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Bounds for graphs of given girth and generalized polygons

Lakdere Benkherouf and Vasyl Ustimenko

Communicated by V.V. Kirichenko

Dedicated to V. V. Kirichenko on the occasion of his 60th birthday

Abstract. In this paper we present a bound for bipartite graphs with average bidegrees η and ξ satisfying the inequality $\eta \geq \xi^{\alpha}$, $\alpha \geq 1$. This bound turns out to be the sharpest existing bound. Sizes of known families of finite generalized polygons are exactly on that bound. Finally, we present lower bounds for the numbers of points and lines of biregular graphs (tactical configurations) in terms of their bidegrees. We prove that finite generalized polygons have smallest possible order among tactical configuration of given bidegrees and girth. We also present an upper bound on the size of graphs of girth $g \geq 2t+1$. This bound has the same magnitude as that of Erdös bound, which estimates the size of graphs without cycles C_{2t}.

1. Introduction

Let Γ be a simple graph (undirected, no multiple edges, no loops) and let F be a family of graphs none of which is isomorphic to a subgraph of Γ. In this case we say that Γ is F-free.

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Let P be a certain graph theoretical property. By $ex_P(v,F)$ we denote the greatest number of edges of F-free graph on v-vertices, which satisfies property P.

If property P is trivial, that is, valid for all simple graphs we shall omit index P and write $ex(v,F)$.

Extremal graph theory contains several important results on $ex(v,F)$, where F is a finite collection of cycles of different length (see [2], [24]).

The following unpublished result of Paul Erdös is often refereed to as The Even Circuit Theorem (see [2], [24]).

Let C_n denote the cycle of length n. Then

$$\mathrm{ex}(v, C_{2k}) \leq Cv^{1+1/k},$$

where C is positive independent constant. For a proof of this result and its generalization: see [25], [26].

In [24] the upper bound

$$\mathrm{ex}(v, C_3, C_4, \ldots, C_{2k+1}) \leq (1/2)^{1+1/k} v^{1+1/k} + O(v)$$

was established for all integers $k \geq 1$.

In this paper we will prove that

$$\mathrm{ex}(v.C_3, C_4, \ldots, C_{2k}) \leq 1/2v^{1+1/k} + O(v),$$

and obtain the following bound

$$\mathrm{ex}_{P(m)}(C_3, C_4, \ldots C_{2t}) \leq 1/2^{1+1/(m+1)t} v^{1+1/(m+1)t} + O(v) \text{ for even } k,$$

and

$$\mathrm{ex}_{P(m)}(C_3, C_4, \ldots, C_{2t}) \leq v^{1+1/m(t+2)+t-1} + O(v) \text{ for all odd } k,$$

where $m > 1$ is a real number and $P(m)$ is a property: graph is bipartite with average bidegrees η and ξ satisfying inequality $\eta \geq \xi^m$.

Studies of $\mathrm{ex}_{P(m)}(v, C_3, \ldots, C_t)$ is motivated by some problems in Operation Research, Theory of Communication Networks and Cryptography. Among graphs satisfying $P(m)$ are tactical configurations, that is, biregular bipartite graphs.

In section 1 we shall establish some lower bounds for the numbers of points and lines of tactical configurations. In section 2 we shall consider tactical configurations of minimal order. This is a natural generalization of the well known cages (see [3] and further references). Section 3 is devoted to upper bounds for size of tactical configurations. In section 4 we shall develop an important technique for computing walks on bipartite graphs with given average bidegrees. This method allow us to generalize results of section 3 for more general case of graphs with the property $P(m)$. We shall establish $\mathrm{ex}(v, C_3, \ldots, C_{2k})$ and repeat the result of Erdös and Simonovits on $\mathrm{ex}(v, C_3, C_4, \ldots, C_{2k+1})$ in the last section.

2. Some inequalities for tactical configurations

A *tactical configuration* introduced by E. H. Moore [15] almost century ago is a rank two incidence structure $\Delta = \Delta(l, p, a, b)$ consisting of l lines and p points in which each line is incident to a points and each point is incident to b lines. We denote the incidence graph of Δ by $\Gamma = \Gamma(\Delta)$, though when no confusion arise we may abuse terminology and refer to Γ as a tactical configuration. We call bipartite graphs the incidence graphs of the incidence structures. If the structure is a tactical configuration, then the incidence graphs are called biregular with bidegrees a, b.

We shall assume that the graph $\Gamma(\Delta)$ has order $v = l + p$ (number of vertices), and size $e = la = pb$ (number of edges). We also mean, as usual, that the girth g of a graph is the length of its minimal cycle.

The following lemma gives a lower bound on the number of points in a tactical configuration of girth $\geq 2k$. It gives also a lower bound for the number of lines.

Lemma 1. *Let $\Gamma = \Gamma(\Delta)$ with $\Delta(l, p, s + 1, r + 1)$ of girth $\geq 2k$. Then the following inequalities hold*

1. *If $k = 2t + 1$, then*

$$(1 + r)\sum_{i=0}^{t}(rs)^i \leq p \tag{2.1}$$

$$(1 + s)\sum_{i=0}^{t}(rs)^i \leq l \tag{2.2}$$

2. *If $k = 2t$, then*

$$(1 + r)\sum_{i=0}^{t-1}(rs)^i + (rs)^t \leq p \tag{2.3}$$

$$(1 + s)\sum_{i=0}^{t-1}(rs)^i + (rs)^t \leq l \tag{2.4}$$

Proof. The approach we adopt in the proof has its root in the Theory of Branching Process in Applied Probability, see for example Karlin and Taylor [7]. The idea is to consider an arbitrary vertex v and count the number of vertices at a given distance d, $d \leq [g/2]$, where g is the girth.

Let us assume that we start counting from a point $v = p$. The pass of length $d \leq k$ between two chosen vertices is unique, because of absence of cycles of length $2d$. Thus, we may use the idea of branching processes to count the number of vertices at distance d from p.

If l_{2h+1} refers to the number of lines at distance $2h+1$ from p and p_{2h+2} refers to the number of points at distance $2h+2$ from p, then $l_1 = r+1$, $p_2 = (r+1)s$, $l_3 = (r+1)sr$, ... Finally, we get

$$l_{2h+1} = (s+1)r^h s^h \tag{2.5}$$

$$p_{2h+2} = (r+1)r^h s^{h+1} \tag{2.6}$$

where $h = 0, 1, \dots, t$.

Summing (2.5) from $h = 0$ to t in case of odd t gives (2.2). Summing (2.6) from $h = 0$ to t in case of even t gives (2.3). By interchanging points and lines in (2.5) and (2.6)together with parameters r and s and using the same a argument as above we obtain (2.1) and (2.4). This completes the proof. □

Remark 1. *If $t+1 = s+1 = k$, then the order of the graph is $v = 2p = 2l$ and the inequalities in Lemma 1 are equivalent to the well known Tutte's inequality for arbitrary regular graph.*

$$v \geq 2(1 + (k-1) + \dots (k-1)^{(g-2/2)})$$

3. Minimal configurations with prescribed girth and their applications

The well known assignment problem in Operations Research is equivalent to finding the tactical configuration of given bidegees $r+1$ and $s+1$ of minimal order. It is an important special case of the Transport Problem (see, for instance Taha [16]). There is a well known efficient algorithm to solve this problem. In many cases this algorithm can be modified to solve efficiently assignments problem with additional restrictions. In our case this translate to the problem of finding a tactical configuration with minimal number of vertices among graphs satisfying a the list of restrictions.

let us consider the case when the precise list of restrictions is the absence of cycles of length $4, 6, \dots 2k-2$. One can notice that the incidence graph of tactical configuration does not have cycles of odd length and the last requirement is equivalent to inequality $g \geq 2k$. Unfortunately there is no known modification of existing assignments problem algorithms or other methods for the efficient solution of this problem.

Let $v(r, s, g)$ refers to the minimal order of a tactical configuration with bidegrees $s+1$ and $r+1$ and girth $g \geq 2k$, that is, the solution of the variant of assignment problem as above.

The problem of testing whether or not given tactical configuration $\Delta(r, s, l, p)$ of the girth g is a solution of the problem, that is, checking the condition $p + l = v(s, r, g)$ can be a very difficult one. The computation of function $v(r, s, g)$ is a hard problem of Applied Combinatorics.

We may assume, without loss of generality, that $r \geq s$. It is clear that if both inequalities (2.1) and (2.3) above turn out to be equalities, then our test is trivial and Δ is the solution of our variant of assignment problem. In this special situation we will use term "extra special configuration". In such a configuration we have the "best possible solution" of the problem. Of course , in the case of small bidegrees and girth we may easily find examples where tactical configuration Δ is not an extraspecial, but $p + l = v(r + 1, s + 1, g)$

We will use term "extraspecial" for graphs of extraspecial tactical configurations and regular graphs (not necessarily bipartite) of Tutte's bound order.

It is important that the totality of extraspecial configurations is non empty. Generalized m-gons were defined by J. Tits in 1959 (see [18], [19] and the survey [17]) as a tactical configurations of bidegrees $s+1$ and $t+1$ of girth $2m$ and diameter m. The pair (s, t) is known as order of generalized m-gon.

The following result is well known (see [3])

Theorem 1. *A finite generalized n-gon of order (s, t) has $n \in \{3, 4, 6, 8, 12\}$ unless $s = t = 1$. If $s > 1$ and $t > 1$, then*

1. $n \neq 12$
2. *If $n = 4$, then $s \leq t^2$, $t \leq s^2$;*
3. *If $n = 6$, then st is a square and $s \leq t^3$, $t \leq s^3$;*
4. *If $n = 8$, then $2st$ is a square and $s \leq t^2$, $t \leq s^2$;*

This is the original Feit-Higman theorem [6] combined with well known inequalities.

The known examples of generalized n-gons of bidegrees ≥ 3 and $m \in \{3, 4, 6, 8\}$ are rank 2 incidence graphs of geometries of finite simple groups of Lie type. The regular incidence graphs are $m = 3$ (group $A_2(q)$), $m = 4$ (group $B_2(q)$ or $C_2(q)$), $m = 6$ (group $G_2(q)$). In all cases $s = r = q$, where q is prime power.

The biregular but not regular generalized n-gons have parameters $s = q^\alpha$ and $t = q^\beta$, where q is some prime power. The list is below:

1. $n = 4$

   $s = q, r = q^2$ and q is arbitrary prime power,

   $s = q^2, r = q^3$ and q is arbitrary prime power

2. $n = 6$

   $s = q^2$, $t = q^3$ and $q = 3^{2k+1}$, $k > 1$

3. $n = 8$

   $s = q$, $t = q^2$ and $q = 2^{2k+1}$.

Besides finite generalized polygons related to simple groups of Lie type, which we consider above, there are important "nonclassical examples": nondezargezian projective plane, nonclassical generalized quadragons and hexagons (see [17] and further references).

Theorem 2. *Finite generalized polygons are extraspecial configurations.*

Proof. The order of regular generalized m-gons of degree $q+1$ is $1 + q + q^2 + \cdots + q^{m-1}$ and reaches the Tutte's bound for graphs of girth $m-2$. The finite irregular tactical configurations which are generalized polygons have to be of even diameter $m = 2k$. If their degrees are $r+1$ and $s+1$ then the numbers of points p and number of lines l can be computed by the formulas

$$\begin{aligned} p &= 1 + r + rs + r^2 s + r^2 s^2 + ... + r^k s^k + r^{k+1} s^k, \\ l &= 1 + s + sr + s^2 r + s^2 r^2 + ... + s^k r^{k+1} + s^{k+1} r^{k+1}, \end{aligned}$$

where k have to be an element of $\{2, 3, 4, 6\}$. They are at bounds (2.1) and (2.2) for points and lines.

Thus finite generalized m-gone is a perfect cage configuration. □

Application in Operations Research need explicit constructions of tactical configurations of given girth and bi-degrees of "small size", that is, close to bounds (3.1)-(3.4). General constructions of that kind are presented in [21].

3.1. Cages and $v(r, s, g)$ for $r = s$

We shall next examine the function $v(k, k, g)$ and regular extraspecial configurations. A cage (see [3]) is a $k = t + 1$-regular graph of given girth with the minimal number $v(k, g)$ of vertices. As it follows from definitions of functions $v(r, s, g)$, which is the minimal order of tactical configuration with bidegrees $r + 1$ and $s + 1$ of girth (see section 2) and $v(k, g)$

$$v(t, t, g) \geq v(t + 1, g)$$

Remark. We use same name for two functions but number of variables shall allow to distinguish them.

If we are dealing with t-regular extraspecial configuration, then $v(t, t, g)$ is same as $v(t, g)$ which achieves Tutte's bound.

The cage whose number of vertices is equal to this bound and whose girth is odd is called Moore graph. The only Moore's graph of degree 2 are $2n + 1$-gons. An m-gon is just a totality of vertices (points) and edges (lines) of ordinary cycle of length m with the natural incidence. A Moore graph of degree $k \geq 3$ has diameter 2 and $k \in \{3, 7, 51\}$.

We are interested in the case of even girth because tactical configurations are bipartite graphs and have no odd cycles. When the degree is 2, then we have a $2n$-gone which is an example of extraspecial configurations. In fact, the $(2, g)$-cage is the g-circuit, and $v(g, 2) = g$.

Let us list some well known families of cages of even girth.

(i) the $(k, 4)$-cage is the complete bipartite graph $K_{k,k}$ and $v(k, 4) = 2k$.

If $k = q + 1$ for a prime power q, then

(ii) a $(k, 6)$-cage is the incidence graph of a projective plane $PG(2, q)$, and $v(k, g) = 2(q^2 + q + 1)$;

(iii) a $(k, 8)$-cage is the incidence graph of a generalized quadrangle $CQ(q, q)$, and $v(k, g) = 2(q^3 + q^2 + q + 1)$;

(iv) a $(k, 12)$-cage is the incidence graph of a generalized hexagon $GH(q, q)$, and $v(k, q) = 2(q + 1)(q^4 + q^2 + 1)$

The $(3, 8)$-cage is the Tutte - Coxeter graph (v=30) [20].

One has $v(7, 6) = 90$ and the unique $(7, 6)$ cage was independently found in [8], [5]. Finally, there are 3 distinct $(3, 10)$- cages, all of them are bipartite [9], and $v(3, 10) = v(2, 2, 10) = 70$.

The problem of determining $v(k,g)$ was posed in 1959 by F. Kartesi who noticed that $v(3,5)=10$ was realized by the Petersen graph. Sachs showed that $v(k,g)$ is finite and Erdös and Sashs gave the upper bound. This bound was improved in [10] for the best known general bound see [14]. For the case of bipartite graphs similar problem had been considered in [12]. A lower bound is given by Tutte's formula.

Applications in Operations Research, Cryptography, Networking also need constructions of regular graphs of a given gorth with the lowest known order. There are some interesting examples of cubic graphs of that kind (see [22] and further references).

4. Bounds for the size of tactical configurations

The minimization problem for the order of a graph with prescribed bidegrees r, s and girth g is equivalent to the maximization of the size (number of edges) of a graph with parameters r, s and g. The maximal number of edges of the graph of order v without cycles C_{2k} is estimated by Erdös Even Circuit Theorem.

Let $ex(v,n)$ be, as usual, the greatest number of edges (size) in a graph on v vertices, which contains no cycles C_3, C_4, …, C_n.

As it was mentioned in the introduction, from Erdös' Even Circuit Theorem and its modifications (see [2]) it follows that

$$ex(v,2k) \leq Cv^{1+1/k} \tag{4.1}$$

where C is a positive constant.

In the case of tactical configuration with the restriction on bidegrees it is possible to get a stronger bounds than the one given by the Even Cycle Theorem.

Let us consider some corollaries of the Lemma 1. Without loss of generality we will assume $r=a^m, s=a$, where $m \geq 1$

In case of $k=2t$, we may omit all terms of the left hand side of (1.3) and (1.4) except highest terms, $a^{mt}a < p$, and $a^t a^{mt} < l$.

Adding last inequalities

we get $a^{(m+1)t} < v/2$, or $a < (v/2)^{1/((m+1)t)}$. We also have $l(a+1)=e$ or $la = e-l$. Thus $e-l < l(v/2)^{1/((m+1)t)}$.

Put v instead of l to get $e < v(v/2)^{(1/((m+1)t)} + v$, which leads to the next theorem

Theorem 3.

$$e \leq (1/2)^{(1/(m+1)t)} v^{(1+1/(m+1)t)} + v \tag{4.2}$$

Remark 2. *If $m = 1$ the magnitude of right hand side is same as that of Erdös Even Circuit Theorem, but the constant is better. The constant has monotonic dependence on m, and is always < 1. If $m > 1$, then (2.4) is stronger than Erdos inequality in magnitude. Of course (2.4) is applicable only to bipartite biregular graphs.*

Let us consider the case $k = 2t + 1$. If we discard some of the summands on the left hand side of (2.1) we get $r^t s^t + r^{t+1} s^t < p$. Set as before $r = a^m$, and $s = a$ to get $a^{mt+1}(a^m + 1) < p$. Also, $l(p + 1) = p(a^m + 1) = e$ gives $a^{mt+t}(l/p)(a + 1)l < p$ or $a^{mt+t}(a + 1)l < p^2 = l^2(a + 1)^2/(a^m + 1)^2$.
Simplifying last inequality we obtain

$$a^{mt+t}(a^m + 1)^2/(a + 1) < l.$$

Note that the function $f(a) = (a^m + 1)^2/(a + 1)$ is increasing.

Thus $f(a-1)a^{mt+t} < l$ or $a^{mt+t-1}[(a-1)^2+1] < l$. The last inequality then leads to $(a - 1)^{mt+2m+t-1} < l$ or $a - 1 < e^{(mt+2m+t-1)^{-1}}$. But we know that $l(a + 1) = e$. So $l(a - 1) = e - 2l$, and multiplication of two sides of the last inequality by l produces

$$e < l^{l+(m(t+2)+t-1)^{-1}} + 2l.$$

Order $v = p + l$ is $\geq l$, thus substitution of v instead of l gives us a slightly weaker inequality.

Theorem 4.

$$e \leq v^{1+1/(m(t+2)+t-1)} + 2v \tag{4.3}$$

Remark 3. *If $m = 1$, then the above bound has the same magnitude as that of Erdös bound in Even Circuit Theorem, but the constant is better than in (3.1). In fact we can improve the constant by substitution $l = v/2$ into inequality 3 to get.*

$$e \leq (1/2)^{1+1/(2t+1)} v^{1+1/(2t+1)} + v \tag{4.4}$$

If $m > 1$, then magnitude of (3.3) is better than that of Erdös bound.

Remark 4. *Theorems 3 and 4 give slightly better bounds than the upper bounds given in [21] (better constants but the same magnitude). This, we shall generalize for graphs with average bidegrees in next section.*

5. Bipartite graphs with given average bidegrees

Here, we assume that we have a random tactical configuration $\Delta = \Delta(l, p, a(\omega), b(\omega))$ consisting of l lines and p points laid out as a Branching Process. We shall assume, without loss of generality, that level zero consists of some line x_0, say. This line is incident to m points with probability $p(m)$, where $E(M) = \eta + 1$, where M denotes the random variable representing the outcomes m and $\eta \geq 1$ and is known. Here, E denotes the usual expectation operator.

Now, let ${X_n}^l$,(${X_n}^p$) be the number of lines (points, respectively) at level n. We shall assume from level 1 onwards that each line is incident to $a(\omega)$ points with probability $p(a(\omega))$, where $a(\omega)$ takes the integer values $0, ..., p$, with $E(a) = \eta$. Similarly, we have each point is incident to $b(\omega)$ lines with probability $p(b(\omega))$, with $E(b) = \xi$,where $\xi \geq 1$ and known.

If the girth of our graph is $> 2t$, then there is at most one pass between any two vertices at a distance $\leq t$. Points of level k are precisely at distance $2k + 1$ from the initial line. The line of level k are at distance $2k$. Thus, computation of$X_k l$, $2k \leq k$ and X_k^p can be done by branching process.

We have

$$\begin{aligned} X_0^p &= M, \\ X_n^p &= \sum_{i=1}^{X_n^l} Z_i \\ X_n^l &= \sum_{j=1}^{X_{n-1}^l} Y_j \end{aligned}$$

where $Z_i's$ are i.i.d random variables, with mean η and variance σ_Z^2, corresponding to $a(\omega)$. The variables Y_j are i.i.d random variables corresponding to $b(\omega)$, with mean ξ and variance σ_Y^2. We shall be interested in finding a closed form for the means and the variances of the random variables X_n^p, X_n^l defined above.

The next two lemmas provide an answer to our query.

Lemma 2. $X_0^l = 1$,

(i) $E[X_n^p] = (\eta + 1)(\eta\xi)^n, \;\; n = 1, ...$

Proof. The proof is standard: see Karlin and Taylor [7] for similar ideas. Note that

$$E[X_n^p] = E[E[X_n^p | X_n^l]].$$

Now, consider $E[X_n^p | X_n^l = x] = E[\sum_{i=1}^{x} Z_i] = xE[Z] = \eta x$,
because of the independence of Z_i. Hence

$$E[X_n^p] = \eta E[X_n^l]. \tag{5.1}$$

Now, we compute $E[X_n{}^l]$. Again $E[X_n^l] = E[E[X_n^l | X_{n-1}^p]]$. But,

$$E[X_n^l | X_{n-1}^p = x] = E[\sum_{j=1}^{x} Y_j] = \xi x,$$

by the independence, whence

$$E[X_n^l] = \xi E[X_{n-1}^p]. \tag{5.2}$$

by independence. Hence, combining (5.1) and (5.2) we get

$$E[X_n^p] = \eta\xi E[X_{n-1}^p] = (\eta\xi)^n E[X_0^p]$$

But $E[X_0^p] = \eta + 1$. Hence,

$$E[X_n^p] = (\eta + 1)(\eta\xi)^n. \tag{5.3}$$

To show part (ii) note that $E[X_n^l] = \xi E[X_{n-1}^p] = (\eta+1)\xi(\eta\xi)^{n-1}$, by (5.3), as required. □

The next lemma gives a bound on the variances of the random variables X_n^p and X_n^l.

Lemma 3.

(i) $Var(X_n^p) \leq \tilde{V}\left\{(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-1}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\}$

(ii) $Var(X_n^l) \leq \tilde{V}\left\{\xi(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-2}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\}$,

where $\tilde{V} = \max\{Var(X), Var(Z), Var(X_0^p), Var(X_0^l)\}$.

Proof. We shall only prove (i). The proof of (ii) is similar. Note that

$$Var(X_n^p) = E[(X_n^p)^2] - (E[X_n^p])^2. \tag{5.4}$$

Let us compute $E[(X_n^p)^2]$. We have $E[(X_n^p)^2] = E[E[(X_n^p)^2 | X_n^l]]$, and

$$\begin{aligned} E[(X_n^p)^2 | X_n^l &= x] = E\left[(\sum_{i=1}^{x} Z_i)^2\right] = Var\left[\sum_{i=1}^{x} Z_i\right] + \left(E\left[\sum_{i=1}^{x} Z_i\right]\right)^2 \\ &= xVar(Z) + (x\eta)^2, \end{aligned}$$

by independence. Hence

$$E[(X_n^p)^2] = Var(Z)E[X_n^l] + \eta^2 E[(X_n^l)^2]. \tag{5.5}$$

Now, we are required to compute $E[(X_n^l)^2]$. The same argument used above gives

$$E[(X_n^l)^2] = Var(Y)E[X_{n-1}^p] + \xi^2 E[(X_{n-1}^p)^2]. \tag{5.6}$$

Combining (5.4), (5.5), and (5.6) gives

$$\begin{aligned} Var(X_n^p) &= Var(Z)E[X_n^l]+ \\ &\quad +\eta^2 \left\{Var(Y)E[X_{n-1}^p] + \xi^2 E[(X_{n-1}^p)^2]\right\} - (E[X_n^p])^2. \end{aligned}$$

This can be shown to be equal to:

$$Var(Z)E[X_n^l] + \eta^2 Var(Y)E[X_{n-1}^p] + (\eta\xi)^2 Var(X_{n-1}^p).$$

Using the previous lemma the above is less than or equal to:

$$\max\{Var(Y), Var(Z)\}(\eta+1)(\eta\xi)^{n-1}(\xi+\eta^2) + (\eta\xi)^2 Var(X_{n-1}^p).$$

Now, we use induction to get

$$Var(X_n^p) \leq \tilde{V}\left\{(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-1}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\},$$

where $\tilde{V} = \max\{Var(X), Var(Z), Var(X_0^p), Var(X_0^l)\}$, as required. □

The next lemma (which is a direct consequence of lemmas 3 and 4 and Chebeychev inequality: see [7]) gives confidence intervals for both X_n^p and X_n^l.

(i) The confidence interval for X_n^p is

$$(\eta+1)(\eta\xi)^n \pm k s_p,$$

for some nonnegative $k > 0$ and

$$s_p = (\tilde{V}\left\{(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-1}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\})^{1/2}. \tag{5.7}$$

(ii) The confidence interval for X_n^l is

$$(\eta +)\xi(\eta\xi)^{n-1} \pm k' s_l,$$

for some nonnegative $k > 0$, and

$$s_l = (\tilde{V}\left\{\xi(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-2}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\})^{1/2}. \tag{5.8}$$

Remark 5. *Note that (4.7) shows that the order of X_n^p is at most $O((\eta+1)(\eta\xi)^n)$, while the order of X_n^l is at most $O((\eta+1)\xi(\eta\xi)^{n-1})$.*

6. On the size of general bipartite graphs

In this section we will consider upper bounds for the size of bipartite graphs G_i of increasing order $v = v_i$ without cycles of girth $g > 2k$ satisfying inequality $\eta_i \geq \xi_i^m$, where $m \geq 1$ is some positive real number.

and superlinear size without cycles C_{2k}.

We have a free choice which partition set is the point set. So we may assume that $\eta_i \geq \xi_i$ (the average degree for points is greater than or equal to average degree of lines). Thus our result for $m = 1$ estimates size of general bipartite graphs of a given girth.

Theorem 5. *Let G_i, $i = 1, \ldots$ be a family of bipartite graphs without even cycles $C_4, \ldots, C_{2k}$ such that average degrees η_i and ξ_i of lines and points satisfy the inequality: $\eta_i \geq {\xi_i}^m$. Then, we have:*

(i) $e \leq (1/2)^{(1/(m+1)t)} p v^{(1/(m+1)t)} + O(v)$

(ii) $e \leq p^{1+1/(m(t+2)+t-1)} + O(v)$

hold for cases $k = 2t$ and $k = 2t+1$ respectively.

Proof. Let G_i, $i = 1, \ldots$ be a family of graphs satisfying the restrictions on the bidegrees and the girths as above. It follows from [12], that the size of the bipartite graphs of a given girth, with the restrictions on the bidegrees as stated above, is superlinear function Cv^α, $\alpha > 1$ of order v. Thus, we may assume that function ξ_i is unbounded. Else, we may bound the number of edges of the graphs by a linear expression in v. In fact, we shall conduct all computations up to $O(v)$. We will also keep the notations of the previous section: η_i and ξ_i will be the average degrees for the lines and the points respectively. Without loss of generality we may assume $\eta_i \geq \xi_i$. $i = 1, \ldots$.

Let us consider case $k = 2t$. In this case, there is not more then one pass of length $\leq 2t$ between two given elements at distance $2t$. Hence, we may apply result (4.8) to get, if ξ is "sufficiently large"

$$X_t^l = (\eta+1)\xi(\eta\xi)^{t-1} - C_1\eta^{(t-1/2)}\xi^{n-3/2} \leq l.$$

The expression one left hand side gives us the number of lines at distance $2t$ from chosen line, where C_1 is some constant. If we swap points and lines together with their average degrees we get"

$$(\xi+1)\eta(\eta\xi)^{t-1} - (\xi\eta)^{t-1} \leq p.$$

Addition of last two inequalities gives us

$$2(\eta\xi)^t + [(\eta+\xi)(\eta\phi)^{t-1} - C_1(\eta\xi)^{t-1}(\eta/\xi)^{1/2} - (\eta\xi)^{t-1}] \leq (p+l) = v$$

when ξ is sufficiently large, expression in parenthesis is positive and we are getting

$$2(\eta\xi)^t < v.$$

For $\xi = a$ and $\eta \geq a^m$ we may write

$$a < (v/2)^{1/(m+1)^t}.$$

Analogously to similar case for biregular graphs we are getting $pa \leq p(v/2)^{1/(m+1)^t}$. The last inequality together with $p(a+1) = e$ gives us the following bound for the size e

$$e < p(v/2)^{1/(m+1)t} + O(v).$$

Let us consider the case $k = 2t+1$. It follows from (4.8) that X_t^l is at least $(\eta+1)\eta^t\xi^t - C\eta^t\xi^{t-3/2}$, where C is some positive constant. Thus instead of the inequality ${X^l}_t + {X^l}_{t-1} \leq l$ we can write:

$$\eta^{t+1}\xi^t + \eta^t\xi^t + (\eta^t\xi^{t-1} + \eta^{t-1}\xi^{t-1} - C\eta^t\xi^{t-3/2}).$$

For sufficiently large ξ, the expression in brackets in the previous formula will be negative and we get $\eta^{t+1}\xi^t + \eta^t\xi^t < l$. Setting as before $\eta \geq a^m$, $\xi = a$. Thus $a^{mt+1}(a^m+1) < l$. From $p\xi = l\eta$, we get $e = p(a+1) > l(a^m+1)$ and $(a^m+1) \leq p(a+1)/la^{mt+t}(p/l)(a+1)l < l$ or

$$^{mt+t}(a+1)l < l^2 = p^2(a+1)^2/(a^m+1)^2a.$$

Simplifying the last inequality we obtain

$$a^{mt+t}(a^m+1)^2/(a+1) < p.$$

We can notice that the function $f(a) = (a^m+1)^2/(a+1)$ is increasing. Thus $f(a-1)a^{mt+t} < l$ or $a^{mt+t-1}[(a-1)^2+p] < l$. From the last inequality we get $(a-1)^{mt+2m+t-1} < l$ or $a-1 < l^{(mt+2m+t-1)^{-1}}$. We know that $p(a+1) = e$. So, $p(a-1) = e - 2l$ and multiplication of the two sides of the last inequality by l gives $e < p^{p(m(t+2)+t-1)^{-1}} + O(p)$ of lines and the number of pints of G_i satisfy the inequality $\eta_i \geq {\xi_i}^m$ for certain real number $m > 1$. Then, $e \leq (1/2)^{1+1/k} v^{1+1/k} + O(v)$ in the case of k even and $e \leq v^{1+1/k} + O(v)$ if k is odd. □

Remark 6. *The bounds in the theorem 5 are sharp up to constant when we deal with families of generalized $k+1$-gons. In particular for $m > 1$, we have the following list:* $(m = 2, k = 3)$, $(m = 3/2, k = 3$, $(m = 3/2, k = 5)$, $(m = 2, k = 7)$.

7. On the size of general graphs of high girth

In this section, we shall be concerned with the size of graphs of large girth as function of the order.

Theorem 6. *Let $F = \{G_j\}$, $j = 1, \ldots$ be a family of bipartite graphs without of cycles C_i, $3 \leq i \leq n$, $n \geq 3$, that is, a family of graphs of girth $g > n$. Let e and v be the size and the order of graphs from F respectively.*

Then
(i)$e \leq (1/2) v^{1+1/k} + O(v)$ *for* $n = 2k$
(ii) $e \leq (1/2)^{1+1/k} v^{1+1/k} + O(v)$ *for* $n = 2k+1$

Proof. Let η be the average degree of graph G_i. Let us consider the case $n = 2t+1$. If the girth $g \geq 2k+2$, then there is at most 1 pass between vertices at distance k, we can choose adjacent vertices v and u and count the number of passes at distance $\leq t$ via branching process. Let $Y_l(u)$ $(Y_l(w))$be the totality of vertices x of length l, $l \leq k+1$ such that the pass between u and v does not contain w (u respectively). It is clear, that $|Y_l(u)| = |Y_l(v)| = y_l$. We have $|Y_l(u) \cap Y_s(v)| = 0$, because common point for $Y_l(u)$ and $Y_s(w)$ corresponds to cycle of length $l+s+1 \leq 2k+1$. The induced subgraph of G_i with the union of all $Y_l(u)$ and $Y_s(v$ is a tree which is a bipartite graph. Thus, we can estimate the number y_l via the technique of section 5. We need just to take in account that in our case $\eta = \xi$ and at the first step of branching process we have η instead of $\eta + 1$. Thus $y_l \geq \eta^{l-1} - C\eta^{l-2}$, $l = 1, \ldots, k+1$. After summation of the above inequalities and multiplication by 2 we get that the number Ind of all vertices for our tree is at least

$$2\eta^k + 2(1 + \eta + \ldots \eta^{t-1} - C1\eta^{t-2}).$$

If parameter i for G_i and related η are "sufficiently large" then The expression in brackets above will be positive. Thus $\eta k < Ind < (v/2)$,

$\eta < (v/2)^{1/k}$ and $(v/2)\eta(v/2)^{1+1/k}$. But $(v/2)(\eta+1) = e$ or $(v/2)\eta = e - v/2$. Finally $e < (1/2)^{1+1/k}v^{1+1/k} + O(v)$ and the statement (i) of the theorem is proven.

Let us consider the case of $n = 2k$. There is at most one pass between two given vertices at the distance l, $l \leq 2k$ otherwise we have a cycle C_{2l-2}. Thus we may choose a vertex v and count number X_l vertices at distance l from v by branching process with $\eta = \xi$. As it follows from results of section 4 s_p (s_l, respectively) is less then $C\eta^{k-2}$, where k is a highest degree of η in the expression for X_p^n (respectively X_l^n) for appropriate n , where C is a certain constant. Thus

$$X_l \geq (\eta+1)\eta^{k-1} - C\eta^{k-2}.$$

So from $X_l + X_{l-1} \leq v$ we can obtain

$$\eta^k + [2\eta^{k-1} + \eta^{k-2} - C1\eta^{k-2}] \leq v.$$

If η is sufficiently large then the expression in parenthesis is positive and we can write simply $\eta^k \leq v$. We can get $\eta \leq v^{1/k}$. Multiplication by v of both sides of last inequality together with $2e = (\eta+1)v$ gives us $e \leq 1/2v^{1+1/k} + O(v)$. □

8. Conclusion

Let us reformulate main results in terms of ex notations.

We presented an upper bound on the $\mathrm{ex}(v, C_3, \ldots, C_{2n})$ and a bound $\mathrm{ex}_{P(m)}(v, C_3, \ldots, C_t)$, where $P(m)$ is a property of the bipartite graph whose average bidegrees η and ξ satisfy the inequality $\eta \geq \xi^m$, $m \geq 1$. We proved that the sizes of the tactical configurations of finite generalized polygons are exactly on that bound. In fact, we proved that finite generalized polygons have minimal possible order among tactical configurations of the same bidegrees and girth.

Upper bounds for $ex(v, C_{2n})$ are known to be sharp up to constant in case of $n \in \{2, 3, 5\}$. The question on the sharpness of this bound for other n is still open. We conjecture that our bound for the $\mathrm{ex}_{P(m)}(v, C_3, \ldots C_{2m})$ is sharp if and only if (m, n) belongs to the following list: $(2, 3), (3/2, 3), (3/2, 5), (2, 14)$.

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RESEARCH ARTICLE

On dispersing representations of quivers and their connection with representations of bundles of semichains

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Abstract. In the paper we discuss the notion of "dispersing representation of a quiver" and give, in a natural special case, a criterion for the problem of classifying such representations to be tame. In proving the criterion we essentially use representations of bundles of semichains, introduced about fifteen years ago by the author.

1. Introduction

The classical problems of linear algebra on the reduction of the matrix of a linear map (by means of elementary row and column transformations) and the matrix of a linear operator (by means of similarity transformations) to canonical forms can be generalized in the following two ways: by considering a greater number of maps or giving more complicated structure of vector spaces. The first way led finally to the notion of representations of a quiver (P. Gabriel). As examples of a generalization of the second type it may be mentioned the well-known vectorspace problem [1, p. 82], its natural "two-dimensional" analog [2, 3] and a general extension of the classical problem on one linear operator [4, 3]. Clearly one can consider various generalizations of the classical problems combining two indicated ways. In [3] the author consider a common generalization

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of “mixed” type introducing the notion of “dispersing representation of a quiver (with relations)”. In terms of these representations one can formulate many classification problems, among them the problems on representations of posets [5], bundles of semichaines [6], tangles [7], etc. (and also all the above mentioned ones). In this paper we study dispersing representations of (finite and infinite) quivers without relations. In considering criterions of tameness we essentially use a main result on representations of bundles of finitely many semichains [8, 6] and his extension to the case of infinitely many ones (see the last section).

2. Main notions and examples

Throughout the paper, we will keep the right-side notation. All vector spaces over a field k will be finite-dimensional; the category of such spaces will be denoted as usual by $\mathrm{mod}\, k$. Unless otherwise stated, all quivers and posets will be finite. The sign $\coprod$ will denote the direct sum of posets, categories or functors. Singletons will be always identified with the elements themselves.

We first recall the definition of dispersing representations of a quiver [3, Section 10].

Let $\mathcal{A}$ be a Krull-Schmidt category over a field k. By a (right) module over $\mathcal{A}$ we mean as usual a k-linear functor $F : \mathcal{A} \to \mathrm{mod}\, k$. A collection $M = \{M_i\}$ of modules $M_i : \mathcal{A} \to \mathrm{mod}\, k$, where i run through a set X, is said to be an *X-bunch of modules over $\mathcal{A}$*. An X-bunch M is said to be *faithful* if $\mathrm{Ann} M = \bigcap_{i\in X} \mathrm{Ann} M_i = 0$ ($\mathrm{Ann} M_i$ being the annihilator of M_i). We call X-bunches M and M' of modules over $\mathcal{A}$ and $\mathcal{A}'$, respectively, *equivalent* if there exists an equivalence $F : \mathcal{A} \to \mathcal{A}'$ such that, for each $i \in X$, the modules M_i and FM'_i are isomorphic; in this case we write $M \cong M'$ or $(\mathcal{A}, M) \cong (\mathcal{A}', M')$.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a (not necessarily finite) quiver with the set of vertices (points) Γ_0 and the set of arrows Γ_1, and k a field. Fix a Krull-Schmidt category $\mathcal{A}$ over k and a Γ_0-bunch M of modules over $\mathcal{A}$. We call *M-dispersing representation* of Γ (or *dispersing representation with respect to M*, or simply *dispersing representation* if M is fixed) a pair $U = (M(X), f)$ formed by the collection of vector spaces $M(X) = \{M_i(X)|\, i \in \Gamma_0\}$ for an object $X \in \mathcal{A}$ and a collection $f = \{f_\alpha |\, \alpha : i \to j \text{ run through } \Gamma_1\}$ of linear maps $f_\alpha : M_i(X) \to M_j(X)$. A morphism from $U = (M(X), f)$ to $U' = (M(X'), f')$ is determined by a morphism $\varphi : X \to X'$ satisfying $f_\alpha M_j(\varphi) = M_i(\varphi) f'_\alpha$ for each arrow $\alpha : i \to j$. The category of M-dispersing representations of Γ is denoted by $\mathrm{rep}_M \Gamma$; by $\mathrm{rep}^{\mathrm{inv}}_M \Gamma$ we denote the full subcategory of $\mathrm{rep}_M \Gamma$ consisting of all objects $U = (M(X), f)$ with invertible linear maps f_α (α runs over Γ_1).

If we take $\mathcal{A} = \coprod_{i\in\Gamma_0} \mathcal{A}_i$ with $\mathcal{A}_i = \operatorname{mod} k$ for each i, and $M_i = \coprod_{j\in\Gamma_0} M_{ij}$ with $M_{ij} = \delta_{ij}\mathbf{1}_{\mathcal{A}_j} : \mathcal{A}_j \to \operatorname{mod} k$ (δ_{ij} being the Kroneker delta), then the case of usual representations of Γ occurs.

Our notion is naturally generalized to the case of quivers with relations. Moreover, one can take any ring instead of the field k, an arbitrary category instead of the category $\mathcal{A}$ or $\operatorname{mod} k$, etc.

In terms of dispersing representations one can formulate many classification problems.

Example 2.1. Let Γ be the quiver $\overset{1}{\circ}\!\longrightarrow\!\overset{2}{\circ}$ and C a finite poset which is identified with the following category: $\mathrm{Ob}C = C, C(x,y) = \{(x|y)\}$ if $x \le y$ and $C(x,y) = \varnothing$, otherwise; composition is such that $(x|y)(y|z) = (x|z)$. Denote by $\mathcal{C}$ the category $\oplus kC$ (kC being the linearization of C and $\oplus kC$ its additive hull) and by N the module over $\mathcal{C}$ such that $N(x) = k$ for each $x \in C$ and $N(x|y) = \mathbf{1}_k$. Set $\mathcal{A} = \mathcal{B}\coprod\mathcal{C}$ with $\mathcal{B} = \operatorname{mod} k$, and $M_1 = \mathbf{1}_{\mathcal{B}}\coprod\mathbf{0}_{\mathcal{C}}$, $M_2 = \mathbf{0}_{\mathcal{B}}\coprod N$ with the identity module $\mathbf{1}_{\mathcal{B}} : \mathcal{B} \to \operatorname{mod} k$ and the zero ones $\mathbf{0}_{\mathcal{C}} : \mathcal{C} \to \operatorname{mod} k$, $\mathbf{0}_{\mathcal{B}} : \mathcal{B} \to \operatorname{mod} k$. Then the category of $\{M_1, M_2\}$-dispersing representations of Γ is in fact the category of representations of the poset C [5, §4].

A general case of a "decomposable" bunch (as in the example) arise, in other terms, in studying representations of dyadic sets [9, Section 0].

From the point of view of the author, the most interesting cases occur when (in contrast to the previous case) a system M of modules is not "decomposable" or there is a quiver with relations.

Before discussing such examples we give some definitions.

Let $S = (A, *)$ be a (not necessarily finite) poset with involution. By an *S-graded vector space over k* we mean the direct sum $U = \bigoplus_{a\in A} U_a$ of k-vector spaces U_a such that $U_{a^*} = U_a$ for all $a \in A$. A linear map φ of an S-graded space $U = \bigoplus_{a\in A} U_a$ into an S-graded space $U' = \bigoplus_{a\in A} U'_a$ will be called an *S-map* if $\varphi_{a^*a^*} = \varphi_{aa}$ for each $a \in A$ and $\varphi_{bc} = 0$ for each $b, c \in A$ not satisfying $b \le c$, where φ_{xy} denotes the linear map of U_x into U'_y induced by the map φ. The category of S-graded vector spaces over k (with objects the S-graded spaces and with morphisms the S-maps) is denoted by $\operatorname{mod}_S k$ [1]. Because $S = (A, *)$ with trivial involution is naturally identified with A, these definitions involve the case of usual posets. For a poset $A = \coprod_{i=1}^n A_i$, we identify $\operatorname{mod}_A k$ with $\coprod_{i=1}^n \operatorname{mod}_{A_i} k$.

