M^\bullet)$ is decomposable, then the cohomology of $M^\bullet$ is a decomposable vector bundle.*

Proof. We begin by showing that the functor $\mathbf{G}_{\gamma,\boldsymbol{\sigma}} : \mathfrak{M}_{\mathcal{A},\mathcal{B},\mathcal{C}} \to \mathfrak{G}_{m,n}^{gis}$ preserves direct sums, that is, $\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet \oplus M_2^\bullet) \simeq \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet) \oplus \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_2^\bullet)$. In particular, if $M^\bullet$ is decomposable then $R = \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M^\bullet)$ is decomposable.

Indeed, consider a monad $M^\bullet = M_1^\bullet \oplus M_2^\bullet$ given by

$$\mathcal{A}^{a_1+a_2} \xrightarrow{\alpha} \mathcal{B}^{b_1+b_2} \xrightarrow{\beta} \mathcal{C}^{c_1+c_2}$$

where $\alpha = \alpha_1 \oplus \alpha_2$ and $\beta = \beta_1 \oplus \beta_2$ with $\alpha_i \in \mathrm{Hom}(\mathcal{A}^{a_i}, \mathcal{B}^{b_i})$ and $\beta_i \in \mathrm{Hom}(\mathcal{B}^{b_i}, \mathcal{C}^{c_i})$, $i = 1, 2$. We write α_i, β_i as

$$\alpha_i = \sum_{l=1}^{m} A_l^i \otimes \gamma_l \quad \text{and} \quad \beta_i = \sum_{j=1}^{n} B_j^i \otimes \sigma_j, \ i = 1, 2.$$

Then $\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet \oplus M_2^\bullet)$ is the representation

$$\kappa^{a_1 \oplus a_2} \ \begin{array}{c} \xrightarrow{A_1^1 \oplus A_1^2} \\ \vdots \\ \xrightarrow[A_m^1 \oplus A_m^2]{} \end{array} \ \kappa^{b_1 \oplus b_2} \ \begin{array}{c} \xrightarrow{B_1^1 \oplus B_n^2} \\ \vdots \\ \xrightarrow[B_n^1 \oplus B_n^2]{} \end{array} \ \kappa^{c_1 \oplus c_2}$$

and it is clear that

$$\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet \oplus M_2^\bullet) = (\{\kappa^{a_1+a_2}, \kappa^{b_1+b_2}, \kappa^{c_1+c_2}\}, \{A_i^1 \oplus A_i^2\}_{i=1}^m, \{B_j^1 \oplus B_j^2\}_{j=1}^n)$$

$$\simeq (\{\kappa^{a_1}, \kappa^{b_1}, \kappa^{c_1}\}, \{A_i^1\}_{i=1}^m, \{B_j^1\}_{j=1}^n) \oplus (\{\kappa^{a_2}, \kappa^{b_2}, \kappa^{c_2}\}, \{A_i^2\}_{i=1}^m, \{B_j^2\}_{j=1}^n)$$
$$= \mathbf{G}_{\gamma,\boldsymbol{\sigma}}((M_1^\bullet) \oplus \mathbf{G}_{\gamma,\boldsymbol{\sigma}}((M_2^\bullet).$$

For the converse, suppose $R = \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M^\bullet) \simeq R_1 \oplus R_2$. By Lemma 5.2 we know that there are monads $M_i^\bullet$, for $i = 1, 2$, such that $R_i = \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_i^\bullet)$. It follows that

$$\mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M^\bullet) \simeq \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet) \oplus \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_2^\bullet) \simeq \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(M_1^\bullet \oplus M_2^\bullet)$$

hence $M^\bullet$ is decomposable.

The second claim follows easily from the observation that if a monad is decomposable, then so is its cohomology sheaf. □

The completion of the proof of Theorem 1.3 is at hand: if $M^\bullet$ is a monad of the form (25) with (a, b, c) satisfying $q_{m,n}(a, b, c) = a^2 + b^2 + c^2 - mab - nbc > 1$, then the associated quiver representation is decomposable, by Proposition 2.2. This means that $M^\bullet$ itself, and hence its cohomology sheaf, must also be decomposable, as desired.

5.2. Decomposability of bundles vs. decomposability of representations

The last goal of this paper will be to examine under which assumption one does have the converse of the second part of Proposition 5.3, that is, if the cohomology of a monad is decomposable as a vector bundle, then the quiver representation associated to the monad is also decomposable. The difficulty here, of course, is to argue that if the cohomology of a monad of the form (25) decomposes, then its summands are also cohomologies of monads of the same form. Such statement can be proved under the following additional assumptions, and using the cohomological characterisation of monads provided by Theorem 2.11 above.

Let $\mathbf{B} = (\boldsymbol{\mathcal{F}}_0, \cdots, \boldsymbol{\mathcal{F}}_n)$ be an n-block collection generating the bounded derived category $D^b(X)$ of coherent sheaves on X, and let ${}^\vee\mathbf{B}$ its left dual n-block collection, as in the statement of Theorem 2.11. Let $\mathcal{E}$ be a vector bundle on X given by the cohomology of type (6), and assume that $\mathcal{E}$ is decomposable: $\mathcal{E} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$. From Theorem 2.11, since $\mathcal{E}$ has natural cohomology with respect to ${}^\vee\mathbf{B}$ we have

$$\dim \operatorname{Ext}^{n-i-1}(\mathcal{H}_{i_0}^{n-i}, \mathcal{E}) = a,$$

$$\dim \operatorname{Ext}^{n-j}(\mathcal{H}_{j_0}^{n-j}, \mathcal{E}) = b,$$

$$\dim \operatorname{Ext}^{n-k+1}(\mathcal{H}_{k_0}^{n-k}, \mathcal{E}) = c,$$

and $\operatorname{ext}^p(\mathcal{H}_q^m, \mathcal{E}) = 0$ otherwise. Hence for $l = 1, 2$,

$$\dim \operatorname{Ext}^{n-i-1}(\mathcal{H}_{i_0}^{n-i}, \mathcal{E}_l) = a_l,$$

$$\dim \operatorname{Ext}^{n-j}(\mathcal{H}_{j_0}^{n-j}, \mathcal{E}_l) = b_l,$$

$$\dim \operatorname{Ext}^{n-k+1}(\mathcal{H}_{k_0}^{n-k}, \mathcal{E}_l) = c_l, \quad l = 1, 2,$$

where $a = a_1 + a_2$, $b = b_1 + b_2$, and $c = c_1 + c_2$, with $a_l, b_l, c_l \geqslant 0$ and $\operatorname{Ext}^q(\mathcal{H}_p^m, \mathcal{E}_l) = 0$ otherwise.

Let us prove that $\mathfrak{M}_{\mathcal{F}_{i_0}^i, \mathcal{F}_{j_0}^j, \mathcal{F}_{k_0}^k}$ is closed under direct summands. From Lemma 2.10 and Theorem 2.11, $\mathcal{E}_l$ is isomorphic to a Beilinson monad $G_l^\bullet$, $l = 1, 2$, where each G_l^u is given by

$$G_l^u = \bigoplus_{p,q} \operatorname{Ext}^{n-q+u}(\mathcal{H}_p^{n-q}, \mathcal{E}_l) \otimes \mathcal{F}_p^q, \ l = 1, 2.$$

Then we have

$$G_l^u = 0, \ l = 1, 2; \ u < -1, \ u > 1,$$

and

$$G_l^{-1} = \bigoplus_{p,q} \operatorname{Ext}^{n-q-1}(\mathcal{H}_p^{n-q}, \mathcal{E}_l) \otimes \mathcal{F}_p^q = \operatorname{Ext}^{n-i-1}(\mathcal{H}_{i_0}^{n-i}, \mathcal{E}_l) \otimes \mathcal{F}_{i_0}^i \simeq (\mathcal{F}_{i_0}^i)^{a_l}$$

$$G_l^0 = \bigoplus_{p,q} \operatorname{Ext}^{n-q}(\mathcal{H}_p^{n-q}, \mathcal{E}_l) \otimes \mathcal{F}_p^q = \operatorname{Ext}^{n-j}(\mathcal{H}_{j_0}^{n-j}, \mathcal{E}_l) \otimes \mathcal{F}_{j_0}^j \simeq (\mathcal{F}_{j_0}^j)^{b_l}$$

$$G_l^1 = \bigoplus_{p,q} \mathrm{Ext}^{n-q+1}(\mathcal{H}_p^{n-q}, \mathcal{E}_l) \otimes \mathcal{F}_p^q = \mathrm{Ext}^{n-k+1}(\mathcal{H}_{k_0}^{n-k}, \mathcal{E}_l) \otimes \mathcal{F}_{k_0}^k \simeq (\mathcal{F}_{k_0}^k)^{c_l}$$

for $l = 1, 2$. From the definition of Beilinson monad, Definition 2.9, $\mathcal{E}_l$ is isomorphic to the monad

$$(\mathcal{F}_{i_0}^i)^{a_l} \longrightarrow (\mathcal{F}_{j_0}^j)^{b_l} \longrightarrow (\mathcal{F}_{k_0}^k)^{c_l} \tag{29}$$

with $l = 1, 2$ and $a_l, b_l, c_l \geqslant 0$. We have the following cases:

1) If $a_l, b_l, c_l \neq 0$ for $l = 1, 2$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are cohomology of a monad of type (29)
$$\mathcal{E}_1 = H^0(G_1^\bullet),\ \mathcal{E}_2 = H^0(G_2^\bullet).$$

2) If $a_1, b_1, c_1, b_2, c_2 \neq 0$ and $a_2 = 0$, then $\mathcal{E}_1 = H^0(G_1^\bullet)$ and $\mathcal{E}_2$ is given by the short exact sequence
$$0 \longrightarrow \mathcal{E}_2 \longrightarrow (\mathcal{F}_{j_0}^j)^{b_2} \longrightarrow (\mathcal{F}_{k_0}^k)^{c_2} \longrightarrow 0.$$

3) If $a_1, b_1, c_1, a_2, b_2 \neq 0$ and $c_2 = 0$ then $\mathcal{E}_1 = H^0(G_1^\bullet)$ and $\mathcal{E}_2$ is given by the short exact sequence
$$0 \longrightarrow (\mathcal{F}_{i_0}^i)^{a_2} \longrightarrow (\mathcal{F}_{j_0}^j)^{b_2} \longrightarrow \mathcal{E}_2 \longrightarrow 0\ .$$

4) If $a_1, b_1, c_1, b_2 \neq 0$ and $a_2 = c_2 = 0$, then $\mathcal{E}_1 = H^0(G_1^\bullet)$ and $\mathcal{E}_2 \simeq (\mathcal{F}_{j_0}^j)^{b_2}$.
5) If $b_1, c_1, a_2, b_2 \neq 0$ and $a_1 = c_2 = 0$ then
$$0 \longrightarrow \mathcal{E}_1 \longrightarrow (\mathcal{F}_{j_0}^j)^{b_1} \longrightarrow (\mathcal{F}_{k_0}^k)^{c_1} \longrightarrow 0$$
and
$$0 \longrightarrow (\mathcal{F}_{i_0}^i)^{a_2} \longrightarrow (\mathcal{F}_{j_0}^j)^{b_2} \longrightarrow \mathcal{E}_2 \longrightarrow 0\ .$$

And the symmetric cases to cases $2, 3, 4$ and 5.

We have just proved that:

Lemma 5.4. *The category $\mathfrak{M}_{\mathcal{F}_{i_0}^i, \mathcal{F}_{j_0}^j, \mathcal{F}_{k_0}^k}$ is closed under direct summands.*

Suppose $m = \dim \operatorname{Hom}(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0})$, and $n = \dim \operatorname{Hom}(\mathcal{F}^j_{j_0}, \mathcal{F}^k_{k_0})$ and choose $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}$ bases of $\operatorname{Hom}(\mathcal{F}^i_{i_0}, \mathcal{F}^j_{j_0})$ and $\operatorname{Hom}(\mathcal{F}^j_{j_0}, \mathcal{F}^k_{k_0})$, respectively. Let $\mathbf{G}_{\gamma,\boldsymbol{\sigma}}$ be the functor between $\mathfrak{M}_{\mathcal{F}^i_{i_0},\mathcal{F}^j_{j_0},\mathcal{F}^k_{k_0}}$ and $\mathfrak{S}^{\mathrm{gis}}_{m,n}$ described after equation (27). Note that given a monad of type (29) with $a_l = 0$ the associated representation is

$$0 \ \begin{array}{c} \xrightarrow{0} \\ \vdots \\ \xrightarrow[0]{} \end{array} \ \kappa^{b_l} \ \begin{array}{c} \xrightarrow{B^l_1} \\ \vdots \\ \xrightarrow[B^l_n]{} \end{array} \ \kappa^{c_l}$$

which is $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$-globally injective and surjective. If $c_l = 0$, the associated representation is

$$\kappa^{a_l} \ \begin{array}{c} \xrightarrow{A^l_1} \\ \vdots \\ \xrightarrow[A^l_m]{} \end{array} \ \kappa^{b_l} \ \begin{array}{c} \xrightarrow{0} \\ \vdots \\ \xrightarrow[0]{} \end{array} \ 0$$

that is $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$-globally injective and surjective. If $a_l = c_l = 0$, the associated representation is

$$0 \ \begin{array}{c} \xrightarrow{0} \\ \vdots \\ \xrightarrow[0]{} \end{array} \ \kappa^{b_l} \ \begin{array}{c} \xrightarrow{0} \\ \vdots \\ \xrightarrow[0]{} \end{array} \ 0$$

which is also $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$-globally injective and surjective. Hence we can prove the following.

Theorem 5.5. *Let $\mathcal{E}$ be a vector bundle on X given by the cohomology of a monad in $\mathfrak{M}_{\mathcal{F}^i_{i_0},\mathcal{F}^j_{j_0},\mathcal{F}^k_{k_0}}$ and R the associated $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$-globally injective and surjective representation in $\mathfrak{S}^{\mathrm{gis}}_{m,n}$. Then $\mathcal{E}$ is decomposable if and only if R is decomposable.*

Proof. We only need to prove the sufficient condition. If $\mathcal{E} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$ then from Lemma 5.4, $\mathcal{E}_i$, $i = 1, 2$, are cohomologies of monads in $\mathfrak{M}_{\mathcal{F}^i_{i_0},\mathcal{F}^j_{j_0},\mathcal{F}^k_{k_0}}$, therefore $R = \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(\mathcal{E}) \simeq \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(\mathcal{E}_1) \oplus \mathbf{G}_{\gamma,\boldsymbol{\sigma}}(\mathcal{E}_2) = R_1 \oplus R_2$. $\square$

5.3. An example: generalized Horrocks–Mumford monads

As an application of Theorem 5.5, let $X = \mathbb{P}^{2p}$ with $p \geqslant 2$, $\kappa = \mathbb{C}$, and consider the $2p$-block collection

$$\mathbf{B} = (\Omega^{2p}_{\mathbb{P}^{2p}}(2p), \Omega^{2p-1}_{\mathbb{P}^{2p}}(2p-1), \cdots, \Omega^1_{\mathbb{P}^{2p}}(1), \mathcal{O}_{\mathbb{P}^{2p}})$$

generating the bounded derived category $D^b(\mathbb{P}^{2p})$. The complex

$$\mathcal{O}_{\mathbb{P}^{2p}}(-1)^{2p+1} \xrightarrow{\alpha} \Omega^p_{\mathbb{P}^{2p}}(p)^2 \xrightarrow{\beta} \mathcal{O}^{2p+1}_{\mathbb{P}^{2p}} \tag{30}$$

is a monad, for $\alpha = (\alpha_{ij}) \in \wedge^p\mathbb{C}^{2p+1} \otimes \mathrm{Mat}_{2\times 2p+1}(\mathbb{C})$ and $\beta = (\beta_{ij}) \in \wedge^p\mathcal{C}^{2p+1} \otimes \mathrm{Mat}_{2p+1\times 2}(\mathbb{C})$ given by

$$\begin{aligned}\beta_{i1} &= x_{1+i} \wedge x_{2+i} \wedge \cdots \wedge x_{p+i};\\ \beta_{i2} &= x_i \wedge x_{p+1+i} \wedge x_{p+2+i} \wedge \cdots \wedge x_{2p-1+i}\end{aligned}$$

where $i \equiv k(\mathrm{mod}\ 2\mathrm{p}+1)$ and the matrix α is given by

$$\alpha = (\beta Q)^t$$

with

$$Q = \begin{pmatrix} 0 & 1 \\ (-1)^{p-1} & 0 \end{pmatrix}.$$

Note that when $p = 2$, the monad (30) is precisely the one that yields, as its cohomology, the Horrocks–Mumford rank 2 bundle on $\mathbb{P}^4$. For this reason, monads of the form (30) are called *generalized Horrocks–Mumford monads.* The goal of this section is to prove, as an application of Theorem 5.5, that the cohomology of a monad of type (30) is an indecomposable vector bundle of rank $2\left(\binom{2p}{p} - 2p - 1\right)$ on $\mathbb{P}^{2p}$.

To this end, note that one can fix a basis of the vector space $\wedge^p\mathbb{C}^{2p+1}$ so that the quiver representation associated to the morphism

$$\beta \in \mathrm{Hom}(\Omega^{\mathrm{p}}_{\mathbb{P}^{2\mathrm{p}}}(\mathrm{p})^2, \mathcal{O}^{2\mathrm{p}+1}_{\mathbb{P}^{2\mathrm{p}}})$$

is a representation of the Kronecker quiver $K_{\binom{2p+1}{p}}$ of the form $R = (\{\mathbb{C}^2, \mathbb{C}^{2p+1}\}, \{\phi_l\}_{l=1}^{\binom{2p+1}{p}})$ where $4p+2$ elements ϕ_l are elementary matrices of size $(2p+1)\times 2$ for some l and null matrices otherwise. The crucial step is the following result.

Lemma 5.6. *The representation R is simple.*

In particular, it follows from Theorem 3.5 that the kernel bundle $\ker\beta$, whose dual is a Steiner bundle, is also simple.

Proof. Suppose without loss of generality that the ordered basis is the following

$$\{x_{01\cdots p-1}, x_{12\cdots p}, x_{23\cdots p+1}, \cdots, x_{2p0\cdots p-2}, \cdots\}$$

where $x_{i_1 i_2 \cdots i_p} = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$. Then β can be written as

$$\beta = x_{01\cdots p-1} \cdot E_{2p,1} + x_{12\cdots p} \cdot E_{2p+1,1} + x_{23\cdots p+1} \cdot E_{1,1} + \cdots$$

where $E_{i,j} \in \mathrm{Mat}_{(2\mathrm{p}+1)\times 2}(\mathbb{C})$ is an elementary matrix. The associated quiver representation is of the form $R = (\{\mathbb{C}^2, \mathbb{C}^{2p+1}\}, \{\phi_l\}_{l=1}^{\binom{2p+2}{2}})$ where $\phi_1 = E_{2p,1}$, $\phi_2 = E_{2p+1,1}, \varphi_3 = E_{1,1}$ and so on.

Let $R_1 = (\{V_1, V_2\}, \{\psi\}_{l=1}^{\binom{2p+1}{p}})$ be a subrepresentation of R and without loss of generality suppose $V_1 \neq 0$. Then there is $v = (a, b) \in V_1 \subset \mathbb{C}^2$, with $a \neq 0$. The following diagram commutes

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, shift left=2, "\phi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\phi_{\binom{2p+1}{p}}"'] & \mathbb{C}^{2p+1} \\
V_1 \arrow[u, "i_1"] \arrow[r, shift left=2, "\psi_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\psi_{\binom{2p+1}{p}}"'] & V_2 \arrow[u, "i_2"']
\end{tikzcd} \tag{31}$$

and note that the vectors $\{\phi_j(v)\}_{j=1}^{2p+1}$ are linearly independent, hence $V_2 \simeq \mathbb{C}^{2p+1}$. If $R_2 = (\{W_1, W_2\}, \{\gamma_l\}_{l=1}^{\binom{2p+1}{p}})$ is a subrepresentation of R such that $R \simeq R_1 \oplus R_2$, then $W_2 \equiv 0$ and if $W_1 \neq 0$ there is k such that $\phi_k \neq 0$ and $\phi_k \mid_{W_1} \neq 0$. Therefore R is simple, hence indecomposable. $\square$

Since $\alpha = (\beta Q)^t$, the representation of the Kronecker quiver $K_{\binom{2p+1}{p}}$ associated to α is of the form $R' = (\{\mathbb{C}^{2p+1}, \mathbb{C}^2\}, \{\phi'_l\}_{l=1}^{\binom{2p+1}{p}})$, where ϕ'_l are elementary matrices or null matrices (the transpose of ϕ_l up to sign). Hence R' is also simple. By Theorem 5.5, the cohomology of the monad (30) is an indecomposable vector bundle on $\mathbb{P}^{2p}$.

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A group-theoretic approach to covering systems

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Abstract. In this article, we show how group actions can be used to examine the set of all covering systems of the integers with a fixed set of distinct moduli.

1. Introduction

A *(finite) covering system* C, or simply a *covering*, of the integers is a system of t congruences $x \equiv r_i \pmod{m_i}$, with $m_i > 1$ for all $1 \leqslant i \leqslant t$, such that every integer n satisfies at least one of these congruences. The concept of a covering was introduced by Paul Erdős in a paper in 1950 [8], where he used a covering to find an arithmetic progression of counterexamples to Polignac's conjecture that every positive integer can be written in the form $2^k + p$, where p is a prime. Since then, numerous authors have used covering systems to investigate and solve various problems [1–4, 4–7, 9–16, 18–21, 23–30, 32–35, 37–41, 43–49].

Under the restriction that all moduli in a covering are distinct, Erdős made the following statement in [8]:

> *"It seems likely that for every c there exists such a system all the moduli of which are $> c$."*

This conjecture, known as the *minimum modulus problem*, remained unresolved until recently when Bob Hough [20] showed that it is false. Since the minimum modulus in a covering is now known to be bounded

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above, one can naively speculate as to whether a categorization of all covering systems with a fixed minimum modulus might be possible in some way. Admittedly, such a notion seems intractable, if not impossible. But perhaps, a less ambitious task is possible. For example, could an enumeration be given of all coverings with a fixed set of moduli or a fixed least common multiple of the moduli? Recently [31], we have accomplished this goal for a very specific situation involving primitive covering numbers– a notion introduced by Zhi-Wei Sun [47] in 2007. While the methods in [31] are purely combinatorial, we show in this article how certain group actions can be used to examine the set of all covering systems of the integers with a fixed set of distinct moduli.

2. Preliminaries

It will be convenient on occasion to write any covering $C = \{(r_i, m_i)\}$, where $x \equiv r_i \pmod{m_i}$ is a congruence in the covering, simply as $C = [r_1, r_2, \ldots, r_t]$, when the moduli are written as a list $[m_1, m_2, \ldots, m_t]$. We write $\mathrm{lcm}(M)$ to denote the least common multiple of the elements in a set or list of moduli M. We let Γ_M, or simply Γ, if there is no ambiguity, denote the set of all coverings having moduli M. We define a covering C to be *minimal* if no proper subset of C is a covering . We also define a set, or list, of distinct moduli M to be *minimal* if every possible covering using all the elements of M is minimal. A positive integer L is called a *covering number* if there exists a covering of the integers where the moduli are distinct divisors of L greater than 1. A covering number L is called a *primitive covering number* if no proper divisor of L is a covering number. The following two theorems concerning covering numbers, which we state without proof, are due to Zhi-Wei Sun [47].

Theorem 2.1. *Let $p_1, p_2, \ldots, p_r$ be distinct primes, and let $a_1, a_2, \ldots, a_r$ be positive integers. Suppose that*

$$\prod_{0<t<s} (a_t + 1) \geqslant p_s - 1 + \delta_{r,s}, \quad \text{for all } s = 1, 2, \cdots, r, \tag{1}$$

where $\delta_{r,s}$ is Kronecker's delta, and the empty product $\prod_{0<t<1}(a_t + 1)$ is defined to be 1. Then $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is a covering number.

Infinitely many primitive covering numbers can be constructed using Theorem 2.1. We let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Theorem 2.2. *Let $r > 1$ and let $2 = p_1 < p_2 < \cdots < p_r$ be primes. Suppose further that $p_{t+1} \equiv 1 \pmod{p_t - 1}$ for all $0 < t < r-1$, and $p_r \geqslant (p_{r-1}-2)(p_{r-1}-3)$. Then*

$$p_1^{\frac{p_2-1}{p_1-1}-1} \cdots p_{r-2}^{\frac{p_{r-1}-1}{p_{r-2}-1}-1} p_{r-1}^{\left\lfloor \frac{p_r-1}{p_{r-1}-1} \right\rfloor} p_r$$

is a primitive covering number.

It is straightforward to see that Theorem 2.2 produces an infinite set $\mathcal{L}$ of primitive covering numbers, and that every element of $\mathcal{L}$ satisfies (1). In [47], Sun conjectured that every primitive covering number $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are distinct primes, satisfies (1). However, this conjecture is now known to be false [31].

Unless stated otherwise, we assume throughout this article that the moduli in all coverings are distinct, and that all sets of moduli are minimal.

3. Counting the number of coverings without group theory

While it is the main goal of this paper to use group-theoretic techniques to impose some structure on, and examine, the set of all coverings with a fixed list of distinct moduli, there are certain situations when some information can be obtained without the use of group theory. In particular, using a combinatorial approach, a formula was given in [31] for $|\Gamma_M|$, when $L \in \mathcal{L}$ and M is minimal with $\mathrm{lcm}(M) = L$. The following theorem illustrates another situation when $|\Gamma_M|$ can be determined without the use of group theory.

Theorem 3.1. *For $k \geqslant 2$, let*

$$M_k = [2, \quad 2^2, \quad \ldots \quad , 2^k, \quad 3, \quad 2^{k-1} \cdot 3, \quad 2^k \cdot 3].$$

For brevity of notation, let Γ_k denote the set of all coverings using the moduli M_k. Then

$$|\Gamma_k| = 2^{k+1} \cdot 3.$$

Proof. The proof is by induction on k. First let $k = 2$. The set Γ_2 of all possible coverings using the moduli $M_2 = [2, 4, 3, 6, 12]$ is easy to generate using a computer. We get that

$$\begin{aligned}\Gamma_2 = \{&[0,1,0,1,11],[0,1,0,5,7],[0,1,1,3,11],[0,1,1,5,3],\\ &[0,1,2,1,3],[0,1,2,3,7],[1,2,0,2,4],[1,0,0,2,10],\\ &[1,0,0,4,2],[1,0,1,0,2],[1,2,0,4,8],[1,0,1,2,6],\\ &[1,0,2,0,10],[1,2,1,0,8],[1,0,2,4,6],[1,2,1,2,0],\\ &[1,2,2,0,4],[1,2,2,4,0],[0,3,0,1,5],[0,3,0,5,1],\\ &[0,3,1,3,5],[0,3,1,5,9],[0,3,2,1,9],[0,3,2,3,1]\}\,.\end{aligned} \tag{2}$$

Observe that $|\Gamma_2| = 24$, so that the base case is verified. Let $L_k = 2^k \cdot 3$. Assume, by induction, that $|\Gamma_k| = 2^{k+1} \cdot 3$. Let $\widehat{M}_k = \{2, 2^2, \ldots, 2^k, 3, 2^k \cdot 3\}$. Let $\widehat{R}_k$ be a list of residues in a covering in Γ_k corresponding to the moduli $\widehat{M}_k$. There is just one hole modulo L_k left to fill to complete a covering in Γ_k, and this can be done in exactly one way using a residue $r \pmod{2^{k-1} \cdot 3}$. Thus, there are exactly two holes modulo L_{k+1} that need to be filled to complete a covering in Γ_k. These two holes can be filled in exactly two ways using the two moduli 2^{k+1} and $2^{k+1} \cdot 3$ in the following way. We can use either

$$r \pmod{2^{k+1}} \quad \text{and} \quad r + 2^k \cdot 3 \pmod{2^{k+1} \cdot 3},$$

or

$$r + 2^k \cdot 3 \pmod{2^{k+1}} \quad \text{and} \quad r \pmod{2^{k+1} \cdot 3}.$$

Thus, we have shown that $|\Gamma_{k+1}| = 2\,|\Gamma_k| = 2^{k+2} \cdot 3$, and the proof is complete. $\square$

Remark 3.2. Note that when $k = 2$ in Theorem 3.1, we have $L = 12 \in \mathcal{L}$, and so this is a special case addressed in [31].

4. Group theory and covering systems

In this section, we develop a group-theoretic approach to describe a relationship among the elements in Γ, and to help determine $|\Gamma|$. In particular, we investigate when there exist finite groups that act on Γ and we exploit this action to enumerate and categorize the elements of Γ. We let $\mathrm{orb}_G(C)$ and $\mathrm{stab}_G(C)$ denote, respectively, the orbit and stabilizer of $C \in \Gamma$ under the action of some group G. We begin by providing a brief analysis, without general proofs, in the situation when $L \in \mathcal{L}$ and M is minimal.

4.1. A group action in Sun's primitive covering number situation

A formula was given in [31] for $|\Gamma_M|$ when $L \in \mathcal{L}$ and M is minimal with $\mathrm{lcm}(M) = L$. From this formula, a finite group G can be constructed that acts transitively on Γ_M. This formula, and consequently the group G, are quite complicated in general. However, in special situations, G can be described fairly easily. Let

$$L = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{r-1}^{\alpha_{r-1}} p_r \in \mathcal{L}.$$

Under certain restrictions, the formula in [31] for $|\Gamma_M|$ reduces to

$$|\Gamma_M| = \prod_{i=1}^{r} (p_i!)^{\alpha_i} . \tag{3}$$

Remark 4.1. Formula 3 also holds for values of $L \notin \mathcal{L}$. See Table 1.

A consequence of (3) is the existence of a finite group

$$G \simeq (S_{p_1})^{a_1} \times \cdots \times (S_{p_r})^{a_r} , \tag{4}$$

where

$$(S_{p_i})^{a_i} = \underbrace{S_{p_i} \times \cdots \times S_{p_i}}_{a_i-\text{factors}}$$

and S_{p_i} is the symmetric group on p_i letters, that acts transitively on Γ by appropriately permuting the residues. The following example illustrates this process.

An example: $L = 12$ with $M = [m_1, m_2, m_3, m_4, m_5] = [2,4,3,6,12]$

We see easily that $L = 12$ is a primitive covering number satisfying (1).

- $p_1 = 2$
  We seek a group $H_1 \simeq S_2 \times S_2$. We start with the element $h = (12)(34)$. To construct the other three nontrivial elements of H_1, we conjugate h by the elements (24) and (23) to get

  $$H_1 = \{(1), (12)(34), (14)(23), (13)(24)\}$$

- $p_2 = 3$
  We seek a group $H_2 \simeq S_3$. Let

  $$H_2 = \{(1), (12), (23), (13), (123), (132)\} .$$

Therefore, $G = H_1 \times H_2$. We write a covering C as $[r_1, r_2, r_3, r_4, r_5]$, where $r_i \pmod{m_i}$ is a congruence in C. We illustrate the action on the set Γ of all 24 coverings given in (2). As an example, let $C = [1, 2, 1, 0, 8]$. We use the Chinese remainder theorem to decompose the residues on the composite moduli into prime power moduli, and we substitute $p_i^k \pmod{p_i^k}$ for $0 \pmod{p_i^k}$. We also place subscripts on the residues in these decompositions to remind us of the prime power moduli. Thus,

$$C = [1, 2, 1, [2_2, 3_3], [4_4, 2_3]]\,.$$

Let $g = ((14)(23), (123))$. Then

$$g.C = [4, 3, 2, [3_2, 1_3], [1_4, 3_3]] = [0, 3, 2, 1, 9] \in \Gamma,$$

and it is easy to verify that $\operatorname{orb}_G(C) = \Gamma$.

If it is the desire to navigate explicitly among the coverings $C \in \Gamma$ via this action of G, we see from the previous example that the process is somewhat cumbersome. We show in the next section that, for any value of L, there is a more easily-described group that acts on the set of all coverings. The disadvantage is that the action is not always transitive.

