

Symbolic Rees algebras, vertex covers and irreducible representations of Rees cones

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Communicated by B. V. Novikov

ABSTRACT. Let G be a simple graph and let $I_c(G)$ be its ideal of vertex covers. We give a graph theoretical description of the irreducible b -vertex covers of G , i.e., we describe the minimal generators of the symbolic Rees algebra of $I_c(G)$. Then we study the irreducible b -vertex covers of the blocker of G , i.e., we study the minimal generators of the symbolic Rees algebra of the edge ideal of G . We give a graph theoretical description of the irreducible binary b -vertex covers of the blocker of G . It is shown that they correspond to irreducible induced subgraphs of G . As a byproduct we obtain a method, using Hilbert bases, to obtain all irreducible induced subgraphs of G . In particular we obtain all odd holes and antiholes. We study irreducible graphs and give a method to construct irreducible b -vertex covers of the blocker of G with high degree relative to the number of vertices of G .

Introduction

A clutter \mathcal{C} with vertex set $X = \{x_1, \dots, x_n\}$ is a family of subsets of X , called edges, none of which is included in another. The set of vertices and edges of \mathcal{C} are denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$ respectively. A basic example of a clutter is a graph. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K . The *edge ideal* of \mathcal{C} , denoted by $I(\mathcal{C})$, is the ideal of R generated

Partially supported by CONACyT grant 49251-F and SNI, México.

2000 Mathematics Subject Classification: 13F20, 05C75, 05C65, 52B20.

Key words and phrases: *edge ideal, symbolic Rees algebras, perfect graph, irreducible vertex covers, irreducible graph, Alexander dual, blocker, clutter.*

by all monomials $\prod_{x_i \in e} x_i$ such that $e \in E(\mathcal{C})$. The assignment $\mathcal{C} \mapsto I(\mathcal{C})$ establishes a natural one to one correspondence between the family of clutters and the family of square-free monomial ideals. Let \mathcal{C} be a clutter and let $F = \{x^{v_1}, \dots, x^{v_q}\}$ be the minimal set of generators of its edge ideal $I = I(\mathcal{C})$. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. The $n \times q$ matrix with column vectors v_1, \dots, v_q will be denoted by A , it is called the *incidence matrix* of \mathcal{C} .

The *blowup algebra* studied here is the *symbolic Rees algebra*:

$$R_s(I) = R \oplus I^{(1)}t \oplus \cdots \oplus I^{(i)}t^i \oplus \cdots \subset R[t],$$

where t is a new variable and $I^{(i)}$ is the i th symbolic power of I . Closely related to $R_s(I)$ is the *Rees algebra* of I :

$$R[It] := R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t].$$

The study of symbolic powers of edge ideals was initiated in [21] and further elaborated on in [1, 9, 12, 13, 14, 22, 28]. By a result of Lyubeznik [17], $R_s(I)$ is a K -algebra of finite type. In general the minimal set of generators of $R_s(I)$ as a K -algebra is very hard to describe in terms of \mathcal{C} (see [1]). There are two exceptional cases. If the clutter \mathcal{C} has the max-flow min-cut property, then by a result of [13] we have $I^i = I^{(i)}$ for all $i \geq 1$, i.e., $R_s(I) = R[It]$. If G is a perfect graph, then the minimal generators of $R_s(I(G))$ are in one to one correspondence with the cliques (complete subgraphs) of G [28]. We shall be interested in studying the minimal set of generators of $R_s(I)$ using polyhedral geometry. Let G be a graph and let $I_c(G)$ be the Alexander dual of $I(G)$, see definition below. Some of the main results of this paper are graph theoretical descriptions of the minimal generators of $R_s(I(G))$ and $R_s(I_c(G))$. In Sections 1 and 2 we show that both algebras encode combinatorial information of the graph which can be decoded using integral Hilbert bases.

The *Rees cone* of I , denoted by $\mathbb{R}_+(I)$, is the polyhedral cone consisting of the non-negative linear combinations of the set

$$\mathcal{A}' = \{e_1, \dots, e_n, (v_1, 1), \dots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the i th unit vector.