Recall that a *semichain* is by definition a poset A such that every element of A is comparable with all but at most one elements. Obviously, any semichain A can be uniquely represented in the form $A = \bigcup_{i=1}^m A_i$,

[1]When S is infinite and $U \in \operatorname{mod}_S k$, we have $U_a = 0$ for all but finitely many $a \in A$ (because we consider only finite-dimensional vector spaces).

where each A_i (called a *link* of A) consist of either one point or two incomparable points, and $A_1 < A_2 < \cdots < A_m$, where, for subsets X and Y of a poset, $X < Y$ means that $x < y$ for any $x \in X, y \in Y$ (if each A_i consist of one point, the set A is called a *chain*); the number m is called the *length* of A. A semichain A with involution $*$ is called a $*$-*semichain* if $x^* = x$ for every x belonging to the union of all two-point links.

Example 2.2. Let Γ be the quiver with one vertex, one loop φ and one relation $f(\varphi) = 0$, where $f(t) = t^2$, and let $S = (A, *)$ be a poset with involution. Set $\mathcal{A} = \operatorname{mod}_S k$ and denote by $M : \operatorname{mod}_S k \to \operatorname{mod} k$ the natural imbedding module. In the case when S is a $*$-semichain, M-dispersing representations of Γ was classified in [10, §2] (in connection with classifying the modular representations of quasidihedral groups); the case of a chain with involution was considered earlier in [11, §1]. The case, when $\mathcal{A}$ is an arbitrary Krull-Schmidt subcategory in $\operatorname{mod} k$ and $f(t)$ an arbitrary polynomial, is considered in [4, 3].

Finally we consider an example which plays a central role in our consideration.

Example 2.3. Let $S = \{A_1, \ldots, A_n, B_1, \ldots, B_n\}$ be a family of pairwise disjoint semichains; set $A = \coprod_{i=1}^n A_i$ and $B = \coprod_{i=1}^n B_i$. A *bundle of semichains* $A_1, \ldots, A_n, B_1, \ldots, B_n$ is a pair $\overline{S} = (S, *)$, where $*$ is an involution on the set $S_0 = A \coprod B$ such that $x^* = x$ for each x from the union of all two-point links (of the given semichains).

Let $\overline{S} = (S, *)$ be a bundle of semichaines $A_1, \ldots, A_n, B_1, \ldots, B_n$. A representation of the bundle $\overline{S} = (S, *)$ over a field k is a triple (U, V, φ), where

(1) $U = \{U_1, \ldots, U_n\}$ and $V = \{V_1, \ldots, V_n\}$ are collections of k-spaces such that $U_i \in \operatorname{mod}_{A_i} k$, $V_i \in \operatorname{mod}_{B_i} k$ $(i = 1, \ldots, n)$, and the $A \coprod B$-graded space $(\bigoplus_{i=1}^n U_i) \oplus (\bigoplus_{i=1}^n V_i)$ belong to the subcategory $\operatorname{mod}_{(A \coprod B, *)} k$ of $\operatorname{mod}_{A \coprod B} k$;

(2) $\varphi = \{\varphi_1, \ldots, \varphi_n\}$ is a collection of linear maps $\varphi_i \in \operatorname{Hom}_k(U_i, V_i)$, $i = 1, \ldots, n$.

A morphism from

$$(U, V, \varphi) = (\{U_1, \ldots, U_n\}, \{V_1, \ldots, V_n\}, \{\varphi_1, \ldots, \varphi_n\})$$

to

$$(U', V', \varphi') = (\{U'_1, \ldots, U'_n\}, \{V'_1, \ldots, V'_n\}, \{\varphi'_1, \ldots, \varphi'_n\})$$

is determined by a pair (α, β) formed by a collection $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ of A_i-maps α_i: $U_i \to U'_i$ and a collection $\beta = \{\beta_1, \ldots, \beta_n\}$ of B_i-maps β_i: $V_i \to V'_i$ $(i = 1, \ldots, n)$ such that

(3) the $A\coprod B$-map $(\bigoplus_{i=1}^n \alpha_i)\oplus(\bigoplus_{i=1}^n \beta_i)$ of $(\bigoplus_{i=1}^n U_i)\oplus(\bigoplus_{i=1}^n V_i)$ into $(\bigoplus_{i=1}^n U_i')\oplus(\bigoplus_{i=1}^n V_i')$ belong to the subcategory $\mathrm{mod}_{(A\coprod B,*)}\,k$;

(4) $\varphi_i\beta_i=\alpha_i\varphi_i'$ for each $i=1,\dots,n$.

The category of representations of the bundle of semichains $\overline{S}=(S,*)$ is denoted by $\mathcal{B}_k\,(\overline{S})=\mathcal{B}_k\,(S,*)=\mathcal{B}_k\,(A_1,\dots,A_n,B_1,\dots,B_n,*)$.

The definition of representations of bundles of semichaines can be easily rewrited in terms of dispersing representations.

Denote by $\Lambda(n)$ the quiver with the set of vertices

$$\Lambda_0(n)=\{1^-,\dots,n^-,1^+,\dots,n^+\}$$

and the arrows $(i^-,i^+): i^-\to i^+$ for $i=1,\dots,n$. In our new terms, a representation of the bundle $\overline{S}=(S,*)$ is a P-dispersing representation of $\Lambda(n)$ with the category $\mathcal{S}=\mathcal{K}(\overline{S})=\mathrm{mod}_{(A\coprod B,*)}k$ (as $\mathcal{A}$) and the modules $P_i=P_i(\overline{S}):\mathcal{S}\to\mathrm{mod}\,k$ (i run through $\Lambda_0(n)$) to be the composition of the natural embedding of $\mathcal{S}$ in $\mathcal{S}_0=\mathrm{mod}_{A\coprod B}k$ and the projection of $\mathcal{S}_0$ onto $\mathrm{mod}_{A_i}k$ (resp. $\mathrm{mod}_{B_i}k$) for $i=j^-$ (resp. $i=j^+$). Obviously, the category $\mathcal{B}_k\,(\overline{S})$ is isomorphic to the category $\mathrm{rep}_P\Lambda(n)$ with $P=\{P_i|i\in\Lambda_0(n)\}$.

The representations of a bundle of semichains (and the notion of "bundle of semichains" itself) were introduced in [6, §1] (for the first time, in [8]). In these papers the author give (in terms of matrices) a complete classification of the indecomposable representations of an arbitrary bundle of semichains; the classifying is obtained in the explicit and invariant (without "trace" of the method of solution) form.

In special case, when there is only two semichaines, representations of bundles arose under consideration a problem of I. M. Gelfand [12][2], in the classification of the modular representations of quasidihedral groups [13, 10] (see also [6, §2]) and in studying numerous other problems: in studying representations of different classes of quivers with relations and algebras (see e.g. [14, 15, 16, 17, 18]), in the classification of faithful posets of infinite (non-polynomial, in other terminology) growth [19], under consideration representations of posets with involution [20] and equivalence relation [21]. In studying representations of posets with non-singularity conditions [22, 23] there arose representations of bundles of four semichaines. For an arbitrary (even) number of semichaines, representations of bundles arose first in solving the Gelfand problem and its generalizations [6, §3]. Recently the main classification theorem of [6, §1] is used in solving various classification problems of representation theory,

[2]The tameness of the problem under consideration also follows from properties of an algorithm described in [12, §2]), but an inductive answer indicated there (for two semichaines, if one use our terminology) is false.

topology and algebraic geometry (see e.g. [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

The main reason of wide application of representations of bundles of semichains is that, for many classification problems, "most" of tame cases are reduced to them (such a reduction is today the only method of solving many problems of infinite growth, as for representations of quasidihedral groups [10] or partially ordered sets [19]). As will be seen below, this is also true for dispersing representations of quiver.

3. Main result

We assume from now on that k is an algebraic closed field. For a Krull-Schmidt category $\mathcal{A}$ (over k), we denote by $\mathcal{A}_0$ a fixed full subcategory of $\mathcal{A}$ formed by chosen representatives of all isomorphism classes of indecomposables; we will assume throughout this section that $|\mathrm{Ob}\mathcal{A}_0| < \infty$ (for the case $|\mathrm{Ob}\mathcal{A}_0| = \infty$ see the next section).

Let N be a module over $\mathcal{A}$, and define

$$\mathrm{supp}_0 N = \{X \in \mathrm{Ob}\mathcal{A}_0 | \, N(X) \neq 0\};$$

for $X, Y \in \mathrm{Ob}\mathcal{A}_0$, set $N(X,Y) = N(\mathcal{A}_0(X,Y))$. We call N *saturated* if $\dim_k N(X,X) = \dim_k N(X)(\dim_k N(X) - 1)/2 + 1$ for any $X \in \mathrm{supp}_0 N$ and $\dim_k N(X,Y) + \dim_k N(Y,X)$ is equal to 0 or to $\dim_k N(X) \dim_k N(Y)$ for any distinct $X, Y \in \mathrm{supp}_0 N$ (i.e. nonzero $\dim_k N(X,X)$ and $\dim_k N(X,Y) + \dim_k N(Y,X)$ take the greatest possible values). For $X \in \mathrm{Ob}\mathcal{A}_0$, we will denote by N^X the submodule of N generated by $N(X)$. Let $\mathcal{L}(N)$ denote the lattice of submodules in N ordered by inclusion. We call N *lattice-chained* (resp. *lattice-semichained*) if $\mathcal{L}(N)$ is a chain (resp. a semichaine), and *chained* (resp. *semichained*) if in addition it is saturated. Finally, we say that a submodule N' of N is *singular* if it is comparable (in $\mathcal{L}(N)$) to each submodule of N.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver. For an arrow α, denote by $s(\alpha)$ and $e(\alpha)$ its starting point and its endpoint, respectively. By $w^-(i)$ (resp. $w^+(i)$), where $i \in \Gamma_0$, denote the number of arrows α with $s(\alpha) = i$ (resp. $e(\alpha) = i$); put $w(i) = w^-(i) + w^+(i)$. A vertex i is said to be *trivial* if $w(i) = 0$, *outer* if $w(i) = 1$ and *inner* if $w(i) > 1$. The sets of all trivial, outer and inner vertices are denoted by Γ_0^0, Γ_0^1 and Γ_0^2, respectively. Let $M = \{M_i\}$ be a fixed Γ_0-bunch of $\mathcal{A}$-modules. We call M_i *isolated* if $\mathrm{supp}_0 M_i \cap \mathrm{supp}_0 M_j = \varnothing$ for any $j \neq i$. An isolated chained module $M_i \neq 0$ with $\dim_k M_i(X) \leq 1$ for any object $X \in \mathcal{A}_0$ is said to be *elementary*.

We call Γ *M-tame* (resp. *M-wild*) if so is the problem of classifying the objects of the category $\mathrm{rep}_M \Gamma$ [35]; a quiver of *M-finite* (*M-infinite*) *type* is defined similarly. Further, we call Γ M-inv-*wild* if the problem of classifying the object of the category $\mathrm{rep}_M^{\mathrm{inv}}\Gamma$ is wild. In considering these problems, it is obviously sufficient to confine oneself to quivers without trivial vertices.

Our main result is the following theorem.

Theorem 3.1. *Let* Γ *be a finite* (*not necessarily connected*) *quiver without trivial vertices and* $M = \{M_i\}$ *a* Γ_0*-bunch of nonzero* $\mathcal{A}$*-modules without elementary ones for outer vertices. Then* Γ *is* M*-tame if and only if the following conditions hold*:

(1) $w(i) \leq 2$ *for any* $i \in \Gamma_0$;

(2) *the module* M_i *is semichained for each* $i \in \Gamma_0^1$ *and is simple and isolated for each* $i \in \Gamma_0^2$;

(3) $\sum_{i\in\Gamma_0^1} \dim_k M_i(X) \leq 2$ *for each object* $X \in \mathcal{A}_0$; *moreover, when* $\dim_k M_j(X) = \dim_k M_s(X) = 1$ *for* $j \neq s$, *the submodules* $M_j^X \subseteq M_j$ *and* $M_s^X \subseteq M_s$ *are both singular.*

Otherwise, the quiver Γ *is* M-inv-*wild.*

Note that in all cases Γ is of M-infinite type.

Sketch of proof. We may assume $\Gamma_0^1 = \Gamma_0$, because otherwise one can take the new quiver $\vec{\Gamma}$ with $\vec{\Gamma}_0 = \{\alpha^-, \alpha^+ | \, \alpha \in \Gamma_1\}$, $\vec{\Gamma}_1 = \{\overline{\alpha} : \alpha^- \to \alpha^+ | \, \alpha \in \Gamma_1\}$ and the $\vec{\Gamma}_0$-bunch of $\mathcal{A}$-modules $\vec{M}$ with $\vec{M}_{\alpha^-} = M_{s(\alpha)}$, $\vec{M}_{\alpha^+} = M_{e(\alpha)}$ (taking into account that Γ is M-tame iff $\vec{\Gamma}$ is $\vec{M}$-tame). Then (1)–(3) imply that $(\mathcal{A}/\mathrm{Ann}M, M) \cong (\mathcal{K}(\overline{S}), P(\overline{S}))$ for a bundle of semichaines $\overline{S} = (S, *)$ with $S = \{A_\alpha, B_\alpha | \, \alpha \in \Gamma_1\}$, and it follows from [6, §1] that Γ is M-tame (of M-infinite type). The proof of the fact that Γ is M-wild if the condition (1), (2) or (3) does not hold is divided into several steps.

Step 1. Let $S = \{A_1, \ldots, A_n, B_1, \ldots, B_n\}$ be a family of pairwise disjoint posets. We call $*$-*bundle* (or *involution bundle*) *of* these *posets* a pair $\overline{S} = (S, *)$, where $*$ is an involution on $S_0 = A \coprod B$ ($A = \coprod_{i=1}^n A_i$, $B = \coprod_{i=1}^n B_i$). $\overline{S}$ is said to be *nodal* if $x^* \neq x$ implies that x is comparable to any element of his poset. Nonempty A_i or B_i is said to be *elementary* if it is a chain with all elements being involutory to themselves. We say "bundle of semichaines" instead "nodal $*$-bundle of semichaines". *Representations of a* $*$*-bundle* $\overline{S}$ are defined in the same way as those of a bundle of semichains.

We have the following statement: a $*$-bundle $\overline{S}$ of nonempty and nonelementary posets is wild if (a) there is a poset A_i or B_i which is not semichained, or (b) the bundle is not nodal.

Present the idea of the proof. For $x, y \in S_0$, we write $x \sim_* y$ iff $x = y$ or $x^* = y$, and $x — y$ iff, for some i, $x \in A_i, y \in B_i$ or $x \in B_i, y \in A_i$;

put $r_*(x) = |\{y|y \sim_* x\}|$, and, for $X \subseteq S_0$, $r_*(X) = \max_{x\in X} r_*(x)$. The notation $X — Y$ for subsets X, Y of S_0 means that $x — y$ for any $x \in X$, $y \in Y$. A chain $\{1 < 2 < \ldots < p\}$ is denoted by $\langle p\rangle$ and a poset $\langle i\rangle \coprod \ldots \coprod \langle j\rangle$ by $\langle i, \ldots, j\rangle$.

It is proved that (a) or (b) holds iff there is an "alternating" chain $f = \{C — x_1 \sim_* x_2 — \cdots — x_{2m-1} \sim_* x_{2m} — D\}$ $(m \geq 0)$ with $C, D \subset S_0$ such that (c) $C \cong \langle 1, 1\rangle$ and $r_*(C) = 2$, or $C \cong \langle 1, 2\rangle$ and $r_*(C) = 1$, or $C \cong \langle 1, 1, 1\rangle$ and $r_*(C_1) = 1$; (d) $D \cong \langle 1, 1\rangle$ and $r_*(D) = 1$, or $D = \{x_i\}$ with $1 \leq i < 2m$; (e) $x_i \neq x_j$ for any $i \neq j$ (for $m = 0$, $f = \{C — D\}$ with D to be of the first form). The main stage of the proof is to describe all minimal $*$-bundles with a chain f of the above type and construct for each such $*$-bundle a $k\langle x, y\rangle$-representation from the definition of wildness.

Step 2. One can introduce an *$\sim$-bundle of posets* and its *representations* (and define *elementary* posets, etc.) in the same way as those in the case of an involution $*$, replacing everywhere (in particular, in the definition of $\text{mod}_S k$) $*$, or equivalently the equivalence relation $\sim_*$, by an arbitrary equivalence relation $\sim$. It is proved that an $\sim$-bundle $\overline{S}$ of nonempty and nonelementary posets is wild if $r_\sim(S_0) > 2$. The idea of the proof is similar to that in Step 1. The differences are only (besides the taking $\sim$ instead of $\sim_*$) that, in (c), C is only of the form $\{y\}$ with $r_\sim(y) > 2$, that, in (d), D can be (in addition) of this form, and that, in (e), in addition $r_\sim(x_i) = 2$ for any i.

Step 3. Keeping the notation of Step 1, we call *$(*, \circ)$-bundle* (or *biinvolution bundle*) *of* the given *posets* a triple $\overline{\overline{S}} = (S, *, \circ)$, where $*$ and $\circ$ are, respectively, involutions on S_0 and S_0^2 satisfying $(x, y)^\circ = (y, x)^\circ$ for any x, y, $(x, y)^\circ = (x, y)$ for incomparable x, y and the natural conditions 1)–4) of [5, 4.11] if $x \leq y$. Its representations are defined similar to that for a $*$-bundle (for a poset A, $(A, *, \circ)$-graded spaces are $(A, *)$-graded ones; by $(A, *, \circ)$-maps one must mean $(A, *)$-maps φ such that $\varphi_{ab} = \varphi_{cd}$ whenever $(a, b)^\circ = (c, d)$). It is proved that an $(*, \circ)$-bundle of nonempty and nonelementary (respect to $*$) posets is wild if $\circ$ is nontrivial. The idea of the proof is similar to that in Step 2. The difference is only that the role of x with $r_\sim(x) = 1, 2$ is played by x with $r_\circ(x) = 1, 2$, where $r_\circ(x) = 1$ if $(x, y)^\circ = (x, y)$ for any y and $r_\circ(x) = 2$ otherwise.

Step 4. Identifying the modules M_i $(i \in \Gamma_0)$ with their images in $\text{mod}\, k$, it is proved (with the help of not very complicated arguments) that the general case is reduced to the cases of Step 1–3.

It follows from the above that our main result can be reformulated in the following way.

Theorem 3.2. *Let* Γ *and* M *be as in Theorem* 3.1. *Then* Γ *is* M*-tame* *if and only if there is a bundle of semichaines* $\overline{S} = (S, *)$ *with* $S = \{A_\alpha, B_\alpha | \alpha \in \Gamma_1\}$ *such that* $(\mathcal{A}/\mathrm{Ann}M, \overrightarrow{M}) \cong (\mathcal{K}(\overline{S}), P(\overline{S}))$*. Otherwise,* Γ *is* M-inv-*wild.*

For $\sim$-bundles of posets (which include $*$-bundles), we classify tame cases in the general situation.

4. Extensions of the main result

4.1. The main result for $|\mathrm{Ob}\mathcal{A}_0| = \infty$.

Let $\mathcal{A}$ be a Krull-Schmidt category over a field k, with $|\mathrm{Ob}\mathcal{A}_0| = \infty$. The definitions of various types of $\mathcal{A}$-modules, which we gave for $|\mathrm{Ob}\mathcal{A}_0| < \infty$ (at the beginning of Section 2), can be directly transferred to this case. It is easy to see that a module N is chained (resp. semichained) if and only if so is $N|_{\oplus\mathcal{B}}$ (the restriction of N on $\oplus\mathcal{B}$), for any $\mathcal{B}$ to be a full subcategory of $\mathcal{A}_0$ with finite many objects (because an infinite poset is a chaine if and only if all its finite subposets are chaines, and the same is true for semichaines). Using these facts, one can easily prove that the main result of this section is also true for $|\mathrm{Ob}\mathcal{A}_0| = \infty$.

4.2. The main result for infinite quivers.

Our main result remains also true for an infinite quiver Γ, and the proof of this fact can be carried out in the same way as that for finite quivers; moreover, in view of what we said in the preceding section, it suffices to consider the case when $|\mathrm{Ob}\mathcal{A}_0| < \infty$. But here we already need to know that the problem of classifying the representations of a bundle of semichains is tame when the number of ones is infinite (because $|\mathrm{Ob}\mathcal{A}_0| < \infty$, all the semichaines can be assumed to be finite). The intuition tell us that this fact is true and that the representations of such bundle can be classified analogously to that for finitely many semichaines. In this subsection we clarify an explicit solution of this problem.

Let $\overline{S}$ be $S = \{A_i, B_i | \, i \in \mathcal{I}\}$ be a family of pairwise disjoint (finite) semichains, where $\mathcal{I}$ is some set. Put $A = \coprod_{i\in\mathcal{I}} A_i$, $B = \coprod_{i\in\mathcal{I}} B_i$, $S_0 = A \coprod B$. A bundle of semichains A_i, B_i, where i runs through $\in \mathbb{I}$, is defined similar to that for finitely many semichaines: it is a pair $\overline{S} = (S, *)$ with $*$ to be is an involution on S_0 such that $x^* = x$ for each x belonging to the union of all two-point links. In the new situation, the category $\mathcal{B}_k\,(\overline{S})$ of representations of the bundle $\overline{S}$ are defined in the same way as that for finitely many semichaines, and it is a Krull-Schmidt category too.

It is easy to see that a faithful bundle of infinitely many semichaines has only countable many ones, and hence we can confine oneself to the countable case[3]. As usual, $\mathbb{Z}$ denotes the integer numbers and $\mathbb{N}$ the natural ones.

Thus, let $S = \{A_i, B_i | \, i \in \mathbb{N}\}$ be a family of pairwise disjoint (finite) semichains and $\overline{S} = (S, *)$ a bundle of semichains $A_1, A_2, \dots, B_1, B_2, \dots$; recall that $A = \coprod_{i\in\mathbb{N}} A_i$, $B = \coprod_{i\in\mathbb{N}} B_i$ and $S_0 = A \coprod B$. If $R = (U, V, \varphi)$ is a representation of $\overline{S}$ with the dimension-function $d : S_0 \to \mathbb{N} \cup 0$ (sending $x \in A_i$ to $\dim_k(U_i)_x$ and $y \in B_i$ to $\dim_k(V_i)_y$), then the set of all elements $x \in S_0$ such that $d(x) \neq 0$ will be called the *support* of R.

The indecomposable representations with finite supports (or equivalently, with finitely many semichains) were classified in [8, 6]. Here we classify the indecomposable representations (of a bundle of countable many semichains) with infinite supports.

Let, for a semichaine X, $L(X)$ denotes the set of its links (which is ordered in a natural way). Put $L(A) = \cup_{i\in\mathbb{N}} L(A_i)$, $L(B) = \cup_{i\in\mathbb{N}} L(B_i)$, and denote by $L(S)$, or simply L, the union of the sets $L(A)$ and $L(B)$. It is convenient for us to denote elements of L by lower case letters and to identify the one-points links with the points themselves. The number of points of a link $x \in L$ is denoted by $l(x)$.

Define two symmetric binary relations, α and β, on the set L by putting $x\alpha y$ if and only if $x \neq y$, $l(x) = l(y) = 1$ and $x^* = y$, or $x = y$ and $l(x) = 2$; $x\beta y$ if and only if either $x \in L(A_i)$, $y \in L(B_i)$ or $x \in L(B_i)$, $y \in L(A_i)$ for some $i \in \mathbb{N}$.

We now introduce the notion of L-chains of type $(0, +\infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$.

Throughout, all graphs are nonoriented. For a graph C, we denote by C_0 and C_1 the sets of its vertices and edges, respectively. Let $C^{+\infty}$ be the graph with $C_0^{+\infty} = \mathbb{N}$ and $C_1^{+\infty} = \{(i, i+1) | \, i \in \mathbb{N}\}$, $C^{-\infty}$ be the graph with $C_0^{-\infty} = \{-n | \, n \in \mathbb{N}\}$ and $C_1^{-\infty} = \{(-i-1, -i) | \, i \in \mathbb{N}\}$, and C^{∞} be the graph with $C_0^{\infty} = \mathbb{Z}$ and $C_1^{\infty} = \{(i, i+1) | \, i \in \mathbb{Z}\}$. A *countable L-chain* is a function g, defined on a graph $C \in \{C^{+\infty}, C^{-\infty}, C^{\infty}\}$, that associates to each $j \in C_0$ an element $g(j) \in L$ and to each edge $(j, j+1) \in C_1$ a relation $g(j, j+1) \in \{\alpha, \beta\}$ subject to the following conditions: (a) $g(j)$ and $g(j+1)$ satisfy the relation $g(j, j+1)$; (b) $g(j-1, j) \neq g(j, j+1)$; (c) for each $x \in L$, the set $g^{-1}(x) = \{j \in C_0 | \, g(j) = x\}$ is finite. An *isomorphism* of L-chains g and g', defined on C and C', respectively, is an isomorphism $\tau : C \to C'$ such that $g = \tau g'$.

[3] A representation (U, V, φ) of a bundle $\overline{S}$ is called faithful if $(U_i)_x, (V_i)_y \neq 0$ for any $i \in \mathcal{I}$, $x \in A_i$ and $y \in B_i$; the bundle $\overline{S}$ is called faithful if it has a faithful indecomposable representation.

A countable L-chain defined on $C = C^{+\infty}$, $C^{-\infty}$, C^{∞} will be called an *L-chain of type* $(0, +\infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$, respectively.

A countable L-chain g is called *admissible* if $x\alpha y$ for distinct elements $x, y \in L$ and $g(j) = x$ imply the existence of an edge ρ containing the vertex j and satisfying $g(\rho) = \alpha$ (an L-chain of type $(-\infty, +\infty)$ is always admissible), and *symmetric* if there exist a vertex i such that $g(i-s) = g(i+s)$ for any $s \in \mathbb{N}$ (an L-chain of type $(0, +\infty)$ or $(-\infty, 0)$ is always nonsymmetric). The vertex 1 (resp. -1) of an L-chain of type $(0, +\infty)$ (resp. $(-\infty, 0)$) is called *double* if $g(1,2) = \beta$ and $g(1)\alpha g(1)$ in L (resp. $g(-2,-1) = \beta$ and $g(-1)\alpha g(-1)$ in L). We write $d(g) = 1$ if the vertex 1 (resp. -1) is double and $d(g) = 0$, otherwise; for an L-chain of type $(-\infty, +\infty)$, we put $d(g) = 0$.

Denote by $G_1(L)$ the set of admissible nonsymmetric (countable) L-chains. To an $g \in G_1(L)$, we associate the representation $U_1(g)$ if $d(g) = 0$, and the representations $U_1(g), U_2(g)$ if $d(g) = 1$. These representations are defined in the same way as those in [8, 6] for finite many semichains (in these paper we used the language of matrices, but all the results and proofs can be easily rewrited in terms of vector spaces and linear maps).

The representations $U_i(g)$ of the bundle $\overline{S} = (S, *)$ are all indecomposable. Moreover, the following statement holds.

Theorem 4.1. *Let $\overline{S} = (S, *)$ be a bundle of countable many semichains. Choose one representative in each isomorphism class of L-chains of type $(0, +\infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$ belonging to $G_1(L)$. Then the set of representations of the form $U_i(g)$ associated to the chosen L-chains is a complete set of pairwise nonisomorphic indecomposable representations with infinite support.*

The idea of the proof is similar to that in [8] for finite many semichaines.

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RESEARCH ARTICLE

Tiled orders over discrete valuation rings, finite Markov chains and partially ordered sets. I

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Abstract. We prove that the quiver of tiled order over a discrete valuation ring is strongly connected and simply laced. With such quiver we associate a finite ergodic Markov chain. We introduce the notion of the index $in\,A$ of a right noetherian semiperfect ring A as the maximal real eigen-value of its adjacency matrix. A tiled order Λ is integral if $in\,\Lambda$ is an integer. Every cyclic Gorenstein tiled order is integral. In particular, $in\,\Lambda = 1$ if and only if Λ is hereditary. We give an example of a non-integral Gorenstein tiled order. We prove that a reduced $(0,1)$-order is Gorenstein if and only if either $in\,\Lambda = w(\Lambda) = 1$, or $in\,\Lambda = w(\Lambda) = 2$, where $w(\Lambda)$ is a width of Λ.

1. Introduction

This is the first part of an article dedicated to tiled orders over discrete valuation rings and their relations with finite Markov chains and partially ordered sets.

These orders appear in various parts of ring theory and representation theory (see [9], [14], [15], [24] – [35], [39], [42] – [50], [52] – [58]).

All rings are associative with $1 \neq 0$. $R = R(A)$ denotes the Jacobson radical of a ring A. A ring A is said to be indecomposable if A cannot be decomposed into a direct product of two rings.

In section 2 we recall the basic facts about semiperfect rings. In section 3 we show that an indecomposable semiprime right noetherian semiperfect semidistributive ring is either simple artinian or a tiled order

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Λ over a discrete valuation ring. When writing "$SPSD$-ring" we mean a semiperfect semidistributive ring [31]. Thus, the tiled orders over a discrete valuation rings are, exactly, the noetherian (but not artinian) prime $SPSD$-rings. For tiled order Λ we introduce Λ-lattices and define a duality for completely decomposable Λ-lattices. We also remind the notion of an exponent matrix $\mathcal{E}(\Lambda)$ of a tiled order Λ.

In section 4 we prove that the quiver $Q(\Lambda)$ of tiled order Λ is strongly connected simply laced and give a formula for its adjacency matrix $[Q(\Lambda)]$. We introduce the notion of the index $in\, A$ of a right noetherian semiperfect ring A as the maximal real eigen-value of the adjacency matrix $[Q(A)]$ of the quiver $Q(A)$.

In section 5 for the quiver of an arbitrary tiled order a finite ergodic Markov chain is constructed. In particular, such Markov chains are associated with finite posets. We remind that Markov chain is called ergodic if it is possible to go from its every state to any other state. An ergodic Markov chain is cyclic if each state can be entered only at certain periodic intervals, and it is called regular otherwise.

According to this terminology, a poset shall be called cyclic if associated Markov chain is cyclic and regular otherwise. We observe that linearly ordered set (chain) is cyclic and that a poset, having an isolated element, is regular.

In section 6 with any finite partially ordered set (poset) P we associate a reduced $(0,1)$-order $\Lambda(P)$ and conversely, for any $(0,1)$-order Λ we define a poset P_Λ such that $P_{\Lambda(P)} = P$ and $\Lambda(P_\Lambda) = \Lambda$ (see [57], [49]). The following theorem is proved: *a reduced $(0,1)$-order Λ is Gorenstein if and only if P_Λ is an ordinal power of either a singleton or an antichain with two elements.*

Section 7 is devoted to quivers of Gorenstein orders. We note that the quiver $Q(\Lambda)$ of a cyclic reduced Gorenstein tiled order Λ with the permutation $\sigma(\Lambda)$ in general does not contains a simple cycle of length n, where $n = |<\sigma(\Lambda)>|$.

A tiled order Λ is called integral if $in\,\Lambda$ is an integer. A cyclic Gorenstein tiled order is integral ([45], Theorem 3.4.). In particular, $in\,\Lambda = 1$ if and only if Λ is hereditary.

In conclusion, we give an example of a non-integral Gorenstein tiled order.

The reader is referred to [1] and [41] for information on artinian algebras and their quivers. We recommend [6], [10], [13], [18], [22], [37], [42], [49], [52] for general theory of finite dimensional algebras, ring theory and their applications in representation theory. Applications of linear algebra in graph theory and the theory of Markov chains can be found in [3], [11], [16], [17], [23], [36], [38].

2. Semiperfect rings

The basic facts about semiperfect rings, which were introduced by H.Bass in 1960, can be found in [13], [22], [37], [18]. In this paper we denote by A a semiperfect ring and by $R = R(A)$ its Jacobson radical.

An idempotent $e \in A$ is said to be local if eAe is local ring.

Theorem 2.1. [40] *A ring A is semiperfect if and only if the identity 1 of A can be decomposed into a sum of pairwise orthogonal local idempotents.*

Let $M_n(B)$ be the ring of all square $n \times n$-matrices over a ring B. Then the ring $\bar{A} = A/R$ is a semisimple artinian. Thus, by Wedderburn-Artin Theorem, we have $\bar{A} = A/R = M_{n_1}(\mathcal{D}_1) \times \ldots \times M_{n_s}(\mathcal{D}_s)$, where $\mathcal{D}_i, \quad i = 1, \ldots, s$, are division rings. In this case, every simple A-module is simple as an $\bar{A}$-module. Let $\bar{1} = \bar{f}_1 + \ldots + \bar{f}_s$ be a decomposition of $\bar{1} \in \bar{A}$ into a sum of central idempotents such that $\bar{f}_i\bar{A} = \bar{A}\bar{f}_i = M_{n_i}(\mathcal{D}_i)$. There exists a decomposition $1 = f_1 + \ldots + f_s$, where $\bar{f}_i = f_i + R$ and $f_if_j = \delta_{ij}f_i, \quad i, j = 1, \ldots, s$ and δ_{ij} is the Kronecker delta (see [37], Chapter 3).

For an A-module M we denote by M^n the direct sum of n copies of M and we set $M^0 = 0$.

Consider $A_A = \bigoplus\limits_{i=1}^{s} f_iA$. Obviously, $f_iA = P_i^{n_i}$, where P_i is an indecomposable projective A-module (principal right A-module), whose multiplicity in the right regular module A_A is n_i, i.e. $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$.

Similarly, ${}_AA = \bigoplus\limits_{i=1}^{s} Af_i$. where $Af_i = Q_i^{n_i}$ and each Q_i is an indecomposable projective left A-module (principal left A-module) with multiplicity n_i in the left regular module ${}_AA$, i.e. ${}_AA = Q_1^{n_1} \oplus \ldots \oplus Q_s^{n_s}$. Any principal right (resp. left) A-module has the form eA (resp. Ae), where e is a local idempotent.

A semiperfect ring A is called *reduced* if A/R is a direct product of division rings. Every semiperfect ring $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ is Morita equivalent to the reduced ring $B = End_A(P_1 \oplus \ldots \oplus P_s)$.

The element $a \in A$ is said to be *central modulo R*, if $a + R$ lies in the centre of A/R.

Definition 2.2. *An idempotent $f \in A$ shall be called the canonical if $\bar{f}\bar{A} = \bar{A}\bar{f} = M_{n_k}(\mathcal{D}_k)$ for some $k = 1, \ldots, s$; $\bar{f} = f + R$*

Equivalently, f is a minimal central modulo R idempotent.

A decomposition $1 = f_1 + \ldots + f_s$ into a sum of pairwise orthogonal canonical idempotents shall be called *a canonical decomposition of identity of a ring A.*

Let I be an (two-sided) ideal of A and $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$. Then $I = \bigoplus\limits_{i,j=1}^{s} I_{ij}$ with $I_{i,j} = f_i I f_j, \quad i,j = 1,\ldots,s$. As follows from [9], one canonical Peirce decomposition of I can be obtained from another one by a simultaneous permutation of lines and columns and the substitution of each Peirce component I_{ij} by $aI_{ij}a^{-1}$, where a is an invertible element of a ring A. In particular, for A and R we have such canonical Peirce decompositions:

$$A = \bigoplus_{i,j=1}^{s} A_{ij}, \; R = \bigoplus_{i,j=1}^{s} R_{ij}, \tag{1}$$

where $R_{ij} = f_i R f_j = A_{ij}$ for $i \neq j$ and R_{ii} is the Jacobson radical of $A_{ii}, \; i = 1,\ldots,s$.

Let M be a right A-module and N a left A-module. We set $top\, M = M/MR$ and $top\, N = N/RN$. Denote $U_i = top\, P_i$ and $V_i = top\, Q_i, \; i = 1,\ldots,s$. It is well-known that $P_1,\ldots,P_s$ $(Q_1,\ldots,Q_s)$ represent, up to isomorphism, all indecomposable right (left) projective A-modules, while $U_1,\ldots,U_s$ $(V_1,\ldots,V_s)$ form a representative set of isomorphism classes of all simple right (left) A-modules. In this case $P_i = P(U_i)$ $(Q_i = P(V_i))$ is a projective cover $U_i\,(V_i)$, $i = 1,\ldots,s$. A projective cover of a finitely generated module M over a semiperfect ring A is built as follows: M/MR is a module over a semisimple artinian ring $\bar{A} = A/R$. Therefore, $\bar{M} = M/MR$ is isomorphic to a finite direct sum of simple A-modules: $\bar{M} = \bigoplus\limits_{j=1}^{s} U_j^{m_j}$. Then $P(M) = P(\bar{M}) = \bigoplus\limits_{j=1}^{s} P(U_j)^{m_j} = \bigoplus\limits_{j=1}^{s} P_j^{m_j}$.

Lemma 2.3. Annihilation Lemma ([9], Lemma 3.1). *Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in A$. For every simple right A-module U_i and for each f_j we have $U_i f_j = \delta_{ij} U_i, \; i,j = 1,\ldots,s$. Similarly, for every simple left A-module V_i and for each f_j, $f_j V_i = \delta_{ij} V_i, \; i,j = 1,\ldots,s$.*

Lemma 2.4. *Let A be a semiperfect ring, e and f – nonzero idempotents of the ring A such that $\bar{e} = \bar{f} \in \bar{A}$. Then there exists an invertible element $a \in A$ such that $f = aea^{-1}$.*

Proof. Denote $W_1 = \bar{e}\bar{A} = \bar{f}\bar{A}$. Obviously, eA and fA are projective covers of a semisimple A-module W_1. Therefore they are isomorphic. Modules $(1-e)A$ and $(1-f)A$ are projective covers of a semiperfect A-module $W_2 = (\bar{1}-\bar{f})\bar{A} = (\bar{1}-\bar{e})\bar{A}$. Consequently, they are isomorphic too. Denote $e_1 = e, \; e_2 = 1-e$ and $f_1 = f, \; f_2 = 1-f$.

The isomorphism $e_iA \simeq f_iA$ is given by suitable element $a_i \in f_iAe_i$ such that $f_ia_i = a_ie_i$ $(i = 1,2)$. Let $a = a_1+a_2$. Then $ae_i = a_ie_i = a_i$ and $f_ia = f_ia_i = a_i$ for $i = 1,2$. We'll show that a is invertible. There exists the element $b_i \in e_iAf_i$ defining the inverse isomorphism $f_iA \simeq e_iA$ to $(i = 1,2)$. Then $a_ib_j = \delta_{ij}f_j$ and $b_ia_j = \delta_{ij}e_i$. Let $b = b_1 + b_2$. We have $ab = \sum\limits_{i=1}^{2} a_ib_i = f_1 + f_2 = 1$ and, consequently, $f_i = ae_ia^{-1}$ and $f = aea^{-1}$. □

Lemma 2.5. *Let $1 = f_1+\ldots+f_s$ be canonical decomposition of identity $1 \in A$ into a sum of pairwise canonical idempotents and g be a central modulo R idempotent. There exists an invertible element $a \in A$ such that $f_{i_1} + \ldots + f_{i_k} = aga^{-1}$.*

Proof. Let $\bar{g}\bar{A} = \bar{A}\bar{g} = M_{n_{i_1}}(\mathcal{D}_{i_1}) \times \ldots \times M_{n_{i_k}}(\mathcal{D}_{i_k})$. Then $f = f_{i_1} + \ldots + f_{i_k}$ is a central modulo R idempotent and $\bar{f}\bar{A} = \bar{g}\bar{A}$. By Lemma 2.4 we have $f = aga^{-1}$.

□

Corollary 2.6. *Each central modulo R idempotent g is a sum of the canonical idempotents and there exists the canonical decomposition of $1 \in A$ into a sum of pairwise orthogonal canonical idempotents such that $1 = g_1 + \ldots + g_k + g_{k+1} + \ldots + g_s$, where $g = g_1 + \ldots + g_k$ and $f = f_{i_1} + \ldots + f_{i_k} = ag_1a^{-1} + \ldots + ag_ka^{-1}$.*

Theorem 2.7. ([9], Theorem 3.3). *Let $1 = f_1+\ldots+f_s = g_1+\ldots+g_t$ be two canonical decompositions of $1 \in A$ into a sum of pairwise canonical idempotents. Then $s = t$ and there exist an invertible element $a \in A$ and a permutation $i \longrightarrow \tau(i)$ of $\{1,\ldots,s\}$ such that $f_i = ag_{\tau(i)}a^{-1}$ for each $i = 1,\ldots,s$.*

3. Noetherian semiprime semiperfect semidistributive rings

Definition 3.1. ([18], p. 73). *A ring A is called indecomposable if A cannot be decomposed into a direct product of two rings.*

Definition 3.2. ([18], p. 74). *A ring A is called a finitely decomposable ring (FD-ring) if it decomposable into a direct product of a finite number of indecomposable rings.*

An important class of FD-rings are right noetherian rings (in particular, all right artinian rings). Semiperfect rings (which may be neither noetherian, no artinian) are also examples of FD-rings.

Theorem 3.3. ([18], Theorem 2.5.11) *A finitely decomposable ring A can be uniquely decomposed into a direct product of a finite number of indecomposable rings, that is if $A = B_1 \times \ldots \times B_s = C_1 \times \ldots \times C_t$ are two of such decompositions then $s = t$ and there is a permutation σ of numbers $\{1, \ldots, t\}$ such that $B_i = C_{\sigma(i)}$ for $i = 1, \ldots, t$.*

A module M is distributive if its lattice of submodules is distributive, i.e. for any submodules K, L, N $K \cap (L + N) = K \cap L + K \cap N$. Clearly, any submodule and any factormodule of a distributive module is a distributive module. A semidistributive module is a direct sum of distributive modules. A ring A is right (left) semidistributive if it is semidistributive as a right (left) module over itself. A ring A is semidistributive if it is both left and right semidistributive (see [52]).

Theorem 3.4. [4] *A module is distributive if and only if the socle of each of its factormodule contains no more then one copy of each simple module.*

Theorem 3.5. ([51], see also [31], Theorem 4). *A semiperfect ring A is right (left) semidistributive if and only if, for any local idempotents e and f of the ring A the set eAf is a uniserial right fAf-module (uniserial left eAe-module).*

Corollary 3.6. [31] *Let A be a semiperfect ring, and $1 = e_1 + \ldots + e_n$ is a decomposition of $1 \in A$ into a sum of mutually orthogonal local idempotents. The ring A is right (left) semidistributive if and only if for any idempotents e_i and e_j $(i \neq j)$ from the above decomposition, the ring $(e_i + e_j)A(e_i + e_j)$ is right (left) semidistributive.*

We write $SPSD$-ring for semiperfect semidistributive ring and $SPSDR$-ring ($SPSDL$-ring) for semiperfect right (left) semidistributive ring.

Recall that a semimaximal ring A is a semiperfect semiprime right noetherian ring A such that for each local idempotent $e \in A$, the ring eAe is a discrete valuation ring (not necessarily commutative) [57]. In the same paper, a description of these rings is given.

Theorem 3.7. *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \ldots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \ldots & \pi^{\alpha_{2n}}\mathcal{O} \\ \ldots & \ldots & \ldots & \ldots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \ldots & \mathcal{O} \end{pmatrix}, \qquad (*)$$

where $n \geq 1$, $\mathcal{O}$ *is a discrete valuation ring with a prime element* π, *the* α_{ij} *are integers such that* $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ *for all* i, j, k ($\alpha_{ii} = 0$ *for any* i).