4.2. A group action in the general situation

In this section, we lift the restriction that L must satisfy (1). Let $\mathbb{Z}_L$ be the additive group of integers modulo L. We define the *holomorph* of $\mathbb{Z}_L$ to be

$$\operatorname{Hol}(\mathbb{Z}_L) = \operatorname{Aut}(\mathbb{Z}_L) \ltimes \mathbb{Z}_L \simeq \mathbb{Z}_L^* \ltimes \mathbb{Z}_L, \tag{5}$$

where $\mathbb{Z}_L^*$ is the group of units in the ring $\mathbb{Z}_L$ of integers modulo L. Note that $|\operatorname{Hol}(\mathbb{Z}_L)| = \phi(L)L$. For brevity of notation, we let $\mathcal{G} = \operatorname{Hol}(\mathbb{Z}_L)$.

Remark 4.2. More typically, a semidirect product is written using the notation $A \rtimes B$. However, it is more convenient here to use the isomorphic group $B \ltimes A$.

Theorem 4.3. *There is a natural (left) action of $\mathcal{G}$ on Γ.*

Proof. Let $g = (a, x) \in \mathcal{G}$ and $C = \left\{(r_i, m_i) \,\middle|\, 1 \leqslant i \leqslant t\right\} \in \Gamma$. Define

$$g.C := \left\{(ar_i + x, m_i) \,\middle|\, 1 \leqslant i \leqslant t\right\}.$$

We first show that $g.C$ is indeed a covering. Let n be any integer. Since C is a covering, there exists j such that

$$a^{-1}(n-x) \equiv r_j \pmod{m_j}.$$

Hence,

$$n \equiv ar_j + x \pmod{m_j},$$

so that n is covered by $g.C$.

Note that $(1,0) \in \mathcal{G}$ is the identity element in $\mathcal{G}$, and that $(1,0).C = C$. Next, let $h \in \mathcal{G}$ with $h = (b,y)$. By the definition of the operation in $\mathcal{G}$, we have that

$$gh = (a,x)(b,y) = (ab, ay + x).$$

Thus,

$$\begin{aligned}
(gh).C &= \left\{(abr_i + ay + x, m_i) \,\middle|\, 1 \leqslant i \leqslant t\right\} \\
&= \left\{(a(br_i + y) + x, m_i) \,\middle|\, 1 \leqslant i \leqslant t\right\} \\
&= g.\left\{(br_i + y, m_i) \,\middle|\, 1 \leqslant i \leqslant t\right\} \\
&= g.\left(h.C\right),
\end{aligned}$$

which completes the proof. □

Theorem 4.4. *Let* $C = \left\{(r_i, m_i) \,\middle|\, 1 \leqslant i \leqslant t\right\} \in \Gamma$. *Then*

$$|\mathrm{orb}_{\mathcal{G}}(C)| \geqslant \kappa(L)\phi(L), \tag{6}$$

where $\kappa(L)$ *denotes the square-free kernel of* L, *and* ϕ *is Euler's totient function. Moreover, equality holds in* (6) *if*

$$\kappa(L)\,(r_i - r_j) \equiv 0 \pmod{\gcd(m_i, m_j)} \quad \textit{for all } i \textit{ and } j. \tag{7}$$

Proof. Let $g = (a,x) \in \mathrm{stab}(C)$. Then $g.C = C$ and hence

$$(a-1)r_i + x \equiv 0 \pmod{m_i}, \tag{8}$$

for all $(r_i, m_i) \in C$. Let p be a prime such that $L \equiv 0 \pmod{p}$, and let

$$C_p = \left\{(r_i, m_i) \in C \,\middle|\, m_i \equiv 0 \pmod{p}\right\}.$$

Since C is a covering, there exist i and j, with $i \neq j$ and $(r_i, m_i), (r_j, m_j) \in C_p$, such that $r_i \not\equiv r_j \pmod{p}$. For this particular pair of congruences in C_p, we have by (8) that

$$(a-1)r_i + x \equiv (a-1)r_j + x \pmod{p}. \tag{9}$$

Rearranging (9) and using the fact that $r_i \not\equiv r_j \pmod{p}$, we get that $a \equiv 1 \pmod{p}$. Thus,

$$a \equiv 1 \pmod{\kappa(L)}. \tag{10}$$

There are exactly $\phi(L)/\phi(\kappa(L)) = L/\kappa(L)$ distinct values of $a \in \mathbb{Z}_L^*$ that satisfy (10). For each such value of a, we claim that there is at most one value of $x \in \mathbb{Z}_L$ that satisfies all congruences in (8). To see this, we fix a and write $a - 1 = z\kappa(L)$ for some integer z with $0 \leqslant z \leqslant L/\kappa(L) - 1$. Then the system of congruences (8) can be rewritten as the following system of congruences in the variable x:

$$x \equiv -z\kappa(L)r_i \pmod{m_i}, \quad \text{for all } (r_i, m_i) \in C. \tag{11}$$

By the generalized Chinese remainder theorem, the system (11) has a solution $x \in \mathbb{Z}_L$, and it is unique, if and only if

$$z\kappa(L)\,(r_i - r_j) \equiv 0 \pmod{\gcd(m_i, m_j)}$$

for all i and j. Thus, we have shown that

$$|\mathrm{stab}_{\mathcal{G}}(C)| \leqslant \frac{L}{\kappa(L)}.$$

Consequently, since $|\mathcal{G}| = \phi(L)L$, we have that

$$|\mathrm{orb}_{\mathcal{G}}(C)| = [\mathcal{G} : \mathrm{stab}_{\mathcal{G}}(C)] \geqslant \kappa(L)\phi(L).$$

Moreover, if

$$\kappa(L)\,(r_i - r_j) \equiv 0 \pmod{\gcd(m_i, m_j)}$$

for all i and j, then

$$z\kappa(L)\,(r_i - r_j) \equiv 0 \pmod{\gcd(m_i, m_j)}$$

for any fixed z and all i and j. Thus, in this case, the system (11) has a unique solution, and so equality holds in (6). $\square$

The following corollary is immediate from Theorem 4.4.

Corollary 4.5. *Let* $C = \left\{ (r_i, m_i) \,\middle|\, 1 \leqslant i \leqslant t \right\} \in \Gamma$*. If* (7) *holds and* $\mathcal{G}$ *acts transitively on* Γ*, then*

$$|\Gamma| = |\mathrm{orb}_{\mathcal{G}}(C)| = \kappa(L)\phi(L). \tag{12}$$

Condition (7) alone is not sufficient to deduce (12). For example, let $L = 36$ and $M = [2, 3, 4, 6, 9, 18, 36]$. Then, computer computations show that $|\Gamma| = 144$ and each $C \in \Gamma$ satisfies (7). Also, there are two orbits of size $\kappa(L)\phi(L) = 72$, so that $\mathcal{G}$ does not act transitively on Γ. Thus, in this case, we see that $|\Gamma| = 2\kappa(L)\phi(L)$.

Corollary 4.6. *If* L *is square-free, then equality holds in* (6) *for all* $C \in \Gamma$*.*

Proof. Since $|\mathrm{orb}_{\mathcal{G}}(C)|$ divides $|\mathcal{G}| = L\phi(L)$, we have that $|\mathrm{orb}_{\mathcal{G}}(C)| \leqslant L\phi(L)$. Since L is square-free, $\kappa(L) = L$, and therefore by Theorem 4.4, we deduce that

$$L\phi(L) \geqslant |\mathrm{orb}_{\mathcal{G}}(C)| \geqslant \kappa(L)\phi(L) = L\phi(L). \qquad \square$$

If we want to utilize Theorem 4.4 to determine $|\Gamma|$, then the question of transitivity of the action of $\mathcal{G}$ on Γ is a main concern. Unfortunately, we have been unable to find a way to determine when this occurs in general. For certain values of L and certain lists M, we used a computer to determine $|\Gamma|$ and $|\mathrm{orb}_{\mathcal{G}}(C)|$. This information is given in Table 1. We denote the number of orbits as #. A complete set of orbit representatives for each example given in Table 1 is available upon request. Note that, in Table 1, $L \in S$ only for $L = 80$ and $L = 90$.

L	M	#	$\lvert\mathrm{orb}(C)\rvert$	$\lvert\Gamma\rvert$
36	[2, 3, 4, 6, 9, 18, 36]	2	72	144
60	[2, 3, 4, 5, 6, 10, 15, 20, 30]	6	480	2880
72	[2, 3, 4, 6, 9, 24, 36, 72]	2	144	288
80	[2, 4, 5, 8, 10, 16, 20, 40, 80]	6	320	1920
90	[2, 3, 9, 5, 6, 10, 15, 18, 30, 45]	12	720	8640
108	[2, 3, 4, 6, 9, 18, 27, 54, 108]	4	216	864
120	[2, 3, 4, 5, 8, 10, 12, 30, 40, 60]	6	960	5760

TABLE 1. Data concerning the action of $\mathcal{G}$ on Γ

The examples in Table 1 are all such that M is minimal, and the cardinality of each orbit under the action of $\mathcal{G}$ is the same. However, there are examples of lists of moduli such that the cardinalities of the orbits are different. Although we cannot make it precise, there seems to be a connection between this difference in the cardinalities of the orbits and the following phenomenon.

Definition 4.7. Let M be a list of moduli such that $\Gamma_M \neq \varnothing$ and, to avoid a trivial situation, that some $C \in \Gamma_M$ is minimal. We say that M is *quasi-minimal* if there exist $C_1, C_2 \in \Gamma_M$ such that C_1 is minimal, but C_2 is not.

We give an example to illustrate that quasi-minimal M do exist.

Example 4.8. *The list*

$$M = [3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120]$$

is quasi-minimal since the covering

$$\begin{aligned} C_1 =& \{(0,3), (0,4), (0,5), (1,6), (6,8), (3,10), (5,12), (11,15), \\ & (7,20), (10,24), (2,30), (34,40), (59,60), (98,120)\} \end{aligned}$$

is minimal, but the covering

$$\begin{aligned} C_2 =& \{(2,3), (0,4), (0,5), (3,6), (2,8), (7,10), (6,12), (1,15), \\ & (19,20), (22,24), (13,30), (0,40), (49,60), (0,120)\} \end{aligned}$$

is not minimal. Note that the elements $(0,40)$ *and* $(0,120)$ *can be removed from* C_2 *and the remaining set* $\widehat{C}_2$ *is a covering; in fact, it is minimal.*

Remark 4.9. The covering C_1 in Example (4.8) is due to Erdős [8], while the covering $\widehat{C}_2$ is due to Krukenberg [33].

To illustrate the possible connection between quasi-minimality and the difference in the cardinalities of the orbits, we give examples of two coverings using M from Example 4.8 where the cardinalities of the orbits under the action of $\mathcal{G}$ are different. The covering

$$\begin{aligned} C_3 =& \{(1,3), (2,4), (0,5), (3,6), (4,8), (1,10), (0,12), (8,15), \\ & (7,20), (8,24), (29,30), (11,40), (17,60), (13,120)\} \end{aligned}$$

is not minimal since removing the set of congruences $\{(11,40), (13,120)\}$ from C_3 gives a covering. Examining the orbit of C_3 under $\mathcal{G}$, we see that $|\mathrm{orb}_{\mathcal{G}}(C_3)| = 3840$. However, the covering

$$\begin{aligned} C_4 =& \{(0,3), (3,4), (3,5), (2,6), (5,8), (6,10), (10,12), (4,15), \\ & (0,20), (17,24), (22,30), (25,40), (37,60), (1,120)\} \end{aligned}$$

is minimal and $|\mathrm{orb}_{\mathcal{G}}(C_4)| = 1920$.

5. Final comments

Until now, no attempt had been made to impose an algebraic structure on the set of all coverings with a fixed list of moduli. While our results do not provide an answer in the most general situation, they do indicate that a rich and useful algebraic structure does indeed exist, and it is worthy of further pursuit.

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RESEARCH ARTICLE

The lower bound for the volume of a three-dimensional convex polytope

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Abstract. In this paper, we provide a lower bound for the volume of a three-dimensional smooth integral convex polytope having interior lattice points. Since our formula has a quite simple form compared with preliminary results, we can easily utilize it for other beneficial purposes. As an immediate consequence of our lower bound, we obtain a characterization of toric Fano threefold. Besides, we compute the sectional genus of a three-dimensional polarized toric variety, and classify toric Castelnuovo varieties.

1. Introduction

Points in $\mathbb{Z}^n$ are called *lattice points* of $\mathbb{R}^n$, and a polytope is said to be *integral* if all its vertices are lattice points. For an integral polytope $\mathcal{P}$, we denote by $\mathrm{vol}(\mathcal{P})$ the volume of $\mathcal{P}$ and by $\partial\mathcal{P}$ the boundary of $\mathcal{P}$, and put $\mathrm{Int}(\mathcal{P}) = \mathcal{P} \setminus \partial\mathcal{P}$. Besides, we define $l(S) = \sharp(S \cap \mathbb{Z}^n)$ for a subset $S \subset \mathbb{R}^n$. In the study of integral polytopes, one of the most significant problem is to compute their volume by using the information of the number of lattice points in them. The following classical theorem gave a clue to the solution of this issue.

Theorem 1.1 (cf. [14]). *Let $\mathcal{P}$ be an integral polygon which is homeomorphic to a closed circle. Then its volume is computed by* $2\mathrm{vol}(\mathcal{P}) = l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 2$.

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In the case where an integral polygon is not homeomorphic to a closed circle, Reeve generalized the above Pick's result by employing the Euler characteristics $\chi(\mathcal{P})$ of the polygon and $\chi(\partial\mathcal{P})$ of its boundary.

Theorem 1.2 (cf. [15]). *Let $\mathcal{P}$ be an n-dimensional integral polygon.*

(i) *If $n = 2$, then* $2\mathrm{vol}(\mathcal{P}) = l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 2\chi(\mathcal{P}) + \chi(\partial\mathcal{P})$.

(ii) *If $n = 3$, then*

$$\begin{aligned} 2k(k^2-1)\mathrm{vol}(\mathcal{P}) &= l(k\mathcal{P}) + l(\mathrm{Int}(k\mathcal{P})) - k(l(\mathcal{P}) \\ &\qquad + l(\mathrm{Int}(\mathcal{P}))) + (k-1)(2\chi(\mathcal{P}) - \chi(\partial\mathcal{P})), \\ l(\partial(k\mathcal{P})) &= k^2 l(\partial\mathcal{P}) - 2(k^2-1) \end{aligned}$$

for any positive integer k, where $k\mathcal{P}$ denotes the dilated polytope $\{k\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{P}\}$.

Moreover, Macdonald established the general formula to compute the volume of an integral polytope of arbitrary dimension in [10]. Concretely, for an n-dimensional integral polytope $\mathcal{P}$,

$$\begin{aligned} (n-1)n!\,\mathrm{vol}(\mathcal{P}) = \sum_{k=1}^{n-1}(-1)^{n-k-1}\binom{n-1}{k}(l(k\mathcal{P}) \\ + l(\mathrm{Int}(k\mathcal{P}))) + (-1)^{n-1}(2\chi(\mathcal{P}) - \chi(\partial\mathcal{P})) \end{aligned} \tag{1}$$

and

$$n!\,\mathrm{vol}(\mathcal{P}) = \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k} l(k\mathcal{P}) + (-1)^n\chi(\mathcal{P}). \tag{2}$$

In addition, some other interesting formulae have been provided by Kołodziejczyk and Reay in [7–9]. One can in principle compute the volume of a polytope by using these results. In fact, however, it is not easy to carry it out. This difficulty comes from the intricate behavior of the number of lattice points in a dilated polytope $k\mathcal{P}$. Therefore, from the application standpoint, it is desirable to find a more simple formula even if not as strong as (1) and (2). Specifically, in this paper, we will give a lower bound for the volume of a three-dimensional integral convex polytope (Theorem 1.4). Although this result gives only an inequality, it is of wide application because of its simplicity. First, Corollary 1.6 provides a characterization of toric Fano threefold. Furthermore, we will apply this corollary to compute the sectional genus of a three-dimensional polarized toric variety and classify so-called Castelnuovo varieties in Section 3 (Theorem 3.8). Before describing our main result, we need to define the *smoothness* of a polytope. A polytope is said to be *convex* if it is a convex hull of a finite number of points in $\mathbb{R}^n$.

Definition 1.3. *Let $\mathcal{P}$ be an n-dimensional integral convex polytope in $\mathbb{R}^n$, and P be a vertex of $\mathcal{P}$. Define $\mathbb{R}_{\geqslant 0}(\mathcal{P}-P)=\{a(Q-P)\in\mathbb{R}^n \mid Q\in\mathcal{P}, a\geqslant 0\}$. If there exists a $\mathbb{Z}$-basis $\{\boldsymbol{m_1},\dots,\boldsymbol{m_n}\}$ of $\mathbb{Z}^n$ such that*

$$\mathbb{R}_{\geqslant 0}(\mathcal{P}-P)=\mathbb{R}_{\geqslant 0}\boldsymbol{m_1}+\cdots+\mathbb{R}_{\geqslant 0}\boldsymbol{m_n},$$

the vertex P is said to be smooth. We say $\mathcal{P}$ is smooth if all its vertices are smooth. An m-dimensional ($m<n$) integral convex polytope $\mathcal{P}'$ in $\mathbb{R}^n$ is said to be smooth if it is smooth with respect to $\mathbb{R}^m$ which is the smallest affine subspace of $\mathbb{R}^n$ including $\mathcal{P}'$.

Theorem 1.4. *Let $\mathcal{P}$ be a three-dimensional smooth integral convex polytope having at least one interior lattice point. Then $3\mathrm{vol}(\mathcal{P})\geqslant l(\mathcal{P})+l(\mathrm{Int}(\mathcal{P}))-4$, and equality holds if and only if $\mathcal{P}$ is a polytope associated to the anti-canonical bundle on a toric Fano threefold.*

Toric Fano threefolds have been already classified into eighteen types in [2] and [17], independently. Namely, there exist eighteen polytopes (see, e.g., (6) in Section 2) whose volume achieves the lower bound in Theorem 1.4. We remark that the conditions of smoothness and $l(\mathrm{Int}(\mathcal{P}))\geqslant 1$ are essential for the above theorem. Indeed, if we remove these conditions, we can easily find counterexamples as follows.

Example 1.5. *For a subset S of $\mathbb{R}^3$, we denote by $\mathrm{Conv}(S)$ the convex hull of S.*

(i) *For a nonsmooth integral convex polytope*

$$\mathcal{P}_1=\mathrm{Conv}(\{(0,0,\pm 1),(2,1,\pm 1),(1,2,\pm 1),(1,1,2)\}),$$

we have $3\mathrm{vol}(\mathcal{P}_1)=21/2<l(\mathcal{P}_1)+l(\mathrm{Int}(\mathcal{P}_1))-4=11$.

(ii) *For a unit cube $\mathcal{P}_2$, we have* $3\mathrm{vol}(\mathcal{P}_2)=3<l(\mathcal{P}_2)+l(\mathrm{Int}(\mathcal{P}_2))-4=4$.

On the other hand, as is well known, the theory of polytopes is closely related to the toric geometry. For an ample line bundle L on an n-dimensional compact toric variety X, there exists an associated n-dimensional integral convex polytope $\square_L$ from which we can read off many invariants of L. A computation of the dimension of a cohomology group can be reduced to counting lattice points in the polytope. For example, we have $h^0(X, L) = l(\Box_L)$ and $h^0(X, L + K_X) = l(\mathrm{Int}(\Box_L))$. The degree of L can be computed as $L^n = n!\mathrm{vol}(\Box_L)$. These relations, for example, tell us that Pick's formula coincides with the Riemann-Roch theorem on a surface. Indeed, if X is two-dimensional, the equalities $\chi(\mathcal{O}_X) = 1$ and $h^0(X, K_X) = h^1(X, K_X) = 0$ hold. Besides, the general theory of toric varieties gives that $h^1(X, L) = h^2(X, L) = 0$ if $|L|$ has no base points. Therefore, we can deform Theorem 1.1 as

$$h^0(X, L) = L^2 - h^0(X, L + K_X) + 2$$
$$\chi(\mathcal{O}_X(L)) = L^2 - h^0(X, K_L) + 2 = \frac{1}{2}L.(L - K_X) + \chi(\mathcal{O}_X).$$

In this manner, properties of polytopes and that of line bundles on a toric variety can be translated each other.

Using the terminology of the algebraic geometry, we can interpret Theorem 1.4 as a theorem about line bundles.

Corollary 1.6. *Let L be an ample line bundle on a three-dimensional smooth compact toric variety X. If $h^0(X, L + K_X) \geqslant 1$, then $L^3 \geqslant 2(h^0(X, L) + h^0(X, L + K_X) - 4)$ holds, and equality holds if and only if X is a toric Fano threefold and $L = -K_X$.*

2. Proof of the main theorem

First of all, we need to introduce several notations. We denote by $H_{f(x,y,z)}$ the plane in $\mathbb{R}^3$ defined by an equation $f(x, y, z) = 0$. For a lattice point P and a polygon F included in a plane $H_{f(x,y,z)}$, we denote by $h(F, P)$ the *lattice distance*. In concrete terms, we define $h(F, P) = |n|$, where n is an integer such that $H_{f(x,y,z)-n}$ passes through P.

Henceforth, the notation $\mathcal{P}$ always denotes a three-dimensional smooth integral convex polytope having interior lattice points. In addition, we denote by $V(\mathcal{P})$ the set of vertices of $\mathcal{P}$, and $E(\mathcal{P})$ the set of points on edges of $\mathcal{P}$. Note that $\mathrm{vol}(\mathcal{P})$, $l(\mathcal{P})$ and $l(\mathrm{Int}(\mathcal{P}))$ do not change even if we perform an affine linear transformation (i.e., a composition of parallel displacements and linear transformations by unimodular matrices).

Lemma 2.1. *If we place $\mathcal{P}$ in $\mathbb{R}^3_{z \geqslant 0}$ so that $\mathcal{P}$ has a face on H_z, then $\mathcal{P} \cap H_{z-1}$ is an integral convex polygon.*

Proof. We can assume that the origin O is a vertex of $\mathcal{P}$. Then O has three adjacent lattice points $(a_1, b_1, 0), (a_2, b_2, 0), (a_3, b_3, c_3) \in E(\mathcal{P})$. Since the vertex O is smooth, we have $c_3 = 1$ by the equality

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} = (a_1 b_2 - a_2 b_1)c_3 = \pm 1.$$

By similar arguments, we see that any edge of $\mathcal{P}$ which is extending from a vertex on H_z but not lying on H_z has a lattice point on H_{z-1}. The assertion follows from this fact. $\square$

Lemma 2.2. *There exists a plane $H_{f(x,y,z)}$ such that the section $\mathcal{T} = \mathcal{P} \cap H_{f(x,y,z)}$ is an integral convex polygon having a smooth vertex and at least one interior lattice point.*

Proof. Without loss of generality, we can assume that O, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are contained in $E(\mathcal{P})$. Put $P_1 = (0,1,1)$, $P_2 = (1,0,1)$ and $P_3 = (1,1,0)$. In the case where $P_1 \notin \mathcal{P}$, the integral convex polygon $\mathcal{P} \cap H_x$ must be a unit triangle by the smoothness of $\mathcal{P}$. Similarly, if P_2 (resp. P_3) is not contained in $\mathcal{P}$, then $\mathcal{P} \cap H_y$ (resp. $\mathcal{P} \cap H_z$) becomes a unit triangle. Since $l(\mathrm{Int}(\mathcal{P})) \geqslant 1$ and every vertex of $\mathcal{P}$ has just three edges, it is required that at least two points of P_1, P_2 and P_3 are contained in $\mathcal{P}$. Hence we can assume (after permuting the coordinates, if necessary) that $P_1, P_2 \in \mathcal{P}$. It is sufficient to consider the case where $(1,1,1) \notin \mathrm{Int}(\mathcal{P})$, because if $(1,1,1) \in \mathrm{Int}(\mathcal{P})$, we can finish the proof by putting $f(x,y,z) = z - 1$.

Let (x_0, y_0, z_0) be an interior lattice point of $\mathcal{P}$. Suppose that $P_3 \in \mathcal{P}$. Since $(1,1,1)$ is not contained in $\mathrm{Int}(\mathcal{P})$, there exist four integers α, β, γ and δ such that $\mathcal{P} \subset \{(x,y,z) \mid \alpha x + \beta y + \gamma z + \delta \geqslant 0\}$ and $\alpha + \beta + \gamma + \delta \leqslant 0$. By the conditions $P_1, P_2, P_3 \in \mathcal{P}$, we have

$$\beta + \gamma + \delta \geqslant 0,\ \alpha + \gamma + \delta \geqslant 0,\ \alpha + \beta + \delta \geqslant 0,$$

which imply that $\alpha, \beta, \gamma \leqslant 0$. Then we obtain a contradiction

$$\begin{aligned} &\alpha x_0 + \beta y_0 + \gamma z_0 + \delta > 0 \\ &\qquad\qquad \delta > -\alpha x_0 - \beta y_0 - \gamma z_0 \geqslant -\alpha - \beta - \gamma. \end{aligned}$$

Thus we see that P_3 is not contained in $\mathcal{P}$. In this case, the face $\mathcal{P} \cap H_z$ is a unit triangle, and the vertex $(1,0,0)$ has three adjacent lattice points O, $(0,1,0)$ and $(a,0,c)$ in $E(\mathcal{P})$. One can check $c = 1$ by the smoothness of the vertex $(1,0,0)$ in a similar way to that in the proof of Lemma 2.1. Thus $\mathcal{P}$ has a face included in the plane $H_{x+y+(1-a)z-1}$

containing three points $(1,0,0), (0,1,0)$ and $(a,0,1)$. Then we obtain $a \geqslant 2$ by the inequality

$$0 > x_0 + y_0 + (1-a)z_0 - 1 > (1-a)z_0.$$

It follows that $(2,0,1)$ is contained in $\mathcal{P}$. Similarly, one can verify that $(0,2,1)$ is contained in $\mathcal{P}$. By the assumption $(1,1,1) \notin \mathrm{Int}(\mathcal{P})$, the section $\mathcal{P} \cap H_{z-1}$ must be a triangle $\mathrm{Conv}(\{(0,0,1),\ (2,0,1),\ (0,2,1)\})$, which implies $a = 2$. Next we focus on the point $(1,1,2)$. Suppose that $(1,1,2)$ is not contained in $\mathrm{Int}(\mathcal{P})$. Then there exist four integers ε, ζ, η and θ such that $\mathcal{P} \subset \{(x,y,z) \mid \varepsilon x + \zeta y + \eta z + \theta \geqslant 0\}$ and $\varepsilon + \zeta + 2\eta + \theta \leqslant 0$. Since $(0,0,1), (2,0,1), (0,2,1) \in \mathcal{P}$, we have

$$\eta + \theta \geqslant 0,\ 2\varepsilon + \eta + \theta \geqslant 0,\ 2\zeta + \eta + \theta \geqslant 0,$$

which imply that $\varepsilon + \eta + \theta \geqslant 0$, $\zeta + \eta + \theta \geqslant 0$ and $\varepsilon + \zeta + \eta + \theta \geqslant 0$. It follows that $\eta \leqslant -\varepsilon - \zeta - \eta - \theta \leqslant 0$, $\varepsilon + \eta \leqslant -\zeta - \eta - \theta \leqslant 0$ and $\zeta + \eta \leqslant -\varepsilon - \eta - \theta \leqslant 0$. Since $\mathcal{P}$ has a face included in $H_{x+y-z-1}$, the inequality $x_0 + y_0 - z_0 - 1 < 0$ holds, which implies a contradiction

$$\begin{aligned} &\varepsilon x_0 + \zeta y_0 + \eta z_0 + \theta > 0 \\ &\qquad\qquad \theta > -\varepsilon x_0 - \zeta y_0 - \eta z_0 \geqslant -(\varepsilon+\eta)x_0 - (\zeta+\eta)y_0 \\ &\qquad\qquad\ \geqslant -\varepsilon - \zeta - 2\eta. \end{aligned}$$

Therefore, we can conclude that $(1,1,2)$ is contained in $\mathrm{Int}(\mathcal{P})$, and $\mathcal{P} \cap H_{x-1}$ is the desired section. $\square$

Let $\mathcal{Q}$ be a three-dimensional integral convex polytope, and Q be a vertex of $\mathcal{Q}$. We define

$$\mu(\mathcal{Q}) = \mathrm{vol}(\mathcal{Q}) - \frac{l(\mathcal{Q}) + l(\mathrm{Int}(\mathcal{Q})) - 4}{3}$$

and $\mathcal{Q}^Q = \mathrm{Conv}((\mathcal{Q} \cap \mathbb{Z}^3) \setminus \{Q\})$, and denote by $\overline{\mathcal{Q}}^Q$ the set of points in $\mathcal{Q}^Q$ which are visible from Q, that is,

$$\overline{\mathcal{Q}}^Q = \{P \in \mathcal{Q}^Q \mid (\text{the segment } PQ) \cap \mathcal{Q}^Q = \{P\}\}.$$

Although $\overline{\mathcal{Q}}^Q$ is not a convex polytope but a set consisting of some faces of $\mathcal{Q}^Q$, we formally define $V(\overline{\mathcal{Q}}^Q) = V(\mathcal{Q}^Q) \cap \overline{\mathcal{Q}}^Q$, $\partial\overline{\mathcal{Q}}^Q = \partial\mathcal{Q} \cap \overline{\mathcal{Q}}^Q$ and $\mathrm{Int}(\overline{\mathcal{Q}}^Q) = \overline{\mathcal{Q}}^Q \setminus \partial\overline{\mathcal{Q}}^Q$. The following proposition tells us the precise difference between $\mathcal{Q}$ and $\mathcal{Q}^Q$, which is a central tool in the induction step in the proof of Theorem 1.4.

Proposition 2.3. *Let Q be a vertex of a three-dimensional integral convex polytope $\mathcal{Q}$, and $F_1, \dots, F_k$ be faces of $\mathcal{Q}^Q$ such that $\overline{\mathcal{Q}}^Q = \bigcup_{j=1}^k F_j$. If we put $a_j = l(F_j)$ and $b_j = l(\mathrm{Int}(F_j))$, then*

$$\mu(\mathcal{Q}) = \mu(\mathcal{Q}^Q) + \frac{1}{6} l(\partial \overline{\mathcal{Q}}^Q) - \frac{2}{3} + \frac{1}{6} \sum_{j=1}^{k} (h(F_j, Q) - 1)(a_j + b_j - 2).$$

Proof. For simplicity, we put $h_j = h(F_j, Q)$. Since $\mathrm{vol}(F_j) = (a_j + b_j - 2)/2$ by Theorem 1.1, we have

$$\begin{aligned}
\mu(\mathcal{Q}) - \mu(\mathcal{Q}^Q) &= \frac{1}{6} \sum_{j=1}^{k} h_j (a_j + b_j - 2) - \frac{1}{3} (l(\mathrm{Int}(\mathcal{Q})) - l(\mathrm{Int}(\mathcal{Q}^Q)) + 1) \\
&= \frac{1}{6} \sum_{j=1}^{k} (h_j - 1)(a_j + b_j - 2) + \frac{1}{6} \sum_{j=1}^{k} (a_j + b_j) \\
&\quad - \frac{1}{3} (l(\mathrm{Int}(\overline{\mathcal{Q}}^Q)) + k + 1).
\end{aligned}$$

To estimate the right-hand value, let us compute $\sum_{j=1}^k (a_j + b_j)$, which means counting lattice points in $\overline{\mathcal{Q}}^Q$ (with several duplications). Indeed, we can write

$$\sum_{j=1}^{k} (a_j + b_j) = \sum_{P \in \overline{\mathcal{Q}}^Q \cap \mathbb{Z}^3} c(P), \tag{3}$$

where $c(P)$ denotes the number of times P is counted in the left-hand side of (3). It is clear that $c(P) = 1$ for $P \in \partial \overline{\mathcal{Q}}^Q \setminus V(\overline{\mathcal{Q}}^Q)$ since there exists a unique face containing P in this case. Let us check $c(P) = 2$ for a point P in $\mathrm{Int}(\overline{\mathcal{Q}}^Q) \setminus V(\overline{\mathcal{Q}}^Q)$. This is clear if P is not on any edge of $\mathcal{Q}^Q$. While if P is on some edge of $\mathcal{Q}^Q$, then there exist two faces of $\overline{\mathcal{Q}}^Q$ containing P. Hence, in this case, P is counted two times in the left-hand side of (3). We next consider points in $V(\overline{\mathcal{Q}}^Q)$. Let P be a point in $V(\overline{\mathcal{Q}}^Q) \cap \mathrm{Int}(\overline{\mathcal{Q}}^Q)$ having s edges. Since s faces contain P, P is counted s times in the left-hand side of (3), that is, $c(P) = s$. Meanwhile, if P is contained in $V(\overline{\mathcal{Q}}^Q) \cap \partial \overline{\mathcal{Q}}^Q$ and t edges of $\overline{\mathcal{Q}}^Q$ extend from P, there exist $t - 1$ faces containing P. It follows that $c(P) = t - 1$. Consequently, we obtain

$$\sum_{j=1}^{k} (a_j + b_j) = 2l(\mathrm{Int}(\overline{\mathcal{Q}}^Q)) + l(\partial \overline{\mathcal{Q}}^Q \setminus V(\overline{\mathcal{Q}}^Q)) + \sum_{s=3}^{s_0} (s-2) m_s + \sum_{t=2}^{t_0} (t-1) n_t,$$

where we define

$$m_s = \sharp\{P \in V(\overline{\mathcal{Q}}^Q) \cap \mathrm{Int}(\overline{\mathcal{Q}}^Q) \mid \text{there exist } s \text{ edges of } \overline{\mathcal{Q}}^Q \\ \text{extending from } P\},$$
$$n_t = \sharp\{P \in V(\overline{\mathcal{Q}}^Q) \cap \partial\overline{\mathcal{Q}}^Q \mid \text{there exist } t \text{ edges of } \overline{\mathcal{Q}}^Q \\ \text{extending from } P\}.$$

Next, to compute the value of k, we take a lattice point $P_0 \notin \mathrm{Conv}(\overline{\mathcal{Q}}^Q)$ such that an integral polytope $\mathcal{Q}_0 = \mathrm{Conv}(\overline{\mathcal{Q}}^Q \cup \{P_0\})$ satisfies $V(\mathcal{Q}_0) = V(\overline{\mathcal{Q}}^Q) \cup \{P_0\}$ (see Fig. 1).