A subset $C \subset X$ is called a *vertex cover* of the clutter \mathcal{C} if every edge of \mathcal{C} contains at least one vertex of C . A subset $C \subset X$ is called a *minimal vertex cover* of the clutter \mathcal{C} if C is a vertex cover of \mathcal{C} and no proper subset of C is a vertex cover of \mathcal{C} . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal primes of the edge ideal $I = I(\mathcal{C})$ and let

$$C_k = \{x_i \mid x_i \in \mathfrak{p}_k\} \quad (k = 1, \dots, s)$$

be the corresponding minimal vertex covers of \mathcal{C} , see [27, Proposition 6.1.16]. Recall that the primary decomposition of the edge ideal of \mathcal{C} is given by

$$I(\mathcal{C}) = (C_1) \cap (C_2) \cap \cdots \cap (C_s),$$

where (C_k) denotes the ideal of R generated by C_k . In particular observe that the height of $I(\mathcal{C})$ equals the number of vertices in a minimum vertex cover of \mathcal{C} . This number is called the *vertex covering number* of \mathcal{C} and is denoted by $\alpha_0(\mathcal{C})$. The *ith symbolic power* of I is given by

$$I^{(i)} = S^{-1}I^i \cap R \text{ for } i \geq 1,$$

where $S = R \setminus \cup_{k=1}^s \mathfrak{p}_i$ and $S^{-1}I^i$ is the localization of I^i at S . In our situation the *ith symbolic power* of I has a simple expression:

$$I^{(i)} = \mathfrak{p}_1^i \cap \cdots \cap \mathfrak{p}_s^i,$$

see [27]. The Rees cone of I is a finitely generated rational cone of dimension $n + 1$. Hence by the finite basis theorem [30, Theorem 4.11] there is a unique irreducible representation

$$\mathbb{R}_+(I) = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap H_{\ell_2}^+ \cap \cdots \cap H_{\ell_r}^+ \quad (1)$$

such that each ℓ_k is in \mathbb{Z}^{n+1} , the non-zero entries of each ℓ_k are relatively prime, and none of the closed halfspaces $H_{e_1}^+, \dots, H_{e_{n+1}}^+, H_{\ell_1}^+, \dots, H_{\ell_r}^+$ can be omitted from the intersection. Here H_a^+ denotes the closed halfspace $H_a^+ = \{x \mid \langle x, a \rangle \geq 0\}$ and H_a stands for the hyperplane through the origin with normal vector a , where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. The *facets* (i.e., the proper faces of maximum dimension or equivalently the faces of dimension n) of the Rees cone are exactly:

$$F_i = H_{e_i} \cap \mathbb{R}_+(I), i = 1, \dots, n + 1, H_{\ell_1} \cap \mathbb{R}_+(I), \dots, H_{\ell_r} \cap \mathbb{R}_+(I).$$

According to [9, Lemma 3.1] we may always assume that $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for $1 \leq k \leq s$, i.e., each minimal vertex cover of \mathcal{C} determines a facet of the Rees cone and every facet of the Rees cone satisfying $\langle \ell_k, e_{n+1} \rangle = -1$ must be of the form $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for some minimal vertex cover C_k of \mathcal{C} . This is quite interesting because this is saying that the Rees cone of $I(\mathcal{C})$ is a carrier of combinatorial information of the clutter \mathcal{C} . Thus we can extract the primary decomposition of $I(\mathcal{C})$ from the irreducible representation of $\mathbb{R}_+(I(\mathcal{C}))$.

Rees cones have been used to study algebraic and combinatorial properties of blowup algebras of square-free monomial ideals and clutters [9, 12, 29]. Blowup algebras are interesting objects of study in algebra and geometry [25].

The ideal of *vertex covers* of \mathcal{C} is the square-free monomial ideal

$$I_c(\mathcal{C}) = (x^{u_1}, \dots, x^{u_s}) \subset R,$$

where $x^{u_k} = \prod_{x_i \in C_k} x_i$. Often the ideal $I_c(\mathcal{C})$ is called the *Alexander dual* of $I(\mathcal{C})$. The clutter $\Upsilon(\mathcal{C})$ associated to $I_c(\mathcal{C})$ is called the *blocker* of \mathcal{C} , see [5]. Notice that the edges of $\Upsilon(\mathcal{C})$ are precisely the minimal vertex covers of \mathcal{C} . If G is a graph, then $R_s(I_c(G))$ is generated as a K -algebra by elements of degree in t at most two [14, Theorem 5.1]. One of the main result of Section 1 is a graph theoretical description of the minimal generators of $R_s(I_c(G))$ (see Theorem 1.7). As an application we recover an explicit description [24], in terms of closed halfspaces, of the edge cone of a graph (Corollary 1.10).