The ring $\mathcal{O}$ is embedded into its classical division ring of fractions $\mathcal{D}$, and (*) denotes the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that $a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}\Lambda e_{jj}$, where $e_{11}, \ldots, e_{nn}$ are matrix units of $M_n(\mathcal{D})$. Thus, Λ is a tiled order over a discrete valuation ring (d.v.r.) ([50], [20]). Obviously, a tiled order Λ over a d.v.r. $\mathcal{O}$ is left noetherian. It is clear that $M_n(\mathcal{D})$ is the classical quotient ring of fractions of Λ.

The following is a decomposition theorem for semiprime $SPSD$-rings. (Compare [5], [13]).

Theorem 3.8. [31] *The following conditions for a semiperfect semiprime right noetherian ring* A *are equivalent:*

a) the ring A *is semidistributive;*

b) the ring A *is a direct product of a semisimple artinian ring and a semimaximal ring.*

Hence, the tiled orders over a discrete valuation rings are, exactly, the noetherian (but not artinian) prime $SPSD$-rings.

Denote by $M_n(Z)$ the ring of all square $n \times n$-matrices over the ring of integers Z. Let $\mathcal{E} \in M_n(Z)$. We shall call a matrix $\mathcal{E} = (\alpha_{ij})$ the exponent matrix if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 0$ for $i = 1, \ldots, n$. A matrix $\mathcal{E}$ is called a reduced exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$.

We shall use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of the ring Λ, i.e. $\Lambda = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where e_{ij} are matrix units. If a tiled order is *reduced*, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$, $i \neq j$, i.e. $\mathcal{E}(\Lambda)$ is reduced.

Definition 3.9. *A right (resp. left)* Λ*-module* M *(resp.* N*) is called a right (resp. left)* Λ*-lattice if* M *(resp.* N*) is a finitely generated free* $\mathcal{O}$*-module.*

For instance, all finitely generated projective Λ-modules are Λ-lattices.

Given a tiled order Λ we denote $Lat_r(\Lambda)$ (resp. $Lat_l(\Lambda)$) the category of right (resp. left) Λ-lattices. We denote by $S_r(\Lambda)$ (resp. $S_l(\Lambda)$) the partially ordered by inclusion set, formed by all Λ-lattices contained in a fixed simple $M_n(\mathcal{D})$-module W (resp. in a left simple $M_n(\mathcal{D})$-module V). Such Λ-lattices are called irreducible.

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a tiled order, W (resp. V) is a simple right (resp. left) $M_n(\mathcal{D})$-module with $\mathcal{D}$-basis $e_1, \dots, e_n$ such that $e_i e_{jk} = \delta_{ij} e_k$ ($e_{ij} e_k = \delta_{jk} e_i$).

Then any right (resp. left) irreducible Λ-lattice M (resp. N), lying in W (resp. in V) is a Λ-module with $\mathcal{O}$-basis $(\pi^{\alpha_1} e_1, \dots, \pi^{\alpha_n} e_n)$, while

$$\begin{cases} \alpha_i + \alpha_{ij} \geq \alpha_j, & \text{for the right case;} \\ \alpha_{ij} + \alpha_j \geq \alpha_i, & \text{for the left case.} \end{cases} \tag{2}$$

Thus, irreducible Λ-lattices M can be identified with integer-valued vector $(\alpha_1, \dots, \alpha_n)$ satisfying (2). We shall write $[M] = (\alpha_1, \dots, \alpha_n)$ or $M = (\alpha_1, \dots, \alpha_n)$.

The order relation on the set of such vectors and the operations on them corresponding to sum and intersection of irreducible lattices are obvious.

Remark 1. Obviously, irreducible Λ-lattices $M_1 = (\alpha_1, \dots, \alpha_n)$ and $M_2 = (\beta_1, \dots, \beta_n)$ are isomorphic if and only if $\alpha_i = \beta_i + z$ for $i = 1, \dots, n$ and $z \in \mathbf{Z}$.

Proposition 3.10. *The posets $S_r(\Lambda)$ and $S_l(\Lambda)$ are anti-isomorphic distributive lattices.*

Proof. As soon Λ is a semidistributive ring, then $S_r(\Lambda)$ (resp. $S_l(\Lambda)$) is distributive lattice ([3], Ch. 1, §6) with respect to sum and intersection of submodules.

Let $M = (\alpha_1, \dots, \alpha_n) \in S_r(\Lambda)$. Then $M^* = (-\alpha_1, \dots, -\alpha_n)^T \in S_l(\Lambda)$. If $N = (\beta_1, \dots, \beta_n)^T \in S_l(\Lambda)$, then $N^* = (-\beta_1, \dots, -\beta_n) \in S_r(\Lambda)$.

Obviously, the operations $*$ are satisfied such conditions:

1. $M^{**} = M$; 2. $(M_1 + M_2)^* = M_1^* \cap M_2^*$; 3. $(M_1 \cap M_2)^* = M_1^* + M_2^*$ in the right case and analogous conditions in the left case. Thus, the map $*$: $S_r(\Lambda) \longrightarrow S_l(\Lambda)$ is the anti-isomorphism.

□

Remark 2. The maps $*$ are defined the duality for irreducible Λ-lattices.

If $M_1 \subset M_2$, $(M_1, M_2 \in S_r(\Lambda))$, then $M_2^* \subset M_1^*$. In this case, the Λ-lattice M_2 is called an overmodule of Λ-lattice M_1 (resp. M_1^* is the overmodule of M_2^*).

Definition 3.11. [30] *The direct sum of irreducible Λ-lattices is called a completely decomposable Λ-lattice.*

Let $L = M_1 \oplus \ldots \oplus M_p$ be a right completely decomposable (c.d.) Λ-lattice and $K = N_1 \oplus \ldots \oplus N_q$ be a left c.d. Λ-lattice. Then $L^* =$

$M_1^* \oplus \ldots \oplus M_p^*$ is a left c.d. Λ-lattice and $K^* = N_1^* \oplus \ldots \oplus N_q^*$ is a right c.d. Λ-lattice.

A tiled order $\Lambda = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$ is a completely decomposable both right and left Λ-lattice lying in $\tilde{\Lambda} = M_n(D)$.

A projective Λ-lattice (= finitely generated projective Λ-module) is a c.d. Λ-lattice.

Definition 3.12. *A completely decomposable Λ-lattice M is called relative injective if $M = P^*$, where P is a projective Λ-lattice.*

Definition 3.13. [28] *A tiled order Λ is called Gorenstein tiled order if Λ^*_Λ is a projective left Λ-lattice.*

Remark 3. Obviously, Λ^*_Λ is projective if and only if ${}_\Lambda\Lambda^*$ is projective right Λ-lattice.

Below Gorenstein tiled orders we often call Gorenstein orders.

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Theorem 3.14. (see [28]). *Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{pq})\}$ be a reduced tiled order; then the following conditions are equivalent:*

(a) Λ is Gorenstein;

(b) there exists a permutation $\sigma = \{i \to \sigma(i)\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.

The permutation σ is denoted by $\sigma(\Lambda)$.

Notice that the permutation $\sigma(\Lambda)$ of a reduced Gorenstein order Λ has no cycles of length 1.

Definition 3.15. *A Gorenstein tiled order Λ is said to be cyclic if its permutation $\sigma(\Lambda)$ is a cycle.*

We denote by $\mathcal{M}_r(\Lambda)$ (resp. $\mathcal{M}_l(\Lambda)$) partially ordered subset of the lattice $S_r(\Lambda)$ (resp. $S_l(\Lambda)$), formed by all projective Λ-modules, lying in $S_r(\Lambda)$ (resp. $S_l(\Lambda)$).

Proposition 3.16. *An irreducible Λ-lattice $M \in S_r(\Lambda)$ (resp. $N \in S_l(\Lambda)$) is projective if and only if it contains exactly one maximal submodule.*

Let $M \in S_r(\Lambda)$ and $M^* \in S_l(\Lambda)$.

Proposition 3.17. *A tiled order Λ is Gorenstein if and only if a restriction of the map $* : S_r(\Lambda) \longrightarrow S_l(\Lambda)$ on $\mathcal{M}_r(\Lambda)$ is an anti-isomorphism between partially ordered sets $\mathcal{M}_r(\Lambda)$ and $\mathcal{M}_l(\Lambda)$.*

In general case, the poset $\mathcal{M}_r(\Lambda)$ and $\mathcal{M}_l(\Lambda)$ also are anti-isomorphic, but this anti-isomorphism cannot be extended to anti-isomorphism of the lattices $S_r(\Lambda)$ and $S_l(\Lambda)$.

Let P be an arbitrary poset. A subset of P is called *a chain* if any two of its elements are related. A subset of P is called *a antichain* if no two distinct elements of the subset are related.

We shall denote a chain of n elements by CH_n and an antichain of n elements by ACH_n.

Theorem 3.18. [8], [19] *Given a poset the minimal number of disjoint chains that together contain all elements of P is equal to the maximal number of elements in an antichain, if this number is finite.*

Definition 3.19. [19] *The maximal number $w(P)$ of elements in an antichain of P is called the width of P.*

The width of $\mathcal{M}_r(\Lambda)$ is called the width of a tiled order Λ and is denoted by $w(\Lambda)$.

Let P be an arbitrary partially ordered set. Then one can construct a new partially ordered set $\tilde{P}$, whose elements are the nonempty subsets of P, consisting of pairwise incomparable elements. If $A, B \in \tilde{P}$, then $A \leq B$ if and only if for any $a \in A$ there exists $b \in B$ such that $a \leq b$. The poset P is naturally embedded in $\tilde{P}$: an element $a \in P$ is mapped into the singleton $\{a\}$.

Example. If $P = ACH_n$, then $\tilde{P}$ is the poset of all non-empty subsets of P partially ordered by inclusion.

Proposition 3.20. [57] *The set $\tilde{\mathcal{M}_r}(\Lambda)$ is a lattice. There is a natural isomorphism of lattices $\tilde{\mathcal{M}_r}(\Lambda)$ (resp. $\tilde{\mathcal{M}_l}(\Lambda)$) and $S_r(\Lambda)$ (resp. $S_l(\Lambda)$), which is identical on $\mathcal{M}_r(\Lambda)$ (resp. $\mathcal{M}_l(\Lambda)$).*

4. Quivers of tiled orders

Following P. Gabriel a finite directed graph Q is called *a quiver*.

Denote by $VQ = \{1, \ldots, s\}$ the set of all vertices of Q and by AQ the set of its all arrows. We shall write $Q = \{AQ, VQ\}$. Denote by $1, \ldots, s$ the vertices of a quiver Q and assume that we have t_{ij} arrows beginning at the vertex i and ending at the vertex j. The matrix

$$[Q] = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1s} \\ t_{21} & t_{22} & \ldots & t_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ t_{s1} & t_{s2} & \ldots & t_{ss} \end{pmatrix}$$

is called *the adjacency matrix* of Q.

Let Q be a quiver. Usually the vertices of Q we will denote by the numbers $1, 2, \dots, s$. If an arrow σ connects a vertex i with a vertex j then i is called its *start vertex* and j its *end vertex*. This will be denoted as $\sigma : i \to j$.

A *path of the quiver* Q from a vertex i to a vertex j is an ordered set of k arrows $\{\sigma_1, \sigma_2, ..., \sigma_k\}$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for $1 < m \leq k$, and moreover, the vertex i is the start vertex of σ_1, while the vertex j is the end vertex of σ_k. The number k of these arrows is called the *length of the path*.

The start vertex i of the arrow σ_1 is called the *start of the path* and the end vertex j of the arrow σ_k is called the *end of the path*. We shall say that the path connects the vertex i with the vertex j and it is denoted by $\sigma_1\sigma_2...\sigma_k : i \to j$.

We remind the definition of the quiver of a right noetherian semiperfect ring A ([18], p. 201).

Let A be a semiperfect right noetherian ring, R its Jacobson radical, $P_1, \dots, P_s$ be all pairwise nonisomorphic projective indecomposable modules. Let the projective cover $P(P_iR)$ of P_iR be:

$$P(P_iR) = \bigoplus_{j=1}^{s} P_j^{t_{ij}}, \ i, j = 1, \dots, s.$$

We assign to $P_1, \dots, P_s$ vertices $1, \dots, s$ and join vertex i with vertex j by t_{ij} arrows. The resulting directed graph is called the quiver of A and denote by $Q(A)$.

Analogously, can be defined the left quiver $Q'(A)$ of a left noetherian semiperfect ring A.

From the definition of a projective cover it follows that $Q(A) = Q(A/R^2)$.

If A is a semiperfect ring such that A/R^2 is right artinian, then we define $Q(A)$ by formula: $Q(A) = Q(A/R^2)$. If A/R^2 is left artinian, then $Q'(A) = Q'(A/R^2)$.

Notice that the quiver of a semiperfect ring is invariant under Morita equivalence.

Proposition 4.1. *Let A be a semiperfect ring such that A/R^2 is left and right artinian. Then:*

(1) if $Q(A)$ has an arrow from i to j then the left quiver $Q'(A)$ has an arrow from j to i;

(2) if $Q(A)$ has an arrow σ_{ij} then there exist the nonzero homomorphisms from P_j to P_i and Q_i to Q_j.

The proof immediately follows from the definition of $Q(A)$.

Denote by Q_u the quiver, obtained from Q, by substituting all arrows from i to j by a single arrow (we allow $i = j$). If Q has no arrows from i to j then neither does Q_u.

Let $\bar{Q}$ be the non-oriented graph obtained from Q by omitting its orientation.

Corollary 4.2. *Let A be a ring such that A/R^2 is right and left artinian. Then $\overline{Q_u(A)} = \overline{Q'_u(A)}$.*

Proof follows from Proposition 4.1.

Theorem 4.3. [31] *If A is an right and left artinian ring with $R^2 = 0$, then the following conditions are equivalent:*

(a) A is semidistributive;

(b) every vertex of $Q(A)$ is connected with another (possibly the same) vertex by at most one arrow, and the left quiver $Q'_u(A)$ can be obtained from $Q_u(A)$ by reversing all arrows.

A ring A is called *semiprimary* if A/R is artinian and R is nilpotent.

Theorem 4.4. [29] *A semiprimary semidistributive ring is right and left artinian.*

Definition 4.5. *A semiperfect ring A is called Q-symmetric if A/R^2 is right and left artinian and $Q'(A)$ can be obtained from $Q(A)$ by reversing all arrows.*

It follows from Theorems 4.3 and 4.4, that every $SPSD$-ring is Q-symmetric.

Proposition 4.6. *For Q-symmetric ring A we have $[Q(A)]^T = [Q'(A)]$.*

Proof follows from Definition 4.5.

Definition 4.7. *Let A be a semiperfect ring such that A/R^2 is right artinian. The index in A of A is the maximal real eigen-value of the adjacency matrix $[Q(A)]$ of $Q(A)$.*

Similarly, can be defined the left index of a semiperfect ring A with left artinian A/R^2. It follows from Proposition 4.6, that the left and right indices of $SPSD$-ring coincides. In particular, this is true for tiled orders over discrete valuation rings.

Definition 4.8. *A quiver is called strongly connected if there is a path between any two vertices. By convention, a one-point graph without arrows will be considered as a strongly connected quiver.*

Definition 4.9. *A quiver Q without multiple arrows and multiple loops is called simply laced, i.e. Q is a simply laced quiver if and only if its adjacency matrix $[Q]$ is a $(0,1)$-matrix.*

Theorem 4.10. ([34], Theorem 4.1). *The quiver $Q(A)$ of right and left noetherian semiprime semiperfect ring A is strongly connected.*

Let I be a two-sided ideal of a tiled order Λ. Obviously,

$$I = \sum_{i,j=1}^{n} e_{ij}\pi^{\mu_{ij}}\mathcal{O},$$

where e_{ij} are matrix units. Denote by $\mathcal{E}(I) = (\mu_{ij})$ the exponent matrix of the ideal I. Suppose that I and J are two-sided ideals of the ring Λ, $\mathcal{E}(I) = (\mu_{ij})$, and $\mathcal{E}(J) = (\nu_{ij})$. It follows easily that $\mathcal{E}(IJ) = (\delta_{ij})$, where $\delta_{ij} = \min\limits_{k}\{\mu_{ik} + \nu_{kj}\}$.

Theorem 4.11. *The quiver $Q(\Lambda)$ of a tiled order Λ over a discrete valuation ring $\mathcal{O}$ is strongly connected and simply laced. If Λ is reduced, then $Q(\Lambda) = \mathcal{E}(R^2) - \mathcal{E}(R)$.*

Proof. Taking into account that Λ is a prime noetherian semiperfect ring it follows, by Theorem 4.10 that $Q(\Lambda)$ is a strongly connected quiver. Let Λ be a reduced order. Then $[Q(\Lambda)]$ is a reduced matrix. We shall use the following notation: $\mathcal{E}(\Lambda) = (\alpha_{ij})$; $\mathcal{E}(R) = (\beta_{ij})$, where $\beta_{ii} = 1$ for $i = 1, \dots, n$ and $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ $(i,j = 1,\dots,n)$; $\mathcal{E}(R^2) = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1\le k\le n}\{\beta_{ik}+\beta_{kj}\}$ for $i,j = 1,\dots,n$. Since, $\mathcal{E}(\Lambda)$ is reduced, we have $\alpha_{ij} + \alpha_{ji} \geq 1$ for $i,j = 1,\dots,n$, i.e. $\gamma_{ii} = \min\limits_{1\le k\le n}\{\beta_{ik} + \beta_{ki}\} = \min\limits_{1\le k\le n,\, k\neq i}\{\beta_{ik} + \beta_{ki}\}$. Hence γ_{ii} is equals 1 or 2. If $i \neq j$, then $\beta_{ij} = \alpha_{ij}$ and $\gamma_{ij} = \min\{\min\limits_{1\le k\le n,\, k\neq i,j}\{\alpha_{ik} + \alpha_{kj}\}, \alpha_{ij}+1\}$, i.e. γ_{ij} equals α_{ij} or $\alpha_{ij}+1$.

With any irreducible Λ-lattice M with $\mathcal{O}$-basis $(\pi^{\alpha_1}e_1,\dots,\pi^{\alpha_n}e_n)$ we relate the n-tuple $[M] = (\alpha_1,\dots,\alpha_n)$. Let us consider now

$$[P_i] = (\alpha_{i1},\dots,0,\dots,\alpha_{in}),$$

$$[P_iR] = (\alpha_{i1},\dots,1,\dots,\alpha_{in}) = (\beta_{i1},\dots,\beta_{in}).$$

Set $[P_iR^2] = (\gamma_{i1},\dots,\gamma_{in})$. Then $\vec{q_i} = [P_iR^2] - [P_iR]$ is a $(0,1)$-vector. Suppose that the positions of the units of $\vec{q_j}$ are $j_1,\dots,j_m$. In view of the Annihilation Lemma, this means that $P_iR/P_iR^2 = U_{j_1} \oplus \ldots \oplus U_{j_m}$. By the definition of $Q(\Lambda)$ we have exactly one arrow from the vertex i to each of $j_1, \ldots, j_m$. Thus, the adjacency matrix $[Q(\Lambda)]$ is:

$$[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R).$$

□

5. Finite Markov chains associated with tiled orders

We remind some facts about the relations between the square matrices and quivers.

Let $B = (b_{ij})$ be an arbitrary real square $n \times n$-matrix, i.e. $B \in M_n(\mathbf{R})$. Using B one construct a simply laced quiver $Q(B)$ by the following way: the set of vertices $VQ(B)$ of $Q(B)$ are the integers $1, \ldots, n$. The set of arrows $AQ(B)$ is defined as follows: there is an arrow from i to j if and only if $b_{ij} \neq 0$.

Let τ be a permutation of the set $\{1, 2, \ldots, n\}$ and let

$$P_\tau = \sum_{i=1}^{n} e_{i\tau(i)},$$

be the permutation matrix, where e_{ij} are matrix units. Clearly, $P_\tau^T P_\tau = P_\tau P_\tau^T = E_n$ is the identity matrix of $M_n(\mathbf{R})$. In particular,

$$D_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

is P_σ, where $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix}$, and $D_n^T = D_n$.

We next remind a concept which is called "irreducible matrix" in [38] and "indecomposable matrix" in [17]. We prefer to use the term "permutationally irreducible matrix" in order to avoid confusion with standard notions of representation theory (see [18], §7.7).

Definition 5.1. ([18], §7.7). *A matrix $B \in M_n(\mathbf{R})$ is called permutationally reducible if there exists a permutation matrix P_τ such that*

$$P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where B_1 and B_2 are square matrices of order less that n. Otherwise, the matrix B is called permutationally irreducible.

It follows from the equality $D_n \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix} D_n = \begin{pmatrix} B_1^{(1)} & 0 \\ B_{21} & B_2^{(2)} \end{pmatrix}$ that B is permutationally irreducible if and only if there exists a permutation matrix P_ν such that

$$P_\nu^T B P_\nu = \begin{pmatrix} B_1^{(1)} & 0 \\ B_{21} & B_2^{(2)} \end{pmatrix},$$

where $B_1^{(1)}$ and $B_2^{(2)}$ are square matrices of order less that n.

Proposition 5.2. [38], [11] *A matrix $B \in M_n(\mathbf{R})$ is permutationally irreducible if and only if the quiver $Q(B)$ is strongly connected.*

Corollary 5.3. *A quiver Q is strongly connected if and only if the matrix $[Q]$ is permutationally irreducible.*

The notion of a subquiver Q_1 of a quiver Q is obvious.

Definition 5.4. *A maximal (with respect to inclusion) strongly connected subquiver of Q is called a strongly connected component of Q.*

Definition 5.5. *By a partition $P(Q, Q_1, \dots, Q_m)$ of a quiver Q into strongly connected components $Q_1, Q_2, \dots, Q_m$ we mean a partition of the set of vertices of Q into disjoint subsets such that the subquivers corresponding to these subsets are strongly connected components of Q.*

Theorem 5.6. (see [18], Theorem 7.7.5). *Every quiver Q has a partition $P(Q, Q_1, \dots, Q_m)$ into strongly connected components $Q_1, Q_2, \dots, Q_m$. The partition is unique up to a renumbering of vertices of Q, that is if $P(Q, Q_1, \dots, Q_m)$ and $P(Q, G_1, \dots, G_n)$ are two such partitions then $m = n$ and there exists a permutation σ of $\{1, \dots, m\}$ such that $Q_i = G_{\sigma(i)}$ for $i = 1, \dots, m$.*

Definition 5.7. (of condensation Q^* of a quiver Q, see [11] and [18], §7.7). *Let $P(Q, Q_1, \dots, Q_m)$ be a partition of a quiver Q into strongly connected components $Q_1, \dots, Q_m$. The condensation Q^* of Q is the quiver, whose vertices are $q_1, \dots, q_m$ corresponding to $Q_1, \dots, Q_m$ and Q^* has an arrow from q_i to q_j if and only if Q has an arrow from a vertex belonging to VQ_i to a vertex from VQ_j ($i \neq j$, $i, j = 1, \dots, m$).*

For basic concepts of Markov chains the reader is referred [23].

Let $P = (p_{ij})$ be the transition matrix for a Markov chain MC_n.

Definition 5.8. *The quiver $Q(MC_n)$ of the Markov chain MC_n is the quiver $Q(P)$ of its transition matrix P.*

Obviously, $Q(MC_n)$ is simply laced quiver.

Definition 5.9. *A square $n \times n$-matrix $P = (p_{ij})$ is called stochastic if P is non-negative and if the sum of the elements of each row of P is 1.*

Thus, every stochastic matrix can be regarded as the transition matrix for a finite (homogeneous) Markov chain and, conversely, the transition matrix for such Markov chain is stochastic.

Let Q be a quiver with the adjacency matrix $[Q] = (q_{ij})$. We shall refer to the eigen-vectors (resp. eigen-values) of $[Q]$ as the eigen-vectors (resp. eigen-values) of the quiver Q.

Definition 5.10. *A quiver Q with $VQ \neq \varnothing$ shall be called Frobenius if it has a positive right eigen-vector.*

Theorem 5.11. (Compare with [16], Ch. 13, §6 and [36], Ch. 7, §4). *For any Frobenius quiver Q there exists a stochastic matrix P such that $Q(P) = Q$.*

Proof. Suppose Q has a positive eigen-vector $\vec{z} = (z_1, z_2, \dots, z_n) > 0$. This means that $z_i > 0$ for $i = 1, \dots, n$.

Let λ be an eigen-value corresponding to the eigen-vector $\vec{z}$, i.e.

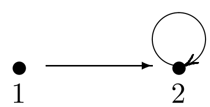
$$[Q]\vec{z} = \lambda\vec{z} \tag{3}$$

We show that $\lambda > 0$. Since $VQ \neq \varnothing$, then $[Q]$ is a non-zero non-negative matrix. Hence, in the left hand side of (3) we have a non-zero non-negative vector, and the vector on its right hand side has non-zero coordinates. Consequently, $\lambda\vec{z} > 0$ and $\lambda > 0$. We consider the diagonal matrix $Z = diag(z_1, z_2, \dots, z_n)$. Then the matrix $P = (p_{ij}) = \lambda^{-1}Z^{-1}[Q]Z$ is stochastic. Indeed, we have $\sum\limits_{j=1}^{n} q_{ij}z_j = \lambda z_i$ and $\sum\limits_{j=1}^{n} p_{ij} = \lambda^{-1}z_i^{-1}\sum\limits_{j=1}^{n} q_{ij}z_j = \lambda^{-1}z_i^{-1}\lambda z_i = 1$. Obviously, $[Q(P)] = [Q]$.

□

As follows from the Perron-Frobenius theorem (see [16], Ch.13 §2 and Corollary 5.3), every strongly connected quiver is Frobenius.

Examples.

(1). Let $P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $Q(P) = \{$ 

$\}$ is a Frobenius quiver.

(2). Let $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$. Then

$$[Q(P)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Obviously, $\chi_{[Q(P)]} = x(x-1)^2(x-2)$ and we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}. \tag{4}$$

Consequently, the quiver of a Markov chain is not necessarily Frobenius.

6. $(0,1)$-orders and finite partially ordered sets

Definition 6.1. *A tiled order*

$$\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$$

is called a $(0,1) -$ *order if* $\mathcal{E}(\Lambda)$ *is a* $(0,1)$*-matrix.*

Therefore, by an $(0,1)$-order we shall always mean a tiled $(0,1)$-order over a discrete valuation ring $\mathcal{O}$.

With a reduced $(0,1)$-order Λ we associate the partially ordered set

$$P_\Lambda = \{1, \dots, n\}$$

with the relation $\leq$ defined by the formula: $i \leq j \Leftrightarrow \alpha_{ij} = 0$.

Obviously, $(P, \leq)$ is a partially ordered set (poset).

Conversely, with any finite poset $P = \{1, \dots, n\}$ we relate the reduced $(0,1)$-matrix $\mathcal{E}_p = (\lambda_{ij})$ by the following way: $\lambda_{ij} = 0 \Leftrightarrow i \leq j$, otherwise $\lambda_{ij} = 1$. Then $\Lambda(P) = \{\mathcal{O}, \mathcal{E}_P\}$ is a reduced $(0,1)$-order.

Proposition 6.2. *Given a reduced* $(0,1)$*-order* Λ*, the width* $w(\Lambda)$ *coincide with the width of the partially ordered set* P_Λ*.*

Proof is obviously.

Definition 6.3. ([3], Ch.1, §3). *By "a covers" b in a poset P, it is meant that $a > x > b$ for no $x \in P$.*

Definition 6.4. ([18], p. 233, see also [33]). *Let $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite poset with an ordering relation $\leq$. The diagram of P is the quiver $Q(P)$ with the set of vertices $VQ(P) = \{1, \ldots, n\}$ and the set of arrows $AQ(P)$ such that in $AQ(P)$ there is an arrow $\sigma : i \to j$ if and only if α_j covers α_i.*

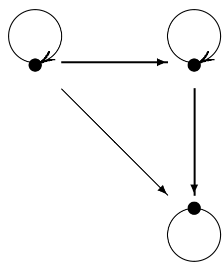
Definition 6.5. ([41], §8.4). *A quiver without oriented cycles is called an acyclic quiver.*

Proposition 6.6. *The condensation Q^* of a quiver Q is an acyclic simply laced quiver.*

Definition 6.7. *An arrow $\sigma : i \to j$ of an acyclic quiver Q is called* **extra** *if there exists a path from i to j of length greater than 1.*

Theorem 6.8. ([33], [18], §7.7). *Let Q be an acyclic simply laced quiver without extra arrows. Then Q is the diagram of some finite poset P. Conversely, the diagram $Q(P)$ of a finite poset P is an acyclic simply laced quiver without extra arrows.*

Example. If $Q =$



then

$$Q^* = \left\{ \begin{array}{ccc} 1 & & 2 \\ \bullet & \longrightarrow & \bullet \\ & \searrow & \downarrow \\ & & \bullet \\ & & 3 \end{array} \right\}$$

In Q^* there is an extra arrow σ_{13}. Deleting it, we obtain

$$\begin{array}{ccccc} 1 & & 2 & & 3 \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

which is the diagram $Q(CH_3)$ of the poset CH_3.

Thus, if we delete from Q^* all extra arrows, then by Theorem 6.8 we obtain the diagram of finite partially ordered set, which shall be denoted by $S(Q)$. In particular, with any matrix $B \in M_n(\mathbf{R})$ we associate the finite poset $S(B) = S(Q(B))$.

Definition 6.9. *Let MC_n be a finite Markov chain. The partially ordered set $SQ(MC_n)$ shall be called the associated poset of MC_n. In particular, if MC_n is ergodic, then $SQ(MC_n)$ consists of one element.*

Definition 6.10. *A finite poset P is called connected if its diagram $Q(P)$ is a connected quiver.*

We give a construction which for a given finite partially ordered set $P = \{p_1, \ldots, p_n\}$ permits to associate a strongly connected quiver without multiple arrows and multiple loops.

Denote by P_{max} (respectively P_{min}) the set of the maximal (respectively minimal) elements of P and by $P_{max} \times P_{min}$ their Cartesian product.

Definition 6.11. *The quiver $\tilde{Q}(P)$ obtained from the diagram $Q(P)$ by adding the arrows σ_{ij} for all $(p_i, p_j) \in P_{max} \times P_{min}$ shall be called the quiver associated with the partially ordered set P.*

Obviously, $\tilde{Q}(P)$ is a strongly connected simply laced quiver.

Theorem 6.12. *The quiver $Q(\Lambda(P))$ coincides with the quiver $\tilde{Q}(P)$.*

Proof. Recall that $[Q(\Lambda(P))] = \mathcal{E}(R^2) - \mathcal{E}(R)$. Suppose that in $Q(P)$ there is an arrow from s in t. This means that $\alpha_{st} = 0$ and there is no positive integer k ($k \neq s, t$) such that $\alpha_{sk} = 0$ and $\alpha_{kt} = 0$. The elements β_{ss} and β_{tt} of the exponent matrix $\mathcal{E}(R) = (\beta_{ij})$ are equal to 1. We have that $\mathcal{E}(R^2) = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1\le k\le n} (\beta_{sk} + \beta_{kt}) = 1$. Thus, in $[Q(\Lambda(P))]$ in the (s,t)-th position we have $\gamma_{st} - \beta_{st} = 1 - \alpha_{st} = 1 - 0 = 1$. Consequently, $Q(\Lambda(P))$ has an arrow from s to t.

Suppose that $p \in P_{max}$. This means that $\alpha_{pk} = 1$ for $k \neq p$. Therefore the entries of the p-th row of $\mathcal{E}(R)$ are all 1, i.e.

$$(\beta_{p1}, \ldots, \beta_{pp}, \ldots, \beta_{pn}) = (1, \ldots, 1, \ldots, 1).$$

Similarly, if $q \in P_{min}$, then the q-th column $(\beta_{1q}, \ldots, \beta_{qq}, \ldots, \beta_{nq})^T$ of $\mathcal{E}(R)$ is $(1, \ldots, 1, \ldots, 1)^T$. Hence, $\gamma_{pq} = 2$, $\beta_{pq} = 1$, and $Q(\Lambda(P))$ has an arrow from p to q. Consequently, we proved that $\tilde{Q}(P)$ is a subquiver of $Q(\Lambda(P))$.

We show now the converse inclusion. Suppose that $\gamma_{pq} = 2$. Then obviously

$$(\beta_{p1}, \dots, \beta_{pp}, \dots, \beta_{pq}) = (1, \dots, 1, \dots, 1)$$

and

$$(\beta_{1q}, \dots, \beta_{qq}, \dots, \beta_{nq})^T = (1, \dots, 1, \dots, 1)^T.$$

Therefore $p \in P_{max}$, $q \in P_{min}$ and there is an arrow, which goes from p to q.

Suppose $\gamma_{pq} = 1$ and $\beta_{pq} = 0$. Consequently, $p \neq q$, $\beta_{pq} = \alpha_{pq} = 0$ and $p < q$. Since $\gamma_{pq} = \min\limits_{1 \leq k \leq n} (\beta_{pk} + \beta_{kp})$, then $\beta_{pk} + \beta_{kq} \geq 1$ for $k = 1, \dots, n$. Thus for $k \neq p, q$ we have $\beta_{pk} + \beta_{kq} \geq 1$ from which we obtain $\alpha_{pk} + \alpha_{kp} \geq 1$. Hence, there is no positive integer $k\,(k \neq p, q)$ such that $\alpha_{pk} = \alpha_{kq} = 0$. This means that in $\tilde{Q}(P)$ there is an arrow from p to q, which proved the opposite inclusion.

$\square$

Definition 6.13. *Index $in\,P$ of a finite partially ordered set P is the maximal real eigen-value of the adjacency matrix of $\tilde{Q}(P)$.*

Thus, $in\,P = in\,\Lambda(P)$.

Examples.

1. The index of finite linearly ordered set CH_n is 1.

2. Let $ACH_n = \left\{ \begin{array}{cccccc} 1 & 2 & 3 & \dots & n-1 & n \\ \bullet & \bullet & \bullet & \dots & \bullet & \bullet \end{array} \right\}$ be an antichain of width n. Clearly, $\tilde{Q}(ACH_n)$ is a complete simply laced quiver with n vertices. Thus $inACH_n = n$.

3. Let $P_{m,n} = (m, m, \dots, m)$ - be a primitive partially ordered set formed by n linearly ordered disjoint sets each of length m. It is easy to verify that $in\,P_{m,n} = \sqrt[m]{n}$.

4. Consider $P_4 = \left\{ \begin{array}{ccc} \bullet & & \bullet \\ \uparrow & \nwarrow\!\!\!\!\nearrow & \uparrow \\ \bullet & & \bullet \end{array} \right\}$. Denote by

$$U_n = \begin{pmatrix} 1 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & 1 \end{pmatrix}$$

the square $n \times n$-matrix whose every entry is 1. Obviously, the adjacency matrix $\tilde{Q}(P_4)$ is $[\tilde{Q}(P_4)] = \begin{pmatrix} 0 & U_2 \\ U_2 & 0 \end{pmatrix}$ and $in\, P_4 = 2$

5. Let $P_{2n} = \left\{ \begin{array}{ccccccccc} 1 & & 3 & & 5 & & 2n-3 & & 2n-1 \\ \bullet & \to & \bullet & \to & \bullet & \dots & \bullet & \to & \bullet \\ & \searrow\!\!\!\!\!\nearrow & & \searrow\!\!\!\!\!\nearrow & & & & \searrow\!\!\!\!\!\nearrow & \\ \bullet & \to & \bullet & \to & \bullet & \dots & \bullet & \to & \bullet \\ 2 & & 4 & & 6 & & 2n-2 & & 2n \end{array} \right\}$. Obviously,

$$[\tilde{Q}(P_{2n})] = \begin{bmatrix} 0 & U_2 & 0 & \dots & 0 \\ 0 & 0 & U_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & U_2 \\ U_2 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and $in\, P_{2n} = 2$.

Let r be a maximal eigen-value of permutationally irreducible non-negative matrix $A = (a_{ij})$. We denote

$$s_i = \sum_{k=1}^{n} a_{ik} \quad (i = 1, 2, \dots, n), \quad s = \min_{1 \le i \le n} s_i, \quad S = \max_{1 \le i \le n} s_i.$$

Proposition 6.14. (see [16], p. 63). *Let A be a permutationally irreducible non-negative matrix. Then $s \le r \le S$ and the equality sign on the left or the right of r holds for $s = S$ only, i.e. holds only when all the "row -sums" $s_1, s_2, \dots, s_n$ are equal.*

Corollary 6.15. *Let A be a $(0,1)$-matrix and $s = k$, $S = k+1$. Then r is not integer.*

Proof is obviously.

Definition 6.16. (see [3], pp. 198-199). *Let X and Y be any two (disjoint) posets. The ordinal sum $X \oplus Y$ of X and Y is the set of all $x \in X$ and $y \in Y$; $x < y$ for all $x \in X$ and $y \in Y$; the relations $x \le x_1$ and $y \le y_1$ $(x, x_1 \in X;\ y, y_1 \in Y)$ have unchanged meanings.*

The ordinal sum is associative, and we can consider the ordinal power $X^{\oplus n} = \underbrace{X \oplus \dots \oplus X}_{n}$ for any poset X.

In particular, $CH_n = CH_1^{\oplus n}$ and $P_{2n} = ACH_2^{\oplus n}$.

If X and Y are finite posets, then $\mathcal{E}_{X\oplus Y} = \begin{pmatrix} \mathcal{E}_X & 0_{m\times n} \\ U_{n\times m} & \mathcal{E}_Y \end{pmatrix}$, where m (resp. n) is a number of elements in X (resp. in Y); $0_{m\times n}$ is $m\times n$-matrix, whose every entry is 0 and $U_{n\times m}$ is $n\times m$-matrix, whose every entry is 1. As usual, $U_{n\times n} = U_n$ and $0_{n\times n} = 0_n$.

Remark. $in\, CH_n = w(CH_n) = 1$ and $in\, P_{2n} = w(P_{2n}) = 2$.

Proposition 6.17. *If $in\, P = 1$, then P is CH_n for some n.*

The proof follows from the Proposition 6.14 and Theorem 4.11.

Proposition 6.18. *For any finite poset P we have:*

$$in\, P \leq w(P).$$

Proof. Let $c_1, \dots, c_m$ be an antichain formed by all minimal elements of P. There are exactly m arrows from a maximal element a to each c_i, $(i = 1, \dots, m)$.

The elements $a_1, \dots, a_k \in P$ which cover $b \in P$ form an antichain. Thus, there are exactly k arrows from b to $a_1, \dots, a_k$. Obviously, $m \leq w(P)$ and $k \leq w(P)$. Let $[\tilde{Q}(P)] = B = (b_{ij})$. Then,

$$S = \max_{1\leq i\leq n} \sum_{j=1}^{n} b_{ij} \leq w(P)$$

and by Proposition 6.14 we have $in\, P \leq w(P)$.

□

Example. A quiver Q with the adjacency matrix $[Q] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is not the quiver associated with a finite poset P.

Theorem 6.19. *Let P is a finite poset. Then $in\, P = w(P) = 2$ if and only if $P = P_{2n} = ACH_2^{\oplus n}$.*

Proof. The equalities $in\, P_{2n} = w(P_{2n}) = 2$ follows from (5, examples).

Let $P = \{p_1, \dots, p_n\}$, $n \geq 3$ and $in\, P = 2$. We show first, that $\tilde{Q}(P)$ has no loops. Let p_n be an isolated element. Then $\{p_1, \dots, p_{n-1}\}$ is the chain CH_{n-1}. One can suppose that

$$p_1 < p_2 < \dots < p_{n-1}.$$

Thus,

$$[\tilde{Q}(P)] = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

We have $s_1 = 1$ and $s_n = 2$. By Corollary 6.15, $1 < in\,P < 2$ and $\tilde{Q}(P)$ has no loops as desired. Consequently, the $(0,1)$-matrix $[Q(P)]$ with $in\,P = 2$ has zero main diagonal and exactly two 1's in each row. Thus, P_{max} consists of two elements: p_{n-1} and p_n.

Denote by P^T the poset anti-isomorphic to P. Obviously, $in\,P = in\,P^T$. Then $in\,P^T = 2$ and P^T has exactly two maximal elements. Moreover, there are exactly two 1's in each row of $[\tilde{Q}(P^T)]$. Thus, one can assume that $P_{min} = \{p_1, p_2\}$, $P_{max} = \{p_{n-1}, p_n\}$. The $(0,1)$-matrix $[\tilde{Q}(P)]$ has zero main diagonal and exactly two 1's each row and in each column. There exists a numeration of $\{p_3, \dots, p_{n-2}\}$ such that $\sigma_{ij} \in AQ(P)$ if and only if $i < j$, $(i = 1, 2, \dots, n-1, n)$. Write $[\tilde{Q}(P)] = B = (b_{ij})$. Obviously, $b_{n-1,n} = b_{n1} = b_{n-1,2} = b_{n2} = 1$. Moreover, $B - \begin{pmatrix} 0_{n-2,2} & 0_{n-2} \\ u_2 & 0_{2,n-2} \end{pmatrix} = [Q(P)]$ is an upper triangular matrix with zero main diagonal. Then,

$$B = \left[\begin{array}{cccccccccc} 0 & 0 & 1 & & * & & & & & \\ 0 & 0 & 1 & & * & & & & & \\ 0 & 0 & 0 & & * & & & & & \\ 0 & 0 & 0 & & 0 & & & & & \\ 0_2 & & & 0_2 & & & & & & \\ & & & & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & & & & & & \\ u_2 & & & 0_2 & \dots & \dots & \dots & \dots & 0_2 & 0_2 \end{array} \right].$$

Obviously, B must have at least 4 columns. If 1 occupies the position $(3,4)$, then one can assume, that it is in $(1,4)$-th position. We have

$$\begin{array}{ccc} 1 & & 3 \\ \bullet & \longrightarrow & \bullet \\ \downarrow & \swarrow & \\ \bullet & & \\ 4 & & \end{array}$$

and the arrow σ_{14} is extra. By Theorem 6.8 it is impossible.

If B has 4 columns, then $P = P_4$ and $\tilde{Q}(P_4) = \begin{bmatrix} 0_2 & u_2 \\ u_2 & 0_2 \end{bmatrix}$. Continuing this process we shall conclude that 1 can not be in $(5,6)$-th position if B has 6 columns, then $P = P_6$. Obviously, in general case, we have $P = P_{2n}$.

□

Remark. Similarly, one can show that if $\mathit{in}\, P = w(P) = m$ and $\tilde{Q}(P)$ has no loops, then $P = ACH_m^{\oplus n}$.

The description of Gorenstein $(0,1)$-order is given in [32], Theorem 2.1. In view of Theorem 6.19 and the definition of ordinal power, we have the following.