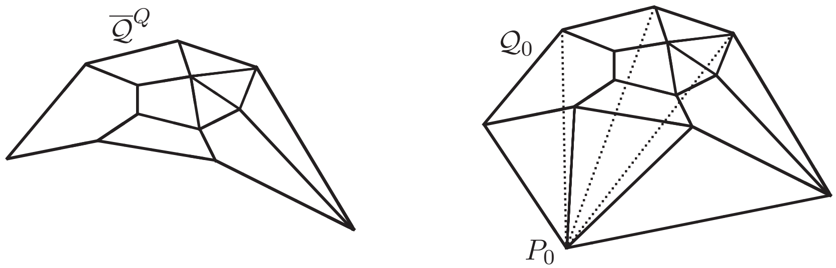

FIGURE 1.

Then the number of vertices, edges and faces of $\mathcal{Q}_0$ are $\sum_{s=3}^{s_0} m_s + \sum_{t=2}^{t_0} n_t + 1$, $\big(\sum_{s=3}^{s_0} sm_s + \sum_{t=2}^{t_0} tn_t\big)/2 + \sum_{t=2}^{t_0} n_t$ and $k + \sum_{t=2}^{t_0} n_t$, respectively. Hence, by Euler's polyhedron formula, we have

$$\sum_{s=3}^{s_0} m_s + \sum_{t=2}^{t_0} n_t + 1 - \frac{\sum_{s=3}^{s_0} sm_s + \sum_{t=2}^{t_0} tn_t}{2} - \sum_{t=2}^{t_0} n_t + k + \sum_{t=2}^{t_0} n_t = 2,$$

which implies that $k = \big(\sum_{s=3}^{s_0}(s-2)m_s + \sum_{t=2}^{t_0}(t-2)n_t\big)/2 + 1$. As a consequence,

$$\begin{aligned}\mu(\mathcal{Q}) - \mu(\mathcal{Q}^{\mathcal{Q}}) &= \frac{1}{6}\Big(l(\partial\overline{\mathcal{Q}}^Q \setminus V(\overline{\mathcal{Q}}^Q)) + \sum_{t=2}^{t_0} n_t\Big) - \frac{2}{3} \\ &\qquad + \frac{1}{6}\sum_{j=1}^{k}(h_j - 1)(a_j + b_j - 2) \\ &= \frac{1}{6} l(\partial\overline{\mathcal{Q}}^Q) - \frac{2}{3} + \frac{1}{6}\sum_{j=1}^{k}(h_j - 1)(a_j + b_j - 2). \qquad \square\end{aligned}$$

By noting that $l(\partial\overline{\mathcal{Q}}^{Q}) \geqslant 3$, $a_j \geqslant 3$ and $b_j \geqslant 0$, we obtain the following corollary.

Corollary 2.4. *Let $\mathcal{Q}$, Q and F_j be as in Proposition 2.3. If there exists a face F_{j_0} of $\mathcal{Q}^Q$ such that $h(F_{j_0}, Q) \geqslant 2$, then $\mu(\mathcal{Q}) \geqslant \mu(\mathcal{Q}^Q)$.*

Let us show the main result. Since the proof is relatively long, we divide it into two parts.

Proof of the inequality in Theorem 1.4. Let $\mathcal{T}$ be a section of $\mathcal{P}$ as in Lemma 2.2. We take a lattice point $P_0 \in V(\mathcal{P}) \setminus \mathcal{T}$ and put $\mathcal{P}_1 = \mathcal{P}^{P_0}$. By carrying out such operation repeatedly, we construct a sequence of integral convex polytopes

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots, \tag{4}$$

where $P_i \in V(\mathcal{P}_i) \setminus \mathcal{T}$ and $\mathcal{P}_{i+1} = \mathcal{P}_i^{P_i}$. It is sufficient to show that, by going through a suitable process, we can find $\mathcal{P}_n$ such that $\mu(\mathcal{P}_0) \geqslant \mu(\mathcal{P}_n)$, $l(\mathcal{P}_n) = l(\mathcal{T}) + 2$ and $\mathcal{T}$ is not a face of $\mathcal{P}_n$ (see Fig. 2).

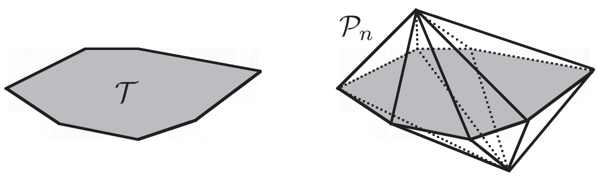

Figure 2.

Indeed, for such a $\mathcal{P}_n$, it follows from Theorem 1.1 that

$$\begin{aligned}
\mu(\mathcal{P}_n) &= \mathrm{vol}(\mathcal{P}_n) - \frac{l(\mathcal{P}_n) + l(\mathrm{Int}(\mathcal{P}_n)) - 4}{3} \\
&\geqslant \frac{2}{3}\mathrm{vol}(\mathcal{T}) - \frac{l(\mathcal{T}) + l(\mathrm{Int}(\mathcal{T})) - 2}{3} \\
&= \frac{2}{3} \cdot \frac{l(\mathcal{T}) + l(\mathrm{Int}(\mathcal{T})) - 2}{2} - \frac{l(\mathcal{T}) + l(\mathrm{Int}(\mathcal{T})) - 2}{3} = 0.
\end{aligned}$$

To verify the existence of $\mathcal{P}_n$, it is sufficient to prove the following claim:

Claim A. *Let i be a nonnegative integer. If $l(\mathcal{P}_i) \geqslant l(\mathcal{T}) + 3$ and $\mathcal{T}$ is not a face of $\mathcal{P}_i$, then we can construct $\mathcal{P}_{i_0}$ ($i_0 > i$) such that $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i_0})$ and $\mathcal{T}$ is not a face of $\mathcal{P}_{i_0}$.*

We define $A = \{Q \in V(\mathcal{P}_i) \setminus \mathcal{T} \mid \mathcal{T}$ is not a face of $\mathcal{P}_i^Q\}$. If there exists a point $Q \in A$ such that $l(\partial\overline{\mathcal{P}_i}^Q) \geqslant 4$, by putting $P_i = Q$, we obtain the inequality $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1})$ by Proposition 2.3. Hence Claim A is true in this case. We thus assume that

$$l(\partial\overline{\mathcal{P}_i}^Q) = 3 \text{ for any } Q \in A. \tag{5}$$

We take a point $Q_0 \in A$. Note that the inequality $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_i^{Q_0}) - 1/6$ follows from Proposition 2.3. We denote by Q_1, Q_2 and Q_3 the vertices of a triangle $\partial\overline{\mathcal{P}_i}^{Q_0}$, and put $\boldsymbol{v_j} = Q_j - Q_0$ for $j = 1, 2, 3$. We define $\varepsilon_j = \max\{\varepsilon \in \mathbb{N} \mid Q_0 + \varepsilon\boldsymbol{v_j} \in \mathcal{P}_i\}$ and $Q_j' = Q_0 + \varepsilon_j\boldsymbol{v_j}$ for $j = 1, 2, 3$.

(i) We first consider the case where $l(\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\})) \geqslant 5$. We put $t = \sharp(\{Q_1, \ Q_2, Q_3\} \cap \mathcal{T})$. If $t = 3$, then $\mathcal{T}$ is a triangle $\mathrm{Conv}(\{Q_1, Q_2, Q_3\})$ whose border has no lattice points except for three vertices. This contradicts the property of $\mathcal{T}$ of having a smooth vertex and interior lattice points. Hence we have $t \leqslant 2$.

(i)–(a) If $t \leqslant 1$, we can assume $Q_1, Q_2 \notin \mathcal{T}$ and $\varepsilon_1 \leqslant \varepsilon_2$. In the case where $\varepsilon_1 \geqslant 2$, we put $P_i = Q_0$, $P_{i+1} = Q_1$ and $P_{i+2} = Q_2$. Then $\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}$ contains $Q_1 + \boldsymbol{v_1}$, Q_2, Q_3 and at least one lattice point in $\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\}) \setminus \{Q_0, \ldots, Q_3\}$. It follows from Proposition 2.3 that $\mu(\mathcal{P}_{i+1}) \geqslant \mu(\mathcal{P}_{i+2})$. Similarly, since $\partial\overline{\mathcal{P}_{i+2}}^{P_{i+2}}$ contains $Q_1 + \boldsymbol{v_1}$, $Q_1 + \boldsymbol{v_2}$, $Q_2 + \boldsymbol{v_2}$, Q_3 and at least one lattice point in $\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\}) \setminus \{Q_0, \ldots, Q_3\}$, we have $\mu(\mathcal{P}_{i+2}) \geqslant \mu(\mathcal{P}_{i+3}) + 1/6$. In sum, Claim A is true by

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+2}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+3}).$$

We next consider the case where $\varepsilon_1 = 1$. Since $Q_1 \in A$, we have $l(\partial\overline{\mathcal{P}_i}^{Q_1}) = 3$ by (5), and more concretely, $Q_1'(= Q_1)$ has three adjacent lattice points $Q_1' - \boldsymbol{v_1}(= Q_0)$, $Q_0 + \alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}$ and $Q_0 + \gamma\boldsymbol{v_1} + \delta\boldsymbol{v_3}$ in $E(\mathcal{P}_i)$, where $\alpha, \gamma \geqslant 0$ and $\beta, \delta \geqslant 1$. We denote by F a face of $\overline{\mathcal{P}_i}^{Q_1}$ containing two points Q_0 and $Q_0 + \alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}$. By an easy computation, we obtain $h(F, Q_1) \geqslant \beta$. If $\beta \geqslant 2$, we can finish the proof by putting $P_i = Q_1$. Indeed, in this case, the inequality $\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1})$ holds by Corollary 2.4. Similar arguments can be carried out for the case where $\delta \geqslant 2$. Let us consider the case where $\beta = \delta = 1$. We can assume $\alpha \geqslant \gamma$ without loss of generality, and have $\alpha + \gamma \geqslant 1$ by the existence of $\mathcal{T}$. If $\alpha + \gamma \geqslant 2$, we put $P_i = Q_0$ and $P_{i+1} = Q_1$. Then, since $\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}$

contains $Q_2 + k\boldsymbol{v_1}$ $(k = 0, \dots, \alpha)$, $Q_3 + k\boldsymbol{v_1}$ $(k = 0, \dots, \gamma)$ and at least one lattice point in $\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\}) \setminus \{Q_0, \dots, Q_3\}$, we obtain $\mu(\mathcal{P}_{i+1}) \geqslant \mu(\mathcal{P}_{i+2}) + (\alpha + \gamma - 1)/6$. Hence

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+2}) + \frac{\alpha + \gamma - 2}{6} \geqslant \mu(\mathcal{P}_{i+2}).$$

Let us consider the remaining case where $\alpha = 1$ and $\gamma = 0$. Note that $\mathcal{P}_i$ has a face containing Q_1, Q_3 and $Q_1 + \boldsymbol{v_2}$ (see Fig. 3).

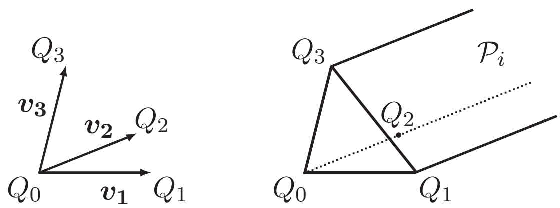

Figure 3.

We define $\zeta_j = \max\{\zeta \in \mathbb{Z}_{\geqslant 0} \mid Q_j + \zeta\boldsymbol{v_2} \in \mathcal{P}_i\}$ for $j = 1, 2, 3$. Considering the properties of $\mathcal{T}$, at least one of $Q_1 + \zeta_1\boldsymbol{v_2}$, $Q_2 + \zeta_2\boldsymbol{v_2}$, and $Q_3 + \zeta_3\boldsymbol{v_2}$ is not on $\mathcal{T}$. Then, by (5), such a point has just three adjacent lattice points in $E(\mathcal{P}_i)$. This implies that $V(\mathcal{P}_i) = \{Q_0, Q_1, Q_3, Q_1 + \zeta_1\boldsymbol{v_2}, Q_2 + \zeta_2\boldsymbol{v_2}, Q_3 + \zeta_3\boldsymbol{v_2}\}$, which contradicts the existence of $\mathcal{T}$.

(i)–(b) If $t = 2$, we can assume that $Q_1 \notin \mathcal{T}$ and $Q_2, Q_3 \in \mathcal{T}$. By the properties of $\mathcal{T}$, we see that $Q_0 + \varepsilon\boldsymbol{v_1}$ is not on $\mathcal{T}$ for $0 \leqslant \varepsilon \leqslant \varepsilon_1$. As we saw in the case (i)-(a), Q_1' has three adjacent lattice points $Q_1' - \boldsymbol{v_1}$, $Q_0 + \alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2}$ and $Q_0 + \gamma\boldsymbol{v_1} + \delta\boldsymbol{v_3}$ in $E(\mathcal{P}_i)$, and the proof is finished by putting $P_i = Q_1'$ in the case where $\beta \geqslant 2$ or $\delta \geqslant 2$. We assume $\beta = \delta = 1$. Since $\mathcal{T}$ has interior lattice points, we see that at least one of $Q_0 + \alpha\boldsymbol{v_1} + \boldsymbol{v_2}$ and $Q_0 + \gamma\boldsymbol{v_1} + \boldsymbol{v_3}$ is not on $\mathcal{T}$. Then this case is equivalent to the case (i)-(a) by regarding Q_1' as Q_0.

(ii) We next consider the case where $l(\mathrm{Conv}(\{Q_0, Q_1, Q_2, Q_3\})) = 4$. If $h(\mathrm{Conv}(\{Q_1, Q_2, Q_3\}), Q_0) \geqslant 2$, we can finish the proof by putting $P_i = Q_0$. Hence we can assume that Q_0 is a smooth vertex, that is, $Q_0 = O$, $Q_1 = (1, 0, 0)$, $Q_2 = (0, 1, 0)$ and $Q_3 = (0, 0, 1)$. Moreover, we assume that every vertex in A is smooth in order to avoid the duplication with the case (i). We denote by L_j the segment Q_0Q_j' $(j = 1, 2, 3)$, and put $u = \sharp\{L_j \mid L_j \cap \mathcal{T} \neq \varnothing, j = 1, 2, 3\}$.

(ii)–(a) In the case where $u \leqslant 1$, we can assume that $L_1 \cap \mathcal{T} = \varnothing$ and $L_2 \cap \mathcal{T} = \varnothing$. Since Q_1' is smooth, it has three adjacent lattice points $(\varepsilon_1 - 1, 0, 0)$, $(\alpha, 1, 0)$ and $(\gamma, 0, 1)$ in $E(\mathcal{P}_i)$. If we put $P_{i+\varepsilon} = (\varepsilon, 0, 0)$

for $\varepsilon = 0, \ldots, \varepsilon_1$, since $\partial\overline{\mathcal{P}_{i+\varepsilon_1}}^{P_{i+\varepsilon_1}}$ contains lattice points $(k, 1, 0)$ with $k = 0, \ldots, \alpha$ and $(l, 0, 1)$ with $l = 0, \ldots, \gamma$, it follows that

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \cdots \geqslant \mu(\mathcal{P}_{i+\varepsilon_1}) - \frac{\varepsilon_1}{6} \geqslant \mu(\mathcal{P}_{i+\varepsilon_1+1}) + \frac{\alpha+\gamma-\varepsilon_1-2}{6}.$$

If $\alpha + \gamma \geqslant \varepsilon_1 + 2$, the proof is finished. We next consider the vertex Q_2' and its three adjacent lattice points $(0, \varepsilon_2 - 1, 0)$, $(1, \beta, 0)$ and $(0, \delta, 1)$ in $E(\mathcal{P}_i)$. Similarly to the case of Q_1', we can finish the proof in the case where $\beta + \delta \geqslant \varepsilon_2 + 2$. Hence we assume that $\alpha + \gamma \leqslant \varepsilon_1 + 1$ and $\beta + \delta \leqslant \varepsilon_2 + 1$. Let (x_0, y_0, z_0) be a lattice point in $\mathrm{Int}(\mathcal{P}_i)$. Since $\mathcal{P}_i$ has a face containing three points $(\varepsilon_1, 0, 0)$, $(\alpha, 1, 0)$ and $(\gamma, 0, 1)$ (resp. $(0, \varepsilon_2, 0)$, $(1, \beta, 0)$ and $(0, \delta, 1)$), we have

$$\begin{gathered} x_0 + (\varepsilon_1 - \alpha)y_0 + (\varepsilon_1 - \gamma)z_0 - \varepsilon_1 < 0 \\ (\text{resp. } (\varepsilon_2 - \beta)x_0 + y_0 + (\varepsilon_2 - \delta)z_0 - \varepsilon_2 < 0). \end{gathered}$$

By noting $x_0, y_0, z_0 \geqslant 1$ and $\alpha, \gamma, \beta, \delta \geqslant 0$, we obtain $(\alpha, \gamma) = (\varepsilon_1 + 1, 0)$ or $(0, \varepsilon_1 + 1)$ and $(\beta, \delta) = (\varepsilon_2 + 1, 0)$ or $(0, \varepsilon_2 + 1)$. Clearly, $\gamma = \delta = 0$ gives a contradiction. Besides, considering the shape of $\mathcal{P}_i$, if either α or β is zero, then the other one also must be zero and $\varepsilon_1 = \varepsilon_2 = 1$. In sum, we have $(\alpha, \gamma) = (\beta, \delta) = (0, 2)$. Then, putting $P_i = Q_0$ and $P_{i+j} = Q_j$ for $j = 1, 2$, we have

$$\begin{aligned} l(\partial\overline{\mathcal{P}_i}^{P_i}) &= l(\mathrm{Conv}(\{(1,0,0), (0,1,0), (0,0,1)\})) = 3, \\ l(\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}) &= l(\mathrm{Conv}(\{(2,0,1), (0,1,0), (0,0,1)\})) = 4, \\ l(\partial\overline{\mathcal{P}_{i+2}}^{P_{i+2}}) &= l(\mathrm{Conv}(\{(2,0,1), (0,2,1), (0,0,1)\})) = 6. \end{aligned}$$

It follows from Proposition 2.3 that $\mu(\mathcal{P}_i) = \mu(\mathcal{P}_{i+1}) - 1/6 = \mu(\mathcal{P}_{i+2}) - 1/6 = \mu(\mathcal{P}_{i+3}) + 1/6$.

(ii)–(b) If $u = 2$, we can assume that $L_1 \cap \mathcal{T} = \varnothing$, $L_2 \cap \mathcal{T} \neq \varnothing$ and $L_3 \cap \mathcal{T} \neq \varnothing$. As we saw in the case (ii)-(a), it is sufficient to consider the case where Q_1' has three adjacent lattice points $(\varepsilon_1 - 1, 0, 0)$, $(\alpha, 1, 0)$ and $(\gamma, 0, 1)$ with $(\alpha, \gamma) = (\varepsilon_1 + 1, 0)$ or $(0, \varepsilon_1 + 1)$. Here we consider only the former case $(\alpha, \gamma) = (\varepsilon_1 + 1, 0)$. The latter case can be shown in a similar way. First, we remark that $\varepsilon_3 = 1$ holds by the condition $\gamma = 0$. This means that $(0, 0, 1)$ is on $\mathcal{T}$. Denote by L_4 the line passing through Q_1' and $(\alpha, 1, 0)$. Since $\mathcal{T}$ has interior lattice points, L_4 does not contain a lattice point on $\mathcal{T}$. Then, by regarding Q_1' as Q_0, this case can be reduced to the case (ii)-(a).

(ii)–(c) Assume $u = 3$. In this case, we see that $Q_j \notin \mathcal{T}$ for $j = 1, 2, 3$ since $\mathcal{T}$ has interior lattice points. It follows that $(2,0,0), (0,2,0), (0,0,2) \in \mathcal{P}_i$. If we put $P_i = Q_0$ and $P_{i+j} = Q_j$ for $j = 1, 2, 3$, then

$$\begin{aligned}
l(\partial\overline{\mathcal{P}_i}^{P_i}) &= l(\mathrm{Conv}(\{(1,0,0),(0,1,0),(0,0,1)\})) = 3,\\
l(\partial\overline{\mathcal{P}_{i+1}}^{P_{i+1}}) &= l(\mathrm{Conv}(\{(2,0,0),(0,1,0),(0,0,1)\})) = 3,\\
l(\partial\overline{\mathcal{P}_{i+2}}^{P_{i+2}}) &= l(\mathrm{Conv}(\{(2,0,0),(0,2,0),(0,0,1)\})) = 4,\\
l(\partial\overline{\mathcal{P}_{i+3}}^{P_{i+3}}) &= l(\mathrm{Conv}(\{(2,0,0),(0,2,0),(0,0,2)\})) = 6.
\end{aligned}$$

Hence we have

$$\mu(\mathcal{P}_i) \geqslant \mu(\mathcal{P}_{i+1}) - \frac{1}{6} \geqslant \mu(\mathcal{P}_{i+2}) - \frac{2}{3} \geqslant \mu(\mathcal{P}_{i+3}) - \frac{2}{3} \geqslant \mu(\mathcal{P}_{i+4}).$$

Since $\mathcal{T}$ has interior lattice points, at least one of $(2,0,0), (0,2,0)$ and $(0,0,2)$ is not on $\mathcal{T}$, that is, $\mathcal{T}$ is not a face of $\mathcal{P}_{i+4}$. $\square$

In order to show the latter part of Theorem 1.4, we require results in the theory of toric varieties and the classification theory of polarized varieties. Hence, in the proof below, we take in advance the contents in the next section although not in the proper order. See Section 3 for precise definitions and notations.

Proof of the equivalency in Theorem 1.4. The classification of toric Fano threefolds has been completed, and they are classified into eighteen types (cf. [2, 17]). For each type X of toric Fano threefolds, the polytope $\square_{-K_X}$ associated to the anti-canonical bundle has just one interior lattice point. Moreover, we can obtain

$$\mathrm{vol}(\square_{-K_X}) = \frac{l(\square_{-K_X})}{3} - 1$$

by steady calculations. We list several examples of them for readers' exercise.

$$\begin{array}{rl}
X & \square_{-K_X}\\
\mathbb{P}^3 & \text{the fourth dilation of a unit three-simplex}\\
\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 & \text{the twice dilation of a unit cube}\\
\mathbb{P}^2\times\mathbb{P}^1 & \mathrm{Conv}(\{(0,0,\pm1),(3,0,\pm1),(0,3,\pm1)\})\\
\Sigma_1\times\mathbb{P}^1 & \mathrm{Conv}(\{(0,0,\pm1),(3,0,\pm1),(1,2,\pm1),(0,2,\pm1)\})\\
\vdots & \quad\vdots
\end{array} \tag{6}$$

Let us show the sufficiency. We first consider the case where $l(\mathrm{Int}(\mathcal{P}))$=1 and $\mathrm{vol}(\mathcal{P}) = l(\mathcal{P})/3 - 1$. Let L be an ample line bundle on a three-dimensional toric variety X whose associated polytope $\square_L$ coincides with $\mathcal{P}$. Then our assumptions are equivalent to two equalities $h^0(X, L + K_X) = 1$ and $L^3 = 2h^0(X, L) - 6$. By noting Lemma 3.7, the sectional genus and the Δ-genus of the polarized variety (X, L) is $g(X, L) = h^0(X, L) - 2$ and $\Delta(X, L) = h^0(X, L) - 3$, respectively. On the other hand, since $\lfloor (L^3 - 1)/(L^3 - \Delta(X, L) - 1) \rfloor = 2$, the above sectional genus coincides with the upper bound in Theorem 3.3. Namely, (X, L) is a Castelnuovo variety in this case. Since (X, L) is a Mukai variety by the remark after Theorem 3.3, we can conclude that X is a Fano variety and $L = -K_X$.

In the remaining part, we prove the inequality $\mathrm{vol}(\mathcal{P}) > (l(\mathcal{P}) + l(\mathrm{Int}(\mathcal{P})) - 4)/3$ under the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. Recall the sequence of integral convex polytopes (4) and Claim A in the proof of Theorem 1.4. Then it is sufficient to show that, by going through a suitable process, we can construct $\mathcal{P}_{i_0}$ such that $\mu(\mathcal{P}) > \mu(\mathcal{P}_{i_0})$ and $\mathcal{T}$ is not a face of $\mathcal{P}_{i_0}$. We place $\mathcal{P}$ in $\mathbb{R}^3$ so that four points O, $Q_1 = (1, 0, 0)$, $Q_2 = (0, 1, 0)$ and $Q_3 = (0, 0, 1)$ are contained in $E(\mathcal{P})$, and define

$$\begin{aligned}
\varepsilon_1 &= \max\{\varepsilon \in \mathbb{N} \mid (\varepsilon, 0, 0) \in \mathcal{P}\}, \\
\varepsilon_2 &= \max\{\varepsilon \in \mathbb{N} \mid (0, \varepsilon, 0) \in \mathcal{P}\}, \\
\varepsilon_3 &= \max\{\varepsilon \in \mathbb{N} \mid (0, 0, \varepsilon) \in \mathcal{P}\},
\end{aligned}$$

$Q'_1 = (\varepsilon_1, 0, 0)$, $Q'_2 = (0, \varepsilon_2, 0)$ and $Q'_3 = (0, 0, \varepsilon_3)$. Then, by the smoothness of $\mathcal{P}$, we see that Q'_1 has three adjacent lattice points $(\varepsilon_1 - 1, 0, 0)$, $(\alpha_1, 1, 0)$ and $(\alpha_2, 0, 1)$ in $E(\mathcal{P})$. Similarly, Q'_2 (resp. Q'_3) has three adjacent lattice points $(0, \varepsilon_2 - 1, 0)$, $(1, \beta_1, 0)$ and $(0, \beta_2, 1)$ (resp. $(0, 0, \varepsilon_3 - 1)$, $(1, 0, \gamma_1)$ and $(0, 1, \gamma_2)$) in $E(\mathcal{P})$. In the case where $\alpha_1 + \alpha_2 \geqslant \varepsilon_1 + 3$, we can take $\mathcal{T}$ so that it does not contain points on the x-axis by using a similar method to that in the proof of Lemma 2.2. If we put $P_i = (i, 0, 0)$ for $i = 0, \ldots, \varepsilon_1$, then $\partial \overline{\mathcal{P}_i}^{P_i}$ is a triangle with vertices $(i + 1, 0, 0)$, Q_2 and Q_3 for $i = 0, \ldots, \varepsilon_1 - 1$, and $\partial \overline{\mathcal{P}_{\varepsilon_1}}^{P_{\varepsilon_1}}$ is a trapezoid $\mathrm{Conv}(\{Q_2, Q_3, (\alpha_1, 1, 0), (\alpha_2, 0, 1)\})$. We thus obtain

$$\begin{aligned}
\mu(\mathcal{P}_0) &\geqslant \mu(\mathcal{P}_1) - \frac{1}{6} \geqslant \cdots \geqslant \mu(\mathcal{P}_{\varepsilon_1}) - \frac{\varepsilon_1}{6} \\
&\geqslant \mu(\mathcal{P}_{\varepsilon_1+1}) + \frac{\alpha_1 + \alpha_2 - \varepsilon_1 - 2}{6} > \mu(\mathcal{P}_{\varepsilon_1+1})
\end{aligned}$$

by Proposition 2.3. Also in the cases where $\beta_1 + \beta_2 \geqslant \varepsilon_2 + 3$ or $\gamma_1 + \gamma_2 \geqslant \varepsilon_3 + 3$, we can finish the proof in essentially the same way.

We assume henceforth that $\alpha_1+\alpha_2 \leqslant \varepsilon_1+2$, $\beta_1+\beta_2 \leqslant \varepsilon_2+2$ and $\gamma_1+\gamma_2 \leqslant \varepsilon_3+2$. Moreover, without loss of generality, we can assume that OQ_1' is the shortest edge of $\mathcal{P}$ and $\alpha_1 \geqslant \alpha_2$. Since $\mathcal{P}$ has a face containing three points Q_1', $(\alpha_1,1,0)$ and $(\alpha_2,0,1)$, the inclusion

$$\begin{aligned}\operatorname{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, x+(\varepsilon_1-\alpha_1)(y-1) \\ +(\varepsilon_1-\alpha_2)(z-1)+\varepsilon_1-\alpha_1-\alpha_2+1 \leqslant 0\}\end{aligned} \tag{7}$$

is derived from the inequality $x+(\varepsilon_1-\alpha_1)y+(\varepsilon_1-\alpha_2)z-\varepsilon_1<0$. Similarly, by considering the vertices Q_2' and Q_3', we obtain

$$\begin{aligned}\operatorname{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, (\varepsilon_2-\beta_1)(x-1)+y \\ +(\varepsilon_2-\beta_2)(z-1)+\varepsilon_2-\beta_1-\beta_2+1 \leqslant 0\},\end{aligned} \tag{8}$$

$$\begin{aligned}\operatorname{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, (\varepsilon_3-\gamma_1)(x-1)+(\varepsilon_3-\gamma_2)(y-1) \\ +z+\varepsilon_3-\gamma_1-\gamma_2+1 \leqslant 0\}.\end{aligned} \tag{9}$$

(i) Assume that $\alpha_1 \leqslant \varepsilon_1$, and let (x_0,y_0,z_0) be a lattice point in $\operatorname{Int}(\mathcal{P})$. By noting $\alpha_1 \geqslant \alpha_2$ and $\alpha_1+\alpha_2 \leqslant \varepsilon_1+2$, we have $x_0=1$ and $\alpha_1+\alpha_2=\varepsilon_1+2$ by (7). If $\alpha_2<\varepsilon_1$, we see that $z_0=1$ by (7) and $y_0=1$ by (8), which contradicts the assumption $l(\operatorname{Int}(\mathcal{P})) \geqslant 2$. We thus have $\alpha_2 \geqslant \varepsilon_1$, and similarly $\beta_2>\varepsilon_2$ and $\gamma_2>\varepsilon_3$. Since $\varepsilon_1=\alpha_1=\alpha_2=2$ in this case, $\varepsilon_2,\varepsilon_3 \leqslant 2$ follows from the shortestness of the edge OQ_1'. Moreover, it follows from $(2,1,0),(2,0,1) \in \mathcal{P}$ that $\beta_1 \geqslant \varepsilon_2$ and $\gamma_1 \geqslant \varepsilon_3$. As a consequence, we have $(\varepsilon_2,\beta_1,\beta_2)=(\varepsilon_3,\gamma_1,\gamma_2)=(1,1,2)$. By the shortestness of the edge OQ_1' again, the faces $\mathcal{P}\cap H_x$ and $\mathcal{P}\cap H_{x-2}$ are one of the four types of polygons as in Fig. 4.

