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RESEARCH ARTICLE

A decomposition theorem for semiprime rings

Marina Khibina

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Abstract. A ring A is called an FDI-ring if there exists a decomposition of the identity of A in a sum of finite number of pairwise orthogonal primitive idempotents. We call a primitive idempotent e artinian if the ring eAe is Artinian. We prove that every semiprime FDI-ring is a direct product of a semisimple Artinian ring and a semiprime FDI-ring whose identity decomposition doesn't contain artinian idempotents.

1. Introduction

In this paper all rings are associative with $1 \neq 0$. Recall that a nonzero idempotent $e \in A$ is called *local* if the ring eAe is local. Obviously, every local idempotent is primitive. The well-known Müller's Theorem [4] gives the following criterion for a ring A to be semiperfect:

A ring is semiperfect if and only if $1 \in A$ can be decomposed into a sum of a finite number of pairwise orthogonal local idempotents.

For every associative ring A with $1 \neq 0$ we prove the theorem:

The following statements for a ring A are equivalent:

(1) *the idempotent $e \in A$ is local*;

(2) *the projective module $P = eA$ has exactly one maximal submodule.*

The following important notion used in the paper is the notion of *finitely decomposable identity ring* (or for short, *FDI-ring*, see [2], p. 77):

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a ring A is called an FDI-ring if there exists a decomposition of the identity $1 \in A$

$$1 = e_1 + e_2 + \ldots + e_n$$

into a sum of finite number of pairwise orthogonal primitive idempotents $e_1, \ldots, e_n$. Obviously, every semiperfect ring and every right Noetherian ring is a FDI-ring.

We call a FDI-ring A piecewise right Artinian if all rings e_iAe_i are right Artinian for $i = 1, \ldots, n$.

We prove that every semiprime FDI-ring A is a direct product of a semisimple Artinian ring and an FDI-ring which is not piecewise right Artinian.

The main working tool of this paper is the notion of a minor of the ring A: *Let A be a ring, P a finitely generated projective A-module which is a direct sum of n indecomposable modules. The ring of endomorphisms $B = E(P)$ of the module P is called a minor of order n of the ring A* (see [1]).

Many properties carry over from the ring to all of its minors. Following [1] we shall say that a property Φ of a ring A is N-minoral property if and only if all its minors whose orders are not greater than a prescribed value N have this property Φ.

The following examples are given in [1].

Example 1.1. An Artinian ring A is semisimple if and only if for any two indecomposable projective A-modules $P_1 \not\simeq P_2$, $Hom_A(P_1, P_2) = 0$ and $Hom_A(P_1, P_1)$ is a division ring. Therefore semisimplisity is a 2-minoral property.

Example 1.2. An Artinian ring A is generalized uniserial (i.e., Artinian serial) if and only if for any indecomposable projective A-modules P_1, P_2, P_3 and for any homomorphisms $\varphi_1 : P_1 \to P_3$ and $\varphi_2 : P_2 \to P_3$, one of the equations: $\varphi_1 = \varphi_2 x$ or $\varphi_2 = \varphi_1 y$ is solvable, where $x : P_1 \to P_2$ and $y : P_1 \to P_3$. Therefore, the property of being generalized uniserial is 3-minoral.

Example 1.3. The property of being hereditary for an order Λ in a semisimple k-algebra $\tilde{\Lambda}$ is 2-minoral.

On other hand, an analogous notion is defined in the paper [3]:

Let $\mathcal{C}$ be a class of rings, and $\mathcal{P}$ a property that rings in $\mathcal{C}$ may or may not have. We say that $\mathcal{P}$ **is k-determined in** $\mathcal{C}$ if a ring Λ in $\mathcal{C}$ has $\mathcal{P}$ if and only if all $e\Lambda e$ have $\mathcal{P}$, for e a sum of at most k pairwise orthogonal primitive idempotents of Λ.

The following two properties are proved in [3].

Proposition 1.4. *The property of being left serial is three-determined in the class of Artinian rings.*

Proposition 1.5. *The property of being hereditary is two-determined in the class $\mathcal{C}$ of orders over complete discrete valuation rings.*

2. Projective modules

Let M be an A-module. We set $rad\, M = M$, M has no maximal submodules, and otherwise, $rad\, M$ denotes the intersection of all maximal submodules of M. We write $R = R(A) = rad\, A_A$, s the Jacobson radical of A.

The following proposition is well-known (see, for example, [2], Proposition 4.2.10, p. 115).