Theorem 6.20. *A reduced* $(0,1)$*-order* Λ *is Gorenstein if and only if* P_Λ *is an ordinal power of either a singleton or an antichain with two elements.*

Theorem 6.21. *A reduced* $(0,1)$*-order* Λ *is Gorenstein if and only if either* $\mathit{in}\, P_\Lambda = w(P_\Lambda) = 1$ *or* $\mathit{in}\, P_\Lambda = w(P_\Lambda) = 2$*. In the first case,* Λ *is hereditary.*

7. Quivers of Gorenstein orders

We observe that the quiver $Q(\Lambda)$ of a cyclic reduced Gorenstein tiled order Λ with the permutation $\sigma(\Lambda)$ not always contains a simple cycle of length n, where $n = |<\sigma(\Lambda)>|$.

For example, consider the cyclic reduced Gorenstein tiled order $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ with the permutation $\sigma(\Lambda)$, where

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}.$$

We compute $[Q(\Lambda)]$.

$$\mathcal{E}(R) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \mathcal{E}(R^2) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Whence,

$$[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that the quiver $Q(\Lambda)$ has simple cycles containing vertex 1 as follows:

$$\left\{ \begin{matrix} 1 & & 3 & & 5 & & 1 \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\},$$

$$\left\{ \begin{matrix} 1 & & 3 & & 5 & & 2 & & 4 & & 1 \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\},$$

$$\left\{ \begin{matrix} 1 & & 3 & & 6 & & 2 & & 4 & & 1 \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\},$$

$$\left\{ \begin{matrix} 1 & & 3 & & 6 & & 2 & & 5 & & 1 \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\},$$

$$\left\{ \begin{matrix} 1 & & 4 & & 1 \\ \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\}, \left\{ \begin{matrix} 1 & & 4 & & 6 & & 3 & & 5 & & 1 \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\},$$

$$\left\{ \begin{matrix} 1 & & 6 & & 2 & & 5 & & 1 \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{matrix} \right\}.$$

Thus the quiver $Q(\Lambda)$ has no cycle of length 6.

Proposition 7.1. *Let $Q(\Lambda)$ be the quiver of a cyclic reduced Gorenstein tiled order Λ with the permutation $\sigma(\Lambda)$ such that $| < \sigma > | = p$ is a prime number; then $Q(\Lambda)$ contains a simple cycle of length p.*

Proof. Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a cyclic reduced Gorenstein tiled order with the permutation $\sigma(\Lambda)$. It can be assumed that

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{pmatrix}.$$

At least one arrow goes out from each vertex of $Q(\Lambda)$. Suppose that an arrow connects vertex 1 with the vertex a in $Q(\Lambda)$, i. e., $q_{1a} = 1$. Since

$$\sigma^{(a-1)} = \begin{pmatrix} 1 & 2 & \cdots & n \\ a & a+1 & \cdots & a-1 \end{pmatrix},$$

we see that $a = \sigma^{(a-1)}(1)$. Using the Main Lemma of [45], we obtain

$$q_{\sigma^{(a-1)}(1)\sigma^{(a-1)}(a)} = q_{\sigma^{(a-1)}(1)\sigma^{2(a-1)}(1)} = 1.$$

Therefore there exists an arrow from a to $\sigma^{2(a-1)}(1)$. As before,

$$q_{\sigma^{2(a-1)}(1)\sigma^{3(a-1)}(1)} = 1, \ldots, q_{\sigma^{k(a-1)}(1)\sigma^{(k+1)(a-1)}(1)} = 1,$$

where k is an arbitrary positive integer.

The permutation $\sigma^{(a-1)}$ generates the cyclic group $< \sigma^{(a-1)} >$ of order $b = n/(a-1,\, n)$. Thus, $\sigma^{b(a-1)}(1) = 1$ and $Q(\Lambda)$ contains the simple cycle

$$\left\{ \begin{matrix} 1 & & \sigma^{(a-1)}(1) & & \sigma^{2(a-1)}(1) & & & & \sigma^{(b-1)(a-1)}(1) & & 1 \\ \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet \end{matrix} \right\}.$$

If $n = p$ is a prime, then $b = p$ and $Q(\Lambda)$ has a simple cycle of length p. □

Suppose that the permutation $\sigma(\Lambda)$ of a reduced Gorenstein order Λ is decomposed into a product of two permutations σ_1 and σ_2 act over non-intersecting sets. To be precise, σ_1 acts over the set $I_1 = \{1, 2, \ldots, n\}$ and σ_2 does over $I_2 = \{n+1, n+2, \ldots, n+m\}$. Let $1 = e_1 + \cdots + e_{m+n}$ be a decomposition $1 \in \Lambda$ into a sum of mutually orthogonal local idempotents. Put $e = e_1 + \cdots + e_n$, $f = 1 - e$, $Q = Q(\Lambda)$, where $Q(\Lambda)$ is the quiver of Λ; $Q_1 = Q(e\Lambda e)$, where $Q(e\Lambda e)$ is the quiver of $e\Lambda e$; $Q_2 = Q(f\Lambda f)$, where $Q(f\Lambda f)$ is the quiver of $f\Lambda f$. Trivially, $e\Lambda e$ and $f\Lambda f$ are also Gorenstein tiled orders with the permutations $\sigma_1(\Lambda) = \sigma(e\Lambda e)$ and $\sigma_2(\Lambda) = \sigma(f\Lambda f)$ respectively. It is easily shown that

i) if there exists an arrow from vertex i to vertex j in $Q(\Lambda)$, where $i, j \in I_1$ or $i, j \in I_2$, then there exists an arrow from vertex i to vertex j in $Q(e\Lambda e)$

or $Q(f\Lambda f)$ respectively;
ii) if $Q(e\Lambda e)$ or $Q(f\Lambda f)$ has no arrow from vertex i to vertex j, where $i, j \in I_1$ or $i, j \in I_2$ respectively, then the quiver $Q(\Lambda)$ has no arrow from vertex i to vertex j.

Proposition 7.2. *Suppose that σ_1 and σ_2 are cycles that do not intersect, whose lengths $| < \sigma_1 > | = m > 2$ and $| < \sigma_2 > | = n > 2$ are mutually prime. Then*

$$[Q] = \begin{pmatrix} [Q_1] & U_{m\times n} \\ U_{n\times m} & [Q_2] \end{pmatrix},$$

where $U_{m\times n} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ *is an $m \times n$-matrix.*

Proof. Since the orders of the permutations σ_1 and σ_2 are pairwise prime, it follows that the order of the permutation $\sigma = \sigma_1 \times \sigma_2$ is equal to mn. By the Main Lemma of [45], $q_{1n+1} = q_{\sigma^t(1)\sigma^t(n+1)} = q_{\sigma_1^t(1)\sigma_2^t(n+1)}$ for any positive integer t. If t varies from 1 to nm, then $\sigma_1^t(1)$ changes m times from 1 to n, $\sigma_2^t(n+1)$ changes n times from $n+1$ to $n+m$. However, among the pairs $(\sigma^t(1), \sigma^t(n+1))$, there are no identical pairs. Therefore, $q_{ij} = q_{1n+1}$ for $i \leq n$, $j > n$. As above, $q_{ij} = q_{n+1,1}$ for $i > n$, $j \leq n$. Since the quiver $Q(\Lambda)$ of any reduced Gorenstein tiled order Λ is strongly connected, it follows that, $q_{ij} = 1$ for $i \leq n$, $j > n$ and for $i > n$, $j \leq n$.

Now suppose that there exists an arrow from vertex i ($i \leq n$) to vertex l ($l \leq n$) in $Q(\Lambda)$. Then $\beta_{ij} + \beta_{jl} > \beta_{il}$ for any $j = 1, \ldots, n+m$. Also, this inequality holds for $j = 1, \ldots, n$, that is there exists an arrow from vertex i to vertex l in $Q(e\Lambda e)$.

Otherwise, suppose that an arrow connects vertex i ($i \leq n$) with vertex l ($l \leq n$) in $Q(e\Lambda e)$. Then $\beta_{ij} + \beta_{jl} > \beta_{il}$ for any $j = 1, \ldots, n$. It is clear that $l \neq \sigma(i)$.

Suppose that $Q(\Lambda)$ has no arrow from i to l. Hence, there exists t such that $n < t \leq n+m$, $\beta_{it} + \beta_{tl} = \beta_{il}$.

For $i \neq l$, we obviously have $\alpha_{it} + \alpha_{tl} = \alpha_{il}$. Adding $\alpha_{l\sigma(i)}$ to both sides of this equality, we obtain

$$\alpha_{it} + \alpha_{tl} + \alpha_{l\sigma(i)} = \alpha_{il} + \alpha_{l\sigma(i)} = \alpha_{i\sigma(i)}.$$

Then, adding $\alpha_{t\sigma(i)}$, we obtain

$$\alpha_{it} + \alpha_{t\sigma(i)} + \alpha_{tl} + \alpha_{l\sigma(i)} = \alpha_{i\sigma(i)} + \alpha_{t\sigma(i)}.$$

Whence, $\alpha_{tl}+\alpha_{l\sigma(i)}=\alpha_{t\sigma(i)}$ or $\beta_{tl}+\beta_{l\sigma(i)}=\beta_{t\sigma(i)}$. At the same time $q_{t\sigma(i)}=1$. This contradiction proves that there is an arrow which connects vertex i with vertex l in $Q(\Lambda)$.

Thus, for $i\neq l$, $Q(e\Lambda e)$ has an arrow from vertex i to vertex l if and only if there exists an arrow from i to l in $Q(\Lambda)$, where $i,l\in I_1$.

By the same argument, an arrow connects vertex t with vertex k in $Q(f\Lambda f)$ iff $Q(\Lambda)$ has an arrow from t to k, where $t,k\in I_2, t\neq k$.

Now let $i=l$; then, by the assumption, $\alpha_{it}+\alpha_{ti}=\alpha_{ti}+\alpha_{it}=1$, that is $Q(\Lambda)$ has no loop at vertex t. Since $q_{ik}=1$ for $k>n$; we have $\alpha_{it}+\alpha_{tk}>\alpha_{ik}$ if $k\neq t$. Adding α_{ti} to both sides of this equality, we have $1+\alpha_{tk}>\alpha_{ti}+\alpha_{ik}\geq\alpha_{tk}$. Whence, $\alpha_{ti}+\alpha_{ik}=\alpha_{tk}$, i. e., $q_{tk}=0$ for all $k>n, k\neq t$. Consequently, the quiver Q_2 has no arrow from t to k, where $k\neq t, k=n+1,\dots,n+m$. This contradicts the fact that the quiver $Q(f\Lambda f)$ is strongly connected. Thus, in $Q(\Lambda)$, there exists an loop at vertex i.

□

Example. Let $\Gamma_\alpha=\{\mathcal{O},\mathcal{E}(\Gamma_\alpha)\}$, where

$$\mathcal{E}(\Gamma_\alpha)=\begin{bmatrix}0&3\alpha&3\alpha&2\alpha&2\alpha\\0&0&3\alpha&\alpha&\alpha\\0&0&0&0&0\\0&\alpha&2\alpha&0&2\alpha\\0&\alpha&2\alpha&2\alpha&0\end{bmatrix},\ \sigma(\Gamma_\alpha)=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}.$$

Then $[Q(\Gamma_\alpha)]=\begin{pmatrix}[Q(T_{3\alpha,3})]&U_{3\times2}\\U_{2\times3}&E_2\end{pmatrix}$, where

$$\mathcal{E}(T_{3\alpha,3})=\begin{bmatrix}0&3\alpha&3\alpha\\0&0&3\alpha\\0&0&0\end{bmatrix}.$$

The example of Γ_α shows that conditions $m>2$ and $n>2$ in Proposition 7.2 are essential.

Definition 7.3. *We remind that a real $s\times s$ - matrix $P=(p_{ij})$ is called doubly stochastic if $\sum\limits_{j=1}^{s}p_{ij}=1$ and $\sum\limits_{i=1}^{s}p_{ij}=1$ for any $i,j=1,\dots,s$.*

Theorem 7.4. (see [45]). *Let $\Lambda=\{\mathcal{O},\mathcal{E}(\Lambda)\}$ be a cyclic reduced Gorenstein tiled order; then $[Q(\Lambda)]=\lambda P$, where λ is a positive integer and P is a doubly stochastic matrix.*

Such matrix λP is called *semimagic* or *semimagic square* (see [12], [36], p. 16).

Corollary 7.5. *A cyclic Gorenstein tiled order is integral.*

Example. (see [45], p. 4238). Let $\Lambda_6 = \{\mathcal{O}, \mathcal{E}(\Lambda_6)\}$, where

$$\mathcal{E}(\Lambda_6) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 3 & 3 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

Λ_6 is a reduced Gorenstein tiled order with

$$\sigma(\Lambda_6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$$

and

$$B = [Q(\Lambda_6)] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then $s = 4$ and $S = 5$ and the order Λ_6 is not integral by Corollary 6.15.

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RESEARCH ARTICLE

On groups of finite normal rank

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Abstract. In this article the investigation of groups of finite normal rank is continued. The finiteness of normal rank of nonabelian p-group G is proved where G has a normal elementary abelian p-subgroup A for which quotient group G/A is isomorphic to the direct product of finite number of quasicyclic p-groups.

A number of authors studied the groups in which finiteness conditions were laid on some systems of their subgroups [1]. Earlier the author investigated the groups of finite F-rank [2], where F was some system of nonabelian finitely generated subgroups of a group and some classes of groups of finite normal rank.

In this article the investigation of groups of finite normal rank is continued.

Definition. *We shall say that a group G has finite normal rank r, if r is a minimal number with the property that for any finite set of elements $g_1, g_2, ..., g_n$ of a group G there are the elements $h_1, h_2, ..., h_m$ of G such that $m \leq r$ and*

$$< h_1, h_2, ..., h_m >^G = < g_1, g_2, ..., g_n >^G .$$

In the case when there is not such number r, the normal rank of group G is considered to be infinite.

We shall use the notation $r_n(G)$ for the normal rank of group G. The special rank of group G is denoted by the generally accepted symbol $r(G)$.

The principal result of this article is the theorem.

Theorem. *Let G be a nonabelian p-group, where p is a prime number. Let A be a normal subgroup of G, which is an elementary abelian p-group. Quotient group G/A is isomorphic to the direct product of l quasicyclic p-groups. If subgroup A can be generated as a G-subgroup by n elements, i.e.*

$$A =< a_1, a_2, ...a_n >^G,$$

and n, l are the finite numbers, then the normal rank of group G is finite and $r_n(G) \leq n + l$.

This result was announced in [3] earlier.

We shall need the following lemma in proof of the theorem.

Lemma. *The normal rank of wreath product of group of prime order p and direct product of l quasicyclic p-groups is equal to $l + 1$.*

Proof. Let A be the basis of wreath product W,$W =< a > wr(X_{j=1}^l P_j)$, where P_j is a quasicyclic p-group. We shall prove at first that for any $b_1, b_2, ..., b_n$ from A there is such element $b \in A$, for which

$$< b_1, b_2, ..., b_n >^G =< b >^G .$$

Since the group $W = \cup_{i=1}^{\infty}(< a > wr(X_{j=1}^l < g_{ji} >)$, $|g_{ji}| = p^i$, then the elements $b_1, b_2, ..., b_n$ are contained in subgroup $V =< a > wr(X_{j=1}^l < g_{ij} >)$ for some number i. The upper central series of subgroup V is

$$E = Z_0 < Z_1 < Z_2 < ... < Z_{lp^i - 1} < Z_{lp^i} < V,$$

where Z_{lp^i} is the basis of wreath product V, factors Z_{k+1}/Z_k, $k = 0, 1, ..., lp^i$ have orders p, factors V/Z_{lp^i} is isomorphic to the direct product of l cyclic groups of orders p^i [4] .

The subgroups $B_k =< b_k >^{X_{j=1}^l < g_{ij} >}$, $k = 1, 2, ..., n$ are normal in group V, therefore intersections $B_k \cap Zj$, $j = 1, 2, ..., lp^i$ are nontrivial. Since the factors Z_{k+1}/Z_k, $k = 0, 1, ..., lp^i$ are cyclic of prime order, then the equalities $B_k \cap Z_q = Z_q$, $q = 0, 1, ..., t_k$, are valid, where $t_k \leq lp^i$.

From here it follows that $B_k = Z_{t_k}$, therefore for any $m_1, m_2 \leq n$ the one from subgroups B_{m_1}, B_{m_2} embeds in another. Consequently the subgroups $B_k, k = 1, 2, ..., n$ form a series of embeded subgroups

$$B_{k_1} < B_{k_2} < ... < B_{k_n} = B,$$

where $B =< b_1, b_2, ..., b_n >^{X_{j=1}^l < g_{ji} >}$. Thererore $B =< b >^{X_{j=1}^l < g_{ji} >}$, where $b = b_{k_n}$. From here follows the equality

$$< b_1, b_2, ..., b_n >^G =< b >^G .$$

Now we shall prove that for any $c_1, c_2, ..., c_r \in W$ the subgroup $C =< c_1, c_2, ..., c_r >^G$ can be generated as G-subgroup by no more than $l + 1$ elements. It is sufficient to consider the case $C_1 \not\leq A$, where $C_1 =< c_1, c_2, ..., c_r >$. Since $C_1A/A \simeq C_1/(C_1 \cap A)$, the subgroup C_1 is finite and $r(C_1A/A) \leq l$, then we can choose the elements $d_1, d_2, ..., d_{s+u}$ such that

$$C =< d_1, d_2, ..., d_s, d_{s+1}, ..., d_{s+u} >^G$$

and $d_i \in A, i = 1, 2, ..., s, d_{s+1}, d_{s+2}, ..., d_{s+u} \notin A, u \leq l$. As we proved, there is the element $d \in A$ for which

$$< d_1, d_2, ..., d_s >^G =< d >^G,$$

therefore $C =< d, d_{s+1}, ..., d_{s+u} >^G$. Consequently the normal rank of wreath product W is no more than $l + 1$.

For proving the equality $r_n(W) = l + 1$ we numerate the elements of subgroup $X_{j=1}^{\infty} P_j$ as $h_1, h_2, ...$ and assume $a^{h_i} = a_i$. According to the structure of subgroup W the subgroup $A_0 =< a_i a_j^{-1}, i, j = 1, 2, ... >$ is normal in W and quotient group W/A_0 is isomorphic to the direct product of a group of prime order p and l quasicyclic p-groups. Since the normal rank of quotient group W/A_0 is equal to $l + 1$ and $r_n(W/A_0) \leq r_n(W)$, where $r_n(W) \leq l + 1$, then we have the equality $r_n(W) = l + 1$. Lemma is proved. □

Proof of the theorem. At first we shall prove that for any finite set of elements $b_1, b_2, ..., b_k$ of A there are the elements $c_1, c_2, ..., c_t$ of A such that $t \leq n$ and $< b_1, b_2, ..., b_k >^G =< c_1, c_2, ..., c_t >^G$. We shall prove at first this statement by the induction on number v of elements $a_1, a_2, ..., a_v$, where $A =< a_1, a_2, ..., a_v >^G$. If $v = 1$ then $A =< a_1 >^G$, therefore group G is isomorphic to some quotient group of wreath product of a group of prime order p and direct product of l quasicyclic p-groups. From the proof of the lemma it follows that there is an element $b \in A$ for which

$$< b_1, b_2, ..., b_k >^G =< b >^G .$$

Let our statement be valid for $u = n - 1$. Let $u = n$ and

$$B =< b_1, b_2, ..., b_k >^G, A_1 =< a_1, a_2, ..., a_{n-1} >^G .$$

If subgroup B is contained in A_1 then according to the inductive assumption there are such elements $c_1, c_2, ..., c_t, t \leq n$ that $B =< c_1, c_2, ..., c_t >^G$. Let now $B \not\leq A_1$. Quotient group G/A_1 is isomorphic to some quotient group of wreath product of a group of prime order p and direct product of l quasicyclic p-groups. From this and isomorphism $BA_1/A_1 \simeq B/B \cap A_1$

it follows by the lemma that there is an element $b \in B$ for which $B/B \cap A_1 =< b(B \cap A_1) >^G$. Consequently for every $b_i, i = 1, 2, ..., k$, there are such integers $n_1, n_2, ..., n_{r_i}$ and the elements $g_1, g_2, ..., g_{r_i}$ of G that the equalities

$$b_i = (b^{n_1})^{g_1}(b^{n_2})^{g_2}...(b^{n_{r_i}})^{g_{r_i}}h_i$$

are valid, where $h_i \in (B \cap A_1)$. Since the element b belongs to the subgroup B then $B =< b, h_1, h_2, ..., h_k >^G$, therefore

$$B =< b >^G < h_1, h_2, ..., h_k >^G . \qquad (1)$$

According to the inductive assumption and inclusion $< h_1, h_2, ..., h_k >^G \leq A_1$ there are such elements $d_1, d_2, ..., d_m$ of A that $m \leq n - 1$ and

$$< h_1, h_2, ..., h_k >^G =< d_1, d_2, ..., d_m >^G .$$

From this equality and (1) it follows that $B =< b, d_1, ..., d_m >^G, m \leq n - 1$. Our statement is proved.

Let now $B =< b_1, b_2, ..., b_k >^G$, where even if one from the elements $b_i, i = 1, 2, ..., k$ does not belong to the subgroup A. Since the subgroup D generated by the elements $b_1, b_2, ..., b_k$ is finite, then the intersection $D \cap A$ is finite too. Therefore there are the elements $c_1, c_2, ..., c_j, j \leq n$, for which $< D \cap A >^G =< c_1, c_2, ..., c_j >^G$. Since quotient group G/A is a direct product of l locally cyclic groups and $DA/A \simeq D/D \cap A$, then there are such elements $c_{j+1}, ..., c_{i+y}$ of D that

$$< D >^G =< c_1, c_2, ..., c_{j+y} >^G,$$

$y \leq l$. Consequently the equality $B =< c_1, c_2, ..., c_{j+y} >^G$ is valid, where $j + y \leq n + l$. The theorem is proved. □

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Radical theory in BCH-algebras

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Abstract. The notion of k-nil radical in BCH-algebras is defined, and related properties are investigated.

1. Introduction

In 1966, Y. Imai and K. Iséki [7] and K. Iséki [8] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Q. P. Hu and X. Li [4, 5] introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. They have studied some properties of these algebras. In 1992, W. P. Huang [6] introduced a nil ideals in BCI-algebras. In [9], E. H. Roh and Y. B. Jun discussed the concept of nil subsets by using nilpotent elements in BCH-algebras. In this paper, we introduce the notion of k-nil radical in BCH-algebras, and study some useful properties. We prove that the k-nil radical of a subalgebra (resp. a (closed, translation, semi-) ideal) is a subalgebra (resp. a (closed, translation, semi-) ideal). Concerning the homomorphisms, we discuss related properties.

2. Preliminaries

By a *BCH-algebra* we shall mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

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(H1) $x * x = 0$,

(H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(H3) $(x * y) * z = (x * z) * y$.

In a BCH-algebra X, the following holds for all $x, y, z \in X$,

(p1) $x * 0 = x$,

(p2) $(x * (x * y)) * y = 0$,

(p3) $0 * (x * y) = (0 * x) * (0 * y)$,

(p4) $x * 0 = 0$ implies $x = 0$,

(p5) $0 * \big(0 * (0 * x)\big) = 0 * x$.

A nonempty subset S of a BCH-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A nonempty subset A of a BCH-algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in A$,

(I2) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

A nonempty subset A of a BCH-algebra X is called a *closed ideal* of X if it satisfies (I2) and

(I3) $0 * x \in A$, $\forall x \in A$.

Note that every closed ideal of a BCH-algebra is a subalgebra, but the converse is not true (see [1]). A mapping $f : X \to Y$ of BCH-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCH-algebras, then $f(0) = 0$.

3. Main Results

Throughout this section X is a BCH-algebra and k is a positive integer. For any elements x and y of X, let us write $x * y^k$ for $(\cdots((x*y)*y)*\cdots)*y$ in which y occurs k times.

Definition 3.1. Let I be a nonempty subset of X. Then the set

$$\sqrt[k]{I} := \{x \in X \mid 0 * x^k \in I\}$$

is called the *k-nil radical* of I.

Lemma 3.2. ([9, Lemmas 3.2 and 3.3]) *For any $x, y \in X$, we have*

(1) $0 * (0 * x)^k = 0 * (0 * x^k)$,

(2) $0 * (x * y)^k = (0 * x^k) * (0 * y^k)$.

Proposition 3.3. *If I and J are nonempty subsets of X, then*

$$\sqrt[k]{I \cup J} = \sqrt[k]{I} \cup \sqrt[k]{J}.$$

Proof. Note that

$$\begin{aligned} x \in \sqrt[k]{I \cup J} &\Leftrightarrow 0 * x^k \in I \cup J \\ &\Leftrightarrow 0 * x^k \in I \quad \text{or} \quad 0 * x^k \in J \\ &\Leftrightarrow x \in \sqrt[k]{I} \quad \text{or} \quad x \in \sqrt[k]{J} \\ &\Leftrightarrow x \in \sqrt[k]{I} \cup \sqrt[k]{J}. \end{aligned}$$

This completes the proof. □

Proposition 3.4. *Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a collection of nonempty subsets of X, where Λ is any index set. Then*

(i) $\sqrt[k]{\bigcap_{\alpha \in \Lambda} I_\alpha} = \bigcap_{\alpha \in \Lambda} \sqrt[k]{I_\alpha}$.

(ii) $\forall \alpha \in \Lambda,\ 0 \in I_\alpha \Rightarrow 0 \in \sqrt[k]{I_\alpha}$.

(iii) $\forall \alpha, \beta \in \Lambda,\ I_\alpha \subseteq I_\beta \Rightarrow \sqrt[k]{I_\alpha} \subseteq \sqrt[k]{I_\beta}$.

Proof. (i) Note that

$$\begin{aligned} x \in \sqrt[k]{\bigcap_{\alpha \in \Lambda} I_\alpha} &\Leftrightarrow 0 * x^k \in \bigcap_{\alpha \in \Lambda} I_\alpha \\ &\Leftrightarrow 0 * x^k \in I_\alpha \quad \text{for all} \quad \alpha \in \Lambda \\ &\Leftrightarrow x \in \sqrt[k]{I_\alpha} \quad \text{for all} \quad \alpha \in \Lambda \\ &\Leftrightarrow x \in \bigcap_{\alpha \in \Lambda} \sqrt[k]{I_\alpha}, \end{aligned}$$

and hence (i) is valid.

(ii) and (iii) are straightforward. □

Proposition 3.5. *If I is a subalgebra of X and $x \in \sqrt[k]{I}$, then $0 * x \in \sqrt[k]{I}$.*

Proof. If $x \in \sqrt[k]{I}$, then $0 * x^k \in I$. Since I is a subalgebra of X, we have $0 * (0 * x)^k = 0 * (0 * x^k) \in I$ by using Lemma 3.2(1). This shows that $0 * x \in \sqrt[k]{I}$. □

Theorem 3.6. *If I is a subalgebra of X, then so is the k-nil radical $\sqrt[k]{I}$ of I.*

Proof. Let $x, y \in \sqrt[k]{I}$. Then $0 * x^k \in I$ and $0 * y^k \in I$. Since I is a subalgebra, it follows from Lemma 3.2(2) that

$$0 * (x * y)^k = (0 * x^k) * (0 * y^k) \in I$$

so that $x * y \in \sqrt[k]{I}$. Hence $\sqrt[k]{I}$ is a subalgebra of X. □

Theorem 3.7. *If I is an ideal of X, then so is the k-nil radical $\sqrt[k]{I}$ of I.*

Proof. Assume that I is an ideal of X. Obviously $0 \in \sqrt[k]{I}$. Let $x, y \in X$ be such that $x*y \in \sqrt[k]{I}$ and $y \in \sqrt[k]{I}$. Then $0*y^k \in I$ and $(0*x^k)*(0*y^k) = 0*(x*y)^k \in I$. Since I is an ideal of X, it follows from (I2) that $0*x^k \in I$ so that $x \in \sqrt[k]{I}$. Hence $\sqrt[k]{I}$ is an ideal of X. □

Lemma 3.8. ([1, Theorem 4]) *Let I be a subalgebra of a BCH-algebra X such that $x * y \in I$ implies $y * x \in I$ for all $x, y \in X$. Then I is a closed ideal of X.*

Theorem 3.9. *For any closed ideal I of a BCH-algebra X, the k-nil radical $\sqrt[k]{I}$ of I is also a closed ideal of X.*

Proof. Let I be a closed ideal of X. Then I is a subalgebra of X, and so $\sqrt[k]{I}$ is a subalgebra of X. Let $x, y \in X$ be such that $x * y \in \sqrt[k]{I}$. Then $0 * (x * y)^k \in I$. Using (H3), (p3), (p5) and Lemma 3.2(2), we have

$$\begin{aligned} 0 * (y * x)^k &= (0 * y^k) * (0 * x^k) \\ &= \big(0 * (0 * (0 * y^k))\big) * (0 * x^k) \\ &= \big(0 * (0 * x^k)\big) * \big(0 * (0 * y^k)\big) \\ &= 0 * \big((0 * x^k) * (0 * y^k)\big) \\ &= 0 * \big(0 * (x * y)^k\big) \in I, \end{aligned}$$

since I is a subalgebra. Hence $y * x \in \sqrt[k]{I}$, and so, by Lemma 3.8, $\sqrt[k]{I}$ is a closed ideal of X. □

Definition 3.10. ([1, Definition 12]) A nonempty subset I of a BCH-algebra X is called a *semi-ideal* of X if it satisfies (I1) and

(I4) $x \leq y$ and $y \in I$ imply $x \in I$

where $x \leq y$ means $x * y = 0$.

Note that every closed ideal is a semi-ideal, but the converse may not be true (see [1]).

Theorem 3.11. *If I is a semi-ideal of X, then so is $\sqrt[k]{I}$.*

Proof. Obviously $0 \in \sqrt[k]{I}$. Let $x, y \in X$ be such that $x \leq y$ and $y \in \sqrt[k]{I}$. Then $0 * y^k \in I$ and $x * y = 0$. These imply that

$$0 = 0 * (x * y)^k = (0 * x^k) * (0 * y^k), \text{ that is, } 0 * x^k \leq 0 * y^k.$$

Since I is a semi-ideal of X, it follows that $0 * x^k \in I$ or equivalently $x \in \sqrt[k]{I}$. Hence $\sqrt[k]{I}$ is a semi-ideal of X. □

Proposition 3.12. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. If S is a nonempty subset of Y, then $\sqrt[k]{f^{-1}(S)} \subseteq f^{-1}(\sqrt[k]{S})$.*

Proof. Let $x \in \sqrt[k]{f^{-1}(S)}$. Then $0 * x^k \in f^{-1}(S)$, and so $0 * f(x)^k = f(0 * x^k) \in S$. Hence $f(x) \in \sqrt[k]{S}$ which implies $x \in f^{-1}(\sqrt[k]{S})$. This completes the proof. □

Theorem 3.13. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. If J is a closed ideal of Y, then $f^{-1}(\sqrt[k]{J})$ is a closed ideal of X containing $\sqrt[k]{f^{-1}(J)}$.*

Proof. The inclusion $\sqrt[k]{f^{-1}(J)} \subseteq f^{-1}(\sqrt[k]{J})$ is by Proposition 3.12. Let $x, y \in f^{-1}(\sqrt[k]{J})$. Then $f(x), f(y) \in \sqrt[k]{J}$, and so $0 * f(x)^k \in J$ and $0 * f(y)^k \in J$. Since J is a subalgebra of Y, it follows from Lemma 3.2(2) that

$$\begin{aligned} f\big(0 * (x * y)^k\big) &= 0 * f(x * y)^k = 0 * \big(f(x) * f(y)\big)^k \\ &= \big(0 * f(x)^k\big) * \big(0 * f(y)^k\big) \in J \end{aligned}$$

so that $0 * (x * y)^k \in f^{-1}(J)$, that is, $x * y \in \sqrt[k]{f^{-1}(J)} \subseteq f^{-1}(\sqrt[k]{J})$. Hence $f^{-1}(\sqrt[k]{J})$ is a subalgebra of X. Now let $a, b \in X$ be such that $a * b \in f^{-1}(\sqrt[k]{J})$. Then $f(a) * f(b) = f(a * b) \in \sqrt[k]{J}$, and so $0 * \big(f(a) * f(b)\big)^k \in J$. Using Lemma 3.2(2), (p5), (H3) and (p3), we have

$$\begin{aligned} 0 * f(b * a)^k &= 0 * \big(f(b) * f(a)\big)^k \\ &= \big(0 * f(b)^k\big) * \big(0 * f(a)^k\big) \\ &= \big(0 * (0 * (0 * f(b)^k))\big) * \big(0 * f(a)^k\big) \\ &= \big(0 * (0 * f(a)^k)\big) * \big(0 * (0 * f(b)^k)\big) \\ &= 0 * \big((0 * f(a)^k) * (0 * f(b)^k)\big) \\ &= 0 * \big(0 * (f(a) * f(b))^k\big) \in J, \end{aligned}$$

because J is a subalgebra. Hence $f(b * a) \in \sqrt[k]{J}$, and so $b * a \in f^{-1}(\sqrt[k]{J})$. Using Lemma 3.8, we know that $f^{-1}(\sqrt[k]{J})$ is a closed ideal of X. □

Theorem 3.14. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. If U is a semi-ideal of Y, then $f^{-1}(\sqrt[k]{U})$ is a semi-ideal of X containing $\sqrt[k]{f^{-1}(U)}$.*

Proof. Obviously $0 \in f^{-1}(\sqrt[k]{U})$. Let $x, y \in X$ be such that $x \leq y$ and $y \in f^{-1}(\sqrt[k]{U})$. Then $x * y = 0$ and $f(y) \in \sqrt[k]{U}$, that is, $0 * f(y)^k \in U$. Using Lemma 3.2(2), we have

$$\big(0*f(x)^k\big)*\big(0*f(y)^k\big) = 0*\big(f(x)*f(y)\big)^k = 0*f(x*y)^k = 0*f(0)^k = 0,$$

and so $0 * f(x)^k \leq 0 * f(y)^k$. Since U is a semi-ideal, it follows that

$$f(0 * x^k) = f(0) * f(x)^k = 0 * f(x)^k \in U$$

so that $0 * x^k \in f^{-1}(U)$, i.e., $x \in \sqrt[k]{f^{-1}(U)} \subseteq f^{-1}(\sqrt[k]{U})$. Therefore $f^{-1}(\sqrt[k]{U})$ is a semi-ideal of X. □

Theorem 3.15. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. Then $f(\sqrt[k]{I}) \subseteq \sqrt[k]{f(I)}$ for every nonempty subset I of X. Moreover, the equality is valid when f is one-to-one.*

Proof. Let $y \in f(\sqrt[k]{I})$. Then there exists $x \in \sqrt[k]{I}$ such that $f(x) = y$. Hence $0 * x^k \in I$ and

$$0 * y^k = f(0) * f(x)^k = f(0 * x^k) \in f(I),$$

and so $y \in \sqrt[k]{f(I)}$. Thus $f(\sqrt[k]{I}) \subseteq \sqrt[k]{f(I)}$. Assume that f is one-to-one and let $y \in \sqrt[k]{f(I)}$. Then $y = f(x)$ for some $x \in X$, and

$$f(0 * x^k) = 0 * f(x)^k = 0 * y^k \in f(I).$$

Since f is one-to-one, it follows that $0 * x^k \in f^{-1}\big(f(I)\big) = I$ so that $x \in \sqrt[k]{I}$. Therefore $y = f(x) \in f(\sqrt[k]{I})$. This completes the proof. □

Definition 3.16. [10] A *translation ideal* of X is defined to be an ideal U of X satisfying an additional condition:

$$\forall x, y, z \in X,\ x*y \in U,\ y*x \in U \Rightarrow (x*z)*(y*z) \in U,\ (z*x)*(z*y) \in U.$$

Theorem 3.17. *If U is a translation ideal of X, then so is $\sqrt[k]{U}$.*

Proof. If U is a translation ideal of X, then U is an ideal of X and so $\sqrt[k]{U}$ is an ideal of X (see Theorem 3.7). Let $x, y, z \in X$ be such that $x * y \in \sqrt[k]{U}$ and $y * x \in \sqrt[k]{U}$. Then

$$(0 * x^k) * (0 * y^k) = 0 * (x * y)^k \in U$$

and

$$(0 * y^k) * (0 * x^k) = 0 * (y * x)^k \in U.$$

Since U is a translation ideal, it follows from Lemma 3.2(2) that

$$0 * \big((x * z) * (y * z)\big)^k = \big((0 * x^k) * (0 * z^k)\big) * \big((0 * y^k) * (0 * z^k)\big) \in U$$

and

$$0 * \big((z * x) * (z * y)\big)^k = \big((0 * z^k) * (0 * x^k)\big) * \big((0 * z^k) * (0 * y^k)\big) \in U,$$

and so $(x * z) * (y * z) \in \sqrt[k]{U}$ and $(z * x) * (z * y) \in \sqrt[k]{U}$. Therefore $\sqrt[k]{U}$ is a translation ideal of X. □

Theorem 3.18. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. If U is a translation ideal of Y, then $f^{-1}(\sqrt[k]{U})$ is a translation ideal of X containing $\sqrt[k]{f^{-1}(U)}$.*

Proof. Let $x, y, z \in X$ be such that $x*y \in f^{-1}(\sqrt[k]{U})$ and $y*x \in f^{-1}(\sqrt[k]{U})$. Then $f(x) * f(y) = f(x * y) \in \sqrt[k]{U}$ and $f(y) * f(x) = f(y * x) \in \sqrt[k]{U}$. Hence

$$\big(0 * f(x)^k\big) * \big(0 * f(y)^k\big) = 0 * \big(f(x) * f(y)\big)^k \in U$$

and

$$\big(0 * f(y)^k\big) * \big(0 * f(x)^k\big) = 0 * \big(f(y) * f(x)\big)^k \in U.$$

Since U is a translation ideal of Y, it follows that

$$\begin{aligned}
& 0 * f\big((x * z) * (y * z)\big)^k \\
= \;& 0 * \big(f(x * z) * f(y * z)\big)^k \\
= \;& \big(0 * f(x * z)^k\big) * \big(0 * f(y * z)^k\big) \\
= \;& \big(0 * (f(x) * f(z))^k\big) * \big(0 * (f(y) * f(z))^k\big) \\
= \;& \big((0 * f(x)^k) * (0 * f(z)^k)\big) * \big((0 * f(y)^k) * (0 * f(z)^k)\big) \in U
\end{aligned}$$

and

$$\begin{aligned}
& 0 * f\big((z * x) * (z * y)\big)^k \\
= \;& 0 * \big(f(z * x) * f(z * y)\big)^k \\
= \;& \big(0 * f(z * x)^k\big) * \big(0 * f(z * y)^k\big) \\
= \;& \big(0 * (f(z) * f(x))^k\big) * \big(0 * (f(z) * f(y))^k\big) \\
= \;& \big((0 * f(z)^k) * (0 * f(x)^k)\big) * \big((0 * f(z)^k) * (0 * f(y)^k)\big) \in U
\end{aligned}$$

so that $f\big((x * z) * (y * z)\big) \in \sqrt[k]{U}$ and $f\big((z * x) * (z * y)\big) \in \sqrt[k]{U}$. Hence $(x * z) * (y * z) \in f^{-1}(\sqrt[k]{U})$ and $(z * x) * (z * y) \in f^{-1}(\sqrt[k]{U})$, completing the proof. □

Let U be a translation ideal of X and define a relation "$\sim$" on X by $x \sim y$ if and only if $x * y \in U$ and $y * x \in U$ for every $x, y \in X$. Then "$\sim$" is a congruence relation on X. By $[x]$ we denote the equivalence class containing x, and by X/U we denote the set of all equivalence classes, that is, $X/U := \{[x] \mid x \in X\}$. Then $(X/U; \odot, [0])$ is a BCH-algebra, where $[x] \odot [y] = [x * y]$ for every $x, y \in X$ (see [10]). If U is a translation ideal of X, then so is $\sqrt[k]{U}$ (see Theorem 3.17). Hence $(X/\sqrt[k]{U}; \odot, [0])$ is a BCH-algebra and $[0] = \sqrt[k]{U}$. For any two BCH-algebras X and Y, the *product BCH-algebra* is defined to be a BCH-algebra $(X \times Y; *, 0)$, where $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$, $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$, and $0 = (0, 0)$ (see [4, 5]).

Lemma 3.19. *Let X and Y be BCH-algberas. For any $(x, y) \in X \times Y$, we have $(0, 0) * (x, y)^k = (0 * x^k, 0 * y^k)$.*

Proof. It is straightforward. □

Theorem 3.20. *Let A and B be nonempty subsets of BCH-algebras X and Y, respectively. Then*

(i) $\sqrt[k]{A} \times \sqrt[k]{B} = \sqrt[k]{A \times B}$,

(ii) if A and B are translation ideals of X and Y respectively, then $\sqrt[k]{A \times B}$ is a translation ideal of $X \times Y$ and

$$\frac{X \times Y}{\sqrt[k]{A \times B}} \cong X/\sqrt[k]{A} \times Y/\sqrt[k]{B}.$$

Proof. (1) We have that

$$\begin{aligned}
\sqrt[k]{A \times B} &= \{(a, b) \in X \times Y \mid (0, 0) * (a, b)^k \in A \times B\} \\
&= \{(a, b) \in X \times Y \mid (0 * a^k,\ 0 * b^k) \in A \times B\} \\
&= \{(a, b) \in X \times Y \mid 0 * a^k \in A,\ 0 * b^k \in B\} \\
&= \{(a, b) \in X \times Y \mid a \in \sqrt[k]{A},\ b \in \sqrt[k]{B}\} \\
&= \{a \in X \mid a \in \sqrt[k]{A}\} \times \{b \in X \mid b \in \sqrt[k]{B}\} \\
&= \sqrt[k]{A} \times \sqrt[k]{B}
\end{aligned}$$

(ii) Obviously $\sqrt[k]{A \times B}$ is a translation ideal of $X \times Y$. Consider natural homomorphisms

$$\pi_X : X \to X/\sqrt[k]{A},\ x \mapsto [x] \ \text{ and } \ \pi_Y : Y \to Y/\sqrt[k]{B},\ y \mapsto [y].$$

Define a mapping $\Phi : X \times Y \to X/\sqrt[k]{A} \times Y/\sqrt[k]{B}$ by $\Phi(x, y) = ([x], [y])$ for all $(x, y) \in X \times Y$. Then clearly Φ is a well-defined onto homomorphism.

Moreover,

$$\begin{aligned}
\mathrm{Ker}\Phi &= \{(x,y)\in X\times Y \mid \Phi(x,y)=([0],[0])\}\\
&= \{(x,y)\in X\times Y \mid ([x],[y])=([0],[0])\}\\
&= \{(x,y)\in X\times Y \mid [x]=[0],\ [y]=[0]\}\\
&= \{(x,y)\in X\times Y \mid x\in \sqrt[k]{A},\ y\in \sqrt[k]{B}\}\\
&= \sqrt[k]{A}\times \sqrt[k]{B} = \sqrt[k]{A\times B}.
\end{aligned}$$

By the homomorphism theorem (see [10, Theorem 3.7]), we have

$$\frac{X\times Y}{\sqrt[k]{A\times B}} = \frac{X\times Y}{\mathrm{Ker}\Phi} \cong X/\sqrt[k]{A}\times Y/\sqrt[k]{B}.$$

□

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RESEARCH ARTICLE

On the finite state automorphism group of a rooted tree

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Abstract. The normalizer of the finite state automorphism group of a rooted homogeneous tree in the full automorphism group of this tree was investigated. General form of elements in the normalizer was obtained and countability of the normalizer was proved.