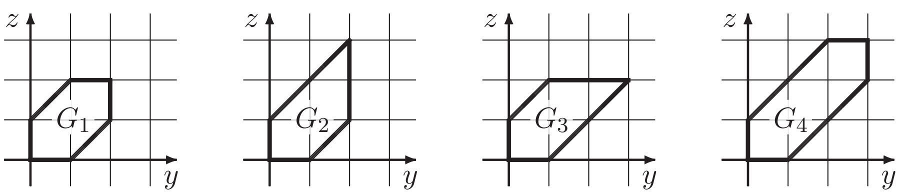

Figure 4.

Then, considering the smoothness of $\mathcal{P}$ and the assumption $l(\operatorname{Int}(\mathcal{P})) \geqslant 2$, there exist only three possibilities $(\mathcal{P}\cap H_x, \mathcal{P}\cap H_{x-1}, \mathcal{P}\cap H_{x-2}) = (G_1,G_4,G_4),(G_4,G_4,G_1),(G_4,G_4,G_4)$. In the first two cases, we have $\operatorname{vol}(\mathcal{P})=9$, $l(\mathcal{P})=27$ and $l(\operatorname{Int}(\mathcal{P}))=2$. On the other hand, in the last case, $\operatorname{vol}(\mathcal{P})=10$, $l(\mathcal{P})=30$ and $l(\operatorname{Int}(\mathcal{P}))=2$. Hence the inequality $\operatorname{vol}(\mathcal{P})>(l(\mathcal{P})+l(\operatorname{Int}(\mathcal{P}))-4)/3$ holds in each case.

(ii) Suppose that $\varepsilon_1+1 \leqslant \alpha_1 \leqslant \varepsilon_1+2$ and $\beta_1, \beta_2 \leqslant \varepsilon_2$. Let (x_0, y_0, z_0) be a lattice point in $\mathrm{Int}(\mathcal{P})$. We have $y_0 = 1$ and $\beta_1+\beta_2 = \varepsilon_2+2$ by (8). Then, since $\varepsilon_1 - \alpha_2 \geqslant \alpha_1 - 2 \geqslant 0$, we have $x_0 = 1$ and $\alpha_1+\alpha_2 = \varepsilon_1+2$ by (7). By the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$, there must be an interior lattice point such that $z_0 \geqslant 2$. It follows that $\alpha_2 = \varepsilon_1$ and $\beta_2 = \varepsilon_2$. These facts, together with the shortestness of OQ_1', immediately give that $\varepsilon_1 = \varepsilon_2 = \alpha_2 = \beta_2 = 1$ and $\alpha_1 = \beta_1 = 2$. By using the shortestness of OQ_1' again, we see that $\mathcal{P} \cap H_x$ and $\mathcal{P} \cap H_y$ are unit squares, which yields a contradiction $l(\mathrm{Int}(\mathcal{P})) = 0$.

(iii) Suppose that $\varepsilon_1+1 \leqslant \alpha_1 \leqslant \varepsilon_1+2$ and $\beta_2 \geqslant \varepsilon_2+1$. Note that $\alpha_1 \geqslant 2$ and $\beta_1 \leqslant 1$ in this case. Then β_1 must be one since $(2,1,0) \in \mathcal{P}$. Hence $\mathcal{P} \cap H_z$ is a trapezoid $\mathrm{Conv}(\{O, Q_1', (\alpha_1, 1, 0), Q_2\})$, which contradicts the shortestness of OQ_1'.

(iv) Suppose that $\varepsilon_1+1 \leqslant \alpha_1 \leqslant \varepsilon_1+2$ and $\beta_2 = 0$. In this case, ε_2 must be one by the smoothness of the vertex Q_3. We thus have $\varepsilon_3 = 1$ and $\gamma_2 = 0$, which imply $\gamma_1 \geqslant 2$ (namely, $(1,0,2) \in \mathcal{P}$) by (9). By noting $(2,0,1) \notin \mathcal{P}$, we see that $\mathcal{P} \cap H_y$ is a trapezoid $\mathrm{Conv}(\{O, Q_1, (1,0,\gamma_1), Q_3\})$, which contradicts the shortestness of OQ_1'.

(v) We finally consider the remaining case where $\varepsilon_1+1 \leqslant \alpha_1 \leqslant \varepsilon_1+2$, $\beta_1 = \varepsilon_2+1$ and $\beta_2 = 1$. If $\varepsilon_1 = 1$, by the shortestness of OQ_1', $\mathcal{P} \cap H_x$ is a unit square, and $\mathcal{P} \cap H_y$ is a unit triangle or a unit square. This contradicts the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. We thus assume $\varepsilon_1 \geqslant 2$. Note that $\alpha_1 = \varepsilon_1+1$ and $\alpha_2 = 1$ in this case. Hence, if (x_0, y_0, z_0) is a lattice point in $\mathrm{Int}(\mathcal{P})$, $x_0 = y_0$ and $z_0 = 1$ follow from (7) and (8). We define $\varepsilon_4 = \max\{\varepsilon \in \mathbb{N} \mid (\varepsilon, \varepsilon_2+\varepsilon, 0) \in \mathcal{P}\}$, and put $Q_4 = (\varepsilon_4, \varepsilon_2+\varepsilon_4, 0)$. By the smoothness of $\mathcal{P}$, the vertex Q_4 has three adjacent lattice points $(\varepsilon_4-1, \varepsilon_2+\varepsilon_4-1, 0)$, $(\delta_1, \delta_1+\varepsilon_2-1, 0)$ and $(\delta_2, \delta_2+1, 1)$ in $E(\mathcal{P})$ (see Fig. 5).

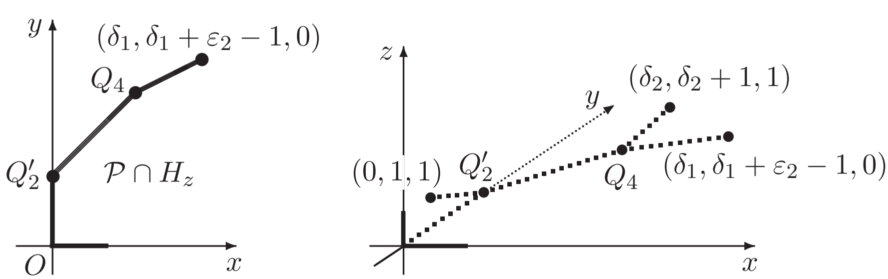

FIGURE 5.

If $\delta_1+\delta_2 \geqslant \varepsilon_4+3$, we put $P_i=(i,\varepsilon_2+i,0)$ for $i=0,\dots,\varepsilon_4$. Then $\partial\overline{\mathcal{P}_i}^{P_i}$ is a triangle with vertices $(i+1,\varepsilon_2+i+1,0)$, $(0,\varepsilon_2-1,0)$ and $(0,1,1)$ for $i=0,\dots,\varepsilon_4-1$, and $\partial\overline{\mathcal{P}_{\varepsilon_4}}^{P_{\varepsilon_4}}$ is a trapezoid

$$\mathrm{Conv}(\{(0,\varepsilon_2-1,0),(\delta_1,\delta_1+\varepsilon_2-1,0),(0,1,1),(\delta_2,\delta_2+1,1)\}).$$

The proof is finished since

$$\begin{aligned}\mu(\mathcal{P}_0) &\geqslant \mu(\mathcal{P}_1)-\frac{1}{6} \geqslant \cdots \geqslant \mu(\mathcal{P}_{\varepsilon_4})-\frac{\varepsilon_4}{6}\\ &\geqslant \mu(\mathcal{P}_{\varepsilon_4+1})+\frac{\delta_1+\delta_2-\varepsilon_4-2}{6} > \mu(\mathcal{P}_{\varepsilon_1+1})\end{aligned}$$

by Proposition 2.3. Finally, let us show that the case where $\delta_1+\delta_2 \leqslant \varepsilon_4+2$ does not occur. Since $\mathcal{P}$ has a face containing three points Q_4, $(\delta_1,\delta_1+\varepsilon_2-1,0)$ and $(\delta_2,\delta_2+1,1)$, the inclusion

$$\begin{aligned}\mathrm{Int}(\mathcal{P}) \subset \{(x,y,z) \mid x,y,z \geqslant 1, &(\varepsilon_4-\delta_1+1)(x-1)-(\varepsilon_4-\delta_1)(y-1)\\ &+(\delta_1\varepsilon_2-\varepsilon_2\varepsilon_4-\delta_1-\delta_2+2\varepsilon_4)(z-1)\\ &-\delta_1-\delta_2+\varepsilon_4+2 \leqslant 0\}\end{aligned}$$

holds. As we have already mentioned, any interior lattice point of $\mathcal{P}$ can be written as $(x_0,x_0,1)$. This contradicts the above inclusion and the assumption $l(\mathrm{Int}(\mathcal{P})) \geqslant 2$. □

3. Application

In this section, we apply our result to the computation of the sectional genus of a polarized toric variety. For an n-dimensional (smooth) complex projective variety X and an ample line bundle L on X, the pair (X,L) is called a (*smooth*) *polarized variety*. We remark that, in the case where L is ample, the associated polytope $\square_L$ is smooth if and only if X is smooth. Let us review the classification theory of polarized varieties before getting to the main subject. We first recall the well-known upper bound for the geometric genus of a smooth curve.

Theorem 3.1 (Castelnuovo's bound, [1]). *Let C be a smooth curve of genus g. Assume that C admits a birational map onto a nondegenerate curve of degree d in $\mathbb{P}^r$. Then*

$$g \leqslant \frac{1}{2}a(a-1)(r-1)+a(d-a(r-1)-1),$$

where $a=\lfloor (d-1)/(r-1)\rfloor$.

A smooth curve is said to be *extremal* if its genus is equal to Castelnuovo's bound, which was studied in [4]. As a higher dimensional extension, Fujita established various invariants for polarized varieties and proved a similar inequality (Theorem 3.3).

Definition 3.2. *For an n-dimensional smooth polarized variety (X, L), we define the sectional genus and the Δ-genus by*

$$\begin{aligned} g(X,L) &= \frac{1}{2}L^{n-1}.((n-1)L+K_X)+1, \\ \Delta(X,L) &= L^n+n-h^0(X,L). \end{aligned}$$

Theorem 3.3 (cf. [5])**.** *Let L be a line bundle on an n-dimensional smooth projective variety X. If $|L|$ has no base points and the associated morphism $\Phi_{|L|}$ is birational on its image, then*

$$g(X,L) \leqslant a\Delta(X,L) - \frac{1}{2}a(a-1)(L^n - \Delta(X,L) - 1),$$

where $a = \lfloor (L^n-1)/(L^n - \Delta(X,L)-1) \rfloor$.

We call (X, L) a *Castelnuovo variety* if its sectional genus achieves the maximum of the above upper bound. Castelnuovo varieties can be roughly classified according to the relation between L^n and $2\Delta(X, L)$. First, the case where $L^n < 2\Delta(X, L)$ has been classified in [5]. If $L^n = 2\Delta(X, L)$, then (X, L) is a Mukai variety (i.e., $K_X \in |(2-n)L|$), which has been classified in [11]. On the other hand, the case where $L^n > 2\Delta(X, L)$ still has many unknown aspects. Two- or three-dimensional polarized toric varieties, which we consider below, contain examples of this case.

In the two-dimensional case, we do not need to use the results in this paper, but only the Riemann-Roch theorem. To see this, let us introduce the notion of a ladder.

Definition 3.4. *Let (X, L) be an n-dimensional polarized variety, and put $X_0 = X$ and $L_0 = L$. A sequence $X_0 \supset X_1 \supset \cdots \supset X_{n-1}$ of (smooth) subvarieties of X is called a (smooth) ladder of (X, L) if $X_i \in |L_{i-1}|$ for each $i \geqslant 1$, where we put $L_i = L|_{X_i}$.*

Theorem 3.5 (cf. [5])**.** *Let (X, L) be an n-dimensional polarized variety having a ladder. If $L^n > 2\Delta(X, L)$, then L is very ample and $g(X, L) = \Delta(X, L)$.*

If X is smooth and L is generated by global sections, by virtue of Bertini's theorem, we obtain a smooth ladder of (X, L) by cutting X_i by a general member of $|L_i|$. Since any ample line bundle on a compact toric variety is generated by global sections, a polarized toric variety always has a smooth ladder. Using these results, let us now consider the two-dimensional polarized toric varieties.

Theorem 3.6. *For a smooth compact toric surface X and an ample line bundle L on X, the polarized variety (X, L) is a Castelnuovo variety with $L^2 \geqslant 2\Delta(X, L) + 2$ unless L is a line in $\mathbb{P}^2$.*

Proof. By the general theory of toric varieties, L is very ample, and we have $p_a(X) = 0$, $h^0(X, L) \geqslant 3$ and $h^1(X, L) = h^2(X, L) = 0$. Hence we obtain $L^2 = 2h^0(X, L) + L.K_X - 2$ by the Riemann-Roch theorem. On the other hand, since $-L.K_X$ is equal to $l(\partial\Box_L)$, we have the inequality $-L.K_X \geqslant 3$, where the equality holds if and only if $(X, L) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Consequently, we have $L^2 \leqslant 2h^0(X, L) - 6$, which implies that $L^2 \geqslant 2\Delta(X, L) + 2$. Then, since $g(X, L) = \Delta(X, L)$ by Theorem 3.5, we can conclude that (X, L) is a Castelnuovo variety. $\Box$

Next, in order to investigate the three-dimensional case, we compute the value of $L^2.K_X$. This computation also can be reduced to a matter of the number of lattice points.

Lemma 3.7. *For a three-dimensional smooth polarized toric variety (X, L), the equality $L^2.K_X = 2(h^0(X, L + K_X) - h^0(X, L) + 2)$ holds.*

Proof. We denote by $D_1, \dots, D_d$ the T_N-invariant divisors of X, and by F_i the face of $\Box_L$ corresponding to D_i. It is known that $K_X \sim -\sum_{i=1}^d D_i$ and $L^2.D_i$ is equal to twice of the area of F_i. Hence the statement of the lemma can be rewritten as

$$\sum_{i=1}^d \mathrm{vol}(F_i) = -l(\mathrm{Int}(\Box_L)) + l(\Box_L) - 2 = l(\partial\Box_L) - 2.$$

Let us compute the left-hand side by using Theorem 1.1 and Euler's polyhedron formula. Denote by v and e the number of vertices and edges of $\Box_L$, respectively. It follows from the smoothness of $\Box_L$ that every vertex of $\Box_L$ has three edges. Hence we have $3v = 2e$ and

$$\begin{aligned}\sum_{i=1}^d \mathrm{vol}(F_i) &= \frac{1}{2}\sum_{i=1}^d (l(F_i) + l(\mathrm{Int}(F_i)) - 2) = \frac{1}{2}(2l(\partial\Box_L) + v) - d \\ &= l(\partial\Box_L) - v + e - d = l(\partial\Box_L) - 2.\end{aligned}$$

$\Box$

Theorem 3.8. *Let X be a three-dimensional smooth compact toric variety, and L be an ample line bundle on X. Assume that $(X, L) \not\simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Then a polarized variety (X, L) is a Castelnuovo variety if and only if*

(i) *X is a Fano variety and $L \sim -K_X$, or*

(ii) *$h^0(X, L + K_X) = 0$ and $h^0(X, 2L + K_X) \leqslant h^0(X, L) - 4$.*

We provide several additional explanations. The former case has been classified into eighteen types in [2] and [17]. Besides, $L^3 = 2\Delta(X, L)$ holds in this case. On the other hand, $L^3 > 2\Delta(X, L)$ holds in the case (ii), and it is known that three-dimensional polarized toric varieties with $h^0(X, L + K_X) = 0$ (not necessarily assume the latter inequality) can be classified into five types (see [13]). We will see further details of this classification after the proof. Incidentally, more generally, Fukuma classified n-dimensional polarized varieties (not necessarily toric) with $h^0(X, (n-2)L + K_X) = 0$ in [6].

Proof of Theorem 3.8. If we rewrite the inequality in Theorem 3.1 by using Lemma 3.7, we see that (X, L) is a Castelnuovo variety if and only if

$$(a^2 + a - 2)h^0(X, L) = 2(2a^2 + a - 3 + (a-1)L^3 - h^0(X, L + K_X)). \tag{10}$$

On the other hand, we have $h^0(X, L) \leqslant (L^3 - 1)/a + 4$ by the definition of a. By combining this inequality with (10), we see that

$$2ah^0(X, L + K_X) \geqslant (a-1)(a-2)(L^3 - 1) \tag{11}$$

holds if (X, L) is a Castelnuovo variety.

(i) We first consider the case where $h^0(X, L + K_X) \geqslant 1$. In order to prove the sufficiency, we assume that (X, L) is a Castelnuovo variety. Note that $a \geqslant 2$ by (10). By Corollary 1.6 and Lemma 3.7, we have

$$\begin{aligned} h^0(X, L{+}K_X) &\leqslant h^0(X, L{+}K_X) + \frac{1}{4}(L^3 - 2h^0(X, L) - 2h^0(X, L{+}K_X) + 8) \\ &= \frac{1}{4}L^3 - \frac{1}{2}h^0(X, L) + \frac{1}{2}\left(\frac{1}{2}L^2.K_X + h^0(X, L) - 2\right) + 2 \\ &= \frac{1}{4}L^3 + \frac{1}{4}L^2.K_X + 1 \leqslant \frac{1}{4}L^3, \end{aligned}$$

where the last inequality follows from the fact that $-L^2.K_X$ is twice the sum of areas of faces of $\square_L$. Then the inequality (11) induces $(2a^2 - 7a +$

4)$L^3 \leqslant 2(a-1)(a-2)$. If $a \geqslant 3$, we have $\mathrm{vol}(\square_L) = L^3/6 \leqslant 2/3$, which clearly contradicts the assumption $l(\mathrm{Int}(\square_L)) \geqslant 1$. Hence we obtain $a = 2$. Then, since

$$h^0(X, L+K_X) = L^3 - 2h^0(X, L) + 7 \geqslant 2h^0(X, L+K_X) - 1$$

by (10) and Corollary 1.6, we have $h^0(X, L+K_X) = 1$ and $L^3 = 2\Delta(X, L)$. Hence (X, L) is a Mukai variety, which means that $(X, L) \simeq (X, -K_X)$ is a Fano variety. Such varieties, so-called toric Fano three-folds, have been classified into eighteen types in [2] and [17] independently (see (6) in Section 2). Hence the necessity is checked by computing. For all types, in practice, we can confirm that $a = 2$ and the equality (10) holds.

(ii) Assume that $h^0(X, L+K_X) = 0$ (equivalently, $l(\mathrm{Int}(\square_L)) = 0$). In this case, Theorem 1.2 gives

$$\begin{aligned} 12\mathrm{vol}(\square_L) &= l(\square_{2L}) + l(\mathrm{Int}(\square_{2L})) - 2l(\square_L) \\ &= l(\partial\square_{2L}) + 2l(\mathrm{Int}(\square_{2L})) - 2l(\square_L) \\ &= 4l(\partial\square_L) + 2l(\mathrm{Int}(\square_{2L})) - 2l(\square_L) - 6 \\ &= 2l(\square_L) + 2l(\mathrm{Int}(\square_{2L})) - 6, \end{aligned}$$

which implies that $L^3 = h^0(X, L) + h^0(X, 2L+K_X) - 3$. Therefore, the inequality in the statement is equivalent to $L^3 \leqslant 2h^0(X, L) - 7$. If (X, L) is a Castelnuovo variety, we have $a \leqslant 2$ by (11). In the case where $a = 1$, it is clear that $L^3 < 2h^0(X, L) - 7$ by the definition of a. On the other hand, if $a = 2$, the condition (10) is equivalent to the equality $L^3 = 2h^0(X, L) - 7$. Conversely, if $L^3 \leqslant 2h^0(X, L) - 7$, we have

$$a = \begin{cases} 1 & (L^3 < 2h^0(X, L) - 7), \\ 2 & (L^3 = 2h^0(X, L) - 7) \end{cases}$$

by definition. In either case, we can easily check that (X, L) satisfies (10). $\square$

By virtue of [13, Proposition 2.3], we can see the detailed structure of (X, L) in the case (ii) in Theorem 3.8. If $h^0(X, L+K_X) = 0$, then X is one of the following five types.

(a) a toric $\mathbb{P}^1$-bundle over a smooth toric surface.

(b) $(X, L) \simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$ $(k = 1, 2, 3)$.

(c) a toric $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ over $\mathbb{P}^1$.

(d) a blow-up of $\mathbb{P}^3$ at T_N-invariant i points $(i = 1, 2, 3, 4)$.

(e) a blow-up of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ at T_N-invariant i points $(i = 1, 2)$.

In each case, the polytope $\square_L$ is as follows. For simplicity, we put $\boldsymbol{m_1} = (1, 0, 0)$, $\boldsymbol{m_2} = (0, 1, 0)$ and $\boldsymbol{m_3} = (0, 0, 1)$, and assume $a \geqslant b \geqslant c$ in the cases (c) and (e).

(a) $\mathrm{Conv}(F_0 \cup F_1)$, where F_0 and F_1 are parallel smooth faces of distance one such that they define the same two-dimensional smooth fan.

(b) $\mathrm{Conv}(\{O, k\boldsymbol{m_1}, k\boldsymbol{m_2}, k\boldsymbol{m_3}\})$.

(c) $\mathrm{Conv}(\{O, \boldsymbol{m_1}, \boldsymbol{m_2}, (1, 0, a), (0, 1, b), (0, 0, c)\})$ or $\mathrm{Conv}(\{O, 2\boldsymbol{m_1}, 2\boldsymbol{m_2}, (2, 0, 2a - c), (0, 2, 2b - c), (0, 0, c)\})$.

(d) $\mathrm{Conv}(\{k\boldsymbol{m_1}, k\boldsymbol{m_2}, k\boldsymbol{m_3} \mid k = 1, 3\})$, $\mathrm{Conv}(\{\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}, 3\boldsymbol{m_1}, 3\boldsymbol{m_2}, 2\boldsymbol{m_3}, (1, 0, 2), (0, 1, 2)\})$, $\mathrm{Conv}(\{\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}, 3\boldsymbol{m_1}, 2\boldsymbol{m_2}, 2\boldsymbol{m_3}, (1, 2, 0), (0, 2, 1), (1, 0, 2), (0, 1, 2)\})$ or $\mathrm{Conv}(\{k\boldsymbol{m_1}, k\boldsymbol{m_2}, k\boldsymbol{m_3}, (2, 1, 0), (2, 0, 1), (1, 2, 0), (0, 2, 1), (1, 0, 2), (0, 1, 2) \mid k = 1, 2\})$.

(e) a polytope obtained from $\mathcal{Q}$ by cutting of a unit three-simplex at one of O, $2m_1$ and $2m_2$, where $\mathcal{Q}$ denotes the latter polytope in the case (c),
a polytope obtained from the above one by cutting of a unit three-simplex at one of $(2, 0, 2a - c)$, $(0, 2, 2b - c)$ and $(0, 0, c)$.

By computing, we can check $6\mathrm{vol}(\square_L) \leqslant 2l(\square_L) - 8$, that is, (X, L) is a Castelnuovo variety in the latter four cases except for $\mathrm{Conv}(\{O, m_1, m_2, m_3\})$ (in which case L is a hyperplane in $\mathbb{P}^3$). On the other hand, in the case (a), the value of $\mathrm{vol}(\square_L)$ varies greatly depending on the shapes of F_0 and F_1. See the following examples.

Example 3.9. *Let (X, L) be a three-dimensional polarized toric variety of type (a).*

(a_1) *If $F_i = \mathrm{Conv}(\{(i, 0, 0), (i, 3, 0), (i, 3, 3), (i, 0, 3)\}$, then $6\mathrm{vol}(\square_L) = 2l(\square_L) - 10$.*

(a_2) *If $F_i = \mathrm{Conv}(\{(i, 0, 0), (i, 2, 0), (i, 3, 1), (i, 3, 2), (i, 2, 3), (i, 1, 3), (i, 0, 2)\})$, then $6\mathrm{vol}(\square_L) = 2l(\square_L) - 7$.*

(a_3) *If $F_i = \mathrm{Conv}(\{(i, 1, 0), (i, 2, 0), (i, 3, 1), (i, 3, 2), (i, 2, 3), (i, 1, 3), (i, 0, 2), (i, 0, 1)\})$, then $6\mathrm{vol}(\square_L) = 2l(\square_L) - 6$.*

In the first two cases, (X, L) is a Castelnuovo variety, while the last one is not.

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RESEARCH ARTICLE

Cyclic left and torsion-theoretic spectra of modules and their relations

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Abstract. In this paper, strongly-prime submodules of a cyclic module are considered and their main properties are given. On this basis, a concept of a cyclic spectrum of a module is introduced. This spectrum is a generalization of the Rosenberg spectrum of a noncommutative ring. In addition, some natural properties of this spectrum are investigated, in particular, its functoriality is proved.

Introduction

In this paper, we consider strongly-prime ideals and modules. The concept of strongly-prime ideal was introduced by Beachy in [1]. Also in that paper the author introduced and investigated the concept of a strongly-prime module. Independently, the concept of strongly-prime module and submodule were introduced and investigated by Dauns in his paper [3]. Also, the strongly-prime modules were investigated by Algirdas Kaučikas in [2], where the author studied strongly-prime submodules of cyclic modules, but he did not study the concept of the Rosenberg spectrum for modules. The concept of pre-order on ideals was introduced by Rosenberg, and this concept is a basic one in the definition of cyclic spectrum, whose functoriality is investigated in this paper. Also we consider the notion and some properties of torsion-theoretic spectra of rings and modules. The notion and main properties of torsion-theoretic spectra were introduced by Golan in [5]. The main result of this paper is the

Key words and phrases: strongly-prime ideal, strongly-prime module, cyclic spectrum, torsion-theoretic spectrum, localizations.

proof of the fact that there exists mapping from the cyclic spectrum to the torsion-theoretic spectrum of module is continuous and surjective.

1. Strongly-prime ideals and modules

Let R be an associative ring with $1 \neq 0$. To have a reference, recall some necessary concepts from the ring theory that are related to the concept of spectrum of a noncommutative ring.

A left ideal $\mathfrak{p}$ of a ring R is called *prime*, if for every $x, y \in R$, $xRy \subseteq \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Clearly, any left prime ideal is two-sided if and only if it is prime in the classical way. Set of all two-sided prime ideals is denoted by $\mathrm{Spec}(R)$ and is called a (prime) *spectrum* of a ring R.

Recall the definition of a *pre-order* $\leqslant$ on the set of left ideals of ring R in the following way: $\mathfrak{a} \leqslant \mathfrak{b}$ for left R-ideals $\mathfrak{a}$ and $\mathfrak{b}$ if and only if there exists a finite subset V of ring R such that $(\mathfrak{a} : V) \subseteq \mathfrak{b}$. A left prime ideal $\mathfrak{p}$ of a ring R is called a *left Rosenberg point* if $(\mathfrak{p} : x) \leqslant \mathfrak{p}$ for any $x \in R \backslash \mathfrak{p}$, [8]. The set of all left Rosenberg points of a ring R is called a left *Rosenberg spectrum* of R and is denoted by $\mathrm{spec}(R)$.

The space $\mathrm{spec}(R)$ may by defined in another way: this is the set of all strongly prime left ideals. Recall that left ideal $\mathfrak{p}$ of the ring R is called *strongly-prime*, if for every $x \in R \backslash \mathfrak{p}$ there exist a finite set V of ring R such that $(\mathfrak{p} : Vx) = \{r \in R \colon rVx \subseteq \mathfrak{p}\} \subseteq \mathfrak{p}$. Clearly, every strongly-prime left ideal of a ring R is a prime left ideal and every maximal left ideal is strongly-prime. It is known that if R is noetherian, then $\mathrm{Spec}(R) \subseteq \mathrm{spec}(R)$.

Now let us recover the information about corresponding analogues of the above concepts for left modules over a ring R.

The concept of strongly-prime module can be given in two ways.

A nonzero left module M over a ring R is called *strongly-prime*, if for any nonzero $x, y \in M$ there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq R$ such that $\mathrm{Ann}_R\{a_1x, a_2x, \ldots, a_nx\} \subseteq \mathrm{Ann}_R\{y\}$, $(ra_1x = ra_2x = \cdots = ra_nx = 0)$, $r \in R$ implies $ry = 0$.

In [1], the authors introduced such a concept of strongly-prime submodule. A nonzero left module M over a ring R is called *strongly-prime*, if for any nonzero $x \in M$ there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq R$ such that $\mathrm{Ann}_R\{a_1x, a_2x, \ldots, a_nx\} = 0$. If in this concept we put $M = R$, we obtain the concept of a strongly-prime ring. Such strongly-prime rings were studied in [4].

A submodule P of some module M is called *strongly-prime*, if the quotient module M/P is a strongly-prime R-module. The set of all strongly-prime submodules of module M is called the left prime spectrum of M and is denoted by spec(M). In particular, a left ideal $\mathfrak{p} \subset R$ is called *strongly-prime* if the quotient module $R/\mathfrak{p}$ is a strongly-prime R-module. In terms of elements, left ideal $\mathfrak{p} \subset R$ is strongly-prime if for every $u \notin \mathfrak{p}$ there exists such elements $\{a_1, ..., a_n\} \subseteq R$ and a natural number $n = n(u)$ such that $ra_1u, \ldots, ra_nu \in \mathfrak{p},\ r \in R$ implies $r \in \mathfrak{p}$.

2. Preorder on the set of modules and cyclic left spectrum of module

It is easy to see that if R is a left noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(R)$, then $R/\mathfrak{p}$ is a left noetherian prime ring. This implies that it is sufficient to prove that in a left noetherian prime ring R zero ideal belongs to spec(R). But taking into account the assumption that R is a prime Goldie ring, for any $0 \neq x \in R$ any two-sided ideal RxR is essential, thus there exists a regular element $a = \sum_{i=1}^{n} r_i x s_i \in RxR$ (Using Goldie theorem). Let $V = \{r_1, \ldots, r_n\}$ and $y \in (0 : Vx)$, then $ya = \sum y r_i x s_i = 0$. Since a is regular, it follows that $y = 0$, hence $0 \in \mathrm{spec}(R)$ indeed.

Clearly, it is necessary to demonstrate how to calculate prime left ideals in an easy example. For this purpose we use the following example.

Example 1. Consider the matrix ring $R = M_2(k)$ over a (commutative) field k.

It is well known that $\mathrm{Spec}(R) = \{0\}$. Let L be a nonzero left R-ideal and $0 \neq r \in L$. Since all nonzero left ideals of the ring R are maximal, $L = Rr$. Multiplying r by the matrix units e_{11} and e_{12} resp., it easily follows that we may assume r to be of the form $r = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$, for some nonzero string $\left(\begin{smallmatrix} a & b \end{smallmatrix}\right) \in k^2$. One thus finds $L = R\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) = [a, b]_k$.

Moreover, $[a, b]_k = [a', b']_k$ if and only if there exists such $c \in k$ that $a = ca'$ and $b = cb'$. Then $\mathrm{spec}(R) = \{[a, b]_k \mid a, b \in k\}$ may be identified with the projective line P_k^1 (with "generic point" $(0) = [0, 0]_k$). (See [6])

As in [8] we introduce a preorder $\leqslant$ on the set of all left ideals by putting $K \leqslant L$ for a pair of left R-ideals L and K if and only if there exists a finite subset V of the ring R such that $(K : V) \subseteq L$.