Proposition 2.1. *If P is a nonzero projective A-module, then $rad\, P = P \cdot rad\, A \neq P$.*

Theorem 2.2. *Suppose that $P = eA$ $(e^2 = e \neq 0)$ has exactly one maximal submodule. Then the idempotent e is local. Conversely, if e is a local idempotent and $P = eA$, then PR is the unique maximal submodule of P.*

Proof. Suppose that $P = eA$ has exactly one maximal submodule M. Then by Proposition 2.1 $M = PR$. For any $\varphi : P \to P$ either $Im\, \varphi = P$ or $Im\, \varphi \subseteq PR$.

In the first case, since P is projective, we have $P \simeq Im\, \varphi \oplus Ker\, \varphi$ which implies $Ker\, \varphi = 0$. So, φ is an automorphism.

In the second case φ is non-invertible. Obviously, all non-invertible elements of $Hom_A\,(P, P) \simeq eAe$ form an ideal and therefore the ring eAe is local.

Conversely, let e be a local idempotent of the ring A and $\pi : A \to \bar{A}$ be the natural epimorphism of A into $\bar{A} = A/R$ (R is the Jacobson radical of A). We denote $\pi(a) = \bar{a}$. Suppose $1 \neq e$. We have $1 = e + f$ and $ef = fe = 0$. Obviously, $\bar{f}\bar{A}$ is a proper right ideal in $\bar{A}$. So, it is contained in a maximal right ideal $\tilde{I}$ if $\bar{A}$. We will show that $\bar{e}\bar{A} \cap \tilde{I} = 0$, otherwise $(\bar{e}\bar{A} \cap \tilde{I})^2 \neq 0$.

Since $\bar{A}$ is a semiprimitive ring then $(\bar{e}\bar{A} \cap \tilde{I})^2 = 0$. There exists $\bar{e}\bar{a} \in \tilde{I}$ and $\bar{e}\bar{a}\bar{e}\bar{a} \neq 0$. So, $\bar{e}\bar{a}\bar{e} \neq 0$. Since eAe is a local ring and $rad\,(eAe) = eRe$, then $\bar{e}\bar{A}\bar{e}$ is a division ring. Therefore, there is an element $\bar{e}\bar{x}\bar{e} \in \bar{e}A\bar{e}$ such that $\bar{e}\bar{a}\bar{e}\bar{x}\bar{e} = \bar{e}$ and $\bar{e} \in \tilde{I}$. Thus $\bar{1} \in \tilde{I}$. We get a contradiction. Therefore $\bar{e}\bar{A} \cap \tilde{I} = 0$ and $\bar{A} = \bar{e}\bar{A} \oplus \tilde{I}$. Since $\tilde{I}$ is maximal ideal in $\bar{A}$ then $\bar{e}\bar{A}$ is simple and PR is the unique maximal submodule in $P = eA$. $\square$

Let A be an FDI-ring with the following decomposition of identity $1 \in A$:

$$1 = e_1 + \ldots + e_n.$$

We may assume that all rings $e_i A e_i$ are local for $i = 1, \ldots, k$ and the rings $e_j A e_j$ are non-local for $j = k+1, \ldots, n$. Put $e = e_1 + \ldots + e_k$ and $f = 1 - e$. Let $eAf = X$, $fAe = Y$ and

$$A = \begin{pmatrix} eAe & X \\ Y & fAf \end{pmatrix} \qquad (*)$$

be the corresponding two-sided Peirce decomposition of A. By Müller's Theorem the ring eAe is semiperfect.

We shall call the decomposition $(*)$ *standard two-sided Peirce decomposition of a FDI-ring A.*

3. Piecewise right Artinian semiprime rings are semisimple Artinian

Recall that a ring A is called *semiprime* if A does not contain nonzero nilpotent ideals. We shall need the following lemma.

Lemma 3.1. *Let e be a nonzero idempotent of a ring A. For any nilpotent ideal I of the ring eAe there exists a nilpotent ideal I of A such that $e\tilde{I}e = I$.*

Proof. Let $f = 1 - e$ and $\tilde{I} = I + IeAf + fAeI + fAeIeAf$. It is clear that $\tilde{I}$ is the nilpotent ideal. □

Corollary 3.2. *Let e be a nonzero idempotent of a semiprime ring A. Then the ring eAe is semiprime.*

Definition 3.3. *A ring A with the Jacobson radical R is called* semiprimary *if A/R is semisimple Artinian and R is nilpotent.*

Theorem 3.4. *A piecewise right Artinian ring A is semiprimary.*

Proof. Obviously, A is semiperfect. Let $1 = e_1 + \ldots + e_n$ be the decomposition of $1 \in A$ into the sum of a finite number of pairwise orthogonal local idempotents. Let $R = rad\, A_A$ be the Jacobson radical of A. Then $e_i R e_i = rad\,(e_i A e_i)$ is either zero or nilpotent. By induction on n it is easy to see, that R is a nilpotent ideal. So, A/R is semisimple Artinian and A is semiprimary. □

Example 3.5. Let

$$A = \left\{ \left(\begin{array}{cc} \alpha & \beta \\ 0 & \alpha \end{array} \right) \mid \alpha \in \mathbb{Q}, \beta \in \mathbb{R} \right\}.$$

Obviously, A is a local semiprimary ring which is not right or left Artinian.