1. Introduction

Automorphism groups of rooted trees are studied strongly last years in connection with their application in geometric group theory, theory of dynamic systems, ergodic and spectral theory, and that they also contain various interesting subgroups with extremal properties. In particular, there are free constructions among them, various constructions of groups of intermediate growth, etc (see [GNS] and its bibliography).

Among subgroups of automorphism group of a rooted tree the finite state automorphism group arise the big interest [Su].

In the paper [NS] the number of problems on the finite state automorphism group of a rooted tree was posed. This work partially solves one of these problems. In the paper the normalizer of the finite state automorphism group of a rooted tree in the full automorphism group of this tree was investigated. General form of elements in normalizer was obtained and countability of normalizer was proved. According to

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[L, LN] this normalizer is isomorphic to the automorphism group of the finite state automorphism group of a rooted tree.

2. Preliminary

Definition 1. *A synchronous automaton is a set* $A = \langle X_I, X_O, Q, \pi, \lambda\rangle$, *where*

1. X_I *and* X_O *are finite sets (respectively, the input and the output alphabets),*
2. Q *is a set (the set of internal states of the automaton),*
3. $\pi : X_I \times Q \longrightarrow Q$ *is a mapping (transition function), and*
4. $\lambda : X_I \times Q \longrightarrow X_O$ *is a mapping (output function).*

Automaton A is finite if $|Q| < \infty$.

Henceforth, we will consider the automata whose input and output alphabets coincide. Let $X = X_I = X_O$ be a finite alphabet, X^* be the set of all words over X, X^ω be the set of all ω-words (infinite words) over X.

A permutation of the set X^* or X^ω is called a *(finitely) automatic* if it is caused by a (finite) automaton over alphabet X. All finitely automatic permutations form subgroup of the group $GA(X)$ of all automatic permutations over X. Let us denote this subgroup by $FGA(X)$.

For the alphabet X we can construct the word tree T_X (see also [GNS]). The vertices of the tree T_X are the elements of the set X^*. Two vertices u and v are incident if and only if $u = vx$ or $v = ux$ for a certain $x \in X$. The vertex $\emptyset$ is the root of the tree.

The group $\mathrm{Aut}T_X$ of all automorphisms T_X is isomorphic to the group $GA(X)$ of all automatic permutations over X.

For every two vertices u, v of the tree T_X (i. e. $u, v \in V(T_X)$) we define the *distance* between u and v, denoted by $d(u, v)$, to be equal to the length of the path connecting them.

For rooted tree T_X with the root $v_0 = \emptyset$ and an integer $n \geq 0$ we define the level number n (the sphere of the radius n) as the set

$$V_n = \{v \in V(T_X) : d(v_0, v) = n\}.$$

Let us say that vertex v of rooted tree T_X lies under vertex w, if path, that connects vertice v and v_0, contains vertex w.

Let us denote by T_v the full subtree consisting of all vertices, that lie under the vertex v with the root v.

Let $G \leq \mathrm{Aut}T_X$ be an automorphism group of the rooted tree T_X. Then for every vertex v of the tree T_X and a nonnegative integer n:

1. The group of all automorphisms $g \in G$ fixing every vertex outside the subtree T_v is called *the vertex group* (or *the rigid stabilizer* of the vertex) and is denoted by $\mathsf{rist}\, v$.
2. The group of all automorphisms fixing all vertices of the level number n is denoted by $\mathsf{stab}_G(n)$ or just $\mathsf{stab}(n)$ and is called *the level stabilizer*.

An automorphism group G is said to be *level-transitive* if it acts transitively on all the levels of the rooted tree T_X.

An automorphism group G is said to be *weakly branch* if it is level-transitive and for every vertex v of the tree the vertex group is nontrivial.

Statement 1. *[LN] If G is a weakly branch group then the centralizer $C_{\mathrm{Aut}\, T_X}(G)$ of G in the automorphism group $\mathrm{Aut}\, T_X$ is trivial.*

In the word tree T_X every subtree T_v, where $v \in V(T_X)$, can be naturally identified with the whole tree T_X by the map:

$$\pi_v : x_1 x_2 \dots x_n x_{n+1} \dots x_m \mapsto x_{n+1} x_{n+2} \dots x_m,$$

where $x_1 x_2 \dots x_n = v$.

So, if $g \in \mathsf{stab}(n)$ then the action of g on T_v for every $v \in V_n$ can be identified by π_v with the isometry g_v of T_X defined by the equality

$$\pi_v\left(u^g\right) = \left(\pi_v\left(u\right)\right)^{g_v}.$$

The isometry g_v is called *the state of g in v* or *the restriction of g on v.*

When $g \in \mathsf{stab}(n)$, we write $g = (g_{v_1}, g_{v_2}, \dots, g_{v_{r^n}})^{(n)}$, where

$$\{v_1, v_2, \dots, v_{r^n}\} = V_n,\ r = |X|.$$

Let T_X^n be the subtree of the rooted tree T_X, that consists of all vertices on a distance no greater than n from the root. Then the group $\mathrm{Aut}T_X^n$ is naturally embedded in the group $\mathrm{Aut}T_X$ and the latter is decomposed into semidirect product

$$\mathrm{Aut}T_X = \mathsf{stab}(n) \leftthreetimes \mathrm{Aut}T_X^n.$$

So for each $g \in \mathrm{Aut}T_X$ we can write

$$g = g_n a_g = (g_{v_1}, g_{v_2}, \dots, g_{v_{r^n}})^{(n)} a_g, \tag{1}$$

where $g_n \in \mathsf{stab}(n)$, and $a_g \in \mathrm{Aut}T_X^n$.

By the state of an element g_n in the vertex $v \in V_n$ we mean the state of an element $g \in \mathrm{Aut}T_X$ in the vertex v.

An automorphism $g \in \mathrm{Aut}T_X$ is called *finite state* automorphism if the set of all its states is finite.

All finite state automorphisms form a subgroup of the group $\mathrm{Aut}T_X$. The group $FGA(T_X)$ of all finite state automorphisms T_X is isomorphic to the group $FGA(X)$ of all finitely automatic permutations.

End is an infinite sequence of vertices $(v_0, v_1, v_2, \ldots)$, $v_k \in V_k$ such that $d(v_k, v_{k+1}) = 1$ for every nonnegative integer k. Every ω-word determines an end of the tree T_X. Conversely every end of the tree T_X determines some ω-word.

An ω-word (end) w is called *periodic* if there exists the word $v \in X^*$ such that $w = v \cdot v \cdot v \cdot \ldots = v^\omega$. We say that w is *ultimately periodic* if there exist words $u, v \in X^*$ such that $w = u \cdot v^\omega$.

Let X^{up} be the set of all ultimately periodic words over alphabet X (of the ends of the tree T_X).

Lemma 2. *[Su]*

1. *The set X^{up} is an orbit of the group $FGA(X)$.*

2. *The action of the group $FGA(T_X)$ is faithful on this orbit.*

3. *The permutation group $(FGA(T_X), X^{up})$ is an imprimitive group and its domain of imprimitivity are intersections of domains of imprimitivity of permutation group $(\mathrm{Aut}T_X, X^\omega)$ with the set X^{up}.*

3. Main results

In the paper the normalizer $N = N_{\mathrm{Aut}T_X}(FGA(T_X))$ of the group $FGA(T_X)$ in the group $\mathrm{Aut}T_X$ of all automorphisms of rooted tree T_X, $|X| \geq 2$ is investigated.

As it was shown in [L] (see also [LN]) the normalizer N is isomorphic to the automorphism group of the group $A = FGA(T_X)$.

In the paper the next results on the structure of normalizer (of automorphism group) have been obtained:

Theorem 3. *Let $g \in N$. For every ultimately periodic end u the sequence of states $\{g_{(n)} \mid n \in \mathbb{N}\}$ that correspond to the end u (i.e. states in vertices pertinent to this end) is ultimately periodic.*

Theorem 4. *For an element $g \in N$ there exist $m, k \in \mathbb{N}$, $a, b \in FGA(T_X)$ and $h \in N$ such that*

$$h = (h, ..., h)^{(m)} a,$$

$$g = (h, ..., h)^{(k)} b.$$

Corollary 1. *The normalizer $N = N_{\mathrm{Aut}T_X}(FGA(T_X))$, $|X| \geq 2$, is countable.*

4. Proofs

Let $|X| = r \geq 2$, and let $u_0 = 00\ldots$ be an end of the tree T_r.

Lemma 5. *An element of the group N turn an ultimately periodic end to an ultimately periodic one. That is $N : X^{up} \to X^{up}$.*

Proof. Since X^{up} is an orbit of the group A, it is sufficient to prove the statement for one ultimately periodic end. Let us consider, for example, the end u_0.

Let w be not an ultimately periodic end. Suppose there is $g \in N$ which turn the end w to the end u_0.

Let $a = (a, 1, ..., 1)^{(1)}\tau$ lie in A where τ is a cyclic permutation of order $r - 1$ with 0 as fixed point. Therefore, $u_0^a = u_0$, and u_0 is the only fixed end of the element a.

We have $gag^{-1} : w \to w$. Since g acts on set of ends as permutation, we have that the end w is the only fixed end of the element gag^{-1}.

Since $w \notin X^{up}$, among subtrees with roots in the vertices of the end w there are infinitely many different subtrees. That is, $gag^{-1} \notin A$. We have contradiction. □

This lemma implies

Corollary 2. *1. The set X^{up} is an orbit of the group N.*

2. Action of the group N is faithful on this orbit.

Let $g \in N$, and

$$g = g_n a_g = (g_{v_1}, g_{v_2}, \ldots, g_{v_{r^n}})^{(n)} a_g$$

be decomposition (1) for g where $\{v_1, v_2, \ldots, v_{r^n}\} = V_n$.

Lemma 6. *Let $g \in N$. For each V_n the elements $g_{v_1}, g_{v_2}, \ldots, g_{v_{r^n}}$ are contained in the same left (right) coset of A.*

Proof. We can assume $a_g = 1$. Let $v_i, v_j \in V_n$ and $A \ni b : v_i \to v_j$ be such that $b_n = 1$. We have

$$(b^g)_{v_j} = g_{v_i}^{-1} g_{v_j}.$$

Since $b^g \in A$ then $g_{v_i}^{-1} g_{v_j} \in A$ for all $v_i, v_j \in V_n$. □

Corollary 3. *For an element $g \in N$ there exists $a \in A$ such that $ga \in \mathsf{stab}(n)$ and $(ga)_{v_i} = (ga)_{v_1}$ for all $i = 2, ..., r$.*

Let T be a rooted tree. We will denote by $k_n(v)$ the number of vertices belonging to V_{n+1} and adjacent to v for each integer $n \geq 0$ and $v \in V_n$. A tree is *spherically homogeneous* if $k_n(v)$ does not depend on $v \in V_n$. If k_n does not depend on n too then the tree is called *homogeneous.* For example word tree T_X is homogeneous.

For spherically homogeneous tree the sequence $\chi = \langle k_0, k_1, \ldots \rangle$ is called *tree index* and such a tree is denoted by T_χ. We will use denotation $\bar{k} = \{k, k, ...\}$ for homogeneous tree.

For denotation of vertices of the tree T_χ we will use two indices: first one is the number of the level containing this vertex, second one is the number of this vertex (in the lexigraphic ordering) among the all vertices of the given level.

We will need the next fact

Lemma 7. *The group* $\operatorname{Aut} T_\chi$ *contains finitely generated weakly branch subgroups for all $\chi = \langle k_1, k_2, ... \rangle$ ($k_i \geq 2$).*

Proof. The group $\operatorname{Aut} T_{\bar{2}}$ contains finitely generated weakly branch subgroups, for example, the Grigorchuk 2-group Gr is a such one [GNS].

The natural embeddings $\{0, 1\}$ in $\{0, ...k_i - 1\}$ define the natural embedding $T_{\bar{2}}$ in T_χ and the group $\operatorname{Aut} T_{\bar{2}}$ is being ebedded in $\operatorname{Aut} T_\chi$.

Let us define $h = h_1 \in \operatorname{Aut} T_\chi$ recurrently

$$h_i = (h_{i+1}, 1, ..., 1)^{(1)} \sigma_i$$

where σ_i is the cyclic permutation $(v_{i2}, ..., v_{ik_i})$.

Let $H = \langle Gr, h \rangle$. The group H acts level-transitively on T_χ. We use induction by level number n. The group Gr acts transitively on $\{v_{11}, v_{12}\} \subset V_1$ and h cyclically permutes the vertices $v_{12}, ..., v_{1k_i}$. Thus H acts transitively on the first level. Let H acts transitively on V_n. It is sufficient to prove that for the level number $n+1$ the group H acts transitively on the vertices that are adjacent to the vertex v_{n1} from level number n. In this case the proof is similar to the proof for the level number one with substitution $h^{k_1 \ldots k_n}$ for h.

Therefore H is a level-transitive subgroup of T_χ.

Since there are vertices with infinite rigid stabilizers in G on each level we conclude that rigid stabilizer in H of each vertex is infinite.

Thus, H is a finitely generated weakly branch subgroup of group T_χ. □

Remark 1. *For homogeneous tree $T_{\bar{k}}$ group H is contained in the group $FGA(T_{\bar{k}})$.*

Proof of theorem 3. Let $|X| = r$. Since the group $FGA(X)$ acts transitively on X^{up}, it is sufficient to prove the theorem only for one ultimately periodic end u_0, and $g : u_0 \to u_0$.

1. $r = 2$.

   Let $\alpha_i \in A$ $(i = 1, ..., k)$ such that $\alpha_i = (\alpha_i, a_i)^{(1)}$ where $a_1, ..., a_k$ are elements genereting a weakly branch group H (for example, Grigorchuk group). Then

   $$\alpha^g : u_0 \longrightarrow u_0, \tag{2}$$

   $$(\alpha^g)_{v_{n2}} = a_i^{g_{v_{n2}}} \tag{3}$$

   where $v_{n2} \in V_n$ and $v_{n2} = 00\dots01$.

   Since $\alpha_i^g \in A$ and taking into account (2) we conclude that sequences $\{a_i^{g_{v_{n2}}} \mid n \in \mathbb{N}\}$ are ultimately periodic for $i = 1, ..., k$. Therefore there are $p_i, n_0 \in \mathbb{N}$ such that for $i = 1, ..., k$ and $n \geq n_0$ the next equality holds

   $$a_i^{g_{v_{n+p_i,2}}} = a_i^{g_{v_{n2}}}.$$

   Thus

   $$g_{v_{n+p_i,2}} g_{v_{n2}}^{-1} \in C_{\mathrm{Aut}\, T_2}(\langle a_i \rangle).$$

   Taking $p = gcd(p_1, ..., p_k)$ we have

   $$a_i^{g_{v_{n+p,2}}} = a_i^{g_{v_{n2}}},$$

   $$g_{v_{n+p,2}} g_{v_{n2}}^{-1} \in C_{\mathrm{Aut}\, T_2}(\langle a_i \rangle)$$

   for $i = 1, ..., k$ and $n \geq n_0$. Therefore using (1) we have

   $$\begin{aligned} g_{v_{n+p,2}} g_{v_{n2}}^{-1} \in \textstyle\bigcap_{i=1}^k C_{\mathrm{Aut}\, T_2}(\langle a_i \rangle) &= C_{\mathrm{Aut}\, T_2}(\langle a_1, ..., a_k \rangle) = \\ &= C_{\mathrm{Aut}\, T_2}(H) = 1 \end{aligned}$$

   for $n \geq n_0$.

   Thus $\{g_{v_{n2}} \mid n \in \mathbb{N}\}$ is ultimately periodic, and, taking into account (2), we have that $\{g_{v_{n1}} \mid n \in \mathbb{N}\}$ is ultimately periodic too.

2. $r > 2$.

   Let $\alpha_i \in A$ $(i = 1, ..., k)$ such that $\alpha_i = (\alpha_i, a_i, ..., a_i)^{(1)}\sigma$ where $a_1, ..., a_k$ are elements genereting a weakly branch group H (such group exists by statement 7), and σ is the permutation on r points: $\sigma = (0)(123...r-1)$.

   Denote $(1, a_i, ..., a_i)^{(1)}\sigma$ by b_i $(i = 1, ..., k)$.

   All elements $g_{v_{n1}}, \alpha_1, b_1, ..., \alpha_k, b_k$ act naturally on T_χ where $\chi = \{r-1, r, r, ...\}$ that is from the tree T_r truncate the subtree $T_{v_{10}}$.

   For α_i, b_i (i=1,...,k) the next equations hold:

$$\alpha^g : u_0 \longrightarrow u_0, \tag{4}$$

$$(\alpha^g)_{v_{n1}}|_{T_\chi} = b_i^{g_{v_{n1}}}|_{T_\chi} \tag{5}$$

   where $v_{n1} \in V_n$ and $v_{n1} = 00\dots00$.

   Since $\alpha_i^g \in A$ and taking into account (4)we get that sequences $\{b_i^{g_{v_{n1}}}|_{T_\chi} \mid n \in \mathbb{N}\}$ are ultimately periodic for $i = 1, ..., k$. Therefore there are $p_i, n_0 \in \mathbb{N}$ such that for $i = 1, ..., k$ and $n \geq n_0$ the next equality holds

$$b_i^{g_{v_{n+p_i,1}}}|_{T_\chi} = b_i^{g_{v_{n1}}}|_{T_\chi}.$$

   Thus

$$(g_{v_{n+p_i,1}} g_{v_{n1}}^{-1})|_{T_\chi} \in C_{\operatorname{Aut} T_\chi}(\langle b_i|_{T_\chi}\rangle).$$

   Taking $p = gcd(p_1, ..., p_k)$ we have

$$b_i^{g_{v_{n+p,1}}}|_{T_\chi} = b_i^{g_{v_{n1}}}|_{T_\chi},$$

$$(g_{v_{n+p,1}} g_{v_{n1}}^{-1})|_{T_\chi} \in C_{\operatorname{Aut} T_\chi}(\langle b_i|_{T_\chi}\rangle)$$

   for $i = 1, ..., k$ and $n \geq n_0$. Therefore in virtue of (1) and that $H_1 = \langle b_1|_{T_\chi}, ..., b_k|_{T_\chi}\rangle$ is weakly branch subgroup of the group $\operatorname{Aut} T_\chi$ we have

$$\begin{aligned}(g_{v_{n+p,1}} g_{v_{n1}}^{-1})|_{T_\chi} &\in \textstyle\bigcap_{i=1}^k C_{\operatorname{Aut} T_\chi}(\langle b_i|_{T_\chi}\rangle) = \\ &= C_{\operatorname{Aut} T_\chi}(\langle b_1|_{T_\chi}, ..., b_k|_{T_\chi}\rangle) = C_{\operatorname{Aut} T_\chi}(H_1) = 1\end{aligned}$$

   for $n \geq n_0$.

   Thus $\{g_{v_{n1}}|_{T_\chi} \mid n \in \mathbb{N}\}$ is ultimately periodic, and we have by (4) that $\{g_{v_{n1}} \mid n \in \mathbb{N}\}$ is ultimately periodic too.

□

Proof of theorem 4. It follows from the corollary 2 that there is $b_1 \in A$ such that $gb_1 : u_0 \longrightarrow u_0$. The sequence $\{(gb_1)_{v_{n1}} \mid n \in \mathbb{N}\}$ is ultimately periodic by the theorem 3. Therefore there is $k \in \mathbb{N}$ such that $\{(gb_1)_{v_{n1}} \mid n \geq k\}$ is periodic.

Let us denote by $h = (gb_1)_{v_{n1}}$. There is $b_2 \in A$ such that

$$gb_1b_2 = (h, ..., h)^{(k)}$$

by the corollary 3. For h we have $h : u \longrightarrow u$, and the sequence $\{h_{v_{n1}} \mid n \in \mathbb{N}\}$ is periodic. Let this period be m.

There is $a_1 \in A$ such that

$$ha_1 = (h, ..., h)^{(m)}$$

by the corollary 3. Let us denote by $a = a_1^{-1}$, $b = (b_1b_2)^{-1}$. We have

$$h = (h, ..., h)^{(m)}a,$$

$$g = (h, ..., h)^{(k)}b,$$

and statement is proved.

□

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RESEARCH ARTICLE

Virtual endomorphisms of groups

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1. Introduction

A virtual endomorphism of a group G is a homomorphism from a subgroup of finite index $H \leq G$ into G. Similarly a virtual automorphism (an almost automorphism) is an isomorphism between subgroups of finite index.

Virtual automorphisms (commensurations) appear naturaly in theory of lattices of Lie groups (see [Mar91]). Virtual endomorphism of more general sort appear in theory of groups acting on rooted trees. Namely, if an automorphism g of a rooted tree T fixes a vertex v, then it induces an automorphism $g|_v$ of the rooted subtree T_v, "growing" from the vertex v. If the rooted tree T is regular, then the subtree T_v is isomorphic to the whole tree T, and $g|_v$ is identified with an automorphism of the tree T, when we identify T with T_v. It is easy to see that the described map $\phi_v : g \mapsto g|_v$ is a virtual endomorphism of the automorphism group of the tree T (the domain of this virtual endomorphism is the stabilizer of the vertex v).

The described virtual endomorphisms are the main investigation tools of the groups defined by their action on regular rooted trees. Historically the first example of such a group was the Grigorchuk group [Gri80]. Later many other interesting examples where constructed and investigated [GS83a, GS83b, BSV99, SW02]. One of the common feachures of

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these groups is that they are preserved under the virtual endomorphism ϕ_v, i.e., that the restriction $g|_v$ of any element of the group also belongs to the group. Another important property is that the virtual endomorphism ϕ_v contracts the length of the elements of the groups. (Here length of an element of a finitely generated group is the length of the representation of the element as a product of the generators and their inverses.) The groups with the first property are called *self-similar*, or *state-closed*. The groups with the second property are called *contracting*. The contraction property helps to argue by induction on the length of the group elements.

The notion of a self-similar group is very similar to the classical notion of a self-similar set, so that in some cases self-similar groups are called *fractal groups*. See the survey [BGN02], where the analogy and the connections between the notions of self-similar set and self-similar group are studied.

Recentely, the connections became more clear, after the notions of a *limit space* of a contracting group and the notion of an *iterated monodromy group* were defined [Nekc, Nekb, BGN02]. The limit space is a topological space $\mathcal{J}_G$ together with a continuous map $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$, which is naturally associated to the contracting self-similar group. The limit space has often a fractal appearence and the map s is an expanding map on it, which defines a self-similarity structure of the space.

On the other hand, the iterated monodromy groups are groups naturally associated to (branched) self-coverings $s : \mathcal{X} \to \mathcal{X}$ of a topological space. They are always self-similar, and they are contracting if the map s is expanding. In the latter case, the limit space of the iterated monodromy group is homeomorphic to the Julia set of the map s, with the map s on the limit space conjugated with the restriction of s onto the Julia set.

In the present paper we try to collect the basic facts about the virtual endomorphisms of groups. Since the most properties of self-similar groups are related with the dynamics of the associated virtual endomorphism, the main attention is paid to the dynamics of iterations of one virtual endomorphisms.

For a study of iterations of virtual endormorphisms of index 2 and virtual endomorphisms of abelian groups, see also the paper [NS01]. Many results of [NS01] are generalized here.

The structure of the paper is the following. Section "Virtual endomorphisms" introduces the basic definitions and the main examples of virtual endomorphisms. This is the only section, where semigroups of virtual endomorphisms and groups of virtual automorphisms (commensurators) are discussed.

In the next section "Iterations of one virtual endomorphism" we define the main notions related to the dynamics of virtual endomorphisms. This is the *coset tree* and different versions of the notion of invariant subgroup.

Section "Bimodule associated to a virtual endomorphism" is devoted to ring-theoretic aspects of virtual endomorphisms of groups. Every virtual endomorphism of a group defines a bimodule over the group algebra. Many notions related to virtual endomorphisms of groups have their analogs for bimodules over algebras. For instance, in Subsection "Φ-invariant ideals" we study the analogs of the notion of a subgroup invariant under a virtual endomorphism. The rôle of composition of virtual endomorphisms is played by tensor products of bimodules, which are studied in Subsection "Tensor powers of the bimodule". The last subsection introduces, using the language of bimodules, the *standard actions* of a group on a regular tree, defined by a virtual endomorphism. In this way we show that the action of a self-similar group is defined, up to conjugacy, only by the associated virtual endomorphism. More on bimodules, associated to virtual endomorphism is written in [Neka].

The last section is devoted to the the notion of a contracting virtual endomorphism. We give different definitions of the contraction property, define the *contraction coefficient* (or the *spectral radius*) of a virtual endomorphism, and prove the basic properties of groups, posessing a contracting virtual endomorphism. For example, we prove that such groups have an algorithm, solving the word problem in a polynomial time. This was observed for the first time by R. Grigorchuk for a smaller class of groups (see, for example [Gri80, Gri83]), but we show in this paper, that his algorithm works in the general case.

We use the standard terminology and notions from the theory of groups acting on rooted trees. The reader can find it in [GNS00, BGN02, Gri00, Sid98]. We use left actions here, so that the image of a point x under the action of a group element g is denoted $g(x)$. Respectively, in the product g_1g_2, the element g_2 acts first.

2. Virtual endomorphisms

2.1. Definitions and main properties

Definition 2.1. Let G_1 and G_2 be groups. A *virtual homomorphism* $\phi : G_1 \dashrightarrow G_2$ is a homomorphism of groups $\phi : \mathrm{Dom}\,\phi \to G_2$, where $\mathrm{Dom}\,\phi \leq G_1$ is a subgroup of finite index, called the *domain* of the virtual homomorphism. A *virtual endomorphism* of a group G is a virtual homomorphism $\phi : G \dashrightarrow G$.

The index $[G_1 : \mathrm{Dom}\,\phi]$ is called the *index of the virtual endomorphism* $\phi : G_1 \dashrightarrow G_2$ and is denoted $\mathrm{ind}\,\phi$.

By $\mathrm{Ran}\,\phi$ we denote the image of $\mathrm{Dom}\,\phi$ under ϕ.

We say that a virtual endomorphism ϕ *is defined* on an element $g \in G$ if $g \in \mathrm{Dom}\,\phi$.

If $H \leq G$ is a subgroup of finite index, then the *identical virtual homomorphism* $id_H : G \dashrightarrow G$ with the domain H is naturally defined.

A composition of two virtual homomorphisms $\phi_1 : G_1 \dashrightarrow G_2$, $\phi_2 : G_2 \dashrightarrow G_3$ is defined on an element $g \in G_1$ if and only if ϕ_1 is defined on g and ϕ_2 is defined on $\phi_1(g)$. Thus, the domain of the composition $\phi_2 \circ \phi_1$ is the subgroup

$$\mathrm{Dom}\,(\phi_2 \circ \phi_1) = \{g \in \mathrm{Dom}\,\phi_1 : \phi_1(g) \in \mathrm{Dom}\,\phi_2\} \leq G_1.$$

Proposition 2.1. *Let $\phi_1 : G_1 \dashrightarrow G_2$ and $\phi_2 : G_2 \dashrightarrow G_3$ be two virtual homomorphisms. Then*

$$[\mathrm{Dom}\,\phi_1 : \mathrm{Dom}\,(\phi_2 \circ \phi_1)] \leq [G_2 : \mathrm{Dom}\,\phi_2] = \mathrm{ind}\,\phi_2.$$

If ϕ_1 is onto, then

$$[\mathrm{Dom}\,\phi_1 : \mathrm{Dom}\,(\phi_2 \circ \phi_1)] = [G_2 : \mathrm{Dom}\,\phi_2]\,.$$

Proof. We have $[\mathrm{Ran}\,\phi_1 : \mathrm{Dom}\,\phi_2 \cap \mathrm{Ran}\,\phi_1] \leq \mathrm{ind}\,\phi_2$ and we have here equality in the case when ϕ_1 is onto. Let $T = \{\phi_1(h_1), \phi_1(h_2), \dots \phi_1(h_d)\}$ be a left coset transversal for $\mathrm{Dom}\,\phi_2 \cap \mathrm{Ran}\,\phi_1$ in $\mathrm{Ran}\,\phi_1$. Then for every $g \in \mathrm{Dom}\,\phi_1$ there exists a unique $\phi_1(h_i) \in T$ such that $\phi_1(h_j)^{-1}\phi_1(g) = \phi_1(h_i^{-1}g) \in \mathrm{Dom}\,\phi_2$. This is equivalent to $h_i^{-1}g \in \mathrm{Dom}\,(\phi_2 \circ \phi_1)$ and the set $\{h_1, h_2, \dots, h_d\}$ is a left coset transversal of $\mathrm{Dom}\,(\phi_2 \circ \phi_1)$ in G_1. Thus,

$$[G_2 : \mathrm{Dom}\,\phi_2] = [\mathrm{Ran}\,\phi_1 : \mathrm{Dom}\,\phi_2 \cap \mathrm{Ran}\,\phi_1]\,.$$

□

Corollary 2.2. *A composition of two virtual homomorphisms is again a virtual homomorphism.* □

Consequently, the set of all virtual endomorphisms of a group G is a semigroup under composition. This semigroup is called the *semigroup of virtual endomorphisms* of the group G and is denoted $VE(G)$.

Corollary 2.2 also implies that the class of groups as a class of objects together with the class of virtual homomorphisms as a class of morphisms form a category, which will be called the *category of virtual homomorphisms*.

Commensurability

Let $\phi : G_1 \dashrightarrow G_2$ be a virtual homomorphism. If $H \le G_2$ is a subgroup, then by $\phi^{-1}(H)$ we denote the set of such elements $g \in \mathrm{Dom}\,\phi$ that $\phi(g) \in H$.

If H is a subgroup of finite index then $\phi^{-1}(H) = \mathrm{Dom}\,(id_H \circ \phi)$, thus $\phi^{-1}(H)$ is a subgroup of finite index in G_1.

Lemma 2.3. *For every virtual homomorphism $\phi : G_1 \dashrightarrow G_2$ and for every subgroup of finite index $H \le G_2$ the equality*

$$id_H \circ \phi = \phi \circ id_{\phi^{-1}(H)}$$

holds.

Proof. An element $g \in G_1$ belongs to the domain of $id_H \circ \phi$ if and only if $\phi(g) \in H$, i.e., if and only if $g \in \phi^{-1}(H)$. This implies that the domains of the virtual endomorphisms $id_H \circ \phi$ and $\phi \circ id_{\phi^{-1}(H)}$ coincide. They are equal on its domains to the virtual homomorphism ϕ, so they are equal each to the other. □

Definition 2.2. Let $\phi : G_1 \dashrightarrow G_2$ be a virtual homomorphism and let $H \le G_1$ be a subgroup of finite index. Then the *restriction* of ϕ onto H is the virtual homomorphism $\phi|_H : G_1 \dashrightarrow G_2$ with the domain $\mathrm{Dom}\,\phi \cap H$ such that $\phi|_H(g) = \phi(g)$ for all $g \in \mathrm{Dom}\,\phi \cap H$. In other words, $\phi|_H = \phi \circ id_H$.

Two virtual homomorphisms $\phi_1 : G_1 \dashrightarrow G_2$ and $\phi_2 : G_1 \dashrightarrow G_2$ are said to be *commensurable* (written $\phi_1 \approx \phi_2$) if there exists a subgroup of finite index $H \le G_1$ such that $\phi_1|_H = \phi_2|_H$.

For example, any two identical virtual endomorphisms id_{H_1} and id_{H_2} are commensurable.

Proposition 2.4. *The relation of commensurability is a congruence on the category of virtual homomorphisms. In particular, it is a congruence on the semigroup $VE(G)$.*

Proof. Let $\phi_1, \phi_2, \psi_1, \psi_2$ be virtual homomorphisms such that $\phi_i \approx \psi_i$ for $i = 1, 2$. Then there exist subgroups of finite index H_i such that $\phi_i \circ id_{H_i} = \psi_i \circ id_{H_i}$. Lemma (2.3) implies:

$$\phi_1 \circ id_{H_1} \circ \phi_2 \circ id_{H_2} = \phi_1 \circ \phi_2 \circ id_{\phi_2^{-1}(H_1)} \circ id_{H_2} = \phi_1 \circ \phi_2 \circ id_{\phi_2^{-1}(H_1) \cap H_2},$$

and

$$\psi_1 \circ id_{H_1} \circ \psi_2 \circ id_{H_2} = \psi_1 \circ \psi_2 \circ id_{\psi_2^{-1}(H_1) \cap H_2}.$$

Thus,

$$\phi_1 \circ \phi_2 \circ id_{\phi_2^{-1}(H_1)\cap H_2} = \psi_1 \circ \psi_2 \circ id_{\psi_2^{-1}(H_1)\cap H_2}.$$

Multiplying the last equality from the right by id_H, where $H = \phi_2^{-1}(H_1)\cap \psi_2^{-1}(H_1)\cap H_2$, we get $\phi_1\circ\phi_2\circ id_H = \psi_1\circ\psi_2\circ id_H$, thus $\phi_1\circ\phi_2 \approx \psi_1\circ\psi_2$. □

We will denote by Commen the category with groups as objects and commensurability classes of virtual homomorphisms as morphisms. Proposition 2.4 shows that this category is well defined.

Definition 2.3. The quotient of the semigroup $VE(G)$ by the congruence "≈" is called the *restricted semigroup of virtual endomorphisms* and is denoted $RVE(G)$.

The semigroup $RVE(G)$ is the endomorphism semigroup of the object G in the category Commen.

Example. It is easy to see that every virtual endomorphism of $\mathbb{Z}^n$ can be extended uniquely to a linear map $\mathbb{Q}\otimes\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ and that two extensions $\mathbb{Q}\otimes\phi_1$ and $\mathbb{Q}\otimes\phi_2$ are equal if and only if the virtual endomorphisms are commensurable. Consequently, the semigroup $RVE(\mathbb{Z}^n)$ is isomorphic to the multiplicative semigroup $\mathrm{End}(\mathbb{Q}^n)$ of rational $n\times n$-matrices.

Let us describe the isomorphisms in the category Commen.

Definition 2.4. A virtual homomorphism $\phi : G_1 \dashrightarrow G_2$ is called *commensuration* if it is injective and $\mathrm{Ran}\,\phi$ is a subgroup of finite index in G_2.

Two groups are said to be *commensurable* if there exists a commensuration between them.

Thus, two groups are commensurable if and only if they have isomorphic subgroups of finite index. The identical virtual endomorphisms id_H are examples of commensurations.

If a virtual homomorphism ϕ is a commensuration, then it has an inverse $\phi^{-1} : G_2 \dashrightarrow G_1$, such that $\phi\circ\phi^{-1} = id_{\mathrm{Ran}\,\phi}$ and $\phi^{-1}\circ\phi = id_{\mathrm{Dom}\,\phi}$.

It is easy to see that two groups are isomorphic in the category Commen if and only if they are commensurable. The respective isomorphism will be the commensuration.

Definition 2.5. *Abstract commensurator* of a group G is the group of commensurability classes of commensurations of the group G with itself.

We denote the abstract commensurator of a group G by $Comm(G)$. From the definitions follows that it is the automorphism group of the object G in the category Commen.

Proposition 2.5. *The abstract commensurator $Comm(G)$ is a group and is isomorphic to the group of invertible elements of the semigroup $RVE(G)$.*
If the groups G_1 and G_2 are commensurable, then the semigroups $RVE(G_1)$ and $RVE(G_2)$ and the groups $Comm(G_1)$ and $Comm(G_2)$ are isomorphic. □

Remarks. If H is a subgroup of a group G, then its *commensurator* is the group of those elements $g \in G$ for which the subgroups H and $g^{-1}Hg$ are commensurable. Two subgroups H_1, H_2 are said to be commensurable if the intersection $H_1 \cap H_2$ has finite index both in H_1 and in H_2.

For applications of the notions of commensurators of subgroups and abstract commensurators of groups in the theory of lattices of Lie groups see the works [Mar91, AB94, BdlH97].

Examples. 1) It is easy to see that the abstract commensurator of the group $\mathbb{Z}^n$ is $\mathrm{GL}(n, \mathbb{Q})$, i.e., the automorphism group of the additive group $\mathbb{Q}^n$.

2) An example very different from the previous is the Grigorchuk group. It is proved by C. Roever [Röv02] that the abstract commensurator of the Grigorchuk group is finitely presented and simple. It is generated by its subgroup isomorphic to the Grigorchuk group and a subgroup, isomorphic to the Higmann-Thompson group.

More on commensurators see the paper [MNS00].

Conjugacy

Definition 2.6. Two virtual homomorphisms $\phi, \psi : G_1 \dashrightarrow G_2$ are said to be *conjugate* if there exist $g \in G_1$ and $h \in G_2$ such that $\mathrm{Dom}\,\phi = g^{-1} \cdot \mathrm{Dom}\,\psi \cdot g$ and $\psi(x) = h^{-1}\phi(g^{-1}xg)h$ for every $x \in \mathrm{Dom}\,\psi$.

If the virtual homomorphism ϕ is onto, then every its conjugate is also onto and is of the form $\psi(x) = h^{-1}\phi(g^{-1}xg)h = \phi(f^{-1}xf)$, where $f = gh'$ for $h' \in \phi^{-1}(h)$.

2.2. Examples of virtual endomorphisms

Self-coverings

Let $\mathcal{M}$ be an arcwise connected and locally arcwise connected topological space, and suppose $\mathcal{M}_0$ is its arcwise connected open subset. Let $F : \mathcal{M}_0 \to \mathcal{M}$ be a d-fold covering map.

Take an arbitrary basepoint $t \in \mathcal{M}$. Let $t' \in \mathcal{M}_0$ be one of its preimages under F and let ℓ be a path, starting at t and ending at t'.

For every loop γ in $\mathcal{M}$, based at t (i.e., for every element γ of the fundamental group $\pi(\mathcal{M}, t)$) there exists a unique path γ', starting at t' and such that $F(\gamma') = \gamma$. The set G_1 of the elements $\gamma \in \pi(\mathcal{M}, t)$ for which γ' is again a loop is a subgroup of index d in $\pi(\mathcal{M}, t)$ and is isomorphic to $\pi(\mathcal{M}_0, t')$.

The *virtual endomorphism, defined by the map F* is the virtual endomorphism of the group $\pi(\mathcal{M}, t)$ with the domain G_1 which is defined as

$$\phi(\gamma) = \ell\gamma'\ell^{-1}.$$

Proposition 2.6. *Up to a conjugacy, the virtual endomorphism ϕ of the group $\pi(\mathcal{M})$ defined by F does not depend on the choice of t, t' and ℓ.*

Proof. Let us take some basepoint r (possibly $r = t$), some its preimage r' under F and some path ℓ' in $\mathcal{M}$, connecting r with r'. Let σ be a path from r to t in $\mathcal{M}$, realizing an isomorphism $\gamma \mapsto \sigma^{-1}\gamma\sigma$ of the group $\pi(\mathcal{M}, r)$ with the group $\pi(\mathcal{M}, t)$. Let ϕ' be the virtual endomorphism of $\pi(\mathcal{M}, r)$ defined by r' and ℓ'. Let $x \in \pi(\mathcal{M})$ be an element, corresponding to a loop γ at r. Then x corresponds to the loop $\sigma^{-1}\gamma\sigma$ at t. Suppose that x belongs to the domain of ϕ'. Then $\phi'(x) = \ell'\gamma'\ell'^{-1}$, where γ' is the F-preimage of γ, starting at r'. The loop at t, representing $\phi'(x)$ is then $\sigma^{-1}\ell'\gamma'\ell'^{-1}\sigma$.

Let σ' be the F-preimage of the path σ, starting at r'. Then its end t'' is an F-preimage of t, possibly different from t'. Take some path ρ in $\mathcal{M}_0$ starting at t' and ending at t''. Then $F(\rho)$ is a loop based at t. We get also the loop $h = \ell\rho\sigma'^{-1}\ell'^{-1}\sigma$ at t. Denote by h the element $F(\rho)^{-1}$ of the fundamental group $\pi(\mathcal{M}, t)$. Then, in $\pi(\mathcal{M}, t)$:

$$\phi(h^{-1}xh) = \phi(F(\rho)\sigma^{-1}\gamma\sigma F(\rho)^{-1}) = \ell\rho\sigma'^{-1}\gamma'\sigma'\rho^{-1}\ell^{-1},$$

since $\rho\sigma'^{-1}\gamma'\sigma'\rho^{-1}$ is a loop, starting at t, whose F-image is

$$F(\rho)\sigma^{-1}\gamma\sigma F(\rho)^{-1}.$$

Therefore (see Figure 1)

$$\begin{aligned}
&\phi'(g) = \sigma^{-1}\ell'\gamma'\ell'^{-1}\sigma = \\
&\left(\sigma^{-1}\ell'\sigma'\rho^{-1}\ell^{-1}\right)\cdot\left(\ell\rho\sigma'^{-1}\ell'^{-1}\sigma\right)\cdot\sigma^{-1}\ell'\gamma'\ell'^{-1}\sigma\cdot\left(\sigma^{-1}\ell'\sigma'\rho^{-1}\ell^{-1}\right)\cdot \\
&\cdot\left(\ell\rho\sigma'^{-1}\ell'^{-1}\sigma\right) = \left(\sigma^{-1}\ell'\sigma'\rho^{-1}\ell^{-1}\right)\cdot\left(\ell\rho\sigma'^{-1}\gamma'\sigma'\rho^{-1}\ell^{-1}\right)\cdot \\
&\cdot\left(\ell\rho\sigma'^{-1}\ell'^{-1}\sigma\right) = g^{-1}\phi(h^{-1}xh)g,
\end{aligned}$$

where $g = \ell\rho\sigma'^{-1}\ell'^{-1}\sigma$, so that the virtual endomorphisms ϕ and ϕ' are conjugate. □

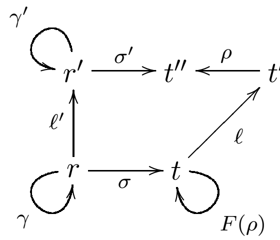

Figure 1:

Example. Take the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and define its double-fold self-covering F, induced by the map $x \mapsto 2x$ on $\mathbb{R}$. Let us take a basepoint $t = 0$. It has two preimages under F: one is 0 and another is 1/2. Take $t' = 0$ and let ℓ be trivial path at 0. The fundamental group of the circle is isomorphic to $\mathbb{Z}$ and is generated by the loop, which is the image of the segment $[0, 1]$ in $\mathbb{T}^1$. It is easy to see that the virtual endomorphism of $\mathbb{Z}$, defined by F is the map $n \mapsto n/2$, defined on the subgroup of even numbers.

The virtual endomorphisms of groups are group-theoretical counterparts of self-coverings of topological spaces. More on relations between dynamics of virtual endomorphisms and dynamics of self-coverings of topological spaces, see the paper [Nekb].

Stabilizers in automorphism groups of graphs

Let Γ be a locally finite graph and let G be a group acting on Γ by automorphisms so that its action on the vertices of Γ is transitive.