Let us try to establish a preorder on the modules. Let R be a regular module over itself with generator 1. Then $M = R \cdot 1$ is a cyclic module.

Theorem 1. *Every cyclic module is isomorphic to the quotient module of a regular module by the annihilator of a generator* $R \cdot m = R/\mathrm{Ann}(m)$*, where* $\mathrm{Ann}(m)$ *is the left annihilator of a generator* m*.*

Consider some submodules of a cyclic module M which is presented as $Rm = R/\operatorname{Ann}(m)$ for the generator m. Let L, K be some submodules. We can represent $L = \mathfrak{A}/\operatorname{Ann}(m)$ and $K = \mathfrak{B}/\operatorname{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of a ring R. Then we define $L = \mathfrak{A}/\operatorname{Ann}(m) \leqslant K = \mathfrak{B}/\operatorname{Ann}(m)$ if and only if $\mathfrak{A} \leqslant \mathfrak{B}$ as the Rosenberg ideals. All properties are carried out. Thus the spectrum of a cyclic module is the set of all ideals that are in the spectrum of ring R.

It is well known that any module is the sum of its cyclic submodules. Then the *cyclic spectrum* of a arbitrary module M is defined as the union of all spectra of its cyclic submodules. The cyclic spectrum of module M is denoted by $\operatorname{Cspec}(M)$. Then we can define $L \leqslant K \iff \operatorname{Cspec}(L) \subseteq \operatorname{Cspec}(K)$ for all submodules of the module M and obtain a preorder on the family of such submodules.

Example 2. Let $M = \{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) | a, b \in k\}$ be module of columns with height 2 over ring $R = M_2(k)$, where k is commutative field.

This module is cyclic with generator $e = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, that is, $M = R \times \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then $\operatorname{Ann}(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \{\left(\begin{smallmatrix} 0 & b \\ 0 & d \end{smallmatrix}\right) | b, d \in k\}$, thus $M/\operatorname{Ann}(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) \cong \{\left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right) | a, c \in k\}$. The maximal submodule is $\{(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))\}$, hence cyclic spectrum consists of one point.

Lemma 1. *Let L and K be left cyclic R-modules. Then $L \leqslant K$ if and only if there exists a cyclic left R-module X, a monomorphism $X \leftarrowtail L^n$ and an epimorphism $X \twoheadrightarrow K$. In other words, there exists a diagram $(L)^n \leftarrowtail X \twoheadrightarrow K$.*

Proof. Recall the definition of preorder for submodules of a cyclic module. Let L, K be some submodules. We can represent $L = \mathfrak{A}/\operatorname{Ann}(m)$ and $K = \mathfrak{B}/\operatorname{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring R. Then we define $L = \mathfrak{A}/\operatorname{Ann}(m) \leqslant K = \mathfrak{B}/\operatorname{Ann}(m)$ iff $\mathfrak{A} \leqslant \mathfrak{B}$ as Rosenberg ideals. Thus consider two cyclic modules L and K. They are fully represented by their ideals $\mathfrak{A}$ and $\mathfrak{B}$. Than if $\mathfrak{A} \leqslant \mathfrak{B}$ by the definition, than there exists a finite subset $V \subseteq R$, such that $(\mathfrak{A} : V) \leqslant \mathfrak{B}$. Put $V = \{v_1, \ldots, v_n\}$ and let $X = R\vec{v}$ be a cyclic module, where $\vec{v} = \{v_1, \ldots, v_n\} \in (L)^n$. Than we have

$$(0 : \vec{v}) = \cap_{i=1}^{n} (\mathfrak{A} : v_i) = (\mathfrak{A} : V) \subseteq \mathfrak{B},$$

which implies that there exists a surjection $X \twoheadrightarrow K$.

On the other hand, assume that there exists a diagram $(L)^n \leftarrowtail^{\alpha} X \twoheadrightarrow^{\beta} K$. Thus we can find such element $x \in X$, that $\beta(x) = \vec{1}$. Put

$\alpha(x) = (\vec{v_1}, \dots, \vec{v_n}) \in (L)^n$, where $(\vec{v_1}, \dots, \vec{v_n}) \in \mathfrak{A}$ for some $v_i \in R$. Put $V = \{v_1, \dots, v_n\}$ and than we have

$$(\mathfrak{A} : V) = \cap_{i=1}^{n} (\mathfrak{A} : v_i) = (0 : \vec{v}) = (0 : x) \subseteq \mathfrak{B},$$

so $\mathfrak{A} \leqslant \mathfrak{B}$ and $L \leqslant K$. □

Usually from the preorder $\leqslant$ we obtain an *equivalence relation* $\sim$ as follows: $K \sim L$ iff $K \leqslant L$ and $L \leqslant K$. The equivalence class of the submodule L will be denoted by $[L]$.

Lemma 2 (11)**.** *If $\mathfrak{P}$ is a strongly-prime module, then for any element $x \in M$ the following properties are equivalent:*

(1) $x \notin \mathfrak{P}$;
(2) $(\mathfrak{P} : x) \leqslant \mathfrak{P}$;
(3) $(\mathfrak{P} : x) \in [\mathfrak{P}]$.

Lemma 3. *Let M be cyclic module. If $\mathfrak{P} \in \mathrm{Cspec}(M)$ and if L and K are submodules such that $L \cap K \leqslant \mathfrak{P}$, then either $L \leqslant \mathfrak{P}$ or $K \leqslant \mathfrak{P}$.*

Proof. Let $L \nleqslant \mathfrak{P}$ and $K \nleqslant \mathfrak{P}$ and let $L \cap K \leqslant \mathfrak{P}$. Thus, by the definition, there exist ideals $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{p}$ of the ring R, such that $L = \mathfrak{A}/\operatorname{Ann}(m)$, $K = \mathfrak{B}/\operatorname{Ann}(m)$ and $P = \mathfrak{p}/\operatorname{Ann}(m)$. Then there exists a finite subset V of the ring R, such that $(\mathfrak{A} \cap \mathfrak{B} : V) \subseteq \mathfrak{p}$. Since $\mathfrak{A} \nleqslant \mathfrak{p}$, this implies that $(\mathfrak{A} : F) \nsubseteq \mathfrak{p}$ for some finite subset F of the ring R. Thus, if we take $F = V$, we obtain the fact, that $(\mathfrak{A} : V) \nsubseteq \mathfrak{p}$. Now, if $x \in (\mathfrak{A} : V) - \mathfrak{p}$, then there exists a finite set $W \subseteq R$ with the property that $(\mathfrak{p} : Wx) \subseteq \mathfrak{p}$. Since $K \nleqslant \mathfrak{p}$, we have $\mathfrak{b} \nleqslant \mathfrak{p}$, get fact that $(\mathfrak{B} : F) \nleqslant \mathfrak{p}$ for any finite set $F \subseteq R$. In particular, this holds for $F = WxV$, thus we can find an element $y \in (\mathfrak{B} : WxV) - \mathfrak{p}$. Finally, $x \in (\mathfrak{A} : V)$ implies that $yWxV \subseteq \mathfrak{B}$, and y belongs to the set $(\mathfrak{B} : WxV)$. Certainly, $yWxV \subseteq \mathfrak{B}$, then $yWxV \subseteq \mathfrak{A} \cap \mathfrak{B}$ and $yWx \subseteq (\mathfrak{A} \cap \mathfrak{B} : V) \subseteq \mathfrak{p}$. Thus, $y \in (\mathfrak{p} : Wx) \subseteq \mathfrak{p}$, that contradicts to the fact, that $y \notin \mathfrak{p}$. □

Similarly

Lemma 4. *If $\mathfrak{P} \in \mathrm{Cspec}(R)$ and if L and K are submodules such that $LK \leqslant \mathfrak{P}$, then either $L \leqslant \mathfrak{P}$ or $K \leqslant \mathfrak{P}$.*

Recall the operation of multiplication of the submodules of cyclic module $R/\mathfrak{c}$. Any submodule of cyclic module can be viewed as the quotient-module of some left ideal by some other left ideal. Let we have two such submodules $L \cong \mathfrak{a}/\mathfrak{c}$ and $K \cong \mathfrak{b}/\mathfrak{c}$. Then $L \cdot K = \mathfrak{a}/\mathfrak{c} \cdot \mathfrak{b}/\mathfrak{c} = \mathfrak{a}\mathfrak{b}/\mathfrak{c}$.

Lemma 5. *Let $\mathfrak{P}$ and $\mathfrak{Q}$ be strongly-prime submodules of the cyclic module M. Then the following holds:*

(1) *If $\mathfrak{P} \sim \mathfrak{Q}$, then $\mathfrak{P} \cap \mathfrak{Q}$ is a strongly-prime module and $\mathfrak{P} \sim \mathfrak{P} \cap \mathfrak{Q}$;*

(2) *If $\mathfrak{P} \cap \mathfrak{Q}$ is a strongly-prime module, then either $\mathfrak{P} \subseteq \mathfrak{Q}$ or $\mathfrak{P} \supseteq \mathfrak{Q}$ or $\mathfrak{P} \sim \mathfrak{Q}$.*

Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be strongly-prime submodules of a cyclic module M. Thus, for every submodule of a cyclic module there exist ideals $\mathfrak{P} = \mathfrak{p}/\operatorname{Ann}(m)$ and $\mathfrak{Q} = \mathfrak{q}/\operatorname{Ann}(m)$, where $\mathfrak{P} \leqslant \mathfrak{Q}$ if and only if $\mathfrak{p} \leqslant \mathfrak{q}$ as Rosenberg ideals. Similarly, we can formulate the definition of equivalence relation. Thus let $\mathfrak{p} \sim \mathfrak{q}$ and $x \notin \mathfrak{p} \cap \mathfrak{q}$. Let $x \notin \mathfrak{p}$, thus there exists a finite subset $V \subseteq R$, such that $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. If $x \notin \mathfrak{q}$, then $(\mathfrak{q} : Wx) \subseteq \mathfrak{q}$ for some finite subset W of the ring R. Let $U = V \cup W$, then $(\mathfrak{p} \cap \mathfrak{q} : Ux) \subseteq \mathfrak{p} \cap \mathfrak{q}$. If $x \in \mathfrak{q}$, then $(\mathfrak{q} : Vx) = R$, hence $(\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p}$. Since $\mathfrak{p} \sim \mathfrak{q}$ by the assumption, $\mathfrak{p} \leqslant \mathfrak{q}$, and thus $(\mathfrak{p} : U) \subseteq \mathfrak{q}$ for some finite subset $U \subseteq R$, and since we may assume that $1 \in U$, we obtain

$$(\mathfrak{p} \cap \mathfrak{q} : UVx) = ((\mathfrak{p} \cap \mathfrak{q} : Vx) : U) \subseteq (\mathfrak{p} : U) \subseteq \mathfrak{q}.$$

Moreover, since $V \subseteq UV$, we also have

$$(\mathfrak{p} \cap \mathfrak{q} : UVx) \subseteq (\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p},$$

hence $(\mathfrak{p} \cap \mathfrak{q} : UVx) \subseteq \mathfrak{p} \cap \mathfrak{q}$, thus $\mathfrak{p} \cap \mathfrak{q}$ is a strongly prime ideal. Clearly $\mathfrak{p} \cap \mathfrak{q} \leqslant \mathfrak{p}$. On the other hand, since $\mathfrak{p} \leqslant \mathfrak{q}$, there exists a finite subset $V \subseteq R$, with $(\mathfrak{p} : V) \subseteq \mathfrak{q}$. We may obviously assume that $1 \in V$, thus we have $(\mathfrak{p} : V) \subseteq \mathfrak{p}$. Hence $(\mathfrak{p} : V) \subseteq \mathfrak{p} \cap \mathfrak{q}$, so $\mathfrak{p} \leqslant \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{p} \sim \mathfrak{p} \cap \mathfrak{q}$.

Let us now assume that $\mathfrak{p} \cap \mathfrak{q}$ is a strongly-prime ideal while $\mathfrak{p} \nsubseteq \mathfrak{q}$ and $\mathfrak{p} \nsupseteq \mathfrak{q}$. Sinc such a $\mathfrak{p} \nsubseteq \mathfrak{q}$ there exists an element $x \in \mathfrak{p} - \mathfrak{q}$. Thus $x \notin \mathfrak{p} \cap \mathfrak{q}$ and we may find a finite subset $V \subseteq R$ such that $(\mathfrak{p} \cap \mathfrak{q} : Vx) \subseteq \mathfrak{p} \cap \mathfrak{q}$. Since $(\mathfrak{p} : Vx) = R$, this yields $(\mathfrak{q} : Vx) \subseteq \mathfrak{p} \cap \mathfrak{q} \subseteq \mathfrak{p}$, hence $\mathfrak{p} \leqslant \mathfrak{q}$. By symmetry $\mathfrak{p} \geqslant \mathfrak{q}$, and thus $\mathfrak{p} \sim \mathfrak{q}$. □

We easy obtain the following corollary:

Corollary 1. *Let $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ be a finite family of strongly-prime modules, such that $\mathfrak{P}_1 \sim \dots \sim \mathfrak{P}_n$, then $\cap_{i=1}^{n} \mathfrak{P}_i$ is a strongly-prime module and $\mathfrak{P}_1 \sim \cap_{i=1}^{n} \mathfrak{P}_i$.*

For any left module M, it's submodule N is called *strongly two-sided*, if left annihilator of every element of N is two-sided ideal. Clearly, new submodule is two-sided. Thus the set of such submodules is not empty, because the zero submodule is strongly two-sided submodule. The sum of all strongly two-sided submodules is called the *bound* of the submodule N. In other words, the *bound of the module* is the largest submodule among those that have two-sided left annihilators for all their elements. In the case when $M = N$ we are talking about the concept of a *bound of the module*. As follows, the *bound* of the module M is the largest strongly two-sided submodule of the module M. Denote the bound of a submodule N by $b(N)$, the bound of the module M by $b(M)$.

Lemma 6. *For every strongly-prime left submodule $\mathfrak{P}$ of the module M we have $b(\mathfrak{p}) \in \operatorname{Cspec}(M)$.*

Proof. Let $x, y \in M$ by elements, such that $xRy \subseteq b(\mathfrak{P})$. Assume that $y \notin b(\mathfrak{P})$. Then there exists such an element $s \in R$ with $ys \notin \mathfrak{P}$. For every $r \in R$, $(xr)R(ys) \subseteq (xRy)s \subseteq b(\mathfrak{P})s \subseteq b(\mathfrak{P}) \subseteq \mathfrak{P}$. Hence $rx \in \mathfrak{P}$. Thus $xR \subseteq b(\mathfrak{P})$, which proves the assertion. $\square$

Lemma 7. *If $L \leqslant K$ are left R-modules, then $b(L) \subseteq b(K)$. Conversely, if R is a left noetherian fully-bounded ring, and if $b(L) \subseteq b(K)$, then $L \leqslant K$.*

Proof. Since $L \leqslant K$, there exists a representation $L = \mathfrak{A}/\operatorname{Ann}(m)$ and $K = \mathfrak{B}/\operatorname{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring R. Then $\mathfrak{A} \leqslant \mathfrak{B}$. Thus there exist a finite subset $V \subseteq R$, that $(\mathfrak{A} : V) \subseteq \mathfrak{B}$. Then for every elements $r \in b(L)$ and $s \in R$, we have $rs \in \mathfrak{A}$, therefor $r \in (\mathfrak{A} : s)$. Thus $r \in (\mathfrak{A} : V) = \cap_{s \in V} (\mathfrak{A} : s)$. Since the former is contained in $\mathfrak{B}$, we have $b(L) \subseteq K$, hence $b(L) \subseteq b(K)$.

On the other hand, if R is a left noetherian fully-bounded ring, then there exists a finite subset $V = \{v_1, \ldots, v_n\} \subseteq R$ such that $b(L) = \cap_{i=1}^{n} (\mathfrak{A} : v_i) = (\mathfrak{A} : V)$. Hence $(\mathfrak{A} : V) = b(\mathfrak{A}) \subseteq b(\mathfrak{B}) \subseteq \mathfrak{B}$, and $\mathfrak{A} \leqslant \mathfrak{B}$, therefore $L \leqslant K$. $\square$

Corollary 2. *Let L and K be left modules such that $L \sim K$, then $b(L) = b(K)$. Moreover if R is a left noetherian fully-bounded ring, then the converse is also true.*

3. Functoriality of cyclic spectrum of module

The cyclic spectrum construction can be regarded as a contravariant functor from the category of modules to the category of sets,

$$\operatorname{CSpec}\colon \operatorname{Mod} \to \operatorname{Set}.$$

A contravariant functor CSpec is a rule assigning to each module M over an associative ring R the set CSpec(M), the cyclic spectrum, i.e. the set of submodules that are related in that spectrum, and to each module homomorphism $f\colon M_1 \to M_2$ the map of sets

$$\begin{gathered}\mathrm{Cspec}(M_1) \to \mathrm{Cspec}(M_2),\\ P \mapsto f^{-1}(P).\end{gathered}$$

Consider the endomorphism ring $E = \mathrm{End}(M)$, and also consider the center of that ring, denoted by $C = \{c \in E \mid cr = rc, \forall r \in E\}$. Consider the construction of partial algebra over the ring C. It is the set Q with a reflexive, symmetric binary relation $\perp \subseteq Q \times Q$ (called commeasurability), partial addition and multiplication operations "+" and "$\cdot$", that are functions $I \to Q$, a scalar multiplication operation $E \times Q \to Q$, and elements $0, 1 \in C$, such that the following axioms are satisfied:

(1) for all $q \in Q$, $a \perp 0$ and $a \perp 1$;

(2) the relation $\perp$ is preserved by the partial binary operations: for all $q_1, q_2, q_3 \in Q$, with $q_i \perp q_j$ $(1 \leqslant i, j \leqslant 3)$ and for all $\lambda \in C$, one has $(q_1 + q_2) \perp q_3$, $(q_1 \cdot q_2) \perp q_3$ and $(\lambda q_1) \perp q_2$;

(3) if $q_i \perp q_j$ for $1 \leqslant i, j \leqslant 3$, then the values of all polynomials in q_1, q_2 and q_3 form a commutative algebra.

Commeasurability subalgebra of a partial C-algebra Q is a subset $Z \subseteq Q$ consisting of pairwise commeasurable elements that is closed under C-scalar multiplication and the partial binary operations of Q.

Given functors $K\colon \mathcal{A} \to \mathcal{B}$ and $S\colon \mathcal{A} \to \mathcal{C}$, we recall that the (right) Kan extension of S along K is a functor $L\colon \mathcal{B} \to \mathcal{C}$ with a natural transformation $\varepsilon\colon LK \to S$ such that for any other functor $F : \mathcal{B} \to \mathcal{C}$ with a natural transformation $\eta\colon FK \to S$ there is a unique natural transformation $\delta\colon F \to L$, such that $\eta = \varepsilon \circ (\delta K)$.

Theorem 2. *The functor* $\mathrm{Cspec}\colon \mathrm{Mod}^{\mathrm{op}} \to \mathrm{Set}$*, with the identity natural transformation* $\mathrm{Cspec}\,|_{\mathrm{Comm\,Mod}^{\mathrm{op}}} \to \mathrm{CSpec}$ *is the Kan extension of the functor* $\mathrm{Cspec}\colon \mathrm{Comm\,Mod}^{\mathrm{op}} \to \mathrm{Set}$ *along the embedding* $\mathrm{Comm\,Mod}^{\mathrm{op}} \subseteq \mathrm{Mod}^{\mathrm{op}}$.

Proof. Let $F\colon \mathrm{Mod} \to \mathrm{Set}$ be a contravariant functor with a fixed natural transformation $\eta\colon F|_{\mathrm{Comm\,Mod}} \to \mathrm{Spec}$. Consider functor C-Spec: $\mathrm{Comm\,Mod} \to \mathrm{CSpec}$. We need to show that there is a unique natural transformation $\delta\colon F \to \mathrm{CSpec}$, that induces $\eta\colon F|_{\mathrm{Comm\,Mod}} \to \mathrm{CSpec}$ upon a restriction to $\mathrm{Comm\,Mod} \subseteq \mathrm{Mod}$. To construct it, fix ring R and module M over it. For every submodule $N \subseteq M$ over ring R the inclusion $N \subseteq M$ given a morphisms of sets $F(M) \to F(N)$, and η provides a morphisms $\eta_N\colon F(N) \to \mathrm{CSpec}(N)$; these compose to give morphisms $F(M) \to \mathrm{CSpec}(N)$. By naturality of the morphisms involved, these maps of $F(M)$ collectively form a cone over the diagram obtained for submodules of module. By the universal property of limit, there exists a unique arrow making corresponding diagram commutative for all $N \subseteq M$.

Defined morphisms δ_M form the components of a natural transformation $\delta\colon F \to CSec$. By construction, δ induces η when restricted to Comm Mod. Uniquness of δ is guaranteed by the uniqueness of the indicated arrow used to define δ_M above. □

4. Localisations

Recall some definitions. By a *torsion-theoretic spectrum* we mean the space of all prime torsion theories (or prime Gabriel filters of a main ring) in the category of left R-modules with Zarisky topology. Recall that *prime torsion theory* $\pi \in R - tors$ is a torsion theory, for which $\pi = \chi(R/I)$ for some critical ideal I of the ring R, where $R - tors$ is class of all torsion theories of the category R-mod and $\chi(R/I)$ is the torsion theory, cogenerated by module $E(R/I)$. If τ is torsion theory of the category R-mod, then left R-module M is called *torsion free module* if and only if there exist R from M into some member of τ. Class of all torsion free modules for some τ is denote by $\mathfrak{F}_\tau$. Further information about the prime torsion theories can be fund in [5].

Remark 1. The class of all torsion theories R-tors can be partially ordered by setting $\tau \leqslant \tau'$ if and only if $\mathfrak{T}_\tau \subseteq \mathfrak{T}_{\tau'}$, namely, the class of all torsion modules of one torsion theory is contained in the class of all torsion modules of other torsion theory.

Introduce the notion of torsion-theoretic spectrum of a module M. Use the concepts of torsion-theoretic spectrum of a ring R introduced above. Introduce the concept of support of module M: $\mathrm{supp}(M) = \{\sigma | \sigma(M) \neq 0\}$. *Torsion-theoretic spectrum of module* M, $\text{R-Sp}(M)$ is defined as $\text{R-sp}(R) \cap \mathrm{supp}(M)$.

If M is a left R-module, denote by $\xi(M)$ the smallest torsion theory such that M will be a torsion module, by $\chi(M)$ the largest torsion theory, that M will be a torsion-free module. Clearly, $\mathcal{T}_{\chi(M)}$ consists of R-modules N such that $\mathrm{Hom}_R(N, E(M)) = 0$, where $E(M)$ is the injective hull of a module M.

Lemma 8. *If σ is a torsion theory and $\mathfrak{P}$ is a left Rosenberg point of a cyclic module M, then $M/\mathfrak{P}$ is either a σ-torsion module or a σ-torsion free module.*

Proof. Assume that $M/\mathfrak{P} \notin \mathcal{F}_\sigma$. If $\mathfrak{P}$ is a left Rosenberg point, then there exists ideal $\mathfrak{p}$ of a ring R such that $\mathfrak{P} = \mathfrak{p}/\operatorname{Ann}(m)$. Pick an element $0 = \bar{x} \in \sigma(R/\mathfrak{p})$. Thus, there exists a finite subset V of the ring R with $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. Obviously, $V\bar{x} \subseteq \sigma(R/\mathfrak{p})$, hence, for every element $v \in V$ there exists left ideal $L_v \in \mathcal{L}(\sigma)$ such that $L_v vx \subseteq \mathfrak{p}$. Let $L = \cap_{v \in L} L_v$, then $L \in \mathcal{L}(\sigma)$ and $LVx \subseteq \mathfrak{p}$. Hence $L \subseteq (\mathfrak{p} : Vx) \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{L}(\sigma)$, and therefore $M/\mathfrak{P}$ is σ-torsion module. □

Proposition 1. *If M is a fully bounded left noetherian module and $\mathfrak{P} \in \mathrm{Cspec}(M)$, then the torsion theory $\tau_{\mathfrak{P}} = \chi(M/\mathfrak{P})$ cogenerated by module $M/\mathfrak{P}$ is prime.*

Proof. Obviously, $\mathfrak{P} \notin \mathcal{L}(\tau_{\mathfrak{P}})$, therefore $M/\mathfrak{P}$ is a $\tau_{\mathfrak{P}}$-torsion free module. Thus, since $\chi(M/\mathfrak{P})$ is the largest torsion theory for which $M/\mathfrak{P}$ is torsion free module. We have $\chi(M/\mathfrak{P}) \leqslant \tau_{\mathfrak{P}}$. Conversely, assume that $\mathcal{L}(\chi(M/\mathfrak{P})) \nsubseteq \mathcal{L}(\tau_{\mathfrak{P}})$. Take $L \in \mathcal{L}(\chi(M/\mathfrak{P})) - \mathcal{L}(\tau_{\mathfrak{P}})$, then $L \leqslant \mathfrak{P}$. Thus, by the definition, $\mathfrak{A} \leqslant \mathfrak{p}$ for some ideals $\mathfrak{A}$ and $\mathfrak{p}$ of the ring R. Thus there exists a finite subset $U \subseteq R$ such that $\cap_{u \in U}(\mathfrak{A} : u) = (\mathfrak{A} : U) \subseteq \mathfrak{p}$. Hence $\mathfrak{p} \in \mathcal{L}(\chi(M/\mathfrak{P}))$, contradicting the definition of $\chi(M/\mathfrak{P})$. □

The previous statements imply the following result.

Theorem 3. *The mapping $\Phi\colon \mathrm{Cspec}(M) \to \text{M-sp}$, where $\Phi(\mathfrak{P}) = \chi(M/\mathfrak{P})$ is continuous and surjective.*

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Constructing R-sequencings and terraces for groups of even order

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Abstract. The problem of finding R-sequencings for abelian groups of even orders has been reduced to that of finding R*-sequencings for abelian groups of odd orders except in the case when the Sylow 2-subgroup is a non-cyclic non-elementary-abelian group of order 8. We partially address this exception, including all instances when the group has order $8t$ for t congruent to 1, 2, 3 or 4 $\pmod 7$. As much is known about which odd-order abelian groups are R*-sequenceable, we have constructions of R-sequencings for many new families of abelian groups. The construction is generalisable in several directions, leading to a wide array of new R-sequenceable and terraceable non-abelian groups of even order.

1. Introduction

There are several problems, usually arising from methods to construct combinatorial objects, that require elements of a finite group to be listed in a way that satistfies various constraints. In this paper we consider R-sequenceability and terraceability, the combinatorial consequences of which include constructions of graph decompositions, quasi-complete Latin squares and neighbor-balanced designs, among others.

First we look at R-sequenceability and R*-sequenceability; secondly we see how we can relax some of the constraints to give R*-terraces, and

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from there terraces. These results allow us to construct R-sequencings and terraces for many groups that were not previously known to possess them, edging closer to answering the longstanding questions of exactly which groups are R-sequenceable or terraceable.

We often need to consider circular lists, where the first element is taken to be to the right of the last element. As in [10], in such a case we add a hooked arrow $(a_1, a_2, \ldots, a_n \hookleftarrow)$ and calculate subscripts modulo the length of the list. Some groups we will refer to throughout the paper: Let $\mathbb{Z}_r$ be the additively written cyclic group on the symbols $\{0, 1, \ldots, r-1\}$, let D_{2r} be the dihedral groups of order $2r$ defined by

$$D_{2r} = \langle u, v : u^r = e = v^2, vu = u^{-1}v \rangle,$$

let Q_{4r} be the dicyclic group of order $4r$ defined by

$$Q_{4r} = \langle u, v : u^{2r} = e, v^2 = u^r, vu = u^{-1}v \rangle,$$

and let A_4 be the alternating group on 4 symbols.

Let G be a group of order n and let $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1} \hookleftarrow)$ be a circular arrangement of the non-identity elements of G. Define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1} \hookleftarrow)$ by $b_i = a_i^{-1} a_{i+1}$ for each i. If the elements of $\mathbf{b}$ are also all of the non-identity elements of G then $\mathbf{b}$ is a *rotational sequencing* or *R-sequencing* of G and $\mathbf{a}$ is the corresponding *directed rotational terrace* or *directed R-terrace* of G. If G has an R-sequencing then it is said to be *R-sequenceable*. If, in addition, we have that $a_2 a_{n-1} = a_1 = a_{n-1} a_2$ then $\mathbf{a}$ is a *directed R^*-terrace*, $\mathbf{b}$ is an *R^*-sequencing* and G is said to be *R^*-sequenceable*.

Inspired by a map-colouring problem of Ringel, R-sequenceability was introduced by Friedlander, Gordon and Miller in [5]. Various different definitions, all equivalent to the above, are used in the literature; see, for example, [1, 5, 9, 12, 17].

Example 1. The following is a directed R*-terrace for $\mathbb{Z}_{11}$:

$$(5, 6, 9, 3, 7, 4, 2, 1, 8, 10 \hookleftarrow).$$

Its R*-sequencing is

$$(1, 3, 5, 4, 8, 9, 10, 7, 2, 6 \hookleftarrow).$$

Much is known about the R-sequenceability of abelian groups. Friedlander, Gordon and Miller [5] conjecture that the only abelian groups that are not R-sequenceable are those with exactly one involution (which they prove cannot be R-sequenced). For even-order abelian groups the only groups for which the conjecture is open are those with non-cyclic Sylow 2-subgroups of order 8. Some new infinite families of groups whose Sylow 2-subgroups are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ are shown to be R-sequenceable in Section 3, including those of order $8t$ with an R*-sequenceable subgroup of order t with t congruent to 1, 2, 3 or 4 $\pmod 7$.

In the non-abelian case several infinite families of R-sequenceable groups are known, including dihedral and dicyclic groups D_{2n} and Q_{4n} when n is even and $n > 2$. See [11] for a recent survey of results. More are added in Section 3, including groups of the form $H_1 \times H_2 \times \cdots \times H_s \times K$, where each H_i is one of, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, D_8, $\mathbb{Z}_6 \times \mathbb{Z}_2$, D_{12} or A_4 and K is R*-sequenceable.

Again, let G be a group of order n, but now let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a linear arrangement of the elements of G. Define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ by $b_i = a_i^{-1} a_{i+1}$ for each i. If $\mathbf{b}$ contains one occurrence of each involution of G and exactly two occurrences of elements from each set

$$\{g, g^{-1} : g^2 \neq e\}$$

then $\mathbf{a}$ is a *terrace* for G and $\mathbf{b}$ is its associated *2-sequencing*.

Example 2. The following is a terrace for $\mathbb{Z}_{11}$:

$$(0, 2, 1, 8, 10, 5, 6, 9, 3, 7, 4).$$

Its 2-sequencing is

$$(2, 10, 7, 2, 6, 1, 3, 5, 4, 8).$$

Terraces were introduced by Bailey [4] as a tool for constructing quasi-complete Latin squares; similar ideas had been used earlier by Williams [21] (restricted to cyclic groups) and Gordon [7] (in the case of *directed terraces*; those whose 2-sequencings have no repeated entries, in which case they are called *sequencings*). *Bailey's Conjecture* is that all groups other than non-cyclic elementary abelian 2-groups are terraced (it is known that non-cyclic elementary abelian 2-groups cannot be terraced [4]). This was proven for abelian groups in [16] and many nonabelian groups are known to have terraces. See [11] for more details on these topics. In Section 4 we add more groups, including direct products comprised of arbitrarily many non-cyclic, non-dicyclic groups of order 12, an R*-sequenceable group and, optionally, a group of odd order.

For our constructions we are interested in direct and central factors of a group. If a group G can be written as a direct product $H \times K$ then H is a *direct factor* of G. More generally, suppose $H \trianglelefteq G$ and let

$$C_G(H) = \{g \in G : gh = hg \text{ for all } h \in H\}$$

be the centralizer of H in G. If $G = HC_G(H)$ then H is a *central factor* of G. Direct factors are also central factors but central factors are not necessarily direct factors.

In the next section we give the main construction on which all the results rely. In Section 3 we see how it can be used to produce R-sequencings and in Section 4 we consider how it can be adapted to produce terraces.