This example shows that the converse of Theorem 3.4 is not true.

Proposition 3.6. *The property of being semiprimary is* 1*-minoral in the class of FDI-rings.*

Proof is analogous to the proof of Theorem 3.4.

Theorem 3.7. *A semiprimary semiprime ring A is semisimple Artinian.*

Proof. By definition of a semiprime ring we have that $R = 0$ and A is semisimple Artinian. □

Corollary 3.8. *Piecewise right Artinian semiprime ring is semisimple Artinian.*

4. A decomposition theorem for semiprime rings

Recall that a ring A is said *decomposable* if A is a direct product of two rings. Otherwise a ring A is called *indecomposable*.

Definition 4.1 ([2], p.74)**.** *A ring A is called* finitely decomposable *(or, for short, FD-*ring*) if it decomposes into a direct product of a finite number of indecomposable rings.*

Proposition 4.2 ([2], Corollary 2.5.15, p.77)**.** *Any FDI-ring is an FD-ring.*

Obviously, we have the following Proposition.

Proposition 4.3. *Let A be a semiprime FDI-ring. Then A is a finite direct product of semiprime indecomposable FDI-rings.*

We fix the decomposition of the identity $1 \in A$ (where A is an indecomposable semiprime FDI-ring) in a sum

$$1 = e_1 + \ldots + e_n$$

of a finite number of pairwise orthogonal primitive idempotents $e_1, \ldots, e_n$.

Definition 4.4. *A primitive idempotent e shall be called* artinian *if the ring eAe is Artinian.*

Theorem 4.5. *Let A be an indecomposable semiprime FDI-ring. The ring A is isomorphic to the ring $M_n(\mathcal{D})$ if and only if $e_i \in A$ is artinian for some i..*

Proof. Suppose that e_k is artinian and e_j is not artinian for $j > k$. Consider the following minor of the second order

$$B_{k,j} = \begin{pmatrix} e_kAe_k & e_kAe_j \\ e_jAe_k & e_kAe_k \end{pmatrix}$$

for $k > j$. Obviously, e_kAe_k is a division ring. Denote by $R_{k,j}$ the Jacobson radical of $B_{k,j}$. Let $P_1^{(k,j)} = e_kB_{k,j}$ and $P_2^{(k,j)} = e_jB_{k,j}$. By Theorem 2.2 $P_1^{(k,j)}R_{k,j}$ is the unique maximal submodule of $P_1^{(k,j)}$. So, we have:

$$P_1^{(k,j)}R_{k,j} \subset (0, e_kAe_j) \subset P_1^{(k,j)}.$$

Then each element $e_kae_j \in e_kAe_j$ defines a homomorphism $\varphi_k : P_2^{(k,j)} \to P_1^{(k,j)}$ such that $Im\, \varphi_{k,j} \subseteq P_1^{(k,j)}R_{k,j}$, i.e., $e_kae_je_{jh}a_1e_k = 0$ for any $a, a_1 \in A$. Therefore,

$$J = \begin{pmatrix} 0 & e_kAe_j \\ e_jAe_k & e_jAe_k \end{pmatrix}$$

is a nilpotent ideal in $B_{k,j}$. By Lemma 3.1 $e_kAe_j = 0$ and $e_jAe_k = 0$.

Let $h_1 = e_1 + \ldots + e_k$ and $h_2 = e_{k+1} + \ldots + e_n$, $X = hAh_2$ and $Y = h_2Ah_1$. Let

$$A = \begin{pmatrix} h_1Ah_1 & X \\ Y & h_2Ah_2 \end{pmatrix}$$

be the corresponding two-sided Peirce decomposition. As above we have $X = 0$ and $Y = 0$. It follows from indecomposability of A that A is the piecewise Artinian ring and by Theorem 3.7 $A \simeq M_n(\mathcal{D})$, where $M_n(\mathcal{D})$ is a ring of all $n \times n$-matrices with elements in a division ring A. The converse assertion is obvious. □

Corollary 4.6 (A decomposition theorem for semiprime rings)**.** *Every semiprime FDI-ring is a direct product of a semisimple Artinian ring and a semiprime FDI-ring whose identity decomposition doesn't contain artinian idempotents.*

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Contact information

M. Khibina In-t of Engineering Thermophysics, NAS, Ukraine
E-Mail: marina_khibina@yahoo.com

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