Take a vertex v and let G_v be the stabilizer of v in the group G. Let u be another vertex, adjacent to v. Denote by G_{vu} the stabilizer of the vertex u in the group G_v and by G_u the stabilizer of u in G. We obviously have $G_{vu} = G_v \cap G_u$. The group G_{vu} has finite index in G_v and in G_u, not greater than the degree of a vertex in the graph Γ (the degrees of all the vertices of Γ are equal, since G acts on Γ transitively).

Let $g \in G$ be an element such that $g(v) = u$. Then we get a virtual endomorphism $\phi : G_v \dashrightarrow G_v$, with $\mathrm{Dom}\,\phi = G_{vu}$ defined as $\phi(h) = g^{-1}hg$.

This virtual endomorphism is obviously a commensuration. It is proved in [Nek00] that every commensuration can be constructed in such a way.

The proof of the next proposition is straightforward.

Proposition 2.7. *The virtual endomorphism ϕ up to a conjugacy, depends only on the orbit of the edge $\{u, v\}$ with respect to the action of G_v.* □

Self-similar actions

Let X be a finite set, called the *alphabet*. By X^* we denote the set of all finite words over X, i.e., the free monoid, generated by X. We include the empty word $\varnothing$.

Definition 2.7. An action of a group G on the set X^* is *self-similar* if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w) \tag{1}$$

for every $w \in X^*$.

Let us take some faithful self-similar action of G on X^*. Let G_x be the stabilizer of the one-letter word $x \in X^*$. Then there exists a unique $h \in G$ such that $g(xw) = xh(w)$ for all $w \in X^*$. The subgroup G_x has a finite index not greater than $|X|$ in G and the map $\phi_x : G_x \to G : g \mapsto h$ is a homomorphism. In this way we get a virtual endomorphism $\phi_x : G \dashrightarrow G$ of the group G.

The following is straightforward.

Proposition 2.8. *If $x, y \in X$ belong to the same G-orbit, then the virtual endomorphisms ϕ_x and ϕ_y are conjugate.* □

If the self-similar action is faithful then for every $g \in G$ and for every finite word $v \in X^*$ there exist a unique element $h \in G$ such that

$$g(vw) = g(u)h(w)$$

for every $w \in X^*$. The element h is called *restriction of g at v* and is denoted $g|_v$. It is easy to see that the following properties of restriction hold.

$$g|_{v_1 v_2} = (g|_{v_1})|_{v_2} \tag{2}$$
$$(g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v. \tag{3}$$

We will see later, that in some sense all virtual endomorphisms of groups are associated to self-similar actions.

3. Iterations of one virtual endomorphism

3.1. Coset tree

Let ϕ be a virtual endomorphism of a group G. Denote $d = \operatorname{ind}\phi$. We get a descending sequence of subgroups of finite index in G:

$$\operatorname{Dom}\phi^0 = G \geq \operatorname{Dom}\phi^1 \geq \operatorname{Dom}\phi^2 \geq \operatorname{Dom}\phi^3 \geq \dots. \tag{4}$$

We have, by Proposition 2.1, an inequality $[\operatorname{Dom}\phi^n : \operatorname{Dom}\phi^{n+1}] \leq d$ for every $n \geq 0$. Consequently, $[G : \operatorname{Dom}\phi^n] \leq d^n$.

Definition 3.1. The virtual endomorphism ϕ is said to be *regular* if

$$\left[\operatorname{Dom}\phi^n : \operatorname{Dom}\phi^{n+1}\right] = d$$

for every $n \geq 0$.

An example of a non-regular virtual endomorphism is the identical endomorphism id_H for H not equal to the whole group.

On the other hand, from Proposition 2.1 follows that if ϕ is onto, then it is regular. Nevertheless, non-surjective virtual endomorphism can be regular, for example the virtual endomorphism $n \mapsto \frac{3}{2}n$ of the group $\mathbb{Z}$, defined on even numbers, is regular.

Definition 3.2. The *coset tree* $T(\phi)$ of a virtual endomorphism ϕ is the rooted tree whose nth level is the set of left cosets $\{g\operatorname{Dom}\phi^n : g \in G\}$ and two cosets $g\operatorname{Dom}\phi^n$ and $h\operatorname{Dom}\phi^{n+1}$ are adjacent if and only if $g\operatorname{Dom}\phi^n \geq h\operatorname{Dom}\phi^{n+1}$. The root of the coset tree is the vertex

$$1 \cdot \operatorname{Dom}\phi^0 = G.$$

The coset tree $T(\phi)$ is a level-homogeneous tree of branch index

$$([G : \operatorname{Dom}\phi], [\operatorname{Dom}\phi : \operatorname{Dom}\phi^2], [\operatorname{Dom}\phi^2 : \operatorname{Dom}\phi^3], \dots).$$

In particular, it is regular if and only if the virtual endomorphism is regular.

The group G acts on the coset tree by left multiplication:

$$g(h \operatorname{Dom} \phi^n) = gh \operatorname{Dom} \phi^n.$$

This action is obviously an action by automorphisms of the rooted tree and is level-transitive.

Directly from the description follows that the stabilizer of the vertex $1 \cdot \operatorname{Dom} \phi^n$ in the group G is the subgroup $\operatorname{Dom} \phi^n$. The stabilizer of the vertex $g \operatorname{Dom} \phi^n$ is its conjugate subgroup $g \cdot \operatorname{Dom} \phi^n \cdot g^{-1}$.

The *nth level stabilizer* is the subgroup

$$St_n(\phi) = \bigcap_{g \in G} g \cdot \operatorname{Dom} \phi^n \cdot g^{-1},$$

equal to the set of all elements of G, fixing every vertex of the nth level of the coset tree.

The nth level stabilizer is a normal subgroup of finite index in G.

3.2. Invariant subgroups

Definition 3.3. Let ϕ be a virtual endomorphism of a group G. A subgroup $H \leq G$ is said to be

1. *ϕ-semi-invariant* if $\phi(H \cap \operatorname{Dom} \phi) \subseteq H$;
2. *ϕ-invariant* if $H \subseteq \operatorname{Dom} \phi$ and $\phi(H) \subseteq H$;
3. *ϕ^{-1}-invariant* if $\phi^{-1}(H) \leq H$.

Recall that $\phi^{-1}(H) = \{g \in \operatorname{Dom} \phi \ : \ \phi(g) \in H\}$. Note that every ϕ-invariant subgroup is ϕ-semi-invariant.

If a subgroup $H \leq G$ is ϕ-invariant, then it is a subgroup of $\operatorname{Dom} \phi^n$ for every $n \in \mathbb{N}$. On the other hand, the *parabolic subgroup*

$$P(\phi) = \bigcap_{n \in \mathbb{N}} \operatorname{Dom} \phi^n$$

is obviously ϕ-invariant. Thus, the parabolic subgroup is the maximal ϕ-invariant subgroup of G.

Example. Let ϕ be a surjective virtual endomorphism of a group G. Let us show that the center $Z(G)$ of the group G is ϕ-semi-invariant. If $h \in Z(G) \cap \operatorname{Dom} \phi$, then $\phi(h)\phi(g) = \phi(g)\phi(h)$ for every $g \in \operatorname{Dom} \phi$. But the set of elements of the form $\phi(g)$ is the whole group G. Thus, $\phi(h) \in Z(G)$.

Proposition 3.1. *If $H \le G$ is a normal ϕ-semi-invariant subgroup, then the formula*

$$\psi(gH) = \phi(g)H$$

for $g \in \operatorname{Dom}\phi$ gives a well defined virtual endomorphism ψ of the quotient G/H.

Proof. The domain of the map ψ is the image of the subgroup of finite index $\operatorname{Dom}\phi$ under the canonical homomorphism $G \to G/H$ and thus has finite index in G/H. Suppose that $g_1H = g_2H$ for some $g_1, g_2 \in \operatorname{Dom}\phi$. Then $g_1^{-1}g_2 \in H \cap \operatorname{Dom}\phi$, so $\phi(g_1^{-1}g_2) \in H$, thus $\phi(g_1)H = \phi(g_2)H$. □

The virtual endomorphism ψ is called the *quotient of* ϕ by the subgroup H and is denoted ϕ/H.

Proposition 3.2. *The subgroup*

$$\mathcal{C}(\phi) = \bigcap_{n\in\mathbb{N}} St_n(\phi) = \bigcap_{n\in\mathbb{N}} \bigcap_{g\in G} g^{-1} \cdot \operatorname{Dom}\phi^n \cdot g$$

is the maximal among normal ϕ-invariant subgroups of G.

The subgroup $\mathcal{C}(\phi)$ is the kernel of the action of G on the coset tree $T(\phi)$.

Proof. An element $h \in G$ belongs to $\mathcal{C}(\phi)$ if and only if every its conjugate belongs to $\operatorname{Dom}\phi^n$ for every $n \in \mathbb{N}$. From this follows that $\mathcal{C}(\phi)$ is normal and ϕ-invariant, since from $h \in \mathcal{C}(\phi)$ follows that all the conjugates of h and $\phi(h)$ belong to $\mathcal{C}(\phi)$.

On the other hand, if N is a normal, ϕ-invariant subgroup of G, then for every $h \in N$ the element $\phi^n(h)$ belongs to N for all $n \in \mathbb{N}$ and thus, $g^{-1}\phi^n(h)g \in N$ for all $g \in G$ and $n \in \mathbb{N}$. This implies that $h \in \mathcal{C}(\phi)$. □

Definition 3.4. The subgroup $\mathcal{C}(\phi)$ is called the *core* of the virtual endomorphism ϕ or the *ϕ-core* of G. The group G is said to be *ϕ-simple* if its ϕ-core is trivial.

Examples. 1) Let ϕ be the virtual endomorphism $n \mapsto n/2$ of $\mathbb{Z}$, with the domain equal to the set of even numbers. Then the group $\mathbb{Z}$ is obviously ϕ-simple.

2) More generally, if ϕ is a virtual endomorphism of the $\mathbb{Z}^n$, then $\mathbb{Z}^n$ is ϕ-simple if and only if no eigenvalue of the respective linear transformation is an algebraic integer (see [NS01]).

3) For examples of virtual endomorphisms of linear groups with trivial core, see the paper [NS01].

4) It is an open question, if the free group of rank 3 with the generators a, b, c is ϕ-simple, where ϕ is defined on the generators of its domain by the equalities

$$\begin{aligned} \phi(a^2) &= cb \\ \phi(b^2) &= bc \\ \phi(ab) &= c^2 \\ \phi(c) &= a \\ \phi(a^{-1}ca) &= b^{-1}ab. \end{aligned}$$

This question is equivalent to a question of S. Sidki in [Sid00] and originates from an automaton, defined by S. V. Aleshin in [Ale83].

Proposition 3.3. *Let ϕ be a virtual endomorphism of a group G. If $H \leq G$ is a ϕ-invariant normal subgroup, then*

$$\mathcal{C}\,(\phi/H) = \mathcal{C}(\phi)/H.$$

Proof. Let K be a normal ϕ/H-invariant subgroup of G/H and let $\tilde{K}$ be its full preimage in G. Then $\tilde{K}$ is also normal. We have $K \leq \mathrm{Dom}\,(\phi/H)$, so every element of $\tilde{K}$ is a product of an element of $\mathrm{Dom}\,\phi$ and an element of H (see the definition of a quotient of a virtual endomorphism by a normal subgroup). But $H \leq \mathrm{Dom}\,\phi$, thus $\tilde{K} \leq \mathrm{Dom}\,\phi$. Let $\tilde{g} \in \tilde{K}$ be an arbitrary element and let g be its image in K. Then, by definition of ϕ/H, $\phi(\tilde{g})H = (\phi/H)\,(g)$, but $(\phi/H)\,(g) \in K$, so $\phi(\tilde{g}) \in \tilde{K}$ and the subgroup $\tilde{K}$ is ϕ-invariant.

On the other hand, if $\tilde{K}$ is a normal ϕ-invariant subgroup of G, then its image in G/H is also normal and ϕ-invariant.

This implies that the maximal ϕ/H-invariant normal subgroup of G/H is the image of the maximal ϕ/H-invariant normal subgroup of G. □

Corollary 3.4. *The group $G/\,\mathcal{C}(\phi)$ is $\phi/\,\mathcal{C}(\phi)$-simple.* □

In this way new groups can be constructed. We can start from some known group F, define a virtual endomorphism ϕ on it, and get the group $F/\,\mathcal{C}(\phi)$. If the group F is finitely generated, then the domain of ϕ is also finitely generated, and ϕ is uniquely determined by its value on the generators of its domains.

Example. The Grigorchuk group is the group $F/\,\mathcal{C}(\phi)$ for F the free group generated by $\{a, b, c, d\}$ and ϕ defined on the generators of its domain as

$$\begin{aligned} \phi(a^2) &= 1 & & \\ \phi(b) &= a & \phi(a^{-1}ba) &= c \\ \phi(c) &= a & \phi(a^{-1}ca) &= d \\ \phi(d) &= 1 & \phi(a^{-1}da) &= b. \end{aligned}$$

The next proposition shows that we can restrict in such constructions to the case when F is a free group.

Proposition 3.5. *Let ϕ be a virtual endomorphism of a finitely generated group G. Then there exist a virtual endomorphism $\tilde{\phi}$ of a finitely generated free group F, a $\tilde{\phi}$-invariant normal subgroup $K \leq F$ and an isomorphism $\rho : F/K \to G$ such that $\rho \circ \left(\tilde{\phi}/K\right) = \phi \circ \rho$. Then the quotient $F/\mathcal{C}\left(\tilde{\phi}\right)$ is isomorphic to $G/\mathcal{C}(\phi)$.*

Proof. Let $\{g_1, g_2, \dots, g_n\}$ be a finite generating set of the group G. Set F to be the free group of rank n with the free generating set $\{\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_n\}$. Let $\pi : F \to G$ be the canonical epimorphism $\pi\left(\tilde{g}_i\right) = g_i$ and let K be the kernel of π, so that $F/K \cong G$. Denote by ρ the respective isomorphism $\rho : F/K \to G$.

The preimage $\pi^{-1}(\mathrm{Dom}\,\phi)$ is a subgroup of finite index in F, so it is a finitely generated free group. Let $\{h_1, h_2, \dots, h_m\}$ be a free generating set of $\pi^{-1}(\mathrm{Dom}\,\phi)$. We can define a virtual endomorphism $\tilde{\phi}$ of the group F with the domain $\mathrm{Dom}\,\tilde{\phi} = \pi^{-1}(\mathrm{Dom}\,\phi)$ putting $\tilde{\phi}(h_i)$ to be equal to some of the elements of the set $\pi^{-1}\left(\phi(\pi(h_i))\right)$. Then we have $\pi(\tilde{\phi}(g)) = \phi\left(\pi\left(g\right)\right)$ for all $g \in \{h_1, h_2, \dots, h_m\}$, and thus for all $g \in \mathrm{Dom}\,\tilde{\phi}$.

Note that $K \leq \pi^{-1}(\mathrm{Dom}\,\phi) = \mathrm{Dom}\,\tilde{\phi}$, and that for every $g \in K$ we have $\pi(\tilde{\phi}(g)) = \phi(\pi(g)) = 1$, so that $\tilde{\phi}(g) \in K$ and K is $\tilde{\phi}$-invariant. Then the equality $\pi \circ \tilde{\phi} = \phi \circ \pi$ is equivalent to the equality $\rho \circ \left(\tilde{\phi}/K\right) = \phi \circ \rho$.

We have, by Proposition 3.3

$$\mathcal{C}(\phi) = \rho\left(\mathcal{C}\left(\tilde{\phi}/K\right)\right) = \rho\left(\mathcal{C}\left(\tilde{\phi}\right)/K\right),$$

thus ρ induces an isomorphism of $F/\mathcal{C}\left(\tilde{\phi}\right)$ with $G/\mathcal{C}(\phi)$. □

Definition 3.5. Let H be a subgroup of G. Define $\Delta_\phi(H)$ to be the set of all elements $g \in G$ such that for every $h \in G$ the element $h^{-1}gh$ belongs to $\mathrm{Dom}\,\phi$ and $\phi(h^{-1}gh) \in H$.

We write Δ_ϕ^n for the nth iteration of the operation Δ_ϕ.

Note that $\Delta_\phi^n(G)$ is the nth level stabilizer $St_n(\phi)$.

Proposition 3.6. *1. For every subgroup $H \le G$ the subgroup $\Delta_\phi(H)$ is normal and is contained in $St_1(\phi) \le \mathrm{Dom}\,\phi$.*

2. $\phi(\Delta_\phi(H)) \le H$.

3. If H is normal, then the virtual endomorphism ϕ induces a well defined virtual homomorphism $\overline{\phi} : G/\Delta_\phi(H) \dashrightarrow G/H$.

4. If the subgroup H is a normal ϕ-invariant subgroup, then $\Delta_\phi(H)$ is a normal ϕ-invariant subgroup.

5. A normal subgroup H is ϕ-invariant if and only if $H \le \Delta_\phi(H)$.

Proof. The first two claims follow directly from the definitions.

If H is normal, then the equality $\overline{\phi}(g\Delta_\phi(H)) = \phi(g)H$ gives a well defined virtual homomorphism $\overline{\phi} : G/\Delta_\phi(H) \dashrightarrow G/H$, since from $g_1^{-1}g_2 \in \Delta_\phi(H)$ follows that $\phi(g_1^{-1}g_2) \in H$.

If H is normal and ϕ-invariant, then $\phi(h^{-1}gh)$ is defined and belongs to H for every $h \in G$, thus $H \le \Delta_\phi(H)$. But then $\phi(\Delta_\phi(H)) \le H \le \Delta_\phi(H)$, so $\Delta_\phi(H)$ is ϕ-invariant.

If $H \le \Delta_\phi(H)$, then for every $g \in H \le \Delta_\phi(H)$ we have $\phi(g) \in H$, thus H is ϕ-invariant. □

Definition 3.6. For any virtual endomorphism ϕ we define

$$\mathcal{E}_n(\phi) = \Delta_\phi^n(\{1\})$$

and $\mathcal{E}_\infty(\phi) = \cup_{n\ge 0}\mathcal{E}_n(\phi)$.

Proposition 3.6 implies that the subgroups $\mathcal{E}_n(\phi)$ are normal and ϕ-invariant for all $n = 0, 1, \ldots, \infty$. It also implies that $\mathcal{E}_n(\phi) \le \mathcal{E}_{n+1}(\phi)$ for all n.

Note also that if $\mathcal{E}_1(\phi) = \{1\}$, then $\mathcal{E}_n(\phi) = \{1\}$ for all $n = 0, 1, \ldots, \infty$. Therefore, $\mathcal{E}_\infty(\phi) = \{1\}$ if and only if $\mathcal{E}_1(\phi) = \{1\}$.

4. Bimodule associated to a virtual endomorphism

4.1. Permutational G-bimodules and the set $\phi(G)G$

Definition 4.1. Let G be a group. A *(permutational) G-bimodule* is a set M with left and right commuting actions of G on M, i.e., with two maps $G \times M \to M : (g, m) \mapsto g \cdot m$ and $M \times G \to M : (m, g) \mapsto m \cdot g$ such that

1. $1 \cdot m = m \cdot 1 = m$ for all $m \in M$;
2. $(g_1 g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$ and $m \cdot (g_1 g_2) = (m \cdot g_1) \cdot g_2$ for all $g_1, g_2 \in G$ and $m \in M$;
3. $(g_1 \cdot m) \cdot g_2 = g_1 \cdot (m \cdot g_2)$ for all $g_1, g_2 \in G$ and $m \in M$.

Two bimodules M_1, M_2 are *isomorphic* if there exists a bijection $f : M_1 \to M_2$, which agrees with the left and the right actions, i.e., such that $g \cdot f(m) \cdot h = f(g \cdot m \cdot h)$ for all $g, h \in G$ and $m \in M_1$.

We say that the right action is *free* if for any $m \in M$ from $m \cdot g = m$ follows that $g = 1$. The right action is *d-dimensional* if the number of the orbits for the right action is d. The bimodule is *irreducible* if for any two elements $m_1, m_2 \in M$ there exist $g, h \in G$ such that $m_2 = g \cdot m_1 \cdot h$.

Proposition 4.1. *Suppose that M is an irreducible G-bimodule with a free d-dimensional right action. Take some $x \in M$. Let G_1 be the subset of all the elements $g \in G$ for which $g \cdot x$ and x belong to the same orbit of the right action. Then G_1 is a subgroup of index d in G and for every $g \in G_1$ there exists a unique $h \in G$ such that $g \cdot x = x \cdot h$. The map $\phi_x : g \mapsto h$ is a virtual endomorphism of the group G.*

The constructed virtual endomorphism ϕ_x is the *endomorphism, associated to the bimodule M (and the element x).*

Proof. The element h is uniquely defined, since the right action is free. The set G_1 is obviously a subgroup. The fact that the map ϕ_x is a homomorphism from G_1 to G follows directly from the definition of a permutational bimodule. The subgroup G_1 has index d in G, since the right action is d-dimensional, and the bimodule is irreducible. □

Proposition 4.2. *Let M be an irreducible G-bimodule with free d-dimensional right action. Then any two associated virtual endomorphisms ϕ_x and ϕ_y are conjugate. If ϕ is conjugate with an associated virtual endomorphism ϕ_x, then it is also associated to M, i.e., $\phi = \phi_y$ for some $y \in M$.*

Proof. Since the bimodule is irreducible, for every $x, y \in M$ there exist $g, h \in G$ such that $y = g \cdot x \cdot h$. Then for every $f \in G$ we have $f \cdot y = y \cdot \phi_y(f)$, what is equivalent to $fg \cdot x \cdot h = g \cdot x \cdot h\phi_y(f)$, i.e., to $g^{-1}fg \cdot x = x \cdot h\phi_y(f)h^{-1}$. It implies that $\phi_y(f) = h^{-1}\phi_x(g^{-1}fg)h$, i.e., that ϕ_y and ϕ_x are conjugate.

Similar arguments show that if $\phi(f) = h^{-1}\phi_x(g^{-1}fg)h$, then ϕ is the virtual endomorphism, associated to M and $g \cdot x \cdot h \in M$. □

Let us show that the bimodule M is uniquely determined, up to an isomorphism, by the associated virtual endomorphism.

Let ϕ be a virtual endomorphism of a group G. We consider the set $\phi(G)G$ of expressions of the form $\phi(g_1)g_0$, where $g_1, g_0 \in G$. Two expressions $\phi(g_1)g_0$ and $\phi(h_1)h_0$ are considered to be equal if and only if $g_1^{-1}h_1 \in \mathrm{Dom}\,\phi$, and $\phi(g_1^{-1}h_1) = g_0h_0^{-1}$.

Another way to describe this equivalence relation is to say that two expressions $\phi(g_1)g_0$ and $\phi(h_1)h_0$ are equal if and only if there exists an element $s \in G$ such that $sg_1, sh_1 \in \mathrm{Dom}\,\phi$ and $\phi(sg_1)g_0 = \phi(sh_1)h_0$ in G.

It is not hard to prove that the described relation is an equivalence.

Definition 4.2. Let $v = \phi(g_1)g_0$ be an element of $\phi(G)G$ and $g \in G$ be arbitrary. *Right action* of the group G on $\phi(G)G$ is defined by the rule $v \cdot g = \phi(g_1)g_0g$ and the *left action* is defined by $g \cdot v = \phi(gg_1)g_0$.

The actions are well defined, since from $\phi(g_1)g_0 = \phi(h_1)h_0$ follows that

$$\phi\left(g_1^{-1}h_1\right) = \phi\left(\left(gg_1\right)^{-1}\left(gh_1\right)\right) = g_0h_0^{-1} = (g_0g)(h_0g)^{-1},$$

thus $\phi(gg_1)g_0 = \phi(gh_1)h_0$ and $\phi(g_1)g_0g = \phi(h_1)h_0g$.

From the definition directly follows that the left and the right actions commute, i.e., that $(g \cdot v) \cdot h = g \cdot (v \cdot h)$ for all $g, h \in G$ and $v \in \phi(G)G$.

The set $\phi(G)G$ together with the left and right actions of the group G is called the *G-bimodule, associated to the virtual endomorphism ϕ.*

It is easy to see that the bimodule $\phi(G)G$ is irreducible. The right action is free, since from $\phi(g_1)g_0g = \phi(g_1)g_0$ follows that $\phi(g_1^{-1}g_1) = g_0^{-1}g_0g$, thus $g = 1$. The right action is $(\mathrm{ind}\,\phi)$-dimensional, since $\phi(g_1)g_0$ and $\phi(h_1)h_0$ belong to one orbit of the right action if and only if $g_1^{-1}h_1 \in \mathrm{Dom}\,\phi$.

Proposition 4.3. *Let M be an irreducible G-bimodule with free d-dimensional right action and let ϕ be its associated virtual endomorphism. Then the bimodule M is isomorphic to the bimodule $\phi(G)G$.*

Proof. Let us fix some $x_0 \in M$. Let $\phi = \phi_{x_0}$ be the virtual endomorphism associated to M and x_0. Define a map $F : \phi(G)G \to M$ by the rule $\phi(g_1)g_0 = g_1 \cdot x_0 \cdot g_0$.

If $\phi(g_1)g_0 = \phi(h_1)h_0$, then $g_1^{-1}h_1 \cdot x_0 = x_0 \cdot g_0h_0^{-1}$, thus $h_1 \cdot x_0 \cdot h_0 = g_1 \cdot x_0 \cdot g_0$, what implies that the map F is well defined.

On the other hand, if $h_1 \cdot x_0 \cdot h_0 = g_1 \cdot x_0 \cdot g_0$, then $g_1^{-1}h_1 \cdot x_0 = x_0 \cdot g_0h_0^{-1}$, i.e., $\phi(g_1)g_0 = \phi(h_1)h_0$, thus the map F is injective.

Since the bimodule M is irreducible, for every $x \in G$ one can find $g_1, g_0 \in G$ such that $x = g_1 \cdot x_0 \cdot g_0$, so the map F is a bijection.

We have $F(\phi(g \cdot g_1)g_0 \cdot h) = gg_1 \cdot x_0 \cdot g_0h = g \cdot F(\phi(g_1)g_0) \cdot h$, thus the map F agrees with the right and the left multiplications, so it is an isomorphism of the G-bimodules. □

The next is a corollary of Propositions 4.2 and 4.3.

Corollary 4.4. *The G-bimodules $\phi_1(G)G$ and $\phi_2(G)G$ are isomorphic if and only if the virtual endomorphisms ϕ_1 and ϕ_2 are conjugate.* □

Example. 1. Consider a faithful self-similar action of a group G on the set X^*. Let $M = X \times G$ be a direct product of sets. The right action of the group G on M is the natural one:

$$(x \cdot g) \cdot h = x \cdot gh.$$

We write an element (x, g) of M as $x \cdot g$.

If $x \cdot g \in M$ and $h \in G$ then, by the definition of a self-similar action, there exists $h|_x \in G$ such that $h(xw) = h(x)h|_x(w)$ for all $w \in X^*$. We define the left action of G on M by the formula

$$h \cdot x \cdot g = h(x) \cdot h|_x g.$$

The obtained permutational bimodule M is called the *self-similarity bimodule* of the action. It is easy to see that the right action of the self-similarity bimodule is free and $|X|$-dimensional and that the bimodule is irreducible, if the action is transitive on the set X^1.

The self-similarity bimodule M is isomorphic to the permutational bimodule $\phi(G)G$, where ϕ is the virtual endomorphism, associated to the self-similar action.

Example. 2. Let $F : \mathcal{M}_0 \to \mathcal{M}$ be a d-fold covering map, where $\mathcal{M}$ is an arcwise connected and locally arcwise connected topological space and $\mathcal{M}_0$ is its open arcwise connected subset. Let $t \in \mathcal{M}$ be an arbitrary point.

Let L be the set of homotopy classes of the paths starting at t and ending at a point z such that $F(z) = t$. (We consider only the homotopies, fixing the endpoints.) Then the set L is a permutational $\pi_1(\mathcal{M}, t)$-bimodule for the following actions:

1. For all $\gamma \in \pi_1(\mathcal{M}, t)$ and $\ell \in L$:

$$\gamma \cdot \ell = \ell\gamma',$$

where γ' is the F-preimage of γ, which starts at the endpoint of ℓ.

2. For all $\gamma \in \pi_1(\mathcal{M}, t)$ and $\ell \in L$:

$$\ell \cdot \gamma = \gamma\ell.$$

It is not hard to prove that the described permutational bimodule is irreducible, free and d-dimensional from the right. Consequently, it is of the form $\phi(\pi_1(\mathcal{M}, t))\pi_1(\mathcal{M}, t)$, where ϕ is the associated virtual endomorphism. It is the endomorphism, defined by F, as in Subsection 2.2.

4.2. Quotients of a permutational bimodule

Definition 4.3. Let M_i be a permutational bimodule over a group G_i, $i = 1, 2$. The bimodule M_2 is a *quotient* of the bimodule M_1 if there exists a surjective map $p : M_1 \to M_2$ and a surjective homomorphism $\pi : G_1 \to G_2$ such that

$$\pi(g_1) \cdot p(m) \cdot \pi(g_2) = p(g_1 \cdot m \cdot g_2)$$

for all $g_1, g_2 \in G_1$ and $m \in M_1$.

Proposition 4.5. *Let ϕ_1 and ϕ_2 be virtual endomorphisms of the groups G_1 and G_2 respectively. Then the bimodule $\phi_2(G_2)G_2$ is a quotient of the bimodule $\phi_1(G_1)G_1$ if and only if there exists a normal ϕ_1-semi-invariant subgroup $N \leq G_1$ such that G_2 is isomorphic to G_1/N so that ϕ_2 is conjugate to ϕ_1/N.*

Proof. Suppose that the bimodule $\phi_2(G_2)G_2$ is a quotient of the bimodule $\phi_1(G_1)G_1$. Let $\pi : G_1 \to G_2$ be the respective homomorphism and let $p : \phi_1(G_1)G_1 \to \phi_2(G_2)G_2$ be the surjective map. Denote by N the kernel of the homomorphism π.

Replacing, if necessary ϕ_2 by a conjugate virtual endomorphism (see Proposition 4.2), we may assume that $p(\phi_1(1)1) = \phi_2(1)1$. Then

$$p(\phi_1(g_1)g_0) = p(g_1 \cdot \phi_1(1) \cdot g_0) = \pi(g_1)p(\phi_1(1)1)\phi(g_0) = \phi_2(\pi(g_1))\pi(g_0)$$

for all $g_0, g_1 \in G_1$.

If g is an element of $N \cap \mathrm{Dom}\,\phi_1$, then $\phi_1(g)1 = \phi_1(1)g'$ in $\phi_1(G_1)G_1$, where $g' = \phi(g)$, thus $p(\phi_1(g)1) = \phi_2(\pi(g))1 = \phi_2(1)1 = \phi_2(1)\pi(g')$. Hence, $\pi(g') = 1$, i.e., $\phi(g) \in N$ and the subgroup N is ϕ-semi-invariant. If g is an arbitrary element of $\mathrm{Dom}\,\phi_1$, then again

$$p(\phi_1(g)1) = \phi_2(\pi(g))1 = \phi_2(1)\pi(g')$$

for $g' = \phi_1(g)$. Consequently, $\pi(\phi_1(g)) = \phi_2(\pi(g))$, i.e., $\phi_2 = \phi_1/N$.

Suppose now that N is a normal ϕ_1-semi-invariant subgroup of G_1. Let us introduce an equivalence relation on $\phi_1(G_1)G_1$ by the rule:

$$\phi_1(g_1)g_0 \sim \phi_1(h_1)h_0$$

if and only if

$$g_1^{-1}h_1 \in \mathrm{Dom}\,\phi_1 \text{ and } \phi_1(g_1^{-1}h_1)h_0g_0^{-1} \in N.$$

It is easy to see that the defined relation is an equivalence and that the quotient of $\phi_1(G_1)G_1$ has a structure of a permutational bimodule over G_1/N, which is isomorphic to the bimodule $\phi_1/N\big(G_1/N\big)G_1/N$. Then Proposition 4.2 finishes the proof. □

4.3. Bimodules over group algebras

Definition 4.4. Let A be an algebra over a field $\Bbbk$. An A-*bimodule* is a $\Bbbk$-space Φ with structures of left and right A-modules such that the left and the right multiplications commute. In other words, two $\Bbbk$-linear maps $A \otimes_{\Bbbk} \Phi \to \Phi : a \otimes v \mapsto a \cdot v$ and $\Phi \otimes \Bbbk A \to \Phi : v \otimes a \mapsto v \cdot a$ are fixed such that

1. $(a_1a_2) \cdot v = a_1 \cdot (a_1 \cdot v)$ and $v \cdot (a_1a_2) = (v \cdot a_1) \cdot a_2$ for all $a_1, a_2 \in A$ and $v \in \Phi$;

2. $(a_1 \cdot v) \cdot a_2 = a_1 \cdot (v \cdot a_2)$ for all $a_1, a_2 \in A$ and $v \in \Phi$.

If M is a permutational G-bimodule, and $\Bbbk$ is a field, than the left and the right actions of G on M extend by linearity to a structure of $\Bbbk G$-bimodule on the linear space $\langle M\rangle_{\Bbbk}$. Here $\langle M\rangle_{\Bbbk}$ denotes the linear space over the field $\Bbbk$ with the basis M, and $\Bbbk G$ is the group algebra of G over the field $\Bbbk$. The $\Bbbk G$-bimodule $\langle M\rangle_{\Bbbk}$ is called *linear span* of the permutational bimodule M.

In particular, if ϕ is a virtual endomorphism of the group G, then the linear span $\Phi = \Phi_{\Bbbk}$ over $\Bbbk$ of the permutational bimodule $\phi(G)G$ is called the *bimodule, associated to* ϕ. By Φ_R and Φ_L we denote the underlying right and left modules, respectively.

We get directly from Corollary 4.4 the next

Proposition 4.6. *Let ϕ_1 and ϕ_2 be conjugate virtual endomorphisms of a group G. Then the respective associated bimodules Φ_1 and Φ_2 are isomorphic.* □

4.4. Inner product

Definition 4.5. Let Φ be the $\mathbb{C}$-span of the permutational bimodule $\phi(G)G$. The group algebra $\mathbb{C}G$ is equipped with the involution $(\alpha g)^* = \overline{\alpha} g^{-1}$, where $\overline{\alpha}$ is the complex conjugation.

The *inner product* on the bimodule Φ, associated to the virtual endomorphism ϕ is the function $\langle \cdot | \cdot \rangle : \Phi \times \Phi \to \mathbb{C}G$, defined by the conditions:

1. the function $\langle \cdot | \cdot \rangle$ is linear over the second variable;
2. $\langle v_1 | v_2 \rangle = \langle v_2 | v_1 \rangle^*$ for all $v_1, v_2 \in \Phi$;
3. $\langle \phi(g_1)h_1 | \phi(g_2)h_2 \rangle = 0$ if $g_1^{-1}g_2 \notin \mathrm{Dom}\,\phi$ and

$$\langle \phi(g_1)h_1 | \phi(g_2)h_2 \rangle = h_1^{-1}\phi(g_1^{-1}g_2)h_2$$

otherwise.

In general, even if $\Bbbk$ is not equal to $\mathbb{C}$, the last condition of the definition gives a well defined function $\langle \cdot | \cdot \rangle : \phi(G)G \times \phi(G)G \to G \cup \{0\}$, which will be also called *inner product*.

Proposition 4.7. *The equality*

$$\langle v_1 | g \cdot v_2 \rangle = \langle g^{-1} \cdot v_1 | v_2 \rangle \tag{5}$$

holds for all $v_1, v_2 \in \phi(G)G$ and $g \in G$.

If $\langle v_1 | v_2 \rangle \neq 0$ for $v_1, v_2 \in \phi(G)G$, then

$$v_1 \cdot \langle v_1 | v_2 \rangle = v_2. \tag{6}$$

Proof. Let $v_i = \phi(g_i)h_i$ for $i = 1, 2$. Then, for equality (5):

$$\langle v_1 | gv_2 \rangle = h_1^{-1}\phi(g_1^{-1}gg_2)h_2^{-1} = h_1^{-1}\phi\left(\left(g^{-1}g_1\right)^{-1} g_2\right) h_2 = \langle g^{-1} \cdot v_1 | v_2 \rangle .$$

For equality (6):

$$v_1 \cdot \langle v_1 | v_2 \rangle = \phi(g_1)h_1 \cdot h_1^{-1}\phi(g_1^{-1}g_2)h_2 = \phi(g_2)h_2.$$

□

As a corollary, we get, that in the case $\Bbbk = \mathbb{C}$ we have

$$\langle v_1 | a \cdot v_2 \rangle = \langle a^* \cdot v_1 | v_2 \rangle \tag{7}$$

for all $v_1, v_2 \in \Phi$ and $a \in \mathbb{C}G$.

4.5. Standard bases and wreath products

Definition 4.6. A *basis* of a permutational G-bimodule M is an orbit transversal of the right action.

A *standard basis* of the bimodule Φ, associated to a virtual endomorphism ϕ is the set of the form

$$\{\phi(r_1)h_1, \phi(r_2)h_2, \dots, \phi(r_d)h_d\},$$

where $\{r_1, r_2, \dots, r_d\}$ is a left coset transversal of the subgroup $\mathrm{Dom}\,\phi$ in G and $\{h_1, h_2, \dots, h_d\}$ is an arbitrary sequence of elements of the group G.

It is easy to see that the notions of standard basis of the bimodule Φ and standard basis of the permutational bimodule $\phi(G)G$ coincide.

Proposition 4.8. *Every standard basis of the bimodule Φ is a free $\Bbbk G$-basis of the right module Φ_R. In particular, the module Φ_R is a free right $\Bbbk G$-module of dimension* $\mathrm{ind}\,\phi$. □

Note also, that directly from the definitions follows that the standard basis is *orthonormal*, i.e., that $\langle x_i |\, x_j\rangle$ is 0 for $i \neq j$ and 1 for $i = j$.

Since the left and the right multiplications commute, we get a homomorphism

$$\psi_L : \Bbbk G \to \mathrm{End}_{\Bbbk}(\Phi_R) = M_{d\times d}(\Bbbk G)$$

defined by the rule $\psi_L(a)(v) = a \cdot v$. By Proposition 4.8, the algebra $\mathrm{End}_{\Bbbk}(\Phi_R)$ is isomorphic to the algebra $M_{d\times d}(\Bbbk G)$ of $d \times d$-matrices over $\Bbbk G$. Here, as usual $d = \mathrm{ind}\,\phi$. We call the homomorphism ψ_L the *linear recursion, associated to ϕ.*

The linear recursion is computed using the formula in the next proposition, which follows directly from the definitions.

Proposition 4.9. *Let $X = \{x_1 = \phi(r_1)h_1, x_2 = \phi(r_2)h_2, \dots, x_d = \phi(r_d)h_d\}$ be a standard basis of Φ_R. Then for any $g \in G$ and $x_i \in X$ we have*

$$g \cdot x_i = x_j \cdot h_j^{-1} \phi\left(r_j^{-1} g r_i\right) h_i,$$

where j is uniquely defined by the condition $r_j^{-1} g r_i \in \mathrm{Dom}\,\phi$. □

The formula in Proposition 4.9 can be also interpreted as a homomorphism $\psi : G \to Symm(X) \wr G$, where "$\wr$" is the wreath product and $Symm(X)$ is the symmetric group on X. Let us recall at first the notion of a permutational wreath product.

Definition 4.7. Let G be a group and let H be a permutation group of a set X. Then the *(permutational) wreath product* $H \wr G$ is the semi-direct product $H \ltimes G^X$, where H acts on the group G^X by the respective permutations of the direct multiples.

The elements of the wreath product $H \wr G$ are written as products $h \cdot f$, where $f \in G^X$ and h is an element of H. The element f can be considered either as a function from X to G, or as a tuple $(g_1, g_2, \dots g_d)$, if an indexing $X = \{x_1, x_2, \dots, x_d\}$ of the set X is fixed. In the last case the multiplication rule for the elements of $H \wr G$ are the following:

$$h'(g'_1, g'_2, \dots g'_d) \cdot h(g_1, g_2, \dots g_d) = h'h(g'_{h(1)}g_1, g'_{h(2)}g_2, \dots, g'_{h(d)}g_d), \quad (8)$$

where $h(i)$ is the index for which $h(x_i) = x_{h(i)}$.

In the case of a standard basis Proposition 4.9 implies that for every $g \in G$ and $x \in X$ there exist $y \in X$ and $h \in G$ such that $g \cdot x = y \cdot h$. It is easy to see that $x \mapsto y$ is a permutation of the set X. Let us denote this permutation by σ_g. In this way we get a homomorphism $g \mapsto \sigma_g$ of G to the symmetric group $Symm(X)$. The kernel of this homomorphism is the first-level stabilizer $St_1(\phi)$.

Proposition 4.10. *The map*

$$\psi : g \mapsto \sigma_g(h_{i_1}^{-1}\phi(r_{i_1}^{-1}gr_1)h_1, h_{i_2}^{-1}\phi(r_{i_2}^{-1}gr_2)h_2, \dots, h_{i_d}^{-1}\phi(r_{i_d}^{-1}gr_d)h_d),$$

where the sequence $(i_1, i_2, \dots, i_d)$ *is such that* $r_{i_k}^{-1}gr_k \in \operatorname{Dom}\phi$ *for all* $k = 1, 2, \dots, k$ *and* σ_g *is the permutation* $k \mapsto i_k$*, is a homomorphism* $\psi : G \to Symm(X) \wr G$*.*

Proof. If $\psi(g) = \sigma_g(g_1, g_2, \dots g_d)$ and $\psi(h) = \sigma_h(h_1, h_2, \dots, h_d)$ then $hg \cdot x_i = h \cdot x_j \cdot g_i = \sigma_h\sigma_g(x_i) \cdot h_j g_i$, where $x_j = \sigma_g(x_i)$. This agrees with the multiplication formula (8), thus $\psi(hg) = \psi(h)\psi(g)$. □

The obtained homomorphism $\psi : G \to Symm(X) \wr G$ is called the *wreath product recursion* associated to the virtual endomorphism ϕ (and the basis X).

On the other hand, any homomorphism $\psi : G \to Symm(X) \wr G$ is associated to some virtual endomorphism. It is the virtual endomorphism ϕ which is defined on $g \in G$ if and only if $\psi(g) = \sigma(g_1, g_2, \dots, g_d)$, where $\sigma(x_1) = x_1$. If ϕ is defined on g, then $\phi(g) = g_1$. Let us choose a left coset transversal $T = \{r_1, r_2, \dots, r_d\}$ of $\operatorname{Dom}\phi$ such that $r_i = \sigma_i(r_{i1}, \dots, r_{id})$, where $\sigma_i(x_1) = x_i$. Then $Y = \{y_1 = \phi(r_1)r_{11}^{-1}, y_2 = \phi(r_2)r_{21}^{-1}, \dots, y_d = \phi(r_d)r_{d1}^{-1})\}$ is a standard basis of the respective module Φ_R. Then a direct computation shows that the homomorphism ψ is reconstructed back as the wreath product recursion, associated to the virtual endomorphism ϕ and the basis Y.