2. The construction

We present the *main construction* for a circular sequence of the non-identity elements of our target group G, which has order $n = 4mt$ and is of the form $H \times K$ with $|H| = 4m$ and $|K| = t$.

Given a circular sequence $\mathbf{a} = (a_1, a_2, \dots, a_{4m-1} \hookleftarrow)$ of the non-identity elements of H, a permutation $\sigma \in S_{4m-1}$, and a circular sequence $\mathbf{k} = (k_1, k_2, \dots, k_{t-1} \hookleftarrow)$ of the non-identity elements of K with $k_{t-1}k_2 = k_1 = k_2k_{t-1}$, we construct a sequence in $H \times K$ from $4m+1$ subsequences. Let $\mathbf{b} = (b_1, b_2, \dots, b_{4m-1} \hookleftarrow)$ be the quotients associated with $\mathbf{a}$; that is, $b_i = a_i^{-1}a_{i+1}$ for each i. Similarly, let $\ell = (\ell_1, \ell_2, \dots, \ell_{t-1} \hookleftarrow)$ be the quotients associated with $\mathbf{k}$; so $\ell_i = k_i^{-1}k_{i+1}$ for each i.

In practice, $\mathbf{a}$ will always be a directed R-terrace. In the next section $\mathbf{k}$ will be a directed R*-terrace and in Section 4 it will be a weaker object, an "R-terrace".

The first three subsequences each have distinct characteristics. These are followed by $2m-1$ that follow one pattern and then $2m-2$ that follow a slightly different one. The final subsequence has just one element. We define them in turn, noting the internal quotients that they generate as we go, and then consider the quotients generated at the joins.

Note that in calculating the quotients we make use of the condition $k_{t-1}k_2 = k_1 = k_2k_{t-1}$. In particular, we use that $k_2 = \ell_{t-1}$ and $k_{t-1}^{-1} = \ell_1$. Also, recall that for circular lists subscripts are calculated modulo the length of the list.

The first subsequence is

$$(e, k_1), (e, k_2), \dots, (e, k_{t-2})$$

which has internal quotients

$$(e,\ell_1),(e,\ell_2),\dots,(e,\ell_{t-3}).$$

The second subsequence is

$$\begin{aligned}&(a_{\sigma(1)-2m},k_{t-1}),(a_{\sigma(1)-2m+1},k_1),(a_{\sigma(1)-2m+2},k_1),\\&\qquad(a_{\sigma(1)-2m+3},k_1),\dots,(a_{\sigma(1)},k_1),(a_{\sigma(1)+1},k_1),\\&\qquad\qquad(a_{\sigma(1)+2},k_2),(a_{\sigma(1)+3},k_3),\dots,(a_{\sigma(1)+t-2},k_{t-2})\end{aligned}$$

which has internal quotients

$$\begin{aligned}&(b_{\sigma(1)-2m},\ell_{t-1}),(b_{\sigma(1)-2m+1},e),(b_{\sigma(1)-2m+2},e),\\&\qquad(b_{\sigma(1)-2m+3},e),\dots,(b_{\sigma(1)},e),(b_{\sigma(1)+1},\ell_1),\\&\qquad\qquad(b_{\sigma(1)+2},\ell_2),(b_{\sigma(1)+3},\ell_3),\dots,(b_{\sigma(1)+t-3},\ell_{t-3}).\end{aligned}$$

The third subsequence is

$$\begin{aligned}&(a_{\sigma(2)-2m+1},k_{t-1}),(a_{\sigma(1)-2m+2},e),(a_{\sigma(1)-2m+3},e),\\&\qquad(a_{\sigma(1)-2m+4},e),\dots,(a_{\sigma(2)},e),(a_{\sigma(2)+1},e),\\&\qquad\qquad(a_{\sigma(2)+2},k_2),(a_{\sigma(3)+3},k_3),\dots,(a_{\sigma(2)+t-2},k_{t-2})\end{aligned}$$

which has internal quotients

$$\begin{aligned}&(b_{\sigma(2)-2m+1},\ell_1),(b_{\sigma(2)-2m+2},e),(b_{\sigma(2)-2m+3},e),\\&\qquad(b_{\sigma(2)-2m+4},e),\dots,(b_{\sigma(2)},e),(b_{\sigma(2)+1},\ell_{t-1}),\\&\qquad\qquad(b_{\sigma(2)+2},\ell_2),(b_{\sigma(2)+3},\ell_3),\dots,(b_{\sigma(2)+t-3},\ell_{t-3}).\end{aligned}$$

For i in the range $4 \leqslant i \leqslant 2m+2$, the ith subsequence is

$$\begin{aligned}&(a_{\sigma(i-1)},k_{t-1}),(a_{\sigma(i-1)+1},e),(a_{\sigma(i-1)+2},k_2),\\&\qquad(a_{\sigma(i-1)+3},k_3),(a_{\sigma(i-1)+4},k_4),\dots,(a_{\sigma(i-1)+t-2},k_{t-2})\end{aligned}$$

which has internal quotients

$$\begin{aligned}&(b_{\sigma(i-1)},\ell_1),(b_{\sigma(i-1)+1},\ell_{t-1}),(b_{\sigma(i-1)+2},\ell_2),\\&\qquad(b_{\sigma(i-1)+3},\ell_3),(b_{\sigma(i-1)+4},\ell_4),\dots,(b_{\sigma(i-1)+t-3},\ell_{t-3}).\end{aligned}$$

For i in the range $2m+3 \leqslant i \leqslant 4m$, the ith subsequence is

$$(a_{\sigma(i-1)}, k_{t-1}), (a_{\sigma(i-1)+1}, k_1), (a_{\sigma(i-1)+2}, k_2),\\ (a_{\sigma(i-1)+3}, k_3), (a_{\sigma(i-1)+4}, k_4), \dots, (a_{\sigma(i-1)+t-2}, k_{t-2})$$

which has internal quotients

$$(b_{\sigma(i-1)}, \ell_{t-1}), (b_{\sigma(i-1)+1}, \ell_1), (b_{\sigma(i-1)+2}, \ell_2),\\ (b_{\sigma(i-1)+3}, \ell_3), (b_{\sigma(i-1)+4}, \ell_4), \dots, (b_{\sigma(i-1)+t-3}, \ell_{t-3}).$$

Note that the only difference in the structure of these subsequences compared to the previous ones is in the second coordinate of the second element, meaning that the only changes in structure in the quotients are in the second coordinates of the first and second elements. Also note that there are no subsequences of this form when $m = 1$.

The final subsequence consists of the single element (e, k_{t-1}). Of course, this gives rise to no internal quotients.

The quotients generated where the subsequences join are

$$(a_{\sigma(1)-2m}, \ell_{t-2}), (a^{-1}_{\sigma(1)+t-1} a_{\sigma(2)-2m+1}, \ell_{t-2}),\\ (a^{-1}_{\sigma(2)+t-1} a_{\sigma(3)}, \ell_{t-2}), (a^{-1}_{\sigma(3)+t-1} a_{\sigma(4)}, \ell_{t-2}), \dots,\\ (a^{-1}_{\sigma(4m-2)+t-1} a_{\sigma(4m-1)}, \ell_{t-2}), (a^{-1}_{\sigma(4m-1)+t-1}, \ell_{t-2})$$

(the fourth to the penultimate one, inclusive, are excluded when $m = 1$). Finally, (e, ℓ_{t-1}) is the quotient generated between the last subsequence and the first.

When we come to prove that the main construction gives directed R*-terraces and other similar objects, we will see that the permutation σ is responsible for lining up the subsequences in such a way that all of the properties we need are satisfied. In order to do this successfully, we also need constraints on the permutation.

Say that $\sigma \in S_{4m-1}$ is *admissible* if $\sigma(2) = \sigma(1) - 2m$ and

$$\{\sigma(3), \sigma(4), \dots, \sigma(2m+1)\} = \{\sigma(2)+1, \sigma(2)+2, \dots, \sigma(2)+2m-1\}$$

where all calculations are performed modulo $4m-1$.

For a positive integer t, the pair $\mathbf{a}$ and σ are *t-compatible* if the following $4m$ elements are distinct (i.e. are all of H):

$$a_{\sigma(1)-2m},\ a^{-1}_{\sigma(1)+t-1} a_{\sigma(2)-2m+1},\ a^{-1}_{\sigma(4m-1)+t-1}$$

and
$$a_{\sigma(i)+t-1}^{-1}a_{\sigma(i+1)}$$
for each i with $1 < i < 4m-1$.

3. R-sequencings

We can now prove the main results for R-sequencings. Theorem 1 gives the case where G has a direct factor of order a multiple of 4, which is sufficient for the abelian group case, and Theorem 4 gives the variant for a central factor.

Theorem 1. *Let $G = H \times K$ with $|H| = 4m$ and $|K| = t$. If H has an R-sequencing with a t-compatible $\sigma \in S_{4m-1}$ and K is R^*-sequenceable then G is R^*-sequenceable.*

Proof. Let $\mathbf{a}$ be the directed R-terrace of H and $\mathbf{k}$ be the directed R*-terrace of K, with the usual notation for their elements and quotients. Apply the main construction to get a circular sequence of elements in G and their quotients. We check that all elements of H appear with each element of K (with the exception that (e,e) does not appear) in each of the sequence and its quotients.

The elements that appear with k_1 in the sequence are:

$$e, a_{\sigma(1)-2m+1}, a_{\sigma(1)-2m+2}, \ldots, a_{\sigma(1)}, a_{\sigma(1)+1},$$
$$a_{\sigma(2m+2)+1}, a_{\sigma(2m+3)+1}, \ldots, a_{\sigma(4m-1)+1}.$$

When $m > 1$, the admissibility of σ implies that the two sets
$$\{\sigma(2m+2), \sigma(2m+3), \ldots, \sigma(4m-1)\}$$
and
$$\{\sigma(1)+1, \sigma(1)+2, \ldots, \sigma(1)+2m-2\}$$
are equal as each has all of the numbers from 1 to $4m-1$ except for
$$\{\sigma(2), \sigma(2)+1, \sigma(2)+2, \ldots, \sigma(2)+2m\}$$
(recall that these calculations are performed modulo $4m-1$). Applying this to the last $2m-2$ elements we see that the sequence contains all of the elements of H. When $m = 1$ we have the elements $a_{\sigma(1)-1}, a_{\sigma(1)}, a_{\sigma(1)+1}$ which are distinct.

The elements that appear with k_j, for $2 \leqslant j \leqslant t-2$ are:

$$e, a_{\sigma(1)+j}, a_{\sigma(2)+j}, \ldots, a_{\sigma(4m-1)+j}$$

which comprise all of the elements of H.

The elements that appear with k_{t-1} are:

$$a_{\sigma(1)-2m}, a_{\sigma(2)-2m+1}, a_{\sigma(3)}, a_{\sigma(4)}, \ldots, a_{\sigma(4m-1)}.$$

Applying the first clause of the admissibility definition to the first two elements we see that these are all of the non-identity elements of H.

The elements that appear with e are:

$$a_{\sigma(2)-2m+2}, a_{\sigma(2)-2m+3}, a_{\sigma(2)-2m+4}, \ldots, a_{\sigma(2)},$$
$$a_{\sigma(2)+1}, a_{\sigma(3)+1}, a_{\sigma(4)+1}, \ldots, a_{\sigma(2m+1)+1}.$$

Applying the second clause of the admissibility definition to the last $2m-1$ elements we see that these are all of the non-identity elements of H.

Turning to the sequence of quotients, the elements that appear with ℓ_1 are:

$$e, b_{\sigma(1)+1}, b_{\sigma(2)-2m+1}, b_{\sigma(3)}, b_{\sigma(4)}, \ldots, b_{\sigma(2m+1)},$$
$$b_{\sigma(2m+2)+1}, b_{\sigma(2m+3)+1}, \ldots, b_{\sigma(4m-1)+1}.$$

The admissibility of σ implies that

$$\{\sigma(2), \sigma(2m+2), \sigma(2m+3), \ldots, \sigma(4m-1)\} =$$
$$\{\sigma(1)+1, \sigma(2m+2)+1, \sigma(2m+3)+1, \ldots, \sigma(4m-1)+1\}.$$

Coupled with the first clause of the admissibility definition applied to the third element we see that the sequence contains all of the elements of H.

The elements that appear with ℓ_j, for $2 \leqslant j \leqslant t-3$ are:

$$e, b_{\sigma(1)+j}, b_{\sigma(2)+j}, \ldots, b_{\sigma(4m-1)+j}.$$

These are all of the elements of H.

The elements that appear with ℓ_{t-2} are:

$$a_{\sigma(1)-2m},\ a^{-1}_{\sigma(1)+t-1}a_{\sigma(2)-2m+1},\ a^{-1}_{\sigma(2)+t-1}a_{\sigma(3)},$$
$$a^{-1}_{\sigma(3)+t-1}a_{\sigma(4)}, \ldots,\ a^{-1}_{\sigma(4m-2)+t-1}a_{\sigma(4m-1)},\ a_{\sigma(4m-1)+t-1}.$$

As $\mathbf{a}$ and σ are t-compatible, these are all of the elements of H.

The elements that appear with ℓ_{t-1} are:

$$b_{\sigma(1)-2m}, b_{\sigma(2)+1}, b_{\sigma(3)+1}, \ldots, b_{\sigma(2m+1)+1}, \\ b_{\sigma(2m+2)}, b_{\sigma(2m+3)}, \ldots, b_{\sigma(4m-1)}, e.$$

We again use that

$$\{\sigma(2), \sigma(2m+2), \sigma(2m+3), \ldots, \sigma(4m-1)\} = \\ \{\sigma(1)+1, \sigma(2m+2)+1, \sigma(2m+3)+1, \ldots, \sigma(4m-1)+1\}$$

and the first clause of the admissibility definition, this time applied to the first element. Doing so, we see that the sequence contains all of the elements of H.

The elements that appear with e are:

$$b_{\sigma(1)-2m+1}, b_{\sigma(1)-2m+2}, \ldots, b_{\sigma(1)}, b_{\sigma(2)-2m+2}, b_{\sigma(2)-2m+3}, \ldots, b_{\sigma(2)}.$$

Using the first clause of the admissibility definition we see that these are all of the non-identity elements of H.

This shows that our sequence is a directed R-terrace. Finally, observe that the first two elements of our sequence are (e, k_1) and (e, k_2) and the last is (e, k_{t-1}). Therefore, that $\mathbf{k}$ is a directed R*-terrace of K implies that our sequence is a directed R*-terrace of G. □

Theorem 2. *Let A be an abelian group such that $A \equiv S \times T$ where S is a Sylow 2-subgroup that is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ and T has order congruent to 1, 2, 3 or 4 $\pmod 7$. If T is R*-sequenceable then A is R*-sequenceable.*

Proof. To apply Theorem 1 for each desired value of t we require an R-sequencing for $\mathbb{Z}_4 \times \mathbb{Z}_2$ along with a t-compatible σ. The following sequences and permutations do what is required:

$t \equiv 1 \pmod 7$, $\sigma = (1, 6, 7)$,

$$\mathbf{a} = (0,1), (2,1), (1,0), (2,0), (3,1), (3,0), (1,1).$$

$t \equiv 2 \pmod 7$, $\sigma = (1, 4)(2, 7, 5)$,

$$\mathbf{a} = (1,0), (2,0), (1,1), (0,1), (2,1), (3,0), (3,1).$$

$t \equiv 3 \pmod 7$, $\sigma = (1,6,7)$,

$$\mathbf{a} = (0,1),(1,0),(2,0),(1,1),(3,0),(3,1),(2,1).$$

$t \equiv 4 \pmod 7$, $\sigma = (1,7)(2,3,6)$,

$$\mathbf{a} = (1,0),(1,1),(2,0),(3,0),(2,1),(0,1),(3,1).$$

We can therefore construct the R-sequencing for A. □

A computer search has shown that there are no satisfactory $\mathbf{a}$ and σ for other values of $t \pmod 7$. We can now find R-sequencings for new families of abelian groups whose Sylow 2-subgroups are non-cyclic of order 8, the only open cases in the even-order question for abelian groups:

Corollary 1. *Let K be an abelian group with $|K| > 5$. If $|K|$ is congruent to $1, 3, 9$ or $11 \pmod{14}$ and the Sylow 3-subgroups of K are isomorphic to $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^\alpha \times \mathbb{Z}_{27}^\beta \times \mathbb{Z}_{81}^\gamma$ or $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^\alpha \times \mathbb{Z}_{27}^\beta \times \mathbb{Z}_{81}^\gamma \times \mathbb{Z}_{3^t}$, where $t > 1$ and $t \equiv \alpha + \beta \pmod 2$, then $\mathbb{Z}_4 \times \mathbb{Z}_2 \times K$ is R*-sequenceable. In particular, if $|K|$ is congruent to one of 1, 11, 17, 23, 25, 29, 31, or $37 \pmod{42}$ then $\mathbb{Z}_4 \times \mathbb{Z}_2 \times K$ is R*-sequenceable.*

Proof. The group K is R*-sequenceable [5,10,16] and hence we can apply Theorem 1 with $H = \mathbb{Z}_4 \times \mathbb{Z}_2$. The last sentence describes the cases where the Sylow 3-subgroups of K are trivial. □

Further, any progress on finding R*-sequencings for odd-order groups with Sylow 3-subgroups other than those described in Corollary 1 can now be translated directly into solving more even-order cases by the same method. For example, it is known that for any abelian 3-group T there are infinitely many R*-sequenceable abelian groups whose Sylow 3-subgroups are isomorphic to T [10].

Theorem 1 generalises the methods of [8] and [16] which are limited to the cases $H = \mathbb{Z}_2^2$ and $H = \mathbb{Z}_2^3$. However, when $m = 1$ and $t \equiv 0 \pmod 3$ it is impossible to achieve t-compatibility.

In this case, Headley [8] uses a slightly different construction which also works in our more general set-up; we will refer to this as the *Headley construction.* Given a circular sequence $\mathbf{a} = [a_1, a_2, a_3]$ of the non-identity elements of H and a circular sequence $\mathbf{k} = [k_1, k_2, \ldots, k_{t-1}]$ of the elements of K with $k_{t-1}k_2 = k_1 = k_2k_{t-1}$, we again construct a sequence in $H \times K$. The first line, the second and third line combined, and the fourth line each have t elements and the fifth line has $t-1$. Recall that subscripts in **a** are calculated modulo 3; we leave them unreduced in order to make the structure clearer:

$$\begin{array}{l}(e,k_1),(e,k_2),\ldots,(e,k_{t-2}),(a_2,k_{t-1}),(a_1,e),\\ (a_3,k_2),(a_1,k_3),(a_2,k_4),\ldots,(a_{t-6},k_{t-4}),\\ (a_1,k_{t-3}),(a_3,k_{t-2}),(a_3,k_{t-1}),(a_2,k_1),(a_1,k_1),\\ (a_3,k_1),(a_2,k_2),(a_3,k_3),\ldots,(a_{t-3},k_{t-3}),(a_2,k_{t-2}),(a_1,k_{t-1}),(a_3,e),\\ (a_2,e),(a_1,k_2),(a_2,k_3),\ldots,(a_2,k_{t-3}),(a_1,k_{t-2}),(e,k_{t-1}).\end{array}$$

Theorem 3. *Let $G = H \times K$ with $|H| = 4$ and $|K| = t$, where $t \equiv 0 \pmod 3$. If H has an R-sequencing and K is R*-sequenceable then G is R*-sequenceable.*

Proof. Headley's construction as described above gives the required directed R*-terrace when **a** is a directed R-terrace for H and **k** is a directed R*-terrace for K. Checking the sequence and the quotients is a similar (but more straightforward) process to the proof of Theorem 1. □

Following the approach of [15], we may relax the condition that H be a direct factor to it being a central factor if we add in conditions on the directed R*-terrace of its quotient group.

Theorem 4. *Let G be a group of order $4mt$ with central factor H of order $4m$. If H has a directed R-terrace* **a** *with a t-compatible $\sigma \in S_{4m-1}$ and G/H has a directed R*-terrace $[K_1, K_2, \ldots, K_{t-1}]$ such that there are elements $k_2 \in K_2$ and $k_{t-1} \in K_{t-1}$ that commute, then G is R*-sequenceable. If $m = 1$ and $t \equiv 0 \pmod 3$ then the requirement for a t-compatible permutation σ may be dropped.*

Proof. Let $k_1 = k_2k_{t-1}$ and for each i with $3 \leqslant i \leqslant t-2$ choose $k_i \in K_i \cap C_G(H)$ (as $HC_G(H) = G$, the set $K_i \cap C_G(H)$ must be non-empty). Each element of G is expressible in the form hk_i for a unique $h \in H$ and we have that $k_ih = hk_i$ for all i and all $h \in H$.

Now apply the main construction or Headley's construction as appropriate to **a** and $[k_1, k_2, \ldots, k_{t-1}]$, with elements (h, k) of G replaced with hk throughout. Thanks to the commutativity of the elements of H with the k_i, the argument goes through exactly as in Theorem 1 or 3. □

We now turn to which groups it is possible to use in the role of H in Theorems 1 and 4. We present here one possible pair of directed R-terrace **a** and permutation σ for the values of t modulo $4m-1$ for which they exist for the groups D_8, $\mathbb{Z}_6 \times \mathbb{Z}_2$, D_{12} and A_4. The group $\mathbb{Z}_2^2$ with

$t \not\equiv 0 \pmod 3$ is covered in [8] and $\mathbb{Z}_2^3$ is covered in [16]. Recall that cyclic groups of even order and Q_8 and Q_{12} are not R-sequenceable.

These directed R-terraces and permutations were found using the group-theory software package GAP [6].

The case $H = D_8$.

$t \equiv 0 \pmod 7$, $\sigma = (1,6)$, $\mathbf{a} = u, v, u^3, u^2, u^2v, uv, u^3v$.
$t \equiv 1 \pmod 7$, $\sigma = (1,6)$, $\mathbf{a} = v, u^3v, u^2, u, uv, u^3, u^2v$.
$t \equiv 2 \pmod 7$, $\sigma = (1,7)(2,3,6)$, $\mathbf{a} = u, v, u^2, u^3, u^3v, uv, u^2v$.
$t \equiv 3 \pmod 7$, $\sigma = (1,6,7)$, $\mathbf{a} = v, u^3v, u^2, u, uv, u^3, u^2v$.
$t \equiv 4 \pmod 7$, $\sigma = (1,7)(2,3,6)$, $\mathbf{a} = v, u, u^2v, u^2, u^3, uv, u^3v$.
$t \equiv 5 \pmod 7$, $\sigma = (1,5,2)(3,4)$, $\mathbf{a} = v, u^2v, u^3v, u^2, u^3, uv, u$.
$t \equiv 6 \pmod 7$, $\sigma = (1,5,2)$, $\mathbf{a} = v, u, u^2v, u^2, u^3, uv, u^3v$.

The case $H = \mathbb{Z}_6 \times \mathbb{Z}_2$.

$t \equiv 0 \pmod{11}$, $\sigma = (1,7,2)(4,5,6)(10,11)$,

$$\mathbf{a} = (2,0),(4,0),(2,1),(3,0),(5,1),(3,1),(4,1),(1,0),(1,1),(0,1),(5,0).$$

$t \equiv 1 \pmod{11}$, $\sigma = (1,6,3,2,11,8,9,7)(4,5)$,

$$\mathbf{a} = (1,0),(0,1),(2,1),(5,1),(2,0),(4,1),(3,1),(4,0),(5,0),(3,0),(1,1).$$

$t \equiv 2 \pmod{11}$, $\sigma = (1,2,7,8,6)(3,10,5,11,4,9)$,

$$\mathbf{a} = (1,0),(2,1),(2,0),(1,1),(5,1),(3,0),(5,0),(4,0),(0,1),(3,1),(4,1).$$

$t \equiv 3 \pmod{11}$, $\sigma = (1,6,5,2,11,9,8,10,7,3)$,

$$\mathbf{a} = (1,0),(4,0),(2,1),(3,1),(5,0),(0,1),(4,1),(3,0),(2,0),(5,1),(1,1).$$

$t \equiv 4 \pmod{11}$, $\sigma = (2,6,7,8)(3,9)(4,11)(5,10)$,

$$\mathbf{a} = (1,0),(5,0),(0,1),(4,0),(1,1),(3,1),(4,1),(3,0),(2,0),(2,1),(5,1).$$

$t \equiv 5 \pmod{11}$, $\sigma = (1,10,3,8,11)(2,4,9)(5,7)$,

$$\mathbf{a} = (0,1),(1,0),(2,0),(4,1),(4,0),(3,1),(2,1),(5,1),(1,1),(5,0),(3,0).$$

$t \equiv 6 \pmod{11}$, $\sigma = (1,8,9,10,11)(3,7,6)$,

$$\mathbf{a} = (1,0),(2,0),(1,1),(5,0),(4,0),(0,1),(2,1),(5,1),(3,1),(3,0),(4,1).$$

$t \equiv 7 \pmod{11}$, $\sigma = (1,7,5,2)(3,6,4)(8,11)$,

$$\mathbf{a} = (0,1),(2,0),(1,0),(3,0),(3,1),(1,1),(4,1),(5,1),(4,0),(2,1),(5,0).$$

$t \equiv 8 \pmod{11}$, $\sigma = (1,3,9,5,2,8,7,11,4)(6,10)$,

$$\mathbf{a} = (1,0),(0,1),(5,1),(3,0),(4,0),(1,1),(4,1),(2,1),(2,0),(3,1),(5,0)$$

$t \equiv 9 \pmod{11}$, $\sigma = (1,3,2,8,4,10,6)(5,9)(7,11)$,

$$\mathbf{a} = (1,0),(4,0),(2,1),(3,1),(5,0),(0,1),(4,1),(3,0),(2,0),(5,1),(1,1).$$

$t \equiv 10 \pmod{11}$, $\sigma = (1,2,7,8,6,9,3,11,5)(4,10)$,

$$\mathbf{a} = (2,0),(2,1),(0,1),(3,0),(4,0),(5,1),(4,1),(1,1),(5,0),(1,0),(3,1).$$

The case $H = D_{12}$.

$t \equiv 0 \pmod{11}$, $\sigma = (1,7,5,4,6,3,2)(8,9,11,10)$,

$$\mathbf{a} = v, u, u^4v, uv, u^5v, u^3, u^4, u^2, u^2v, u^3v, u^5.$$

$t \equiv 1 \pmod{11}$, $\sigma = (1,4,2,9,7,11,8,5,10,6)$,

$$\mathbf{a} = u^3, v, u, u^3v, uv, u^2v, u^4, u^2, u^5, u^5v, u^4v.$$

$t \equiv 2 \pmod{11}$, $\sigma = (2,6,10,5,11,3,8)(4,7,9)$,

$$\mathbf{a} = u^2, u, u^5, v, u^4v, u^3v, u^3, u^5v, u^2v, u^4, uv.$$

$t \equiv 3 \pmod{11}$, $\sigma = (1,6,2,11,7,5,3,4)(8,10)$,

$$\mathbf{a} = v, u^4v, u, u^2v, u^4, u^5, u^2, uv, u^3v, u^3, u^5v.$$

$t \equiv 4 \pmod{11}$, $\sigma = (1,5)(2,10,9,7,4,3,11,6)$,

$$\mathbf{a} = v, u^5v, u^2, u^4, u^3, u^2v, u^4v, uv, u, u^3v, u^5.$$

$t \equiv 5 \pmod{11}$, $\sigma = (1,10,3,6,8,11,2,4,9)$,

$$\mathbf{a} = v, u^2, u^5, u^5v, u^4, u^3, u^2v, u^4v, u, u^3v, uv.$$

$t \equiv 6 \pmod{11}$, $\sigma = (1,10,11,3,6,5,9,2,4,8)$,

$$\mathbf{a} = u^2, u, u^3, v, u^4, u^5v, u^5, u^3v, u^2v, u^4v, uv.$$

$t \equiv 7 \pmod{11}$, $\sigma = (2,6,9,4,8)(3,10)(5,11)$,

$$\mathbf{a} = v, u^5v, uv, u^3, u^3v, u^2, u^4v, u^2v, u^5, u^4, u.$$

$t \equiv 8 \pmod{11}$, $\sigma = (2,6,10)(3,11,4,9,5,8)$,

$$\mathbf{a} = u, v, u^2v, uv, u^4v, u^2, u^5v, u^4, u^3, u^3v, u^5.$$

$t \equiv 9 \pmod{11}$, $\sigma = (1,4,10,8,5,11,6,3,2,9,7)$,

$$\mathbf{a} = u^2, u^4, v, u^5, u^3v, u^2v, u^3, u, uv, u^4v, u^5v.$$

$t \equiv 10 \pmod{11}$, $\sigma = (2,6,8)(3,9,5,7,10)(4,11)$,

$$\mathbf{a} = u^3, v, u^4v, uv, u^5, u^5v, u, u^2v, u^3v, u^4, u^2.$$

The case $H = A_4$.

$t \equiv 0 \pmod{11}$, $\sigma = (1,6,5)(2,11,7,3,4)(8,10,9)$,

$$\mathbf{a} = (2,3,4), (1,2)(3,4), (1,3,2), (1,3)(2,4), (1,4,2), (1,3,4),$$
$$(1,2,3), (1,4,3), (1,2,4), (2,4,3), (1,4)(2,3).$$

$t \equiv 1 \pmod{11}$, $\sigma = (1,8)(4,5,6)(9,10,11)$,

$$\mathbf{a} = (2,3,4), (1,2,4), (1,4,3), (1,3,4), (1,3,2), (1,2)(3,4),$$
$$(1,2,3), (1,3)(2,4), (2,4,3), (1,4,2), (1,4)(2,3).$$

$t \equiv 2 \pmod{11}$, $\sigma = (1,3)(2,8,4,9,6,10,7)(5,11)$,

$$\mathbf{a} = (1,2)(3,4), (2,3,4), (1,3)(2,4), (1,3,4), (1,4,2), (1,2,3),$$
$$(1,4,3), (1,4)(2,3), (1,2,4), (1,3,2), (2,4,3).$$

$t \equiv 3 \pmod{11}$, $\sigma = (1,9,2,3,7,4,8,11,10)(5,6)$,

$$\mathbf{a} = (2,3,4), (2,4,3), (1,2,4), (1,4,2), (1,2,3), (1,4,3),$$
$$(1,3,4), (1,3,2), (1,4)(2,3), (1,3)(2,4), (1,2)(3,4).$$

$t \equiv 4 \pmod{11}$, $\sigma = (1,5,2,10,8,6,11,7)(3,4)$,

$$\mathbf{a} = (2,3,4),(1,2,4),(1,3,2),(1,3)(2,4),(1,2,3),(1,4,2),$$
$$(1,4,3),(1,3,4),(1,4)(2,3),(2,4,3),(1,2)(3,4).$$

$t \equiv 5 \pmod{11}$, $\sigma = (1,3)(2,8,4,9,7)(5,11)(6,10)$,

$$\mathbf{a} = (2,3,4),(1,2,3),(1,3,4),(1,2)(3,4),(2,4,3),(1,3,2),$$
$$(1,4,2),(1,4,3),(1,2,4),(1,4)(2,3),(1,3)(2,4).$$

$t \equiv 6 \pmod{11}$, $\sigma = (1,2,7,9,4,8,6,10,3)(5,11)$,

$$\mathbf{a} = (2,3,4),(1,2,4),(1,2,3),(1,3,4),(2,4,3),(1,4,3),$$
$$(1,3)(2,4),(1,3,2),(1,4)(2,3),(1,4,2),(1,2)(3,4).$$

$t \equiv 7 \pmod{11}$, $\sigma = (1,8,11)(3,7,5)(4,6)$,

$$\mathbf{a} = (2,3,4),(1,2,4),(1,4,2),(1,2)(3,4),(2,4,3),(1,3)(2,4),$$
$$(1,2,3),(1,3,4),(1,4,3),(1,4)(2,3),(1,3,2).$$

$t \equiv 8 \pmod{11}$, $\sigma = (1,5,3)(2,10,6,11,8,9,7)$,

$$\mathbf{a} = (2,3,4),(2,4,3),(1,2,4),(1,4,2),(1,3)(2,4),(1,3,4),$$
$$(1,2,3),(1,4,3),(1,3,2),(1,4)(2,3),(1,2)(3,4).$$

$t \equiv 9 \pmod{11}$, $\sigma = (1,6,2,11,10,7,4,3)$,

$$\mathbf{a} = (1,2)(3,4),(2,3,4),(1,3,2),(1,4)(2,3),(1,2,3),(1,4,3),$$
$$(2,4,3),(1,2,4),(1,3,4),(1,4,2),(1,3)(2,4).$$

$t \equiv 10 \pmod{11}$, $\sigma = (1,7,3,4,2)(9,10)$,

$$\mathbf{a} = (1,2)(3,4),(2,3,4),(1,3)(2,4),(1,3,4),(2,4,3),(1,2,3),$$
$$(1,4,3),(1,4)(2,3),(1,3,2),(1,2,4),(1,4,2).$$

These directed R-terraces along with Theorems 1, 3 and 4 allow us to show the R-sequenceability of many new groups. The following result of Wang and Leonard extends the scope further still:

Theorem 5. [19] *If K is an R^*-sequenceable group of even order and N is a nilpotent group of odd order then $K \times N$ is R^*-sequenceable.*

Proof. Follows immediately from Corollaries 2 and 6 of [19]. □

Theorem 6. *Groups of the form $H_1 \times H_2 \times \cdots \times H_s \times K \times N$, where each H_i is one of the groups $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, D_8, $\mathbb{Z}_6 \times \mathbb{Z}_2$, D_{12} or A_4, the group K is R^*-sequenceable and N is a nilpotent group of odd order, are R^*-sequenceable.*

Proof. Repeatedly apply Theorem 1 and/or 3 to construct a directed R*-terrace for $H_1 \times H_2 \times \cdots \times H_s \times K$. Apply Theorem 5 to complete the proof. □

Groups that are known to be R*-sequenceable include: abelian groups with non-trivial non-cyclic Sylow 2-subgroups of orders other than 8 [5, 8]; abelian groups of odd order or with Sylow 2-subgroups isomorphic to $\mathbb{Z}_2^3$ whose Sylow 3-subgroups are isomorphic to $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^\alpha \times \mathbb{Z}_{27}^\beta$ or $\mathbb{Z}_3^\alpha \times \mathbb{Z}_9^{\alpha+1} \times \mathbb{Z}_{27}^\beta$ [5, 16]; the abelian groups described in Corollary 1; nonabelian groups whose order is the product of two odd primes [20]; dihedral groups of order $4k$, unless $k < 4$ or $k \equiv 0$ or $1 \pmod 6$ [18]; and dicyclic groups of order congruent to 16 or 32 $\pmod{48}$ [18].