Example. Let us consider the virtual endomorphism $\phi(n) = n/2$ of the group $\mathbb{Z}$. Its domain is the subgroup of even numbers. The coset transversal is in this case, for example, the set $\{0, 1\}$.

Let us write the elements of the group $\mathbb{Z}$ in a multiplicative notation, so that $\mathbb{Z}$ is identified with the infinite cyclic group, generated by an element τ. Then, the coset transversal is written as $\{1, \tau\}$.

Thus we choose the following standard basis of the permutational bimodule $\phi(\mathbb{Z}) + \mathbb{Z}$:

$$X = \{\mathit{0} = \phi(1)1, \mathit{1} = \phi(\tau)1\}.$$

Then the wreath recursion is

$$\psi(\tau) = \sigma(1, \tau),$$

where σ is the transposition $(\mathit{0}, \mathit{1})$ of the set X.

The respective linear recursion is

$$\psi(\tau) = \begin{pmatrix} 0 & \tau \\ 1 & 0 \end{pmatrix}.$$

Proposition 4.11. *The kernel of the wreath product recursion associated to a virtual endomorphism ϕ is equal to $\mathcal{E}_1(\phi)$.*

Proof. An element $g \in G$ belongs to the kernel of ψ if and only if $g \cdot x_i = x_i \cdot 1$ for every $x_i \in X$. Hence, $g \in \ker \psi$ if and only if $g \in St_1(\phi)$ and $h_i^{-1}\phi(r_i^{-1}gr_i)h_i = 1$, i.e., $\phi(r_i^{-1}gr_i) = 1$. But $\{r_i\}$ is the left coset representative system, so $g \in \ker \psi$ if and only if for every $h \in G$ the element $h^{-1}gh$ belongs to $\operatorname{Dom} \phi$ and $\phi(h^{-1}gh) = 1$ □

4.6. Φ-invariant ideals

Definition 4.8. Let Φ be a bimodule over a $\Bbbk$-algebra A and let I be a two-sided ideal of A. Denote by $I \cdot \Phi$ the $\Bbbk$-subspace of Φ spanned by the elements of the form $a \cdot v$, where $a \in I$ and $v \in \Phi$. Analogically, denote by $\Phi \cdot I$ the subspace spanned by the elements $v \cdot a$.

If $I \subset A$ is a two-sided ideal in A then its Φ-*preimage* is the set

$$\Phi^{-1}(I) = \{a \in A \ : \ a \cdot v \in \Phi \cdot I \text{ for all } v \in \Phi\}.$$

Proposition 4.12. *For every two-sided ideal $I \subset A$ the sets $I \cdot \Phi$ and $\Phi \cdot I$ are sub-bimodules of Φ and the set $\Phi^{-1}(I)$ is a two-sided ideal of A.*

Proof. Let $a \in A$ and $v \in I \cdot \Phi$ be arbitrary. Then v is a linear combination over $\Bbbk$ of the elements of the form $b \cdot u$, where $b \in I$ and $v \in \Phi$. Hence, $a \cdot v$ and $v \cdot a$ are linear combinations of the elements of the form $ab \cdot u$ and $b \cdot (u \cdot a)$, respectively. But $ab \in I$, so that $ab \cdot u \in I \cdot \Phi$. The element $b \cdot (u \cdot a)$ belongs to $I \cdot \Phi$ by definition. Thus, $a \cdot v$ and $v \cdot a$ belong to the set $I \cdot \Phi$ and it is a sub-bimodule. The fact that $\Phi \cdot I$ is a sub-bimodule is proved in the same way.

Let $a_1, a_2 \in \Phi^{-1}(I)$ and $a \in A$ be arbitrary. Then for every $v \in \Phi$ we have $a_1 \cdot v, a_2 \cdot v \in \Phi \cdot I$, thus $(a_1 + a_2) \cdot v = a_1 \cdot v + a_2 \cdot v \in \Phi \cdot I$, since $\Phi \cdot I$ is closed under addition. We also have $aa_1 \cdot v = a \cdot (a_1 \cdot v) \in \Phi \cdot I$, since $\Phi \cdot I$ is a left-submodule of Φ; and $a_1 a \cdot v = a_1 \cdot (a \cdot v) \in \Phi \cdot I$, since $a_1 \in \Phi^{-1}(I)$. □

Definition 4.9. An ideal I is said to be Φ-*invariant* if $I \subseteq \Phi^{-1}(I)$. The algebra A is said to be Φ-*simple* if it has no proper Φ-invariant two-sided ideals.

An ideal I is Φ-invariant if and only if $I \cdot \Phi \subseteq \Phi \cdot I$.

Suppose that the ideal I is Φ-invariant. Denote by Φ/I the quotient of the $\Bbbk$-spaces $\Phi/(\Phi \cdot I)$. Then Φ/I has a structure of an A/I-bimodule, defined as

$$(a+I)\cdot(v + \Phi \cdot I) = a\cdot v+\Phi\cdot I, \qquad (v + \Phi \cdot I)\cdot(a+I) = v\cdot a+\Phi\cdot I. \quad (9)$$

It is easy to prove, using Proposition 4.12, that multiplications (9) are well defined.

Example. If Φ is associated to a virtual endomorphism ϕ of a group G and N is a normal ϕ-invariant subgroup of G, then the ideal of $\Bbbk G$ generated by $1 - N$ is Φ-invariant, since

$$\phi((1-g)g_1)g_0 = \phi(g_1)g_0 - \phi(gg_1)g_0 = \\ \phi(g_1)g_0 - \phi(g_1)g_0 \cdot \left(g_0^{-1}\phi\left(g_1^{-1}gg_1\right)g_0\right) = \\ \phi(g_1)g_0\left(1 - g_0^{-1}\phi\left(g_1^{-1}gg_1\right)g_0\right).$$

Consequently, if G is not ϕ-simple, then $\Bbbk G$ is not Φ-simple.

In fact, the operation Φ^{-1} on ideals is an exact analog of the operation Δ_ϕ on the normal subgroups of the group G. Namely, the above formula shows that if H is a normal subgroup, then $\Phi^{-1}\big((1-H)\big) = (1-\Delta_\phi(H))$, where (A) denotes the two-sided ideal of $\Bbbk G$ generated by the set A.

The algebra $\Bbbk G$ needs not to be Φ-simple even if the group G is ϕ-simple. But a Φ-simple quotient of the algebra $\Bbbk G$ can be constructed from a ϕ-simple group by the following construction, which is essentially due to S. Sidki.

Let us define a sequence of ideals in $\Bbbk G$:

$$\mathcal{I}_0 = \{0\}, \quad \mathcal{I}_n = \Phi^{-1}(\mathcal{I}_{n-1}) \text{ for } n \geq 1, \quad \mathcal{I}_\infty = \bigcup_{n\geq 0} \mathcal{I}_n \tag{10}$$

It is easy to see that $\mathcal{I}_{n+1} \supseteq \mathcal{I}_n$, that $\mathcal{I}_n$ are Φ-invariant and that $\Phi^{-1}(\mathcal{I}_\infty) = \mathcal{I}_\infty$. The ideals $\mathcal{I}_n$ and $\mathcal{I}_\infty$ are analogs of the ϕ-invariant subgroups $\mathcal{E}_n(\phi)$, $\mathcal{E}_\infty(\phi)$, defined before. In particular, the ideal $\mathcal{I}_1$ is exactly the kernel of the linear recursion $\psi_L : \Bbbk G \rightarrow \mathrm{End}_{\Bbbk G}(\Phi_R)$, which parallels Proposition 4.11.

Theorem 4.13. *Let G be a ϕ-simple group and let I be a proper Φ-invariant ideal of $\Bbbk G$. Then $I \subseteq \mathcal{I}_\infty$. In particular, the algebra $\Bbbk G/\mathcal{I}_\infty$ is $\Phi/\mathcal{I}_\infty$-simple.*

Let us prove at first the following lemmas.

Lemma 4.14. *Let I be a Φ-invariant ideal of A and let J be a Φ/I-invariant ideal of A/I. Then the full preimage $\tilde{J}$ of J in A is Φ-invariant.*

Proof. Let a belong to $\tilde{J}$. This means that $a + I$ belongs to J. Then for every $v \in \Phi$ the element $(a + I)(v + \Phi \cdot I)$ belongs to $(\Phi/I) \cdot J$, since it belongs to $J \cdot (\Phi/I)$ and J is Φ/I-invariant. But $(a + I)(v + \Phi \cdot I) = a \cdot v + I \cdot \Phi \cdot I \subseteq a \cdot v + \Phi \cdot I$, since I is Φ-invariant. Thus the coset $a \cdot v + \Phi \cdot I$ is a subset of $\Phi \cdot \tilde{J}$, which is the preimage of $(\Phi/I) \cdot J$. In particular, $a \cdot v \in \Phi \cdot \tilde{J}$, and the ideal $\tilde{J}$ is Φ-invariant. $\square$

Lemma 4.15. *Let $\{r_1, r_2, \ldots, r_d\} \subset G$ be a left coset transversal of* $\mathrm{Dom}\,\phi$ *in G. Let I be an ideal of $\Bbbk G$. Then $a = \alpha_1 g_1 + \alpha_2 g_2 + \cdots \alpha_m g_m \in \Bbbk G$, were $\alpha_i \in \Bbbk$ and $g_i \in G$, belongs to $\Phi^{-1}(I)$ if and only if for every $i = 1, 2, \ldots, d$ the sum*

$$a_i = \sum_{g_j r_i \in \mathrm{Dom}\,\phi} \alpha_j \phi(g_j r_i)$$

belongs to I.

Proof. The set $v_i = \{\phi(r_i) \cdot 1\}_{i=1,\ldots,d}$ is a $\Bbbk G$-basis of the right module Φ_R. Consequently, $v \in \Phi$ is an element of $\Phi \cdot I$ if and only if $v = \sum_{i=1}^d v_i \cdot b_i$, where $b_i \in I$. We also obviously have that $a \in \Phi^{-1}(I)$ if and only if $a \cdot v_i \in \Phi \cdot I$ for every $i = 1, \ldots, d$. But

$$a \cdot v_i = \sum_{j=1}^{d} v_j \cdot a_j,$$

where the elements a_j are defined as in the proposition. $\square$

Proof of Theorem 4.13. Choose a left coset transversal

$$\{r_1 = 1, r_2, \dots, r_d\} \subset G$$

of $\mathrm{Dom}\,\phi$ in G. Suppose that I is not a subset of $\mathcal{I}_\infty$. Let $\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_m g_m$ be an element of I not belonging to $\mathcal{I}_\infty$ with the minimal possible m. By Lemma 4.15 the elements $a_i = \sum_{g_j r_i \in \mathrm{Dom}\,\phi} \alpha_i \phi(g_j r_i)$ belong to I. There exists i such that $a_i \notin \mathcal{I}_\infty$, otherwise all $a_i \in \mathcal{I}_n$ for some n and thus $a \in \Phi^{-1}(\mathcal{I}_n) = \mathcal{I}_{n+1} \subseteq \mathcal{I}$. But a was chosen to be the shortest element of $I \setminus \mathcal{I}_\infty$. Thus, the only possibility is that one a_{i_0} is equal to $\sum_{j=1}^m \alpha_j \phi(g_j r_{i_0}) \in I \setminus \mathcal{I}_\infty$ and for all the other i we have $a_i = 0$. Then a_{i_0} is again a minimal element of $I \setminus \mathcal{I}_\infty$ and we can repeat the considerations.

It follows that for any two indices $1 \le i, j \le m$ we have $g_i g_j^{-1} \in \mathrm{Dom}\,\phi$. On the next step we get $\phi(g_i r_{i_0})\phi(g_j r_{i_0})^{-1} = \phi(g_i g_j^{-1}) \in \mathrm{Dom}\,\phi$ and then by induction, that $g_i g_j^{-1} \in \mathrm{Dom}\,\phi^n$ for all $1 \le i, j \le m$ and $n \in \mathbb{N}$. Considering $ga = \sum_{i=1}^m \alpha_i g g_i$ we prove that $g(g_i g_j^{-1})g^{-1} \in \mathrm{Dom}\,\phi^n$. Hence, $g_i g_j^{-1}$ belongs to the core $\mathcal{C}(\phi)$ of virtual endomorphism, which is trivial. Consequently, all g_i are equal, i.e., $m = 1$ and $a = \alpha_1 g_1$ for some $g_1 \in G$. But then $1 \in I$ and $I = \Bbbk G$. Contradiction.

The $\Phi/\mathcal{I}_\infty$-simplicity of $\Bbbk G/\mathcal{I}_\infty$ follows now directly from Lemma 4.14. □

Example. The first paper, where the algebra $\Bbbk G/\mathcal{I}_\infty$ was considered is [Sid97]. It is investigated there for the case of the Gupta-Sidki group [GS83a] and the field $\mathbb{F}_3$.

The Gupta-Sidki group can be defined as the group $G = F/\mathcal{C}(\phi)$, where F is the free group generated by two elements a, b and ϕ is its virtual endomorphism

$$\phi(a^3) = 1, \quad \phi(b) = b, \quad \phi(a^{-1}ba) = a, \quad \phi(a^{-2}ba^2) = a^{-1}.$$

It is proved in [GS83a] that G is a torsion 3-group, i.e., that every its element is of order 3^k. S. Sidki proved that the ring $\Bbbk G/\mathcal{I}_\infty$ for $\Bbbk = \mathbb{F}_3$ is primitive and is just-infinite, i.e., that every its proper quotient is finite-dimensional.

4.7. Tensor powers of the bimodule

We define the set $\phi^n(G)\phi^{n-1}(G)\dots\phi(G)G$, analogically to the set $\phi(G)G$, as the set of formal expressions of the form

$$\phi^n(g_n)\phi^{n-1}(g_{n-1})\cdots g_0,$$

where an expression $\phi^n(g_n)\phi^{n-1}(g_{n-1})\ldots g_0$ is identified with an expression

$$\phi^n(h_{n-1})\phi^{n-1}(h_{n-1})\ldots h_0$$

if and only if there exists $s \in G$ such that

$$\phi\left(\phi\left(\phi\left(sg_n\right)g_{n-1}\right)\ldots g_1\right)g_0 = \phi\left(\phi\left(\phi\left(sh_n\right)h_{n-1}\right)\ldots h_1\right)h_0$$

in G.

The group G acts on the set $\phi^n(G)\phi^{n-1}(G)\ldots G$ on the left by

$$g : \phi^n(g_n)\phi^{n-1}(g_{n-1})\ldots g_0 \mapsto \phi^n(gg_1)\phi^{n-1}(g_{n-1})\ldots g_0$$

and on the right by

$$g : \phi^n(g_n)\phi^{n-1}(g_{n-1})\ldots g_0 \mapsto \phi^n(g_n)\phi^{n-1}(g_{n-1})\ldots g_0 g.$$

It is easy to see that these actions are well defined.

We have the following natural interpretation of the set

$$\phi^n(G)\phi^{n-1}(G)\ldots G$$

in terms of the associated bimodule.

Recall, that if Φ_1 and Φ_2 are two A-bimodules, then their tensor product is the bimodule $\Phi_1 \otimes_A \Phi_2$ which, as a $\Bbbk$-space is the quotient of the $\Bbbk$-tensor product $\Phi_1 \otimes_\Bbbk \Phi_2$ by the $\Bbbk$-subspace, spanned by the elements

$$(v_1 \cdot a) \otimes v_2 - v_1 \otimes (a \cdot v_2),$$

for all $v_1 \in \Phi_1, v_2 \in \Phi_2, a \in A$. The left and the right multiplications are defined by the rules $a_1 \cdot (v_1 \otimes v_2) \cdot a_2 = (a_1 \cdot v_1) \otimes (v_2 \cdot a_2)$. We will denote in the sequel the tensor product $\Phi_1 \otimes_A \Phi_2$ just as $\Phi_1 \otimes \Phi_2$.

Proposition 4.16. *The linear span over the field $\Bbbk$ of the permutational bimodule $\phi^n(G)\ldots\phi(G)G$ is isomorphic to the nth tensor power $\Phi^{\otimes n} = \underbrace{\Phi \otimes \Phi \otimes \cdots \otimes \Phi}_{n \text{ times}}$ of the bimodule Φ associated to the virtual endomorphism ϕ.*

Proof. Consider the map $F_1 : \phi^n(G)\phi^{n-1}(G)\ldots G \to \Phi^{\otimes n}$ defined as

$$F_1\big(\phi^n(g_n)\phi^{n-1}(g_{n-1})\ldots g_0\big) = \phi(g_n)1\otimes\phi(g_{n-1})1\otimes\cdots\otimes\phi(g_2)1\otimes\phi(g_1)g_0.$$

It is easy to see that the map F_1 preserves the left and the right multiplications by the elements of G, thus, it can be extended to a morphism of $\Bbbk G$-bimodules.

On the other hand, the map $F_2 : \Phi^{\otimes n} \to \phi^n(G)\phi^{n-1}(G)\ldots G$ defined as

$$F_2\big(\phi(g_n)h_n \otimes \phi(g_{n-1})h_{n-1} \otimes \cdots \otimes \phi(g_1)h_1\big) = \\ = \phi^n(g_n)\phi^{n-1}(h_n g_{n-1})\phi^{n-2}(h_{n-1}g_{n-2})\ldots\phi(h_2 g_1)h_1$$

also can be extended to a morphism of $\Bbbk G$-bimodules and is inverse to the map F_1. Thus, the maps F_1 and F_2 are isomorphisms of the bimodules. □

4.8. Standard actions

Proposition 4.17. *Let $X = \{x_i = \phi(r_i)h_i\}_{i=1,\ldots,d}$ be a standard basis of the right module Φ_R. Then the set*

$$X^n = \{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \ : \ x_{i_k} \in X\}$$

is a basis of the right module of the bimodule $\Phi^{\otimes n}$.

Proof. The set $\{x \cdot g \ : \ x \in X, g \in G\}$ is a $\Bbbk$-basis of the space Φ. Consequently, the set M_n of the elements of the form $x_{i_1} \cdot g_1 \otimes x_{i_2} \cdot g_2 \otimes \cdots \otimes x_{i_n} \cdot g_n$ is a $\Bbbk$-basis of the tensor power $\Phi^{\otimes_\Bbbk n}$. By Proposition 4.9, every element of M_n can be reduced to the form $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_n} \cdot h$, where h is some element of G. It is easy to see that such reduction is unique, and that two elements of M_n are equal if and only if the respective reductions coincide. From this follows that the set X^n is a basis of the right $\Bbbk G$-module of $\Phi^{\otimes n}$. □

For every $n \geq 1$ we get a homomorphism $\psi^{\otimes n} : \Bbbk G \to \operatorname{End}\Phi_R^{\otimes n}$ coming from the left multiplications seen as endomorphisms of the right module $\Phi_R^{\otimes n}$. For every standard basis X, the set X^n is a free basis of the right module $\Phi_R^{\otimes n}$, and thus, the module $\Phi^{\otimes n}$ is free $|X|^n$-dimensional, and the algebra $\operatorname{End}\Phi_R^{\otimes n}$ is isomorphic to the algebra of $|X|^n \times |X|^n$-matrices over the algebra $\Bbbk G$. The homomorphisms maps $\psi^{\otimes n}$ are called the *iterated linear recursions.*

More generally, the bimodule structure defines natural homomorphisms $\psi_n : \operatorname{End}\Phi_R^{\otimes n} \to \operatorname{End}\Phi_R^{\otimes(n+1)}$. Namely, if g is an endomorphism of the right module $\Phi_R^{\otimes n}$, then its image in $\operatorname{End}\Phi_R^{\otimes(n+1)}$ is the endomorphism $\psi_n(g)$ defined as

$$\psi_n(g)(v_1 \otimes v) = g(v_1) \otimes v,$$

where $v_1 \in \Phi^{\otimes n}$ and $v \in \Phi$.

The defined homomorphism ψ_n agrees with the introduced linear recursions, i.e., $\psi_n \circ \psi^{\otimes n} = \psi^{\otimes(n+1)}$.

Note that the kernel of the iterated linear recursion $\psi^{\otimes n}$ is the ideal $\mathcal{I}_n$.

We will write in many cases the element $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \in X^n$ as a word $x_{i_1} x_{i_2} \ldots x_{i_n} \in X^*$. Then the set X^n is identified with the set of the words of length n over the alphabet X.

It follows from Proposition 4.9 that for every $v \in X^n$ and for every $g \in G$ there exists a unique pair (u, h), where $u \in X^n$ and $h \in G$, such that

$$g \cdot v = u \cdot h. \tag{11}$$

It is easy to see that the map $v \mapsto u$ is a permutation of the set X^n and that in this way we get an action of the group G on the set X^n. Taking union we get an action of G on the set X^*. We will call this action the *standard action of G with respect to the basis X*.

It follows from Equation (11) that the standard actions are self-similar in sense of Definition 2.7.

The element h in (11) is called the *restriction* of g at v and is denoted $g|_v$. The image of a word v under the action of $g \in G$ and the restriction $g|_v$ can be computed inductively using Proposition 4.9. This notion of a restriction is a generalization of the previously defined notion for self-similar actions. In particular, the properties (2) and (3) hold, and if the action is faithful, then the restriction is defined uniquely by the condition that $g(vu) = g(v)g|_v(u)$ for all $u \in X^*$.

Proposition 4.18. *Take any faithful self-similar action of a group G over the alphabet $X = \{x_1, x_2, \ldots, x_d\}$. Let $\phi = \phi_{x_1}$ be the associated virtual endomorphism. Take elements r_i for $i = 1, 2, \ldots, d$ such that $r_i(x_1) = x_i$. Let $h_i = r_i|_{x_1}$. Then $\tilde{X} = \{\tilde{x}_1 = \phi(r_1)h_1^{-1}, \tilde{x}_2 = \phi(r_2)h_2^{-1}, \ldots, \tilde{x}_d = \phi(r_d)h_d^{-1}\}$ is a standard basis of the bimodule Φ, associated to the virtual endomorphism ϕ and the original action of G on X^* coincides with the standard action of G with respect to the basis $\tilde{X}$, i.e.,*

$$g(x_{i_1} x_{i_2} \ldots x_{i_n}) = g(\tilde{x}_{i_1} \tilde{x}_{i_2} \ldots \tilde{x}_{i_n})$$

for every $g \in G$ and $x_{i_1} x_{i_2} \ldots x_{i_n} \in X^*$.

Proof. Let g be an arbitrary element of the group G and let $x_i \in X$ be an arbitrary letter. Let $g(x_i) = x_j$. Then $s_j^{-1} g s_i(x_1) = x_1$, so that $s_j^{-1} g s_i \in \operatorname{Dom} \phi$. For every $v \in X^*$ we have $s_j^{-1} g s_i(x_1 v) = x_1 \phi(s_j^{-1} g s_i)(v)$, by definition of ϕ. Then

$$\begin{aligned} g(x_i v) = g s_i(x_1 h_i^{-1}(v)) = s_j(s_j^{-1} g s_i)(x_1 h_i^{-1}(v)) = \\ = s_j(x_1 \phi(s_j^{-1} g s_i) h_i^{-1}(v)) = x_j h_j \phi(s_j^{-1} g s_i) h_i^{-1}(v) \end{aligned}$$

and the proof is finished by induction on the length of the word v. □

Proposition 4.19. *Let $X = \{x_i = \phi(r_i)h_i\}$ and $Y = \{y_i = \phi(s_i)g_i\}$ be two standard bases of the bimodule Φ. Then the respective standard actions of the group G on X^* and Y^* are conjugate, i.e., there exists a bijection $\alpha : X^* \to Y^*$ such that the equality $\alpha^{-1}g\alpha(v) = g(v)$ holds for every $g \in G$ and $v \in X^*$.*

Proof. Let $v = x_{i_1}x_{i_2}\dots x_{i_n}$ be an arbitrary element of X^*. It follows from Proposition 4.9 that there exists a unique $\alpha(v) \in Y^*$ such that $v = \alpha(v) \cdot h$ for some $h \in G$.

We have $g \cdot v = g(v) \cdot g|_v$ for the standard action on X^*, so that $g \cdot v = \alpha(g(v)) \cdot hg|_v$ for some $h \in G$. On the other hand

$$g \cdot v = g \cdot \alpha(v) \cdot h(v) = g(\alpha(v)) \cdot g|_{\alpha(v)}h(v).$$

Consequently, $\alpha(g(v)) = g(\alpha(v))$. □

Recall, that due to Proposition 4.7, we have for every $x_i \in X$ the equality $x_i = y_j \cdot \langle y_j|\, x_i\rangle$, where y_j is such that $\langle y_j|\, x_i\rangle \neq 0$, i.e., $s_j^{-1}r_i \in \mathrm{Dom}\,\phi$. Therefore, $v = y_{j_1} \cdot \langle y_{j_1}|\, x_{i_1}\rangle \otimes y_{j_2} \cdot \langle y_{j_2}|\, x_{i_2}\rangle \otimes \dots \otimes y_{j_n} \cdot \langle y_{j_n}|\, x_{i_n}\rangle$ for some $y_{j_1}y_{j_2}\dots y_{j_n} \in Y^*$, and the map $\alpha : X^* \to Y^*$ can be more explicitely defined by the recurrent formula

$$\alpha(x_i \otimes v) = y_j \otimes \langle y_j|\, x_i\rangle\,(\alpha\,(v))\,, \tag{12}$$

where $v \in X^*$, $y_j \in Y$ is such that $\langle y_j|\, x_i\rangle \neq 0$, and $\langle y_i|\, x_i\rangle \in G$ acts on $\alpha(v)$ by the standard action of G on Y^*.

Proposition 4.20. *The virtual endomorphism ϕ is regular if and only if the respective standard action is transitive on the sets X^n (is* level transitive*).*

If the virtual endomorphism ϕ is regular, then the standard action is conjugate with the action of the group G on the coset tree of ϕ, i.e., there exists an isomorphism of rooted trees $\Lambda : X^* \to T(\phi)$ such that $\Lambda(g(v)) = g(\Lambda(v))$ for all $v \in X^*$.

Proof. It follows from Proposition 4.19 that if one standard action is level-transitive, then all the other standard actions are level-transitive. Therefore, it is sufficient to prove the proposition for one standard basis, so we can assume that our standard basis contains the element $x_0 = \phi(1)1$. The standard action is level transitive if and only if the index of the stabilizer of the word $x_0x_0\dots x_0 = x_0^n$ is equal to d^n, where $d = |X| = \mathrm{ind}\,\phi$. But the stabilizer of the word $x_0x_0\dots x_0 = x_0^n$ is equal to $\mathrm{Dom}\,\phi^n$.

If the virtual endomorphism ϕ is regular, then the isomorphism $\Lambda : X^* \to T(\phi)$ may be defined as $\Lambda(v) = g \cdot \mathrm{Dom}\,\phi^{|v|}$, where g is such that $g(x_0^n) = v$. □

5. Contracting virtual endomorphisms

5.1. Definitions and basic properties

Let ϕ be a virtual endomorphism of a group G. Choose some standard basis $X = \{x_1 = \phi(r_1)h_1, \ldots, X_d = \phi(r_d)h_d\}$ of the right module Φ_R and consider the standard action of the group G on the space X^*.

Definition 5.1. The standard action is said to be *contracting* if there exists a finite set $\mathcal{N}$ such that for every $g \in G$ there exists $n_0 \in \mathbb{N}$ such that

$$g|_v \in \mathcal{N},$$

for all $v \in X^n$, $n \geq n_0$.

The minimal set $\mathcal{N}$ with the above property is called the *nucleus* of the standard action.

It is easy to see that if the standard action is contracting then it is *finite state*, i.e., for every $g \in G$ the set $\{g|_v \ : \ v \in X^*\}$ is finite.

We will use the following notation. If A and B are two subsets of a group G, then AB is the set of products ab, where $a \in A$ and $b \in B$. The power A^n is a short notation for $\underbrace{A \cdot A \cdots A}_{n \text{ times}}$. If $A \subset G$ and $W \subset X^*$, then $A|_W$ is the set of restrictions $a|_w$, where $a \in A$ and $w \in W$.

Lemma 5.1. *Suppose that the group G is generated by a finite set $S = S^{-1} \ni 1$. Then a standard action of G is contracting if and only if there exists a finite set $\mathcal{N} \subset G$ and a number n such that*

$$(S \cup \mathcal{N})^2|_{X^n} \subseteq \mathcal{N}.$$

Proof. If the action is contracting, then the above condition holds for $\mathcal{N}$ equal to the nucleus. In the other direction, by induction on the length of a group element we prove that for every $g \in G$ there exists $k_0 \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^{nk}$, where $k \geq k_0$. Then the nucleus of the action is a subset of $\mathcal{N}|_{\cup_{0 \leq m \leq n-1} X^m}$. ☐

Proposition 5.2. *Suppose that the virtual endomorphism ϕ is contracting with respect to the standard basis X. Let $A \subset G$ be a finite set. Then the set of all possible $h \in G$ such that*

$$g_1 \cdot x_{i_1} \otimes g_2 \cdot x_{i_2} \otimes \cdots \otimes g_m \cdot x_{i_m} = v \cdot h, \tag{13}$$

for some $g_i \in A$, $x_{i_k} \in X$ and $v \in X^m$, is finite.

Proof. It is sufficient to prove the proposition for some set $A' \supseteq A$, so we assume that the set A contains the nucleus $\mathcal{N}$ of the action and that it is *state-closed*, i.e., that for every $g \in A$ and $v \in X^*$ the restriction $g|_v$ also belongs to A. We can do this, since the action is finite-state.

There exists a number k such that $A^2|_v \subseteq \mathcal{N} \subseteq A$ for every word $v \in X^*$ of length greater or equal to k. It is easy to see that then $A^{2n}|_v \subseteq A^n$ for every $v \in X^k$ and every $n \in \mathbb{N}$.

It is sufficient to find a finite set B such that it contains all h, which appear in Equation (13) for numbers m divisible by k.

We can write

$$g_1 \cdot x_{i_1} \otimes g_2 \cdot x_{i_2} \otimes \cdots \otimes g_m \cdot x_{i_m} = v_1 \cdot h_1 \otimes v_2 \cdot h_2 \otimes \cdots \otimes v_{m/k} \cdot h_{m/k},$$

where $h_i \in G$ and $v_i \in X^k$ for all i. From the fact that A is state-closed follows that $h_i \in A^k$. But then $h_1 \cdot v_2 = h_1(v_2) \cdot h_1|_{v_2}$ and $h_1|_{v_2}$ also belongs to A^k, so $(h_1|_{v_2} h_2)|_{v_3} \in A^{2k}|_{v_3} \subseteq A^k$, and we get an inductive proof of the fact that $v_1 \cdot h_1 \otimes v_2 \cdot h_2 \otimes \cdots \otimes v_{m/k} \cdot h_{m/k} = u \cdot h$ for some $h \in A^k$. □

Directly from Proposition 5.2 we get

Corollary 5.3. *If the standard action is contracting then for any finite set $A \subset G$ there exists a finite set $\Sigma_A \subset G$ such that $A \subseteq \Sigma_A$ and*

$$\Sigma_A|_X \cdot A \subseteq \Sigma_A.$$

Now we are ready to prove that the property of an action to be contracting does not depend on the particular choice of the standard basis.

Proposition 5.4. *If some standard action for a virtual endomorphism ϕ is contracting, then any other standard action for ϕ is contracting.*

Proof. Let $X = \{x_1, x_2, \ldots, x_d\}$ and $Y = \{y_1, y_2, \ldots, y_d\}$ be two standard bases. Then we can permute the vectors in the basis so that there exist $r_i \in G$ such that $y_i = x_i \cdot r_i$. Take $A = \{r_i\}_{i=1,\ldots,d}$. Let Σ_A be as in Corollary 5.3 with respect to the standard action over the alphabet X.

Let $g \in G$ and $y_i \in Y$ be arbitrary. Then $g|_{y_i}$ is defined by the condition $g \cdot x_i \cdot r_i = x_j \cdot r_j g|_{y_i}$. Thus, $g|_{y_i} = r_j^{-1} g|_{x_i} r_i$. It is easy to prove now by induction on n that for every $v \in Y^n$ the restriction $g|_v$ belongs to the set $\Sigma_A^{-1} \cdot g|_u \cdot \Sigma_A$ for some $u \in X^n$. Consequently, the standard action with respect to Y is also contracting with the nucleus a subset of $\Sigma_A^{-1} \cdot \mathcal{N} \cdot \Sigma_A$, where $\mathcal{N}$ is the nucleus of the action on X^*. □

Proposition 5.4 justifies the following notion.

Definition 5.2. A virtual endomorphism ϕ is *contracting* if some (equivalently, if all) respective standard actions are contracting.

The next proposition shows that the contraction can be detected by a finite number of group relation.

Proposition 5.5. *Suppose that the virtual endomorphism ϕ of a finitely generated group G is contracting. Then there exist a finitely presented group F, a contracting virtual endomorphism $\tilde{\phi}$ of F, a normal $\tilde{\phi}$-invariant subgroup N of F and an isomorphism $\rho : G \to F/N$ such that $\rho \circ \phi = \tilde{\phi}/N \circ \rho$.*

Proof. Let us fix some standard basis $X = \{x_i\}_{i=1,\ldots,d}$, where $x_1 = \phi(1)1$ and consider the respective standard action of the group G. Let $\mathcal{N}$ be its nucleus, and let S be a finite symmetric generating set of G, which includes the identity. Since the action is contracting, we may suppose that the set S is state-closed, i.e., that for every $s \in S$ and $x \in X$ the restriction $s|_x$ also belongs to S. We may also suppose that S contains the nucleus $\mathcal{N}$. Let $\tilde{S}$ be a set, which is in a bijective correspondence $\tilde{S} \to S : \tilde{s} \mapsto s$ with the set S. Take the group F generated by the set $\tilde{S}$ and defined by all relations of the form $\tilde{s}_1\tilde{s}_2 = \tilde{s}_3$, where $\tilde{s}_i$ are such that $s_1s_2 = s_3$ in the group G. In other words, the group F is the group defined by all the relations of the length 3, which hold for the generators S of the group G.

Let us define a permutational bimodule M over the group F with the standard basis X by the natural rules:

$$\tilde{s}_1 \cdot x = y \cdot \tilde{s}_2, \text{ if and only if } s_1 \cdot x = y \cdot s_2.$$

Another way to interpret the above construction is to say that we define the wreath product recursion $F \to Symm(X) \wr F$ on the generators of F in the same way as was defined the recursion $G \to Symm(X) \wr G$ on the generators of G.

The only thing to check in order to prove that the bimodule M is well defined, is to prove that if g is a word in generators $\tilde{S}$, representing the trivial element, then $g \cdot x = x \cdot 1$ in M for every $x \in X$. But this follows from the fact that if $s_1s_2 = s_3$ in G, $s_2 \cdot x = y \cdot s_2'$, $s_1 \cdot y = z \cdot s_1'$, for some $s_1, s_2, s_3, s_1' = s_1|_y, s_2' = s_2|_x \in S$ and $x, y, z \in X$, then $s_3 \cdot x = z \cdot s_3'$, $s_1s_2 \cdot x = z \cdot s_1's_2'$, so that $s_3' = s_1's_2'$, where $s_3' = s_3|_x \in S$.

Let $\tilde{\phi}$ be the virtual endomorphism of F, associated to the bimodule M and the element x_1 (recall that x_1 corresponds to $\phi(1)1$).

Directly from the definitions follows that the permutational bimodule $\phi(G)G$ is a quotient of the bimodule M with the natural quotient map $\pi : \tilde{s} \mapsto s : F \to G$ and the map $p : M \to \phi(G)G$ defined as $p(x \cdot g) =$

$x \cdot \pi(g)$. Then by Proposition 4.5, the kernel N of the map π is a $\tilde{\phi}$-semi-invariant subgroup such that $\tilde{\phi}/N$ is conjugated with ϕ. But from the choice of $\tilde{\phi}$ follows that in fact we have $\tilde{\phi}/N = \phi$.

If $g \in N$, then $g \cdot x = x \cdot g|_x$ for every x, since $\pi(g) \cdot x = x \cdot \pi(g)|_x$. Thus, $N \leq \mathrm{Dom}\,\tilde{\phi}$, and N is a normal $\tilde{\phi}$-invariant subgroup.

It remains to prove that the virtual endomorphism $\tilde{\phi}$ is contracting. Since the action of G is contracting, by Lemma 5.1 there exists n such that $S^2|_{X^n} \subseteq S$. But we have included all the relations of the form $s_1 s_2 = s_3, s_i \in S$ into the relations of F and the restrictions of the words in generators of F are computed by the same rules as the restrictions of the words in generators of G. Thus, $\tilde{S}^2|_{X^n} \subseteq \tilde{S}$, and by Lemma 5.1, the action of F is contracting. □

Proposition 5.6. *If a virtual endomorphism ϕ of a group G is contracting and the nucleus of a standard action does not contain non-trivial elements of $\mathcal{C}(\phi)$ then $\mathcal{C}(\phi) = \mathcal{E}_\infty(\phi)$.*

Proof. Let that $g \in \mathcal{C}(\phi)$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $g|_v$ belongs to the nucleus for every $v \in X^n$. But then $g \cdot v = v \cdot g|_v$ and $g|_v \in \mathcal{C}(\phi)$, hence $g|_v = 1$ and $g \in \mathcal{E}_n(\phi)$. □

The following is a direct corollary of Proposition 4.5.

Proposition 5.7. *If a virtual endomorphism ϕ of a group G is contracting, and N is a normal ϕ-semi-invariant subgroup of G, then the virtual endomorphism ϕ/N of the group G/N is also contracting.* □

The next easy fact is proved in [Nekc].

Proposition 5.8. *If a virtual endomorphism ϕ of a group G is contracting and onto, then the group G is generated by the nucleus of the standard action.* □

5.2. Contraction coefficient

If the group is finitely generated, then the contractivity of a virtual endomorphism can be established using a more intuitive definition.

If the group G is finitely generated, then we denote by $l(g)$ the length of the shortest representation of g in a product of the generators and the inverses, for a fixed finite generating set of the group.

Definition 5.3. Let G be a finitely generated group, let ϕ be its virtual endomorphism. Let us fix also a standard self-similar action of G on X^*. The number

$$\rho = \lim_{n\to\infty} \sqrt[n]{\limsup_{l(g)\to\infty} \max_{v\in X^n} \frac{l\left(g|_v\right)}{l(g)}}$$

is called the *contraction coefficient* of the action.

The number

$$\rho_\phi = \lim_{n\to\infty} \sqrt[n]{\limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l\left(\phi^n\left(g\right)\right)}{l(g)}}, \tag{14}$$

is called the *contraction coefficient* (or the *spectral radius*) of the virtual endomorphism ϕ.

Note that the function $\rho_\phi(n) = \limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l(\phi^n(g))}{l(g)}$ is sub-multiplicative, i.e., $\rho_\phi(n+m) \le \rho_\phi(n)\rho(m)$, since

$$\frac{l\left(\phi^{n+m}\left(g\right)\right)}{l(g)} = \frac{l\left(\phi^{n+m}\left(g\right)\right)}{l(\phi^n(g))} \cdot \frac{l\left(\phi^n\left(g\right)\right)}{l(g)},$$

$$\limsup_{g\in \mathrm{Dom}\,\phi^{n+m}, l(g)\to\infty} \frac{l\left(\phi^{n+m}\left(g\right)\right)}{l(\phi^n(g))} \le \limsup_{g\in \mathrm{Dom}\,\phi^m, l(g)\to\infty} \frac{l\left(\phi^m\left(g\right)\right)}{l(g)}$$

and

$$\limsup_{g\in \mathrm{Dom}\,\phi^{n+m}, l(g)\to\infty} \frac{l\left(\phi^n\left(g\right)\right)}{l(g)} \le \limsup_{g\in \mathrm{Dom}\,\phi^n, l(g)\to\infty} \frac{l\left(\phi^n\left(g\right)\right)}{l(g)}.$$

Therefore, from the well-known Polya Lemma, the limit in (14) exists. Similar arguments show that the contraction coefficient ρ of the standard action also exists. Both coefficients are finite, since $\rho_\phi \le \rho$ and ρ is not greater than $\max_{g\in S, x\in X} l(g|_x)$, where S is the generating set.

Note also, that if l_1 and l_2 are the length functions computed with respect to different finite generating sets, then there exists a number $C > 0$ such that $C^{-1}l_2(g) \le l_1(g) \le Cl_2(g)$ for all $g \in G$. From this easily follows that the contraction coefficients computed with respect to l_1 will be the same as the coefficients, computed with respect to l_2.

The following proposition is proved in [Nekc].

Proposition 5.9. *A standard action is contracting if and only if its contraction coefficient is less than one.*

Suppose that the virtual endomorphism ϕ is regular. Then it is contracting if and only if its contraction coefficient is less than one. If it is contracting, then ρ_ϕ is equal to the contraction coefficient of every associated standard action. □

Let w be an infinite word in the alphabet X, i.e., a sequence $x_1x_2\ldots$, $x_i \in X$. If $g \in G$, then by $g(x_1x_2\ldots)$ we denote the word $y_1y_2\ldots$ such that $g(x_1x_2\ldots x_n) = y_1y_2\ldots y_n$ for every n. From the definition of a self-similar action follows that the word $g(x_1x_2\ldots)$ is well defined and that we get in this way an action of G on the set X^ω of infinite words.

Definition 5.4. Let G be a finitely generated group, acting on a set A. *Growth degree* of the G-action is the number

$$\gamma = \sup_{w\in A} \limsup_{r\to\infty} \frac{\log|\{g(w)\ :\ l(g)\le r\}|}{\log r}$$

where $l(g)$ is the length of a group element with respect to some fixed finite generating set of G.

One can show, in the same way as before, that the growth degree γ does not depend on the choice of the generating set of G.

Proposition 5.10. *Suppose that a standard action of a group G on X^* is contracting. Then the growth degree of the action on X^ω is not greater than $\frac{\log|X|}{-\log\rho}$, where ρ is the contraction coefficient of the action on X^*.*

Proof. The statement is more or less classical. See, for instance the similar statements in [Gro81, BG00, Fra70].

Let ρ_1 be such that $\rho<\rho_1<1$. Then there exists $C>0$ and $n\in\mathbb{N}$ such that for all $g\in G$ we have $l(g|_{x_1x_2\ldots x_n})<\rho_1^n\cdot l(g)+C$.

Then cardinality of the set $B(w,r)=\{g(w)\ :\ l(g)\le r\}$, where $w=x_1x_2\ldots\in X^\omega$ is not greater than

$$|X|^n\cdot|\{B\left(x_{n+1}x_{n+2}\ldots,\rho_1^n\cdot r+C\right)|,$$

since the map $\sigma^n : x_1x_2\ldots\mapsto x_{n+1}x_{n+2}\ldots$ maps $B(w,r)$ into

$$B\left(x_{n+1}x_{n+2}\ldots,\rho_1^n\cdot r+C\right)$$

and every point of X^ω has exactly $|X|^n$ preimages under σ^n. The map σ^n is the nth iteration of the shift map $\sigma(x_1x_2\ldots)=x_2x_3\ldots$.