4. Terraces

In this section we follow the approach of [12, 15] whereby we relax the requirement of directedness in the R*-terrace for K and see that R-terraces emerge from the construction. Further, if these R-terraces have an additional property then we may turn them into terraces.

Let G be a group of order n and let $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1}, \hookleftarrow)$ be a circular arrangement of the non-identity elements of G. Define $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1} \hookleftarrow)$ by $b_i = a_i^{-1}a_{i+1}$ for each i. If $\mathbf{b}$ contains one occurrence of each involution of G and exactly two occurrences of elements from each set $\{g, g^{-1} : g^2 \neq e\}$ then $\mathbf{a}$ is a *rotational terrace* or *R-terrace* and $\mathbf{b}$ is a *rotational 2-sequencing* or *R-2-sequencing.*

As in the directed definition, if $a_{i-1}a_{i+1} = a_i = a_{i+1}a_{i-1}$ then $\mathbf{a}$ is an *R*-terrace* and $\mathbf{b}$ is an *R*-2-sequencing.* By re-indexing if necessary, we may assume this value of i in an R*-terrace is 1, in which case the R*-terrace is *standard.*

Given a standard R*-terrace with this notation, suppose there is a value r such that $b_r = a_{r+1}^{-1}$. Then r is a *right match-point* of $\mathbf{b}$. We will require standard R*-terraces whose associated R*-2-sequencings have a right match-point r with $2 \leqslant r \leqslant n-3$. An equivalent object is an extendable terrace: a basic terrace $(e, a_2, \dots, a_n)$ is *extendable* if $a_n = a_2^2$ and $a_{j-1}a_{j+1} = a_j = a_{j+1}a_{j-1}$ for some j with $5 \leqslant j < n$. The circular sequence $(a_1, a_2, \dots, a_{n-1} \hookleftarrow)$ is a standard R*-terrace whose R*-2-sequencing has a right match-point r where $2 \leqslant r \leqslant n-3$ if and only if

$$(e, a_{r+1}, a_{r+2}, \dots, a_{n-1}, a_1, a_2, \dots, a_r)$$

is an extendable terrace [15]. This relationship is illustrated in Examples 1 and 2: the standard R*-terrace in Example 1 has 5 as a match-point which can be used to give the extendable terrace in Example 2.

We can now give the main theorem for constructing terraces.

Theorem 7. *Let $G = H \times K$ with $|H| = 4m$ and $|K| = t$. If H has a directed R-terrace with a t-compatible $\sigma \in S_{4m-1}$ and K has an extendable terrace then G has an extendable terrace.*

Proof. First, from the extendable terrace, construct an R*-terrace $\mathbf{k}$ for K that has a right match-point in position r, where $2 \leqslant r \leqslant t-3$. Let $\mathbf{a}$ be the directed R-terrace and apply the main construction to $\mathbf{a}$ and $\mathbf{k}$. We claim that this gives an R*-terrace for G that has a right match-point in position r and hence that G has an extendable terrace.

Compared to the proof of Theorem 1, all that changes is that some non-involutions $g \in K$ may appear as ℓ_i and ℓ_j, with $i \neq j$, in the R*-2-sequencing of K and, if this is the case for a given g, then g^{-1} does not appear in the R*-2-sequencing of K.

The consequence for our purported R-2-sequencing is that, for any given non-involution $h \in H$, rather than having each of the four elements of the form $(h^{\pm 1}, g^{\pm 1})$ once, we have (h, g) and (h^{-1}, g) twice and neither (h, g^{-1}) nor (h^{-1}, g^{-1}) appears. This does not break the constraints of being an R-2-sequencing. Similarly, if $h \in H$ is an involution then we have (h, g) twice and (h, g^{-1}) does not appear.

Finally, as the first $t-2$ elements of the R*-terrace are

$$(e, k_1), (e, k_2), \dots, (e, k_{t-2}),$$

the right match-point at position r of the R*-2-sequencing is maintained. $\square$

As with the R-sequencing result, we have an analogue for central factors:

Theorem 8. *Let G be a group of order $4mt$ with central factor H of order $4m$. If H has a directed R-terrace* $\mathbf{a}$ *with a t-compatible $\sigma \in S_{4m-1}$ and G/H has an extendable terrace $(H, K_2, K_3, \ldots, K_t)$ with j as the position of the element that is the product of its neighbours and such that there are elements $k_{j-1} \in K_{j-1}$ and $k_{j+1} \in K_{j+1}$ that commute, then G has an extendable terrace. If $m = 1$ and $t \equiv 0 \pmod 3$ then the requirement for a t-compatible permutation σ may be dropped.*

Proof. Turn the extendable terrace for G/H into its equivalent standard R*-terrace and the argument then mirrors that of Theorem 4. It is possible to ensure that the match-point condition is met by careful choice of the k_i. As in Theorem 7, the standard R*-terrace for G that emerges has an equivalent extendable terrace. □

These results allow the construction of terraces for many infinite families of groups for which terraces were not previously known, even more so in conjunction with this powerful result for constructing new terraces from existing ones:

Theorem 9. [2, 3] *Let G be a group with a normal subgroup N. If N has odd order and G/N has a terrace then G has a terrace. If N has odd index and N has a terrace then G has a terrace.*

For example:

Corollary 2. *Let G be of the form $H_1 \times H_2 \times \cdots \times H_s \times K \times N$, where each H_i is one of $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, D_8, $\mathbb{Z}_6 \times \mathbb{Z}_2$, D_{12} or A_4, the group K has an extendable terrace, and $|N|$ is odd. Then $H_1 \times H_2 \times \cdots \times H_s \times K$ has an extendable terrace and G has a terrace.*

Proof. To show that $H_1 \times H_2 \times \cdots \times H_s \times K$ has an extendable terrace, repeatedly apply Theorem 7 or, if $H_i \cong \mathbb{Z}_2^2$, Theorem 8 when necessary. Use Theorem 9 to complete the proof. □

Groups that are known to have an extendable terrace include: $\mathbb{Z}_s$, where $s \geqslant 7$ and s is not twice an odd number [12, 14]; abelian 2-groups of order at least 8 that are not elementary abelian [12, 14]; $\mathbb{Z}_2^s \times \mathbb{Z}_p$ where $s \geqslant 2$ and p is an odd prime [12–15]; non-abelian groups of order 12, 16 or 20 [15]; D_{8s} for $s > 1$ [15]; and these two additional families of groups of orders $8s$, with $s > 1$ [15] (the first are the *semidihedral groups*; the second don't seem to have an accepted name in the literature):

$$\begin{aligned} SD_{8s} &= \langle u, v : u^{4s} = e = v^2,\ vu = u^{2s-1}v \rangle, \\ M_{8s} &= \langle u, v : u^{4s} = e = v^2,\ vu = u^{2s+1}v \rangle. \end{aligned}$$

In [15] it is suggested that groups with many involutions might be the most promising place to look for a counterexamples to Bailey's Conjecture. Many new such groups are now known to be terraced; for example, $\mathbb{Z}_2^r \times D_8^s \times D_{12}^t$ provided that $r \neq 1$ and $t > 0$.

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Algebra and Discrete Mathematics
Volume **20** (2015). Number 2, pp. 317–324 RESEARCH ARTICLE

Quasi-Euclidean duo rings with elementary reduction of matrices

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Abstract. We establish necessary and sufficient conditions under which a class of quasi-Euclidean duo rings coincides with a class of rings with elementary reduction of matrices. We prove that a Bezout duo ring with stable range 1 is a ring with elementary reduction of matrices. It is proved that a semiexchange quasi-duo Bezout ring is a ring with elementary reduction of matrices iff it is a duo ring.

Introduction

The problem of the factorization of square matrices over rings was considered in the mid 1960's and was formulated in this way: (P) to characterize the integral domain R, under which an arbitrary invertible square matrix is a product of elementary matrices. *An elementary matrix* with elements of the ring R is understood as a square matrix of one of the following types:

(1) a diagonal matrix with invertible elements on the main diagonal;

(2) a matrix that differs from the unit matrix by the presence of any nonzero element outside the main diagonal.

If R is a field, according to the Gauss approach, arbitrary invertible matrix under it may be decomposed into the product of elementary matrices and the structure of the general linear group $GL_n(R)$ is thoroughly

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studied (see [5]). The investigation of the integral domain (particularly non-commutative) that satisfies the conditions of the problem (P), started in 1966 with the Cohn's fundamental work [3], who defined these domains as the *general Euclidean* (*GE-rings*, for shortness), due to the fact that Euclidean domains were the first well-known examples of GE-rings and are not the fields. Cohn's work became the reason of the thorough and detailed study of the general and special linear group's structure under different rings. In 1996 Zabavsky B. V. [13] analyzed the rings with the elementary reduction of matrices and set up a problem of investigation of such rings.

A ring R is called *a ring with elementary reduction of matrices* [13] in case of an arbitrary matrix over R possesses elementary reduction, i.e. for an arbitrary matrix A over the ring R there exist such elementary matrices over R , $P_1, \ldots, P_k, Q_1, \ldots, Q_s$ of respectful size that

$$P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(\varepsilon_1, \ldots, \varepsilon_r, 0, \ldots, 0), \tag{1}$$

where $R\varepsilon_{i+1}R \subseteq R\varepsilon_i \cap \varepsilon_i R$ for any $i = 1, \ldots, r-1$.

Since in 1949 Kaplansky [7] established the investigation of the elementary divisors rings (i. e. the rings under which arbitrary matrix resolves itself to the accepted diagonality (1) into invertible matrices of the appropriate sizes), so the problem of finding the necessary and sufficient conditions, whereby a given ring is a ring with elementary reduction of matrices is closely related to the problem of arbitrary square invertible matrices decomposition into the product of elementary ones.

Main results

A ring R is understood as an associative ring with nonzero unit element and $U(R)$ is understood as the group of invertible elements of a ring R. A group generated by elementary matrices of type (2) of order n is called a group of elementary matrices $E_n(R)$, while $GE_n(R)$ is understood as a group of elementary matrices of n order over R.

A right (left) Bezout ring is a ring in which every finitely generated right (left) ideal is principal. *A Bezout ring* [6] is a ring which is both right and left Bezout ring. A ring R is called right Hermite if, for any row (a, b), $a, b \in R$, there exists an invertible matrix P of order 2 over R so that $(a, b)P = (d, 0)$, where $d \in R$. Left Hermite rings can be defined by analogy. If the ring is left and right Hermite, then it is called *Hermite ring* [7]. A ring is said to be a right (left) duo ring if any right (left) ideal of this ring is a 2-sided ideal. If the ring is both left and right duo ring, then it is called *duo ring* [4].

A ring R is called a right (left) quasi-duo ring, if any right (left) maximal ideal in R is a two-sided ideal. If the ring is both left and right quasi-duo ring, then it is called *quasi-duo ring* [11].

We say that a ring R has quasi-algorithm, if the function $\varphi : R \times R \to W$ (where W is some ordinal) is given so that for any $a, b \in R$ $(b \neq 0)$ one can find elements $q, r \in R$ such as $a = bq + r$ and $\varphi(b, r) < \varphi(a, b)$. If one can find some quasi-algorithm on R then the ring R is called *quasi-Euclidean* [1].

A ring R is said to have *stable range 1*, if for any $a, b \in R$ satisfying $aR + bR = R$, there exists such $t \in R$ that $a + bt$ is an invertible element in R [12]. A ring R is said to have *idempotent stable range 1*, if for any $a, b \in R$ satisfying $aR + bR = R$, there exists such idempotent $e \in R$ that $a + be$ is invertible [2].

A ring R is called *an exchange ring* if for any element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$ [9].

Proposition 1. *A right quasi-Euclidean ring is right Hermite ring.*

Proof. By Theorem 8 [1] for any elements $a, b \in R$, $a \neq 0$, there exists a finite divisible chain, that

$$b = aq_1 + r_1,\ a = r_1 q_2 + r_2,\ \dots,\ r_{n-2} = r_{n-1} q_n + r_n,\ r_{n-1} = r_n q_{n+1}.$$

Then

$$(a, b) \begin{pmatrix} 1 & -q_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -q_3 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & -q_{n+1} \\ 0 & 1 \end{pmatrix} = (r_n, 0).$$

So for any elements $a, b \in R$ there exists matrix $P \in GE_2(R)$, that $(a, b)P = (r_n, 0)$. Therefore, R is right Hermite ring. □

Lemma 1. *Let R be a duo ring. Then for any matrix $E \in E_n(R)$, there exists such matrix $E' \in E_n(R)$, that*

$$diag(d, \dots, d) \cdot E = E' \cdot diag(d, \dots, d).$$

Proof. The proof follows from the fact, that if R is a duo ring, for any element $a \in R$, there exists such an element $a' \in R$, that $da = a'd$. □

Proposition 2. *A quasi-Euclidean duo ring R is a ring with elementary reduction of matrices if and only if a matrix of the form*

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R),$$

where $aR + bR + cR = R$ admits elementary reduction.

Proof. The necessity is obvious. To prove the sufficiency, we consider the case where $aR + bR + cR = dR$, $d \notin \mathcal{U}(R)$. By virtue of Proposition 1, there exist such elements $a_1, b_1, c_1 \in R$, that

$$a = da_1,\ b = db_1,\ c = dc_1 \quad \text{and} \quad a_1R + b_1R + c_1R = R.$$

Then

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}.$$

Since the matrix $A = \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}$ admits elementary reduction, there exist such elementary matrices $P_1, \ldots, P_k, Q_1, \ldots, Q_s \in E_2(R)$ of respectful size, that

$$P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(\varepsilon_1, \varepsilon_2), \tag{2}$$

where $\varepsilon_1 R \cap R\varepsilon_1 \supseteq R\varepsilon_2 R$.

Multiply equation (2) by the matrix $diag(d, d)$ we obtain:

$$diag(d, d) \cdot P_1 \cdots P_k \cdot A \cdot Q_1 \cdots Q_s = diag(d\varepsilon_1, d\varepsilon_2).$$

According to Lemma 1 there exist such matrices $P_1', \ldots, P_k' \in E_2(R)$, that

$$P_1' \cdots P_k' \cdot diag(d, d) \cdot A \cdot Q_1 \cdots Q_s = diag(d\varepsilon_1, d\varepsilon_2).$$

Therefore, the matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ also admits elementary reduction.

The proof is completed by induction of the order of matrices. $\square$

Theorem 1. *Let R is quasi-Euclidean duo ring in which any noninvertible element belongs to most countable set of maximal ideals of R. Then R is a ring with elementary reduction of matrices.*

According to the Proposition 2, the proof of this theorem repeats the proof given in [14] in the case of commutative rings, therefore we do not give it.

Corollary 1. *A quasi-Euclidean duo ring in which the set of maximal ideals is at most countable is a ring with elementary reduction of matrices.*

Theorem 2. *A Bezout duo ring with stable range 1 is a ring with elementary reduction of matrices.*

Proof. By Theorem 2 [15] ring R is a right Hermite ring. It remains to be proven that the ring R is a ring with elementary reduction of matrices. According to the Proposition 1, it is sufficient to prove theorem for matrices

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M_2(R),$$

where $aR + bR + cR = dR$ for any element $d \in R$. Obviously, there exist such elements $a_1, b_1, c_1 \in R$, that

$$a = da_1, b = db_1, c = dc_1 \quad \text{and} \quad a_1R + b_1R + c_1R = R.$$

Then

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}.$$

Since R is a right Bezout ring of stable range 1, then for elements $a_1, b_1, c_1 \in R$ there exists such elements $s, t \in R$, that $a_1s + b_1 + c_1t = u \in U(R)$. Multiplying the last equality from the left side on the element d we get that $da_1s + db_1 + dc_1t = as + b + ct = du$. Since R is a duo ring, there exists such element $s' \in R$, that $as = s'a$, then $s'a + b + ct = du$. Considering the matrices $P_1, P_2, P_3 \in GE_2(R)$

$$P_1 = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

We have

$$P_1P_2AP_3 = \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & s' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & c \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} s'a + b + ct & c \\ a & 0 \end{pmatrix} = \begin{pmatrix} du & dc_1 \\ da_1 & 0 \end{pmatrix} = B$$

Then the matrix B and, hence, the matrix A obviously admits elementary reduction. Therefore, R is a ring with elementary reduction of matrices. $\square$

Corollary 2. *A semilocal quasi-Euclidean duo ring is a ring with elementary reduction of matrices.*

Theorem 3. *Let R is Hermite duo ring and, for any $a, b \in R$ ($b \neq 0$), there exists such $s \in R$, that $mspec(s) = mspec(a) \backslash mspec(b)$. Then R is a ring with elementary reduction of matrices.*

Proof. Let $a, b \in R$ be such elements, that $aR + bR = dR$, where $d \in R$. Note that the case $d \notin U(R)$ is irrelevant. Otherwise, there exist elements $a_1, b_1 \in R$, such that $a = da_1$, $b = db_1$ and $a_1R + b_1R = R$. As a result, we obtain $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$. By the fact that R is a duo ring, which would also imply the existence of mutually prime elements, thus there exists such $a_1', b_1' \in R$, that

$$(a, b) = (a_1', b_1') \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}.$$

Therefore, it is sufficient to prove the statement of the theorem for mutually prime elements. Thus, assume that $aR + bR = R$. It is obvious that

$$mspec(a) \cap mspec(b) = \{0\}. \tag{3}$$

Using the statement of the theorem, there exists element $s \in R$, which belongs to all maximal ideals of the ring R, except for maximal ideals of the set $mspec(a)$, that is, we have the following equality

$$mspec(s) = mspec(0) \backslash mspec(a).$$

It is obvious that

$$mspec(s) \cap mspec(a) = \{0\}. \tag{4}$$

Let us consider the element $a + bs \in R$ and assume that $a + bs \in \mathcal{M}$, where $\mathcal{M}$ is a maximal ideal of the ring R. There are the following possible cases:

1) $a \in \mathcal{M}$ and $b \in \mathcal{M}$ contradicts with the condition (3).
2) $a \in \mathcal{M}$ and $s \in \mathcal{M}$ contradicts with the condition (4).

Therefore, our initial assumption was incorrect and the condition $a + bs \in \mathcal{U}(R)$ is valid. It also implies that R is a ring of a stable range 1. Due to Theorem 2 a duo ring R is a ring with an elementary reduction of matrices. $\square$

We will denote the Jacobson radical of a ring R by $J(R)$. A ring R is said to be a semiexchange ring [8] if the factor ring $R/J(R)$ is an exchange ring.

Theorem 4. *A semiexchange Bezout duo ring is a ring with elementary reduction of matrices.*

Proof. Let R be a semiexchange Bezout duo ring. Since all idempotent elements of a duo ring belong to its center, due to Theorem 12 [2], then $\bar{R} = R/J(R)$ is a ring with idempotent stable range 1. Since a stable range 1 lifts modulo $J(R)$, we obtain the result that a ring R also has a stable range 1. Then, according to Theorem 2, R is a ring with elementary reduction of matrices. $\square$

Theorem 5. *Let R be semiexchange quasi-duo Bezout ring. Then R is a ring with elementary reduction of matrices if and only if it is a duo ring.*

Proof. As it was mentioned at the beginning and is proven in [10] being a quasi-duo elementary divisor ring implies the duo ring condition, so the necessity is proven.

Sufficiency follows from Theorem 4. $\square$

Corollary 3. *A distributive semiexchange Bezout ring is a ring with elementary reduction of matrices if and only if it is a duo ring.*

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RESEARCH ARTICLE

A morphic ring of neat range one

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Abstract. We show that a commutative ring R has neat range one if and only if every unit modulo principal ideal of a ring lifts to a neat element. We also show that a commutative morphic ring R has a neat range one if and only if for any elements $a, b \in R$ such that $aR = bR$ there exist neat elements $s, t \in R$ such that $bs = c$, $ct = b$. Examples of morphic rings of neat range one are given.

The notion of principal ideals being uniquely generated first appeared in Kaplansky's classic paper [4]. He had raised the question of when a ring R satisfies the property of being uniquely generated. He remarked that for commutative rings, the property holds for principal ideal rings and artinian rings. In the case of a left quasi morphic ring the property of being uniquely generated is equivalent to that a ring has stable range one. The concept of a neat range one ring is introduced by the first named author in [9]. In this paper we show that for a commutative morphic ring the condition of a neat range one is equivalent to the a uniquely generated weak condition relation with a neat elements.

Throughout this paper we assume that R is a commutative ring with an identity element. To make the paper almost self-contained, we recall basic definitions and some results used later. We recall that:

(i) R is a *Bezout ring*, if each finitely generated ideal of R is principal, see [10].

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(ii) Two rectangular matrices A and B are *equivalent* if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$, see [10].

(iii) The ring R is *Hermite* if every rectangular matrix A over R is equivalent to an upper or a lower triangular matrix, see [10].

(iv) R is *an elementary divisor ring* if every square n by n matrix A with coefficients in R can be converted to a diagonal matrix $diag(a_{11}, \ldots, a_{nn})$ such that every a_{ii} divides $a_{i+1,i+1}$, see [4].

(v) a ring R is *a ring of stable range one*, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $a + bt$ is a unit of R, see Bass [1].

(vi) An element $a \in R$ is defined to be a *clean* element of R, if a can be written as the sum of a unit and an idempotent. The ring R is defined to be a *clean ring*, if every element of R is clean, see [10].

(vii) An element $a \in R$ is defined to be a *neat* element of R, if R/aR is a clean ring. The ring R is defined to be a *neat ring*, if every elements in a ring R are neat, see [6].

(viii) R is defined to be of *neat range one*, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element of R, see[9].

(ix) An element $a \in R$ is defined to be *morphic*, if $\mathrm{Ann}(a) \cong R/aR$, where $\mathrm{Ann}(a)$ denotes the annihilator of a in R. The ring R is defined to be morphic, if every its element is morphic, see [7].

We recall from [4] that every elementary divisior ring R is both a Bezout ring and a Hermite ring. Note also that unity elements of R are neat elements and, hence, every ring of stable range one is a ring of neat range one.

In our next result we need the following definition.

Definition 1. (a) An element $a \in R$ is a unit modulo a principal ideal cR if $ax - 1 \in cR$ for some $x \in R$.

(b) A unit $a \in R$ modulo a principal ideal cR lifts to a neat element, if $a - t \in bR$ for a neat element $t \in R$.

Proposition 1. *Let R be a commutative ring. Then the following are equivalent:*

1) R has a neat rang one;

2) Every unit lifts to a neat element modulo every principal ideal.

Proof. We assume that R has neat range one. Let $a, b, c \in R$ be such that $ab - 1 \in cR$, i.e. b is a unit modulo the principal ideal cR. We show that there exists a neat element $t \in R$ such that $b - t \in R$.

Let $x \in R$ be such that $ab - 1 = cx$. Then $ab - cx = 1$. Since R has neat range one, there exists an element $s \in R$ and a neat element $t \in R$ such that $b - cs = t$. Therefore $b - t \in cR$ where t is a neat element in R.

To prove the implication (2) $\Rightarrow$ (1), assume that every unity of R lifts to a neat element modulo every principal ideal. We show that R has a neat range one. Let $a, b, c \in R$ such that $ab + cd = 1$. Then $ab - 1 \in cR$. Therefore, by our hypothesis there exists a neat element $t \in R$ such that $b - t \in cR$. Thus $b - t = cx$ for some $x \in R$ i.e. $b + c(-x) = t$ is a neat element i.e. R has neat range one. □

Proposition 2. *A morphic ring is a ring of neat range one if and only if for any pair of elements $a, b \in R$ such that $aR = bR$ there are neat elements $s, t \in R$ such that $as = b$ and $a = bt$.*

Proof. In view of Proposition 1 it suffices to show that every unit lifts to a neat element modulo every principal ideal in R.

Let x be a unit that lifts to a neat element modulo the principal ideal yR, i.e there exists $z \in R$ such that $zx - 1 \in yR$. We would like to show that there exists a neat elements $t \in R$ such that $x - t \in yR$. Since R is a morphic, there exists a, b such that $yR = \mathrm{Ann}(a)$ and $xaR = \mathrm{Ann}(b)$.

Obviously, $xR \subset \mathrm{Ann}(ab)$ and $yR \subseteq \mathrm{Ann}(ab)$.

Since $zx-1 \in yR$, we have $xR+yR = R$ and $xR+yR = \mathrm{Ann}(ab)$. Then $ab = 0$ and $a \in \mathrm{Ann}(b)$. Also we have $\mathrm{Ann}(b) = xaR \subseteq aR$. Therefore $\mathrm{Ann}(b) = xaR = aR$. Under the assumption on the ring there exists a neat element $t \in R$ such that $xa = ta$. This implies that $(x - t)a = 0$. We have $x - t \in \mathrm{Ann}(a) = yR$. Thus from Proposition 1, the R has neat range one.

Let $aR = bR$. Then there exist $x, y \in R$ such that $a = bx$, $b = ay$. Therefore $b = bxy$, $b(1 - xy) = 0$. This shows that $1 - xy \in \mathrm{Ann}(b)$.

Now $xy + (1 - xy) = 1$ where $xy \in xR$ and $1 - xy \in (1 - xy)R$. Therefore $xR + (1 - xy)R = R$. Since R is assumed to have neat range one, there exists $s \in R$ such that $x + (1 - xt)s = t$ is a neat element in R. Since $1 - xy \in \mathrm{Ann}(b)$, we have $(x + (1 - xy)s)b = tb$, $xb = tb$ where $xb = a$. Thus $a = tb$ for some neat element $t \in R$. Similarly we have $b = sa$, for some neat element $s \in R$, which completes the proof. □

Theorem 1. *If R is an elementary divisor ring, then R is a ring of neat range one.*

Proof. By [8] for any elements $a, b, c \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $s = a + bt = uv$, where $uR + cR = R$,

$vR+(1-c)R$, $uR+vR=R$. Let $\overline{u}=u+sR$, $\overline{v}=v+sR$. Since $uR+vR=R$, one has $ux+vy=1$ and $\overline{u}^2\overline{x}=\overline{u}$, $\overline{v}^2\overline{y}=\overline{v}$, where $\overline{x}=x+sR$, $\overline{y}=y+sR$. Let $\overline{vy}=\overline{e}$, obviously $\overline{e}^2=\overline{e}$ and $\overline{1}-\overline{e}=\overline{ux}$. Since $uR+cR=R$, we obtain $\overline{ce}\overline{\beta}=\overline{e}$, for some element $\overline{\beta}\in R/sR$. Similarly, $(\overline{1}-\overline{c})\overline{\alpha}(\overline{1}-\overline{e})=\overline{1}-\overline{e}$ for some element $\overline{\alpha}\in R/sR$. We proved that for any element $\overline{c}=c+sR$ there exists an idempotent $\overline{e}$ such that $\overline{e}\in\overline{c}\overline{R}$ and $\overline{1}-\overline{e}\in(\overline{1}-\overline{c}\overline{R})$. We have proved that R/sR is a clean ring [6] which completes the proof. □

As a consequence we obtain the following result.

Theorem 2. *If R is an elementary divisor domain and $a\in R\setminus\{0\}$, then the factor-ring R/aR is a morphic ring of neat range one.*

Proof. Since every elementary divisor domain is a Bezout ring [4], by [9] R/aR is a morphic ring. Since every homomorphic image of an elementary divisor ring is an elementary divisor ring, by Theorem 3, R/aR is a morphic ring of neat range one, which completes the proof. □

We say that R has almost stable range one if every finite proper homomorphic image R has stable range one. By [5] a Bezout ring of almost stable range one is an elementary divisor ring.

A well-known Henriksen example of a Bezout domain, namely $R=\mathbb{Z}+x\mathbb{Q}[x]$ (see [2]; for a general theorem on pullbacks of Bezout domains [3]), R is an elementary divisor that does not have almost stable range one [8].

Let R be an elementary divisor domain which is not of almost stable range one. Then there exists an element $a\in R$ such that in the factor-ring $\overline{R}=R/aR$ there exist elements $\overline{b},\overline{c}\in\overline{R}$ such that $\overline{bR}=\overline{c}\overline{R}$. There exist noninvertible neat elements $\overline{s},\overline{t}\in R$ such that $\overline{b}\overline{s}=\overline{c}$, $\overline{c}\overline{t}=\overline{b}$.

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Free abelian dimonoids

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Abstract. We construct a free abelian dimonoid and describe the least abelian congruence on a free dimonoid. Also we show that free abelian dimonoids are determined by their endomorphism semigroups.

1. Introduction

The notion of a dimonoid was introduced by Jean-Louis Loday in [1]. An algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* if for all $x, y, z \in D$ the following conditions hold:

$$
\begin{aligned}
&(D_1) \qquad (x \dashv y) \dashv z = x \dashv (y \vdash z),\\
&(D_2) \qquad (x \vdash y) \dashv z = x \vdash (y \dashv z),\\
&(D_3) \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z).
\end{aligned}
$$

If operations of a dimonoid coincide, the dimonoid becomes a semigroup.

Dimonoids and in particular dialgebras have been studied by many authors (see, e.g., [2]–[5]), they play a prominent role in problems from the theory of Leibniz algebras. The first result about dimonoids is the description of a free dimonoid [1]. T. Pirashvili [4] introduced the notion of a duplex which generalizes the notion of a dimonoid and constructed a free duplex. Free dimonoids and free commutative dimonoids were

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investigated in [6] and [7] respectively. Free normal dibands and other relatively free dimonoids were described in [8], [9]. In this paper we study free abelian dimonoids.

The paper is organized as follows. In Section 2 we give necessary definitions and examples of abelian dimonoids. In Section 3 we construct a free abelian dimonoid and, in particular, consider a free abelian dimonoid of rank 1. In Section 4 we define the least congruence on a free dimonoid such that the corresponding quotient-dimonoid is isomorphic to the free abelian dimonoid. In Section 5 we prove that free abelian dimonoids are determined by their endomorphisms.

2. Examples of abelian dimonoids

It is well-known that a non-empty class H of algebraic systems is a variety if the Cartesian product of any sequence of H-systems is a H-system, every subsystem of an arbitrary H-system is a H-system and any homomorphic image of an arbitrary H-system is a H-system (Birkhoff [10]).

A dimonoid $(D, \dashv, \vdash)$ we call *abelian* (in the same way as a digroup in [11]) if for all $x, y \in D$,

$$x \dashv y = y \vdash x.$$

The class of all abelian dimonoids satisfies the conditions of Birkhoff's theorem and therefore it is a variety. A dimonoid which is free in the variety of abelian dimonoids will be called a *free abelian dimonoid*.

It should be noted that the class of all abelian dimonoids does not coincide with the class of all commutative dimonoids [7] (both operations of such dimonoids are commutative). For example, a non-singleton left zero and right zero dimonoid [9] is abelian but not commutative.

Let Z be the set of all integers, $E = \{\lambda, \mu\}$ be an arbitrary two-element set. Define two binary operations $\dashv$ and $\vdash$ on $Z \times E$ as follows:

$$(m, x) \dashv (n, y) = \begin{cases} (m+n+1, x), & y = \lambda, \\ (m+n-1, x), & y = \mu, \end{cases}$$

$$(m, x) \vdash (n, y) = \begin{cases} (m+n+1, y), & x = \lambda, \\ (m+n-1, y), & x = \mu. \end{cases}$$

Proposition 1. *The algebra $(Z \times E, \dashv, \vdash)$ is an abelian dimonoid.*

Proof. Let $(m,x),(n,y),(s,z) \in Z \times E$. If $y=z=\lambda$ or $y=z=\mu$, we obtain

$$\begin{aligned}((m,x) \dashv (n,\lambda)) \dashv (s,\lambda) &= (m+n+s+2,x)\\ &= (m,x) \dashv ((n,\lambda) \dashv (s,\lambda)) \quad \text{or}\\ ((m,x) \dashv (n,\mu)) \dashv (s,\mu) &= (m+n+s-2,x)\\ &= (m,x) \dashv ((n,\mu) \dashv (s,\mu))\end{aligned}$$

respectively.

For $y=\lambda$, $z=\mu$ or $y=\mu$, $z=\lambda$, we have

$$\begin{aligned}((m,x) \dashv (n,y)) \dashv (s,z) &= (m+n+s,x)\\ &= (m,x) \dashv ((n,y) \dashv (s,z)).\end{aligned}$$

Therefore, the operation $\dashv$ is associative. Analogously we can show that $\vdash$ is an associative operation too.

Show that the axiom (D_1) holds. If $y=z=\lambda$ or $y=z=\mu$,

$$\begin{aligned}(m,x) \dashv ((n,\lambda) \vdash (s,\lambda)) &= (m+n+s+2,x)\\ &= ((m,x) \dashv (n,\lambda)) \dashv (s,\lambda) \quad \text{or}\\ (m,x) \dashv ((n,\mu) \vdash (s,\mu)) &= (m+n+s-2,x)\\ &= ((m,x) \dashv (n,\mu)) \dashv (s,\mu).\end{aligned}$$

For $y=\lambda$, $z=\mu$ or $y=\mu$, $z=\lambda$, we obtain

$$\begin{aligned}(m,x) \dashv ((n,y) \vdash (s,z)) &= (m+n+s,x)\\ &= ((m,x) \dashv (n,y)) \dashv (s,z).\end{aligned}$$

The axiom (D_3) is checked similarly. Now we consider the axiom (D_2). Let $x=z=\lambda$ or $x=z=\mu$. Then

$$\begin{aligned}(m,\lambda) \vdash ((n,y) \dashv (s,\lambda)) &= (m+n+s+2,y)\\ &= ((m,\lambda) \vdash (n,y)) \dashv (s,\lambda) \quad \text{or}\\ (m,\mu) \vdash ((n,y) \dashv (s,\mu)) &= (m+n+s-2,y)\\ &= ((m,\mu) \vdash (n,y)) \dashv (s,\mu).\end{aligned}$$

If $x=\lambda, z=\mu$ or $x=\mu, z=\lambda$, then

$$\begin{aligned}(m,x) \vdash ((n,y) \dashv (s,z)) &= (m+n+s,y)\\ &= ((m,x) \vdash (n,y)) \dashv (s,z),\end{aligned}$$

which completes the verification of (D_2).

The fact that $(Z \times E, \dashv, \vdash)$ is abelian can be checked immediately. $\square$

An element e of an arbitrary dimonoid $(D, \dashv, \vdash)$ is called a *bar-unit* (see, e.g., [1]) if for all $g \in D$,

$$e \vdash g = g = g \dashv e.$$

In contrast to monoids a dimonoid may have many bar-units. For example, for the dimonoid from Proposition 1 we have

$$\begin{aligned} (-1, \lambda) \vdash (m, x) = (m, x) = (m, x) \dashv (-1, \lambda), \\ (1, \mu) \vdash (m, x) = (m, x) = (m, x) \dashv (1, \mu) \end{aligned}$$

for any $(m, x) \in Z \times E$. Thus, $(-1, \lambda)$ and $(1, \mu)$ are bar-units. Moreover, another bar-units of $(Z \times E, \dashv, \vdash)$ do not exist.

Let G be an arbitrary additive abelian group, $X_1, X_2, \dots, X_n$ $(n \geqslant 2)$ be non-empty subsets of G and $X_\alpha = G$ for some $\alpha \in \{1, 2, \dots, n\}$. For all $t = (t_1, t_2, \dots, t_n) \in \prod_{i=1}^n X_i$ we put $t^+ = t_1 + t_2 + \ldots + t_n$.

Take arbitrary $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \prod_{i=1}^n X_i$ and define two binary operations $\dashv_\alpha$ and $\vdash_\alpha$ on $\prod_{i=1}^n X_i$ by

$$\begin{aligned} x \dashv_\alpha y &= (x_1, \dots, x_\alpha + y^+, \dots, x_n), \\ x \vdash_\alpha y &= (y_1, \dots, y_\alpha + x^+, \dots, y_n). \end{aligned}$$

Proposition 2. *For every $\alpha \in \{1, 2, \dots, n\}$ the algebra $(\prod_{i=1}^n X_i, \dashv_\alpha, \vdash_\alpha)$ is an abelian dimonoid.*

Proof. Let $x, y, z \in \prod_{i=1}^n X_i$. Then

$$\begin{aligned} (x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \dots, x_\alpha + y^+, \dots, x_n) \dashv_\alpha (z_1, z_2, \dots, z_n) \\ &= (x_1, \dots, x_\alpha + y^+ + z^+, \dots, x_n) \\ &= (x_1, x_2 \dots, x_n) \dashv_\alpha (y_1, \dots, y_\alpha + z^+, \dots, y_n) \\ &= x \dashv_\alpha (y \dashv_\alpha z), \\ (x \vdash_\alpha y) \vdash_\alpha z &= (y_1, \dots, y_\alpha + x^+, \dots, y_n) \vdash_\alpha (z_1, z_2, \dots, z_n) \\ &= (z_1, \dots, z_\alpha + x^+ + y^+, \dots, z_n) \\ &= (x_1, x_2 \dots, x_n) \vdash_\alpha (z_1, \dots, z_\alpha + y^+, \dots, z_n) \\ &= x \vdash_\alpha (y \vdash_\alpha z). \end{aligned}$$

Thus, operations $\dashv_\alpha$ and $\vdash_\alpha$ are associative.

Show that axioms $(D_1)-(D_3)$ hold:

$$
\begin{aligned}
(x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_\alpha + y^+ + z^+, \ldots, x_n) \\
&= (x_1, x_2, \ldots, x_n) \dashv_\alpha (z_1, \ldots, z_\alpha + y^+, \ldots, z_n) \\
&= x \dashv_\alpha (y \vdash_\alpha z), \\
(x \vdash_\alpha y) \dashv_\alpha z &= (y_1, \ldots, y_\alpha + x^+, \ldots, y_n) \dashv_\alpha (z_1, z_2, \ldots, z_n) \\
&= (y_1, \ldots, y_\alpha + z^+ + x^+, \ldots, y_n) \\
&= (x_1, x_2 \ldots, x_n) \vdash_\alpha (y_1, \ldots, y_\alpha + z^+, \ldots, y_n) \\
&= x \vdash_\alpha (y \dashv_\alpha z), \\
(x \dashv_\alpha y) \vdash_\alpha z &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) \vdash_\alpha (z_1, z_2, \ldots, z_n) \\
&= (z_1, \ldots, z_\alpha + x^+ + y^+, \ldots, z_n) \\
&= x \vdash_\alpha (y \vdash_\alpha z).
\end{aligned}
$$

Therefore, $(\prod_{i=1}^n X_i, \dashv_\alpha, \vdash_\alpha)$ is a dimonoid. Moreover,

$$x \dashv_\alpha y = (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) = y \vdash_\alpha x$$

for all $x, y \in \prod_{i=1}^n X_i$. □

Let $(S, \circ)$ be an arbitrary semigroup. A semigroup $(S, *)$, where $x * y = y \circ x$ for all $x, y \in S$, is called a *dual semigroup* to $(S, \circ)$.

A semigroup $(S, \circ)$ is called *left commutative* (respectively, *right commutative*) if it satisfies the identity $x \circ y \circ a = y \circ x \circ a$ (respectively, $a \circ x \circ y = a \circ y \circ x$).

Proposition 3. *Let $(S, \circ)$ be an arbitrary right commutative semigroup and $(S, *)$ be a dual semigroup to $(S, \circ)$. Then the algebra $(S, \circ, *)$ is an abelian dimonoid.*

Proof. The proof follows from Lemma 3 of [9]. □

Proposition 4. *Let $(S, *)$ be an arbitrary left commutative semigroup and $(S, \circ)$ be a dual semigroup to $(S, *)$. Then the algebra $(S, \circ, *)$ is an abelian dimonoid.*

Proof. The proof follows from Lemma 4 of [9]. □

An important example of abelian dimonoids is the class of abelian digroups (see [11]). The idea of the notion of a digroup first appeared in the work of Jean-Louis Loday [1].

3. The free abelian dimonoid

Let X be an arbitrary set and N be the set of all natural numbers. Denote by $FCm(X)$ the free commutative monoid on X with the identity ε. Words of $FCm(X)$ we write as $w = w_1^{\alpha_1} w_2^{\alpha_2} \ldots w_n^{\alpha_n}$, where $w_1, w_2, \ldots, w_n \in X$ are pairwise distinct, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in N \cup \{0\}$. Here $w_i^0, 1 \leqslant i \leqslant n$, is the empty word ε and $w^1 = w$ for all $w \in X$.

We put

$$FAd(X) = X \times FCm(X)$$

and define two binary operations $\dashv$ and $\vdash$ on $FAd(X)$ as follows:

$$\begin{aligned}(x, u) \dashv (y, v) &= (x, uyv),\\ (x, u) \vdash (y, v) &= (y, xuv).\end{aligned}$$

Note that for every element t of an arbitrary abelian dimonoid $(D, \prec, \succ)$ the degrees

$$t_{\prec}^{n} = \underbrace{t \prec t \prec \ldots \prec t}_{n}, \qquad t_{\succ}^{n} = \underbrace{t \succ t \succ \ldots \succ t}_{n}$$

coincide. Therefore, we will write t^n instead of $t_{\prec}^{n}$ $(= t_{\succ}^{n})$.

Theorem 1. *The algebra $(FAd(X), \dashv, \vdash)$ is the free abelian dimonoid.*

Proof. Let $(x, u), (y, v), (z, w) \in FAd(X)$. Then

$$\begin{aligned}((x, u) \dashv (y, v)) \dashv (z, w) &= (x, uyv) \dashv (z, w)\\ &= (x, uyvzw) = (x, u) \dashv ((y, v) \dashv (z, w)),\\ ((x, u) \vdash (y, v)) \vdash (z, w) &= (y, xuv) \vdash (z, w)\\ &= (z, yxuvw) = (x, u) \vdash ((y, v) \vdash (z, w)).\end{aligned}$$

Thus, operations $\dashv$ and $\vdash$ are associative. In addition,

$$\begin{aligned}((x, u) \dashv (y, v)) \dashv (z, w) &= (x, uyvzw)\\ &= (x, u) \dashv (z, yvw) = (x, u) \dashv ((y, v) \vdash (z, w)),\\ ((x, u) \vdash (y, v)) \dashv (z, w) &= (y, xuvzw)\\ &= (x, u) \vdash (y, vzw) = (x, u) \vdash ((y, v) \dashv (z, w)),\\ ((x, u) \dashv (y, v)) \vdash (z, w) &= (x, uyv) \vdash (z, w)\\ &= (z, yxuvw) = (x, u) \vdash ((y, v) \vdash (z, w)).\end{aligned}$$

So, $(FAd(X), \dashv, \vdash)$ is a dimonoid and, obviously, it is abelian.

For all $(t,w) \in FAd(X)$, where $w = w_1^{\alpha_1} w_2^{\alpha_2} \dots w_n^{\alpha_n}$, we obtain the following representation:

$$(t,w) = (t,\varepsilon) \dashv (w_1,\varepsilon)^{\alpha_1} \dashv \dots \dashv (w_n,\varepsilon)^{\alpha_n}.$$

This representation we call a canonical form of elements of the dimonoid $(FAd(X),\dashv,\vdash)$. It is clear that such representation is unique up to an order of $(w_i,\varepsilon), 1 \leqslant i \leqslant n$. Moreover, $\langle X \times \varepsilon \rangle = (FAd(X),\dashv,\vdash)$.

Show that the dimonoid $(FAd(X),\dashv,\vdash)$ is free abelian. Let $(D',\dashv',\vdash')$ be an arbitrary abelian dimonoid, ξ be any mapping of $X \times \varepsilon$ into D'. Further, we naturally extend ξ to a mapping Ξ of $FAd(X)$ into D' using the canonical representation of elements of $(FAd(X),\dashv,\vdash)$, that is,

$$(t,w)\Xi = (t,\varepsilon)\xi \dashv' ((w_1,\varepsilon)\xi)^{\alpha_1} \dashv' \dots \dashv' ((w_n,\varepsilon)\xi)^{\alpha_n}$$

for any $(t,w) \in FAd(X)$, where $w = w_1^{\alpha_1} w_2^{\alpha_2} \dots w_n^{\alpha_n}$.

It is easy to see that Ξ is a homomorphism of $(FAd(X),\dashv)$ into $(D',\dashv')$. Using that $(D',\dashv',\vdash')$ is an abelian dimonoid too, we obtain

$$\begin{aligned}((t,u) \vdash (s,v))\Xi &= ((s,v) \dashv (t,u))\Xi \\ &= (s,v)\Xi \dashv' (t,u)\Xi = (t,u)\Xi \vdash' (s,v)\Xi\end{aligned}$$

for all $(t,u),(s,v) \in FAd(X)$. □

Observe that the cardinality of a set X is the *rank* of the constructed free abelian dimonoid $(FAd(X),\dashv,\vdash)$ and this dimonoid is uniquely determined up to an isomorphism by $|X|$.

Now we consider the structure of a free abelian dimonoid of rank 1.

Lemma 1. *Operations of the free abelian dimonoid* $(FAd(X),\dashv,\vdash)$ *coincide if and only if* $|X| = 1$.

Proof. Assume that operations of $(FAd(X),\dashv,\vdash)$ coincide and $x,y \in X$ are distinct. Then for all $u,v \in FCm(X)$,

$$(x,u) \dashv (y,v) = (x,uyv) \neq (y,xuv) = (x,u) \vdash (y,v),$$

which contradicts the fact that $\dashv = \vdash$.

Let $X = \{x\}$, then for all $(x,u),(x,v) \in FAd(X)$ we have

$$(x,u) \dashv (x,v) = (x,uxv) = (x,u) \vdash (x,v). \qquad \square$$

Let $(S, \circ)$ be an arbitrary semigroup and $a \in S$. Define on S a new binary operation $\circ_a$ by

$$x \circ_a y = x \circ a \circ y$$

for all $x, y \in S$.

Clearly, $(S, \circ_a)$ is a semigroup, it is called a *variant* of $(S, \circ)$.

Proposition 5. *The free abelian dimonoid $(FAd(X), \dashv, \vdash)$ of rank 1 is isomorphic to the variant $(N^0, +_1)$ of the additive semigroup of all non-negative integers.*

Proof. Let $X = \{x\}$, then $FAd(X) = \{(x, x^n) | n \in N^0\}$. By Lemma 1, for $(FAd(X), \dashv, \vdash)$ we have $\dashv = \vdash$. Define a mapping φ of $(FAd(X), \dashv, \vdash)$ into $(N^0, +_1)$ by

$$\varphi : (x, x^n) \mapsto n$$

for any $(x, x^n) \in FAd(X)$.

It is clear that φ is a bijection. In addition, for all (x, x^n), $(x, x^m) \in FAd(X)$ we obtain

$$\begin{aligned}((x, x^n) \dashv (x, x^m))\varphi &= (x, x^{n+m+1})\varphi = n + m + 1 \\ &= n +_1 m = (x, x^n)\varphi +_1 (x, x^m)\varphi.\end{aligned}$$

□

4. The least abelian congruence

Let $(D, \dashv, \vdash)$ be an arbitrary dimonoid, ρ be an equivalence relation on D which is stable on the left and on the right with respect to each of operations $\dashv, \vdash$. In this case ρ is called a *congruence* on $(D, \dashv, \vdash)$.

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by $\triangle_f$. For a congruence ρ on a dimonoid $(D, \dashv, \vdash)$ the corresponding quotient-dimonoid is denoted by $(D, \dashv, \vdash)/\rho$. A congruence ρ on a dimonoid $(D, \dashv, \vdash)$ is called *abelian* if $(D, \dashv, \vdash)/\rho$ is an abelian dimonoid.

As usual N denotes the set of all positive integers, and let $n \in N$. For an arbitrary set X by $\widetilde{X}$ we denote the copy of X, that is, $\widetilde{X} = \{\widetilde{x} \mid x \in X\}$ and put

$$Y_n^{(1)} = \underbrace{\widetilde{X} \times X \times \ldots \times X}_{n}, \quad Y_n^{(2)} = \underbrace{X \times \widetilde{X} \times X \times \ldots \times X}_{n},$$

$$Y_n^{(3)} = \underbrace{X \times X \times \widetilde{X} \times \ldots \times X}_{n}, \quad \ldots, \quad Y_n^{(n)} = \underbrace{X \times X \times \ldots \times \widetilde{X}}_{n}.$$

We denote the union of n different copies $Y_n^{(i)}, 1 \leqslant i \leqslant n$, of X^n by Y_n and assume $Fd(X) = \bigcup_{n \geqslant 1} Y_n$. Define operations $\prec$ and $\succ$ on $Fd(X)$ as follows:

$$(x_1,\ldots,\widetilde{x}_i,\ldots,x_m) \prec (y_1,\ldots,\widetilde{y}_j,\ldots,y_n) = (x_1,\ldots,\widetilde{x}_i,\ldots,x_m,y_1,\ldots,y_n),$$

$$(x_1,\ldots,\widetilde{x}_i,\ldots,x_m) \succ (y_1,\ldots,\widetilde{y}_j,\ldots,y_n) = (x_1,\ldots,x_m,y_1,\ldots,\widetilde{y}_j,\ldots,y_n)$$

for all $(x_1,\ldots,\widetilde{x}_i,\ldots,x_m),(y_1,\ldots,\widetilde{y}_j,\ldots,y_n) \in Fd(X)$.

According to [1], $(Fd(X),\prec,\succ)$ is the *free dimonoid* on X. Elements of $Fd(X)$ are called *words*, $\widetilde{X}$ is the *generating set* of $(Fd(X),\prec,\succ)$.

Let $(Fd(X),\prec,\succ)$ be the free dimonoid on X and $w \in Fd(X)$. The canonical form of $w = (w_1,\ldots,\widetilde{w}_l,\ldots,w_k)$ is its representation in the shape:

$$w = \widetilde{w}_1 \succ \ldots \succ \widetilde{w}_l \prec \ldots \prec \widetilde{w}_k.$$

We call k as the *length* of w and denote it by $l(w)$. For any $x \in X$ by $q_{\widetilde{x}}(w)$ we denote the quantity of all elements $\widetilde{x} \in \widetilde{X}$ that are included in the canonical form $\widetilde{w}_1 \succ \ldots \succ \widetilde{w}_l \prec \ldots \prec \widetilde{w}_k$ of w.

Define a binary relation σ on $Fd(X)$ as follows: $u = (u_1,\ldots,\widetilde{u}_i,\ldots,u_n)$ and $v = (v_1,\ldots,\widetilde{v}_j,\ldots,v_m)$ of $Fd(X)$ are σ-*equivalent* if for all $x \in X$,

$$q_{\widetilde{x}}(u) = q_{\widetilde{x}}(v) \ \textit{and} \ u_i = v_j.$$

We note that $q_{\widetilde{x}}(u) = q_{\widetilde{x}}(v)$ for all $x \in X$ implies $l(u) = l(v)$.

For example, for $u = (a,\widetilde{b},a,c)$, $v = (a,\widetilde{a})$ and $w = (c,a,a,\widetilde{b})$ we have $q_{\tilde{a}}(p) = 2$ for all $p \in \{u,v,w\}$, $l(v) = 2$ and $(u,w) \in \sigma$.

Theorem 2. *The binary relation σ is the least abelian congruence on the free dimonoid $(Fd(X),\prec,\succ)$.*

Proof. It is easy to see that σ is an equivalence relation. Assume that $u = (u_1,\ldots,\widetilde{u}_i,\ldots,u_n), v = (v_1,\ldots,\widetilde{v}_j,\ldots,v_m) \in Fd(X)$ such that $u\sigma v$ and $w = (w_1,\ldots,\widetilde{w}_k,\ldots,w_l) \in Fd(X)$. Then

$$\begin{aligned} u \prec w &= (u_1,\ldots,\widetilde{u}_i,\ldots,u_n,w_1,\ldots,w_l),\\ v \prec w &= (v_1,\ldots,\widetilde{v}_j,\ldots,v_m,w_1,\ldots,w_l),\\ u \succ w &= (u_1,\ldots,u_n,w_1,\ldots,\widetilde{w}_k,\ldots,w_l),\\ v \succ w &= (v_1,\ldots,v_m,w_1,\ldots,\widetilde{w}_k,\ldots,w_l). \end{aligned}$$

Since $u_i = v_j$ and

$$q_{\widetilde{x}}(u \prec w) = q_{\widetilde{x}}(v \prec w), \qquad q_{\widetilde{x}}(u \succ w) = q_{\widetilde{x}}(v \succ w)$$

for any $x \in X$, we have $(u \prec w)\sigma(v \prec w)$ and $(u \succ w)\sigma(v \succ w)$. Analogously we can show that $(w \prec u)\sigma(w \prec v)$ and $(w \succ u)\sigma(w \succ v)$. Thus, σ is a congruence.

In addition, we note that $(u \prec v)\sigma(v \succ u)$ for all $u, v \in Fd(X)$, therefore $(Fd(X), \prec, \succ)/\sigma$ is abelian. A class of $(Fd(X), \prec, \succ)/\sigma$ which contains w we denote by $[w]$.

Further, we show that the quotient-dimonoid $(Fd(X), \prec, \succ)/\sigma$ is isomorphic to the free abelian dimonoid $(FAd(X), \dashv, \vdash)$ (see Theorem 1).

Define a mapping φ of $(Fd(X), \prec, \succ)/\sigma$ into $(FAd(X), \dashv, \vdash)$ by

$$[w]\varphi = (w_k, w_1 \dots w_{k-1}w_{k+1} \dots w_l)$$

for all words $w = (w_1, \dots, \widetilde{w}_k, \dots, w_l) \in Fd(X)$ with $l(w) \geqslant 2$, and $[w]\varphi = (w_1, \varepsilon)$ for any $w = \widetilde{w}_1 \in Fd(X)$. It is clear that φ is a bijection.

For all $[u], [v] \in (Fd(X), \prec, \succ)/\sigma$, where $u = (u_1, \dots, \widetilde{u}_i, \dots, u_n)$, $v = (v_1, \dots, \widetilde{v}_j, \dots, v_m)$, we have

$$\begin{aligned}([u] \prec [v])\varphi &= [(u_1, \dots, \widetilde{u}_i, \dots, u_n, v_1, \dots, v_m)]\varphi \\ &= (u_i, u_1 \dots u_{i-1}u_{i+1} \dots u_n v_1 \dots v_m) \\ &= (u_i, u_1 \dots u_{i-1}u_{i+1} \dots u_n) \dashv (v_j, v_1 \dots v_{j-1}v_{j+1} \dots v_m) \\ &= [u]\varphi \dashv [v]\varphi.\end{aligned}$$

Since dimonoids $(Fd(X), \prec, \succ)/\sigma$ and $(FAd(X), \dashv, \vdash)$ are abelian,

$$([u] \succ [v])\varphi = ([v] \prec [u])\varphi = [v]\varphi \dashv [u]\varphi = [u]\varphi \vdash [v]\varphi$$

for all $[u], [v] \in (Fd(X), \prec, \succ)/\sigma$.

Thus, $(Fd(X), \prec, \succ)/\sigma$ is free abelian and the composition $\eta^\natural \circ \varphi$, where $\eta^\natural : (Fd(X), \prec, \succ) \to (Fd(X), \prec, \succ)/\sigma$ is the natural homomorphism, is an epimorphism of $(Fd(X), \prec, \succ)$ on $(FAd(X), \dashv, \vdash)$ inducing the least abelian congruence on $Fd(X)$. From the definition of $\eta^\natural \circ \varphi$ it follows that $\triangle_{\eta^\natural \circ \varphi} = \sigma$. $\square$

5. Determinability

One of the venerable algebraic problems the first instance of which was considered by E. Galois (see [12]) is the determinability of an algebraic structure by its endomorphism semigroup. The determinability problem for free algebras in a certain variety was raised by B. Plotkin [13]. For free groups this problem was solved by E. Formanek [14]. An analogous problem for free semigroups and free monoids was decided in [15].

Some characteristics for the enomorphism monoid of a free dimonoid of rank 1 were obtained in [16]. Determinability of free trioids by their endomorphism semigroups was proved in [17].

Recall that an algebra A of some class Ω *is determined* by its endomorphism semigroup in the class Ω if for any algebra $B \in \Omega$ the condition $End(A) \cong End(B)$ implies $A \cong B$. Note that the converse implication is obvious.

Let $\mathfrak{F}_X = (FAd(X), \dashv, \vdash)$ be the free abelian dimonoid on X and $(t, u) \in FAd(X)$, $u = u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n}$. From Theorem 1 it follows that an arbitrary endomorphism $\Xi \in End(\mathfrak{F}_X)$ has form:

$$(t, u)\Xi = (t, \varepsilon)\xi \dashv ((u_1, \varepsilon)\xi)^{\alpha_1} \dashv \dots \dashv ((u_n, \varepsilon)\xi)^{\alpha_n},$$

where $\xi : X \times \varepsilon \to FAd(X)$ is any mapping.

An endomorphism $\theta_{(t,u)} \in End(\mathfrak{F}_X)$ we call *constant* if $(x, \varepsilon)\theta_{(t,u)} = (t, u)$ for all $x \in X$.

Lemma 2.

(i) *An endomorphism f of the free abelian dimonoid $\mathfrak{F}_X$ is constant if and only if $\psi f = f$ for all $\psi \in Aut(\mathfrak{F}_X)$.*

(ii) *An endomorphism f of the free abelian dimonoid $\mathfrak{F}_X$ is constant idempotent if and only if $f = \theta_{(x,\varepsilon)}$ for some $x \in X$.*

Proof. (i) Suppose that an endomorphism $f \in End(\mathfrak{F}_X)$ is constant and $\psi \in Aut(\mathfrak{F}_X)$. Then $f = \theta_{(t,u)}$ for some $(t, u) \in FAd(X)$, in addition,

$$(x, \varepsilon)(\psi\theta_{(t,u)}) = ((x, \varepsilon)\psi)\, \theta_{(t,u)} = (t, u) = (x, \varepsilon)\theta_{(t,u)}$$

for any $x \in X$. Thus, $\psi\theta_{(t,u)} = \theta_{(t,u)}$.

Conversely, let $\psi f = f$ for all $\psi \in Aut(\mathfrak{F}_X)$ and some $f \in End(\mathfrak{F}_X)$. For fixed $x \in X$ we obtain

$$(x, \varepsilon)f = (x, \varepsilon)\,(\psi f) = ((x, \varepsilon)\psi)\, f = (y, \varepsilon)f,$$

where $(y, \varepsilon) = (x, \varepsilon)\psi$. Since $\{(x, \varepsilon)\psi \mid \psi \in Aut(\mathfrak{F}_X)\} = X \times \varepsilon$, we have $(a, \varepsilon)f = (b, \varepsilon)f$ for all $a, b \in X$. From here $f = \theta_{(t,u)}$ for $(t, u) = (x, \varepsilon)f$.

(ii) Let $f \in End(\mathfrak{F}_X)$ be a constant idempotent endomorphism. Then $f = \theta_{(x,u)}$, $(x, u) \in FAd(X)$, and $\theta^2_{(x,u)} = \theta_{(x,u)}$. Since $\theta_{(x,u)\theta_{(x,u)}} = \theta^2_{(x,u)}$, we have

$$\theta_{(x,u)} = \theta_{(x,u)}\theta_{(x,u)} = \theta_{(x,u)\theta_{(x,u)}} = \theta_{(x,u^{l(u)+1}x^{l(u)})}.$$

It means that $(x, u) = (x, u^{l(u)+1}x^{l(u)})$, whence $l(u) = 0$, i.e., $u = \varepsilon$. Clearly, $\theta^2_{(x,\varepsilon)} = \theta_{(x,\varepsilon)}$ for all $x \in X$. $\square$

Theorem 3. *Let $\mathfrak{F}_X = (FAd(X), \dashv, \vdash)$ and $\mathfrak{F}_Y = (FAd(Y), \dashv, \vdash)$ be free abelian dimonoids such that $End(\mathfrak{F}_X) \cong End(\mathfrak{F}_Y)$. Then $\mathfrak{F}_X$ and $\mathfrak{F}_Y$ are isomorphic.*

Proof. Let Ψ be an arbitrary isomorphism of $End(\mathfrak{F}_X)$ into $End(\mathfrak{F}_Y)$. In according to the statements of Lemma 2 for some constant idempotent endomorphism $\theta_{(x,\varepsilon)}, x \in X$, of the free abelian dimonoid $\mathfrak{F}_X$ and for all $\alpha \in Aut(\mathfrak{F}_X)$, we have $\alpha\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)}$. Taking into account that Ψ is a homomorphism, we obtain

$$\theta_{(x,\varepsilon)}\Psi = \left(\alpha\theta_{(x,\varepsilon)}\right)\Psi = \alpha\Psi\ \theta_{(x,\varepsilon)}\Psi.$$

Since $Aut(\mathfrak{F}_X)\Psi = Aut(\mathfrak{F}_Y)$, by the statement (i) of Lemma 2 we have $\theta_{(x,\varepsilon)}\Psi$ is a constant endomorphism of $\mathfrak{F}_Y$. Then $\theta_{(x,\varepsilon)}\Psi = \theta_{(y,v)}$ for some $(y,v) \in FAd(Y)$, in addition, $\theta_{(y,v)}$ is an idempotent of $End(\mathfrak{F}_Y)$. By the statement (ii) of Lemma 2, $v = \varepsilon'$, where ε' is the empty word of $FCm(Y)$ (see Section 3).

Define a map $\xi : X \to Y$ putting $x\xi = y$ if and only if $\theta_{(x,\varepsilon)}\Psi = \theta_{(y,\varepsilon')}$. It is clear that ξ is a bijection. Thus, abelian dimonoids $\mathfrak{F}_X$ and $\mathfrak{F}_Y$ are isomorphic. □

Using similar arguments, the fact that the free dimonoid also is uniquely determined up to an isomorphism by its endomorphism semigroup can be proved.

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