Let $k=\left[\frac{\log r}{-n\log\rho_1}\right]+1$. Then $\rho_1^{nk}\cdot r<1$ and the number of the points in the ball $B(w,r)$ is not greater than

$$|X|^{nk}\cdot\left|B\left(\sigma^{nk}\left(w\right),R\right)\right|,$$

where

$$R=\rho_1^{nk}\cdot r+\rho_1^{n(k-1)}\cdot C+\rho_1^{n(k-2)}\cdot C+\cdots+\rho_1^n\cdot C+C<1+\frac{C}{1-\rho_1^n}.$$

But $|B(u,R)|$ for all $u\in X^\omega$ is less than $K_1=|S|^R$, where S is the generating set of G (we assume that $S=S^{-1}\ni 1$). Hence,

$$\begin{aligned}|B(w,r)| &< K_1\cdot|X|^{n\left(\frac{\log r}{-n\log\rho_1}+1\right)}=\\ &=K_1\cdot\exp\left(\frac{\log|X|\log r}{-\log\rho_1}+n\log|X|\right)=K_2\cdot r^{\frac{\log|X|}{-\log\rho_1}},\end{aligned}$$

where $K_2=K_1\cdot|X|^n$. Thus, the growth degree is not greater than $\frac{\log|X|}{-\log\rho_1}$ for every $\rho_1\in(\rho,1)$, so it is not greater than $\frac{\log|X|}{-\log\rho}$. □

Lemma 5.11. *Let ϕ be a contracting virtual endomorphism of a ϕ-simple infinite finitely generated group G. Then the contraction coefficient of its standard action is greater or equal to $1/\operatorname{ind}\phi$.*

Proof. Consider the standard action on the set X^* for a standard basis X, containing the element $x_0 = \phi(1)1$. Then the parabolic subgroup $P(\phi) = \cap_{n\geq 0} \operatorname{Dom}\phi^n$ is the stabilizer of the word $w = x_0x_0x_0\ldots \in X^\omega$. The subgroup $P(\phi)$ has infinite index in G, otherwise $\cap_{g\in G} g^{-1}Pg = \mathcal{C}(\phi)$ will have finite index, and G will be not ϕ-simple. Consequently, the G-orbit of w is infinite. Then there exists an infinite sequence of generators $s_1, s_2, \ldots$ of the group G such that the elements of the sequence

$$w, s_1(w), s_2s_1(w), s_3s_2s_1(w), \ldots$$

are pairwise different. This implies that the growth degree of the orbit Gw

$$\gamma = \limsup_{r\to\infty} \frac{|\{g(w)\ :\ l(g) \leq r\}|}{\log r}$$

is greater or equal to 1, thus the growth degree of the action of G on X^ω is not less than 1, and by Proposition 5.10, $1 \leq \frac{\log|X|}{-\log\rho}$. ☐

Proposition 5.12. *If there exists a faithful contracting action of a finitely-generated group G then for any $\epsilon > 0$ there exists an algorithm of polynomial complexity of degree not greater than $\frac{\log|X|}{-\log\rho} + \epsilon$ solving the word problem in G.*

Proof. We assume that the generating set S is symmetric (i.e., that $S = S^{-1}$) and contains all the restrictions of all its elements, so that always $l(g|_v)$ is not greater than $l(g)$.

We will denote by F the free group generated by S and for every $g \in F$ by $\hat{g}$ we denote the canonical image of g in G.

Let $1 > \rho_1 > \rho$. Then $\rho_1 \cdot |X| > 1$, since by Lemma 5.11, $\rho \cdot |X| \geq 1$. There exist n_0 and l_0 such that for every word $v \in X^*$ of the length n_0 and every $g \in G$ of the length $\geq l_0$ we have

$$l\left(g|_v\right) < \rho_1^n l(g).$$

Assume that we know for every $g \in F$ of the length less than l_0 if $\hat{g}$ is trivial or not. Assume also that we know all the relations $g \cdot v = u \cdot h$ for all $g, l(g) \leq l_0$ and $v \in X^{n_0}$.

Then we can compute in $l(\hat{g})$ steps, for any $g \in F$ and $v \in X^n$, the element $h \in F$ and the word $u \in X^{n_0}$ such that $\hat{g} \cdot v = u \cdot \hat{h}$. If $v \neq u$ then we conclude that $\hat{g}$ is not trivial and stop the algorithm. If for all $v \in X^{n_0}$ we have $v = u$, then $\hat{g}$ is trivial if and only if all the obtained restrictions $\hat{h} = \hat{g}|_v$ are trivial. We know, whether $\hat{h}$ is trivial if $l(h) < l_0$. We proceed further, applying the above computations for those h, which have the length not less than l_0.

But $l(h) < \rho_1^n l(g)$, if $l(g) \geq l_0$. So on each step the length of the elements becomes smaller, and the algorithm stops in not more than $-\log l(g)/\log \rho_1$ steps. On each step the algorithm branches into $|X|$ algorithms. Thus, since $\rho_1 \cdot |X| > 1$, the total time is bounded by

$$l(g)\left(1 + \rho_1 \cdot |X| + (\rho_1 \cdot |X|)^2 + \cdots + (\rho_1 \cdot |X|)^{[-\log l(g)/\log \rho_1]}\right) <$$
$$\frac{l(g)}{\rho_1 \cdot |X| - 1}\left((\rho_1 \cdot |X|)^{1-\log l(g)/\log \rho_1} - 1\right) =$$
$$\frac{l(g)\rho_1 \cdot |X|}{\rho_1 \cdot |X| - 1}\left((\rho_1 \cdot |X|)^{-\log l(g)/\log \rho_1} - (\rho_1 \cdot |X|)^{-1}\right) =$$
$$C_1 l(g)\left(\exp\left(\log l(g)\left(\frac{\log |X|}{-\log \rho_1} - 1\right)\right) - C_2\right) =$$
$$= C_1 l(g)^{-\log |X|/\log \rho_1} - C_1 C_2 l(g),$$

where $C_1 = \frac{\rho_1 \cdot |X|}{\rho_1 \cdot |X| - 1}$ and $C_2 = (\rho_1 \cdot |X|)^{-1}$. □

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Metrizable ball structures

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Abstract. A *ball structure* is a triple (X, P, B), where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X, $x \in B(x, \alpha)$, which is called a ball of radius α around x. We characterize up to isomorphism the ball structures related to the metric spaces of different types and groups.

Following [1, 2], by *ball structure* we mean a triple $\mathbf{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X, which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

Let $\mathbf{B}_1 = (X_1, P_1, B_1)$ and $\mathbf{B}_2 = (X_2, P_2, B_2)$ be ball structures, $f : X_1 \to X_2$. We say that f is a $\succ$-*mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that

$$B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$$

for every $x \in X_1$. If there exists a $\succ$-mapping of X_1 onto X_2, we write $\mathbf{B}_1 \succ \mathbf{B}_2$.

A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. If there exists an injective $\prec$-mapping of X_1 into X_2, we write $\mathbf{B}_1 \prec \mathbf{B}_2$.

A bijection $f : X_1 \to X_2$ is called an *isomorphism* between $\mathbf{B}_1$ and $\mathbf{B}_2$ if f is a $\succ$-mapping and f is a $\prec$-mapping.

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We say that a property $\mathbf{P}$ of ball structures is a *ball property* if a ball structure $\mathbf{B}$ has a property $\mathbf{P}$ provided that $\mathbf{B}$ is isomorphic to some ball structure with property $\mathbf{P}$.

Example 1. *Let* (X, d) *be a metric space,* $\mathbf{R}^+ = \{x \in \mathbf{R} : x \geq 0\}$*. Given any* $x \in X$*,* $r \in \mathbf{R}^+$*, put*

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

A ball structure $(X, \mathbf{R}^+, B_d)$ *is denoted by* $\mathbf{B}(X, d)$*.*

We say that a ball structure $\mathbf{B}$ is *metrizable* if $\mathbf{B}$ is isomorphic to $\mathbf{B}(X, d)$ for some metric space (X, d).

To obtain a characterization (Theorem 1) of metrizable ball structures, we need some definitions and technical results.

A ball structure $\mathbf{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$, $x \in B(y, \alpha)$.

Lemma 1. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be ball structures and let* f *be a* $\prec$*-mapping of* X_1 *onto* X_2*. If* $\mathbf{B}_1$ *is connected, then* $\mathbf{B}_2$ *is connected.*

Proof. Given any $y, z \in X_1$, choose $\alpha \in P_1$ such that $y \in B_1(z, \alpha)$, $z \in B_1(y, \alpha)$. Since f is a $\prec$-mapping, then there exists $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X_1$. Hence, $f(y) \in B_2(f(z), \beta)$ and $f(z) \in B_2(f(y), \beta)$. Since $f(X_1) = X_2$, then $\mathbf{B}_2$ is connected. $\square$

Lemma 2. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be ball structures and let* f *be an injective* $\succ$*-mapping of* X_1 *into* X_2*. If* $\mathbf{B}_2$ *is connected, then* $\mathbf{B}_1$ *is connected.*

Proof. Given any $y, z \in X_1$, choose $\beta \in P_2$ such that $f(y) \in B_2(f(z), \beta)$ and $f(z) \in B_2(f(y), \beta)$. Since f is a $\succ$-mapping, then there exists $\alpha \in P_1$ such that $B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$ for every $x \in X_1$. Since f is injective, then $z \in B_1(y, \alpha)$ and $y \in B_1(z, \alpha)$. Hence, $\mathbf{B}_1$ is connected. $\square$

Let $\mathbf{B} = (X, P, B)$ be a ball structure. For all $x \in X$, $\alpha \in P$, put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}.$$

A ball structure $\mathbf{B}^* = (X, P, B^*)$ is called *dual* to $\mathbf{B}$. Note that $\mathbf{B}^{**} = \mathbf{B}$.

A ball structure $\mathbf{B}$ is called *symmetric* if the identity mapping $i : X \to X$ is an isomorphism between $\mathbf{B}$ and $\mathbf{B}^*$. In other words, $\mathbf{B}$ is symmetric if, for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x, \alpha) \subseteq B^*(x, \beta)$ for every $x \in X$, and vice versa.

Lemma 3. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be ball structures,* $f : X_1 \to X_2$*. If* f *is a* $\prec$*-mapping of* $\mathbf{B}_1$ *to* $\mathbf{B}_2$*, then* f *is a* $\prec$*-mapping of* $\mathbf{B}_1^*$ *to* $\mathbf{B}_2^*$*. If* f *is an isomorphism between* $\mathbf{B}_1$ *and* $\mathbf{B}_2$*, then* f *is an isomorphism between* $\mathbf{B}_1^*$ *and* $\mathbf{B}_2^*$*.*

Proof. Let f be a $\prec$-mapping of $\mathbf{B}_1$ to $\mathbf{B}_2$ and let $\alpha \in P_1$. Choose $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X_1$. Take any element $y \in B_1^*(x, \alpha)$. Then $x \in B_1(y, \alpha)$ and $f(x) \in B_2(f(y), \beta)$. Hence, $f(y) \in B_2^*(f(x), \beta)$ and $f(B_1^*(x, \alpha)) \subseteq B_2^*(f(x), \beta)$. It means that f is a $\prec$-mapping of $\mathbf{B}_1^*$ to $\mathbf{B}_2^*$.

Suppose that f is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. By the first statement, f is a $\prec$-mapping of $\mathbf{B}_1^*$ to $\mathbf{B}_2^*$ and f^{-1} is a $\prec$-mapping of $\mathbf{B}_2^*$ to $\mathbf{B}_1^*$. It follows that f is an isomorphism between $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$. □

Lemma 4. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is symmetric, then* $\mathbf{B}_2$ *is symmetric.*

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Denote by $i_1 : X_1 \to X_1$ and $i_2 : X_2 \to X_2$ the identity mappings. Clearly, f^{-1} is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_1$. By Lemma 3, f is an isomorphism between $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$. By assumption, i_1 is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_1^*$. Since $i_2 = f i_1 f^{-1}$, then i_2 is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_2^*$. □

A ball structure $\mathbf{B} = (X, P, B)$ is called *multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma(\alpha, \beta) \in P$ such that

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$$

for every $x \in X$. Here, $B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha)$ for any $A \subseteq X$, $\alpha \in P$.

Lemma 5. *If a ball structure* $\mathbf{B} = (X, P, B)$ *is multiplicative, then* $\mathbf{B}^*$ *is multiplicative.*

Proof. Given any $\alpha, \beta \in P$, choose $\gamma(\alpha, \beta)$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$. Take any element $z \in B^*(B^*(x, \alpha), \beta)$ and pick $y \in B^*(x, \alpha)$ such that $z \in B^*(y, \beta)$. Then $x \in B(y, \alpha)$ and $y \in B(z, \beta)$, so $x \in B(B(z, \beta), \alpha)$. Since $B(B(z, \beta), \alpha) \subseteq B(z, \gamma(\beta, \alpha))$, then $x \in B(z, \gamma(\beta, \alpha))$. Hence, $B^*(B^*(x, \alpha), \beta) \subseteq B^*(x, \gamma(\beta, \alpha))$ and $\mathbf{B}^*$ is multiplicative. □

Lemma 6. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is multiplicative, then* $\mathbf{B}_2$ *is multiplicative.*

Proof. Denote by $f_1 : X_1 \to X_2$ the isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Fix any $\beta_1, \beta_2 \in P_2$. Since f is a bijection, it suffices to prove that there exists $\beta \in P_2$ such that

$$B_2(B_2(f(x), \beta_1), \beta_2) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$.

Since f is a $\succ$-mapping, then there exist $\alpha_1, \alpha_2 \in P_1$ such that

$$B_2(f(x), \beta_1) \subseteq f(B_1(x, \alpha_1)), B_2(f(x), \beta_2) \subseteq f(B_1(x, \alpha_2))$$

for every $x \in X_1$.

Since $\mathbf{B}_1$ is multiplicative, then there exists $\alpha \in P_1$ such that

$$B_1(B_1(x, \alpha_1), \alpha_2) \subseteq B_1(x, \alpha)$$

for every $x \in X_1$.

Since f is a $\prec$-mapping, then there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$.

Now fix $x \in X_1$ and take any element $f(z) \in B_2(B_2(f(x), \beta_1), \beta_2)$. Pick $f(y) \in B_2(f(x), \beta_1)$ with $f(z) \in B_2(f(y), \beta_2)$. Then $y \in B_1(x, \alpha_1)$, $z \in B_1(y, \alpha_2)$ and $z \in B_1(B_1(x, \alpha_1), \alpha_2)$. Hence, $z \in B_1(x, \alpha)$ and $f(z) \in B_2(f(x), \beta)$. □

For an arbitrary ball structure $\mathbf{B} = (X, P, B)$, we define a preodering $\leq$ on the set P by the rule

$$\alpha \leq \beta \ if \ and \ only \ if \ B(x, \alpha) \subseteq B(x, \beta)$$

for every $x \in X$. A subset P' of P is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$. A *cofinality* $cf\mathbf{B}$ of $\mathbf{B}$ is a minimum of cardinalities of cofinal subsets of P. Thus, $cf\mathbf{B} \leq \aleph_0$ if and only if there exists a cofinal sequence $< \alpha_n >_{n \in \omega}$ in P such that $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \ldots$.

Lemma 7. *If the ball structures* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *are isomorphic, then* $cf\mathbf{B}_1 = cf\mathbf{B}_2$.

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$ and let P_1' be a cofinal subset of P_1. Since f is a $\succ$-mapping, then there exists a mapping $h_1 : P_2 \to P_1'$ such that $B_2(f(x), \beta) \subseteq f(B_1(x, h_1(\beta)))$ for any $x \in X_1$, $\beta \in P_2$. Since f is a $\prec$-mapping, then there exists a mapping $h_2 : P_1' \to P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), h_2(\alpha))$ for any $x \in X_1$, $\alpha \in P_1'$. From the construction of h_1, h_2 we conclude that $h_2(P_1')$ is a cofinal subset of P_2. Hence, $cf\mathbf{B}_2 \leq cf\mathbf{B}_1$. □

Theorem 1. *A ball structure* $\mathbf{B} = (X, P, B)$ *is metrizable if and only if* $\mathbf{B}$ *is connected symmetric multiplicative and* $cf\mathbf{B} \leq \aleph_0$.

Proof. First suppose that $\mathbf{B}$ is isomorphic to $\mathbf{B}(X, d)$ for an appropriate metric space (X, d). Obviously, $\mathbf{B}(X, d)$ is connected symmetric multiplicative and $cf\mathbf{B} \leq \aleph_0$. By Lemma 1, 4, 6, 7 $\mathbf{B}$ has the same properties.

Now assume that $\mathbf{B}$ is connected symmetric multiplicative and $cf\mathbf{B} \leq \aleph_0$. Let $< \alpha_n >_{n\in\omega}$ be a cofinal sequence in P. Put $\beta_0 = \alpha_0$ and choose $\beta_1 \in P$ such that $\beta_1 \geq \alpha_1$, $\beta_1 \geq \beta_0$, $\beta_1 \geq \gamma(\beta_0, \beta_0)$, where γ is a function from definition of multiplicativity. Suppose that the elements $\beta_0, \beta_1, \dots, \beta_n$ have been chosen. Take $\beta_{n+1} \in P$ such that

$$\beta_{n+1} \geq \alpha_{n+1}, \beta_{n+1} \geq \beta_n, \beta_{n+1} \geq \gamma(\beta_i, \beta_j)$$

for all $i, j \in \{0, 1, \dots, n\}$. Then $< \beta_n >_{n\in\omega}$ is a nondecreasing cofinal sequence in P and $B(B(x, \beta_n), \beta_m) \subseteq B(x, \beta_{n+m})$ for all $x \in X$, $n, m \in \mathbf{N}$.

Define a mapping $d : X \times X \to \omega$ by the rule $d(x, x) = 0$ and

$$d(x, y) = min\{n \in \mathbf{N} : y \in B(x, \beta_n), x \in B(y, \beta_n)\}$$

for all distinct elements $x, y \in X$. Since the sequence $< \beta_n >_{n\in\omega}$ is cofinal in P and $\mathbf{B}$ is connected, then the mapping d is well defined. To show that d is a metric we have only to check a triangle inequality. Let x, y, z be distinct elements of X and let $d(x, y) = n$, $d(y, z) = m$. Since $y \in B(x, \beta_n)$ and $z \in B(y, \beta_m)$, then $z \in B(B(x, \beta_n), \beta_m) \subseteq B(x, \beta_{n+m})$. Since $y \in B(z, \beta_m)$ and $x \in B(y, \beta_n)$, then $x \in B(B(z, \beta_m), \beta_n) \subseteq B(z, \beta_{n+m})$. Hence, $d(x, z) \leq n + m$.

Consider the ball structure $\mathbf{B}(X, d)$ and note that

$$B_d(x, n) = B(x, \beta_n) \bigcap B^*(x, \beta_n).$$

Since $\mathbf{B}$ is symmetric, then the identity mapping of X is an isomorphism between $\mathbf{B}$ and $\mathbf{B}(X, d)$. □

Remark 1. *A metric d on a set X is called integer if $d(x, y)$ is an integer number for all $x, y \in X$. It follows from the proof of Theorem 1 that, for every metrizable ball structure* $\mathbf{B} = (X, P, B)$, *there exists an integer metric d on X such that* $\mathbf{B}$ *and* $\mathbf{B}(X, d)$ *are isomorphic.*

Remark 2. *Let* $\mathbf{B} = (X, P, B)$ *be an arbitrary ball structure. Consider a metric d on X defined by the rule $d(x, x) = 0$ and $d(x, y) = 1$ for all distinct elements of X. Then the identity mapping $i : X \to X$ is a $\prec$-mapping of* $\mathbf{B}$ *onto* $\mathbf{B}(X, d)$. *In particular, for every ball structure* $\mathbf{B}$, *there exists a metric space (X, d) such that* $\mathbf{B} \prec \mathbf{B}(X, d)$.

Remark 3. *Let* $\mathbf{B} = (X, P, B)$ *be a connected multiplicative ball structure,* $cf\mathbf{B} \leq \aleph_0$. *Repeating arguments of Theorem 1, we can prove that there exists a metric* d *on* X *such that the identity mapping* $i : X \to X$ *is a* $\prec$*-mapping of* $\mathbf{B}(X, d)$ *onto* $\mathbf{B}$.

Question 1. *Characterize the ball structure* $\mathbf{B} = (X, P, B)$*, which admit a metric* d *on* X *such that the identity mapping* $i : X \to X$ *is a* $\prec$*-mapping of* $\mathbf{B}(X, d)$ *onto* $\mathbf{B}$.

By Remark 2, every ball structure can be strengthened to some mertizable ball structure, so Question 1 asks about ball structure, which can be weekened to metrizable.

Example 2. *Let* $Gr = (V, E)$ *be a connected graph with a set of vertices* V *and a set of edges* E, $E \subseteq V \times V$. *Endow* V *with a path metric* d, *where* $d(x, y)$, $x, y \in V$ *is a length of the shortest path between* x *and* y. *Denote by* $\mathbf{B}(Gr)$ *the ball structure* $\mathbf{B}(V, d)$. *Obviously,* $\mathbf{B}(Gr)$ *is metrizable.*

Our next target is a description of the ball structures, isomorphic to $\mathbf{B}(Gr)$ for an appropriate graph Gr.

Let $\mathbf{B} = (X, P, B)$ be an arbitrary ball structure, $\alpha \in P$. We say that a finite sequence $x_0, x_1, \dots, x_n$ of elements of X is an α-path of length n if $x_{i-1} \in B(x_i, \alpha)$, $x_i \in B(x_{i-1}, \alpha)$ for every $i \in \{1, 2, \dots, n\}$. A ball structure $\mathbf{B}$ is called an *α-path connected* if, for every $\beta \in P$, there exists $\mu(\beta) \in \omega$ such that $x \in B(y, \beta)$, $y \in B(x, \beta)$ imply that there exists an α-path of length $\leq \mu(\beta)$ between x and y. Note that $\mathbf{B}(Gr)$ is 1-path connected for every connected graph Gr.

A ball structure $\mathbf{B} = (X, P, B)$ is called *path connected* if $\mathbf{B}$ is α-path connected for some $\alpha \in P$.

Lemma 8. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is path connected, then* $\mathbf{B}_2$ *path connected.*

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Choose $\alpha \in P_1$ such that $\mathbf{B}_1$ is α-path connected and fix a corresponding mapping $\mu : P_1 \to \omega$. Since f is a $\prec$-mapping, then there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. Since f is a $\succ$-mapping, then there exists a mapping $h : P_2 \to P_1$ such that

$$B_2(f(x), \lambda) \subseteq f(B_1(x, h(\lambda)))$$

for any $x \in X_1, \lambda \in P_2$.

Fix any $\lambda \in P_2$ and suppose that

$$f(x) \in B_2(f(y), \lambda),\ f(y) \in B_2(f(x), \lambda).$$

Since f is a bijection, then $x \in B_1(y, h(\lambda))$, $y \in B_1(x, h(\lambda))$. Since $\mathbf{B}_1$ is α-path connected, then there exists an α-path $x = x_0, x_1, \dots, x_m = y$ of length $\leq \mu(h(\lambda))$. Then $f(x) = f(x_0), f(x_1), \dots, f(x_m) = f(y)$ is a β-path of length $\leq \mu(h(\lambda))$ between $f(x)$ and $f(y)$. □

Theorem 2. *For every ball structure* $\mathbf{B}$*, the following statements are equivalent*

(i) $\mathbf{B}$ *is metrizable and path connected;*

(ii) $\mathbf{B}$ *is isomorphic to a ball structure* $\mathbf{B}(Gr)$ *for some connected graph* Gr*.*

Proof. (ii)⇒(i). Clearly, $\mathbf{B}(Gr)$ is metrizable and path connected. Hence, $\mathbf{B}$ is metrizable and path connected by Lemma 8.

(i)⇒(ii). Fix a path connected metric space (X, d) such that $\mathbf{B}$ is isomorphic to $\mathbf{B}(X, d)$. Then there exists $m \in \omega$ such that (X, d) is m-path connected. Consider a graph $Gr = (X, E)$ with the set E of edges defined by the rule

$$(x, y) \in E \ if \ and \ only \ if \ x \neq y \ and \ d(x, y) \leq m.$$

Since $\mathbf{B}(X, d)$ is path connected, then the graph Gr is connected.

Let d' be a path metric on the graph Gr. By assumption, for every $n \in \omega$, there exists $\mu(n) \in \omega$ such that $d(x, y) \leq n$ implies that there exists a m-path of length $\leq \mu(n)$ in (X, d) between x and y. Hence, $d(x, y) \leq n$ implies $d'(x, y) \leq \mu(n)$. On the other side, $d'(x, y) \leq k$ implies that $d(x, y) \leq km$. Therefore, the identity mapping of X is an isomorphism between the ball structures $\mathbf{B}(X, d)$ and $\mathbf{B}(Gr)$. □

Example 3. *Let* $X = \{2^n : n \in \omega\}$, $d(x, y) = |x - y|$ *for any* $x, y \in X$*. By Theorem 2, there are no connected graphs* Gr *such that* $\mathbf{B}(X, d)$ *is isomorphic to* $\mathbf{B}(Gr)$*.*

Example 4. *Let* d *be an euclidean metric on* $\mathbf{R}^n$*. By Theorem 2, there exists a connected graph* $Gr_n = (\mathbf{R}^n, E_n)$ *such that* $\mathbf{B}(\mathbf{R}^n, d)$ *is isomorphic to* $\mathbf{B}(Gr_n)$*.*

By Remark 2, for every ball structure $\mathbf{B} = (X, P, B)$, there exists a connected graph $Gr = (X, E)$, $E = \{(x, y) : x, y \in X, x \neq y\}$ such that the identity mapping $i : X \to X$ is a $\succ$-mapping of $\mathbf{B}(Gr)$ onto $\mathbf{B}$.

Question 2. *Characterize the ball structure, which admit a* $\succ$*-bijection to the ball structure* $\mathbf{B}(Gr)$ *for an appropriate graph* Gr*.*

A metric d on a set X is called *non-Archimedian* if

$$d(x,z) \leq max\{d(x,y), d(y,z)\}$$

for all $x, y, z \in X$. The following definitions will be used to describe the ball structures isomorphic to $\mathbf{B}(X,d)$ for an appropriate non-Archimeian metric space (X,d).

Let $\mathbf{B} = (X,P,B)$ be an arbitrary ball structure, $x \in X$, $\alpha \in P$. We say that a ball $B(x,\alpha)$ is a *cell* if $B(y,\alpha) = B(x,\alpha)$ for every $y \in B(x,\alpha)$. If (X,d) is a non-Archimedian metric space, then each ball $B(x,r)$, $x \in X$, $r \in \mathbf{R}^+$ is a cell.

Given any $x \in X$, $\alpha \in P$, denote

$$B^c(x,\alpha) = \{y \in X : \textit{ there exists an } \alpha - \textit{path between } x \textit{ and } y\}.$$

A ball structure $\mathbf{B}^c = (X,P,B^c)$ is called a *cellularization* of $\mathbf{B}$. Note that each ball $B^c(x,\alpha)$ is a cell.

We say that a ball structure $\mathbf{B}$ is *cellular* if the identity mapping $i : X \to X$ is an isomorphism between $\mathbf{B}$ and $\mathbf{B}^c$. In other words, $\mathbf{B}$ is cellular if and only if, for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x,\alpha) \subseteq B^c(x,\beta)$ for every $x \in X$ and, for every $\beta \in P$, there exists $\alpha \in P$ such that $B^c(x,\beta) \subseteq B(x,\alpha)$ for every $x \in X$.

A ball structure $\mathbf{B} = (X,P,B)$ is called *directed* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$.

Lemma 9. *If $\mathbf{B} = (X,P,B)$ is a directed symmetric ball structure, then the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}$ onto $\mathbf{B}^c$.*

Proof. Given any $\alpha \in P$, choose $\beta, \gamma \in P$ such that

$$B(x,\alpha) \subseteq B^*(x,\beta) \subseteq B(x,\gamma)$$

for every $x \in X$. Since $\mathbf{B}$ is directed, we may assume that $\beta \leq \gamma$. Take any element $y \in B(x,\alpha)$. Then $x \in B(y,\beta) \subseteq B(y,\gamma)$. Thus, $y \in B(x,\gamma)$, $x \in B(y,\gamma)$. Hence, there exists a β-path of length ≤ 1 between x and y. It means that $y \in B^c(x,\gamma)$, so $B(x,\alpha) \subseteq B^c(x,\gamma)$. □

Lemma 10. *Let $\mathbf{B}_1 = (X_1,P_1,B_1)$ and $\mathbf{B}_2 = (X_2,P_2,B_2)$ be ball structures. If $f : X_1 \to X_2$ is a $\prec$-mapping of $\mathbf{B}_1$ to $\mathbf{B}_2$, then f is a $\prec$-mapping of $\mathbf{B}_1^c$ to $\mathbf{B}_2^c$. If f is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$, then f is a isomorphism between $\mathbf{B}_1^c$ and $\mathbf{B}_2^c$.*

Proof. Given any $\alpha \in P_1$, choose $\beta \in P_2$ such that $f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta)$ for every $x \in X$. Take any $y \in B_1^c(x,\alpha)$ and choose an α-path $x = x_0, x_1, \dots, x_n = y$ between x and y. Then

$$f(x) = f(x_0), f(x_1), \dots, f(x_n) = f(y)$$

is a β-path between $f(x)$ and $f(y)$. Hence, $f(y) \in B_2^c(f(x), \beta)$ and $f(B_1^c(x, \alpha)) \subseteq B_2^c(f(x), \beta)$ for every $x \in X_1$.

Suppose that f is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. By the first statement, f is a $\prec$-mapping of $\mathbf{B}_1^c$ to $\mathbf{B}_2^c$ and f^{-1} is a $\prec$-mapping of $\mathbf{B}_2^c$ to $\mathbf{B}_1^c$. Hence, f is an isomorphism between $\mathbf{B}_1^c$ and $\mathbf{B}_2^c$. □

Lemma 11. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is cellular, then* $\mathbf{B}_2$ *is cellular.*

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Denote by $i_1 : X_1 \to X_1$ and $i_2 : X_2 \to X_2$ the identity mappings. Clearly, f^{-1} is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_1$. By the Lemma 10, f is an isomorphism between $\mathbf{B}_1^c$ and $\mathbf{B}_2^c$. By assumption, i_1 is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_1^c$. Since $i_2 = f i_1 f^{-1}$, then i_2 is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_2^c$. □

Theorem 3. *For every ball structure* $\mathbf{B}$*, the following statements are equivalent*

(i) $\mathbf{B}$ *is metrizable and cellular;*

(ii) there exists a non-Archimedian metric space (X,d) such that $\mathbf{B}$ *is isomorphic to* $\mathbf{B}(X, d)$.

Proof. (ii)⇒(i). Clearly, $\mathbf{B}(X, d)$ is metrizable and cellular. Hence, $\mathbf{B}$ is metrizable and cellular by Lemma 11.

(i)⇒(ii). Fix a metric space (X, d') such that $\mathbf{B}(X, d')$ is cellular and isomorphic to $\mathbf{B}$. Define a mapping $d : X \times X \to \omega$ by the rule

$$d(x, y) = min\{m \in \omega : y \in B^c(x, m)\}.$$

Obviously, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let $x, y, z \in X$ and let $d(x, y) = m$, $d(y, z) = n$, $m \le n$. Then $y \in B^c(x, m)$, $z \in B^c(y, n)$. It follows that there exists a n-path between x and z. Hence, $z \in B^c(x, n)$ and $d(x, z) \le n$. Thus, we have proved that d is a non-Archimedian metric on X.

Since $d(x, y) \le d'(x, y)$, then the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}(X, d)$ to $\mathbf{B}(X, d')$. Since $\mathbf{B}(X, d')$ is cellular, then there exists a mapping $h : \omega \to \omega$ such that $B^c(x, m) \subseteq B(x, h(m))$ for all $x \in X, m \in \omega$. Hence, i is a $\succ$-mapping of $\mathbf{B}(X, d)$ to $\mathbf{B}(X, d')$. Hence, $\mathbf{B}(X, d)$ and $\mathbf{B}(X, d')$ are isomorphic. □

By Remark 2, for every ball structure $\mathbf{B} = (X, P, B)$, there exists a non-Archimedian metric d on X such that the identity mapping of X is a $\succ$-mapping of $\mathbf{B}(X, d)$ to $\mathbf{B}$.

Lemma 12. *For every metric space (X, d), there exists a family $\{\mathcal{P}_n : n \in \omega\}$ of partitions of X with the following properties*

(i) every partition $\mathcal{P}_{n+1}$ is an enlargement of $\mathcal{P}_n$, i.e. every cell of the partition $\mathcal{P}_{n+1}$ is a union of some cells of the partition $\mathcal{P}_n$;

(ii) there exists a function $f : \omega \to \omega$ such that, for every $C \in \mathcal{P}_n$ and every $x \in C$, $C \subseteq B(x, f(n))$;

(iii) for any $x, y \in X$, there exists $n \in \omega$ such that x, y are in the same cell of the partition $\mathcal{P}_n$.

Proof. Fix any well-ordering $\{x_\alpha : \alpha < \gamma\}$ of X. Choose a subset $Y_0 \subseteq X$, $x_0 \in Y_0$ such that the family $\{B(y, 1) : y \in Y_0\}$ is disjoint and maximal. For every $x \in X$, pick a minimal element $f_0(x) \in Y_0$ such that $B(x, 1) \bigcap B(f_0(x), 1) \neq \emptyset$. Put $H(x, 1) = \{z \in X : f_0(z) = f_0(x)\}$ and note that the family $\{H(y, 1) : y \in Y_0\}$ is a partition of X. If $x, z \in H(y, 1)$, then $d(x, y) \leq 2$, $d(x, z) \leq 2$. Therefore, $H(y, 1) \subseteq B(x, 4)$ for every $x \in H(y, 1)$. Put $\mathcal{P}_0 = \{H(y, 1) : y \in Y_0\}$, $f(0) = 4$.

Assume that the partitions $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{n-1}$ have been constructed and the values $f(0)$, $f(1)$, …, $f(n-1)$ have been determined. Choose a subset $Y_n \subseteq X$, $x_0 \in Y_n$ such that the family $\{B(y, n+1) : y \in Y_n\}$ is disjoint and maximal. Define a mapping $f_n : X \to Y_n$ inductively such that f_n is constant on each cell of the partition $\mathcal{P}_{n-1}$. Put $f_n(x) = x_0$ for every $x \in X$ such that $H(x, n) \bigcap B(x_0, n+1) \neq \emptyset$. Then take the minimal element $x \in X$ such that $f_n(x)$ is not determined. Choose the minimal element $y \in Y_n$ such that $B(x, n+1) \bigcap B(y, n+1) \neq \emptyset$. Put $f_n(x) = y$ and $f_W(z) = y$ for every $z \in H(x, n)$. After this transfinite procedure, we denote $H(x, n+1) = \{z \in X : f_n(z) = f_n(x)\}$. Put $\mathcal{P}_n = \{H(y, n+1) : y \in Y_n\}$. Then $\mathcal{P}_n$ is a partition of X and each cell of $\mathcal{P}_n$ is a union of some cells of $\mathcal{P}_{n-1}$. Thus, (i) is satisfied.

If $z \in H(y, n+1)$, then $d(z, y) \leq f(n-1) + 2(n+1)$. Hence, to satisfy (ii), put $f(n) = 2(f(n-1) + 2(n+1))$.

At last, given any $x, y \in X$, choose $m \in \omega$ such that $d(x_0, x) \leq m+1$, $d(x_0, y) \leq m+1$. Thus x, y are in the same cell of the partition $\mathcal{P}_m$ and we have verified (iii). □

Theorem 4. *For every metric space (X,d), there exists a non-Archimedian metric d' on X such that the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}(X, d')$ to $\mathbf{B}(X, d)$.*

Proof. Fix a family $\{\mathcal{P}_n : n \in \omega\}$ of partitions of X, satisfying (i), (ii), (iii) from Lemma 12. Define a mapping $d' : X \times X \to \omega$ by the rule

$$d'(x, y) = min\{n : x \text{ and } y \text{ are in the same cell of } \mathcal{P}_n\}.$$

By (iii), d' is well defined. By (i), d' is a non-Archimedian metric. By (ii), the identity mapping of X is a $\prec$-mapping of $\mathbf{B}(X, d')$ onto $\mathbf{B}(X, d)$. $\square$

Now we consider non-metrizable versions of Lemma 12 and Theorem 4.

Lemma 13. *Let $\mathbf{B} = (X, P, B)$ be a directed symmetric multiplicative ball structure. Then there exists a family $\{\mathcal{P}_\alpha : \alpha \in P\}$ of partitions of X such that*

(i) for every $\alpha \in P$, there exists $\beta \in P$ such that $C \subseteq B(x, \beta)$ for every $C \in \mathcal{P}_\alpha$ and every $x \in C$.

Moreover, if $\mathbf{B}$ is connected then

(ii) for any $x, y \in X$, there exists $\alpha \in P$ such that x, y are in the same cell of the partition $\mathcal{P}_\alpha$.

Proof. Fix any well-ordering of X and denote by x_0 its minimal element. Fix $\alpha \in P$ and choose a subset $Y \subseteq X$, $x_0 \in Y$ such that the family $\{B(y, \alpha) : y \in Y\}$ is disjoint and maximal. For every $x \in X$, pick a minimal element $f(x) \in Y$ such that $B(x, \alpha) \bigcap B(f(x), \alpha) \neq \emptyset$. Put $H(x, \alpha) = \{z \in X : f(z) = f(x)\}$. Then the family $\mathcal{P}_\alpha = \{H(y, \alpha) : y \in Y\}$ is a partition of X.

Since $\mathbf{B}$ is directed and symmetric, then there exists $\alpha' > \alpha$ such that $y \in B(x, \alpha)$ implies $x \in B(y, \alpha')$.

Fix $x \in X$ and take $x' \in B(x, \alpha) \bigcap B(f(x), \alpha)$. Then x, x', $f(x)$ is an α'-path. Hence, for every $z \in H(x, \alpha)$, we can find an α'-path of length 4 between x and z. Using multiplicativity of $\mathbf{B}$, choose $\beta \in P$ such that $y_4 \in B(y_0, \beta)$ for every α'-path y_0, y_1, y_2, y_3, y_4 in X. Then $H(x, \alpha) \subseteq B(x, \beta)$.

Suppose that $\mathbf{B}$ is connected and $x, y \in X$. Since $\mathbf{B}$ is directed, then there exists $\alpha \in P$ such that $x_0 \in B(x, \alpha)$, $x_0 \in B(y, \alpha)$. Hence, x, y belong to the cell $H(x_0, \alpha)$ of the partition $\mathcal{P}_\alpha$. $\square$

Theorem 5. *If a ball structure $\mathbf{B} = (X, P, B)$ is directed symmetric and multiplicative, then there exists a cellular ball structure $\mathbf{B}' = (X, P, B')$ such that the identity mapping of X is a $\prec$-mapping of $\mathbf{B}'$ onto $\mathbf{B}$. Moreover, if $\mathbf{B}$ is connected, then $\mathbf{B}'$ is connected.*

Proof. Use the family of the partitions $\{\mathcal{P}_\alpha : \alpha \in P\}$ from Lemma 13 and put $B'(x, \alpha) = H(x, \alpha)$. Clearly, each ball $B'(x, \alpha)$ is a cell. By (i), the identity mapping of X is a $\prec$-mapping of $\mathbf{B}'$ onto $\mathbf{B}$. If $\mathbf{B}$ is connected, then $\mathbf{B}'$ is connected by (ii). $\square$

Example 5. *Let G be a group and let $Fin_e(G)$ be a family of all finite subsets of G containing the identity e. Given any $g \in G$, $F \in Fin_e(G)$, put $B(g,F) = Fg$. A ball structure $\mathbf{B}(G) = (G, Fin_e(G), B)$ is denoted by $\mathbf{B}(G)$. It is easy to show, that $\mathbf{B}(G)$ is directed connected symmetric and multiplicative.*

Now we apply the above results to the ball structures of groups.

Theorem 6. *Let G be a group. Then a ball structure $\mathbf{B}(G)$ is metrizable if and only if $|G| \leq \aleph_0$.*

Proof. Apply Theorem 1. □

Theorem 7. *For every group G, the following statements are equivalent*

(i) G is finitely generated;

(ii) $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(Gr)$ for some connected graph Gr

Proof. (i)⇒(ii). Let S be a finite set of generators of G. Consider a Cayley graph $Gr = (G, E)$ of G determined by S. By definition, $(x, y) \in E$ if and only if $x \neq y$ and $x = ty$ for some $t \in S \bigcup S^{-1}$. Clearly, the identity mapping of G is an isomorphism between $\mathbf{B}(G)$ and $\mathbf{B}(Gr)$.

(ii)⇒(i). By Theorem 2, there exists $F \in Fin$ such that $\mathbf{B}(G)$ is F-path connected. In particular, for every $g \in G$, there exists a F-path between e and g. Hence, F generates G. □

A group G is called *locally finite* if every finite subset of G generates a finite subgroup.

Theorem 8. *Let G be a group. Then a ball structure $\mathbf{B}(G)$ is cellular if and only if G is locally finite.*

Proof. Let G be locally finite. Denote by Fin_s the family of all finite subgroups of G. Then Fin_s is cofinal in Fin and each ball $B(g,F)$, $F \in Fin_s$ is a cell. Hence, $\mathbf{B}(G)$ is cellular.

Assume that $\mathbf{B}(G)$ is cellular. Note that $B^c(e,F) = gpF$ for every $F \in Fin$, where gpF is a subgroup of G generated by F. Since $\mathbf{B}$ is isomorphic to $\mathbf{B}^c$, then each ball $B^c(g,F)$ is finite. In particular, gpF is finite for every $F \in Fin$. □

Remark 4. *Let G_1, G_2 be countable locally finite group. By [2, Theorem 4], $\mathbf{B}(G_1) \succ \mathbf{B}(G_2)$ and $\mathbf{B}(G_1) \prec \mathbf{B}(G_2)$. By [2, Theorem 5], $\mathbf{B}(G_1)$ and $\mathbf{B}(G_2)$ are isomorphic if and only if, for every finite subgroup F of G_1, there exists a finite subgroup H of G_2 such that $|F|$ is a divisor of $|H|$, and vice versa. A problem of classification up to an isomorphism of ball structures of uncountable locally finite groups is open.*

Theorem 9. *For every countable group G, there exists a non-Archimedian metric d on G with the following property*

(i) for each $n \in \omega$, there exists $F \in Fin$ such that $d(x, y) \leq n$ implies $x \in Fy$.

Proof. Apply Theorem 6 and Theorem 4. □

Theorem 10. *For every group G, there exists a cellular ball structure $\mathbf{B}' = (G, Fin, B')$ such that the identity mapping of G is a $\prec$-mapping of $\mathbf{B}'$ onto $\mathbf{B}(G)$.*

Proof. Apply Theorem 5. □

Question 3. *Characterize the ball structures isomorphic to the ball structures of groups.*

M.Zarichnyi has pointed out that Theorem 1 has a counterpart in the asymptotic topology [3]. This theorem answers the Open Question 1 from [4]. The results of this paper was announced in [5].

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