The symbolic Rees algebra of the ideal $I_c(\mathcal{C})$ can be interpreted in terms of “ k -vertex covers” [14] as we now explain. Let $a = (a_1, \dots, a_n) \neq 0$ be a vector in \mathbb{N}^n and let $b \in \mathbb{N}$. We say that a is a b -*vertex cover* of I (or \mathcal{C}) if $\langle v_i, a \rangle \geq b$ for $i = 1, \dots, q$. Often we will call a b -vertex cover simply a b -*cover*. This notion plays a role in combinatorial optimization [20, Chapter 77, p. 1378] and algebraic combinatorics [14, 15].

The *algebra of covers* of I (or \mathcal{C}), denoted by $R_c(I)$, is the K -subalgebra of $K[t]$ generated by all monomials x^{at^b} such that a is a b -cover of I . We say that a b -cover a of I is *reducible* if there exists an i -cover c and a j -cover d of I such that $a = c + d$ and $b = i + j$. If a is not reducible, we call a *irreducible*. The irreducible 0 and 1 covers of \mathcal{C} are the unit vector e_1, \dots, e_n and the incidence vectors u_1, \dots, u_s of the minimal vertex covers of \mathcal{C} , respectively. The minimal generators of $R_c(I)$ as a K -algebra correspond to the irreducible covers of I . Notice the following dual descriptions:

$$\begin{aligned} I^{(b)} &= (\{x^a \mid \langle a, u_i \rangle \geq b \text{ for } i = 1, \dots, s\}), \\ J^{(b)} &= (\{x^a \mid \langle a, v_i \rangle \geq b \text{ for } i = 1, \dots, q\}), \end{aligned}$$

where $J = I_c(\mathcal{C})$. Hence $R_c(I) = R_s(J)$ and $R_c(J) = R_s(I)$.

In general each ℓ_i occurring in Eq. (1) determines a minimal generator of $R_s(I_c(\mathcal{C}))$. Indeed if we write $\ell_i = (a_i, -d_i)$, where $a_i \in \mathbb{N}^n$, $d_i \in \mathbb{N}$, then a_i is an irreducible d_i -cover of I (Lemma 1.8). Let F_{n+1} be the facet of $\mathbb{R}_+(I)$ determined by the hyperplane $H_{e_{n+1}}$. Thus we have a map ψ :

$$\begin{aligned} \{\text{Facets of } \mathbb{R}_+(I(\mathcal{C}))\} \setminus \{F_{n+1}\} &\xrightarrow{\psi} R_s(I_c(\mathcal{C})) \\ H_{\ell_k} \cap \mathbb{R}_+(I) &\xrightarrow{\psi} x^{a_k} t^{d_k}, \text{ where } \ell_k = (a_k, -d_k) \\ H_{e_i} \cap \mathbb{R}_+(I) &\xrightarrow{\psi} x_i \end{aligned}$$

whose image provides a good approximation for the minimal set of generators of $R_s(I_c(\mathcal{C}))$ as a K -algebra. Likewise the facets of $\mathbb{R}_+(I_c(\mathcal{C}))$ give

an approximation for the minimal set of generators of $R_s(I(\mathcal{C}))$. In Example 1.9 we show a connected graph G for which the image of the map ψ does not generate $R_s(I_c(G))$. For balanced clutters, i.e., for clutters without odd cycles, the image of the map ψ generates $R_s(I_c(\mathcal{C}))$. This follows from [12, Propositions 4.10 and 4.11]. In particular the image of the map ψ generates $R_s(I_c(\mathcal{C}))$ when \mathcal{C} is a bipartite graph. It would be interesting to characterize when the irreducible representation of the Rees cone determine the irreducible covers.

The *Simis cone* of I is the rational polyhedral cone:

$$\text{Cn}(I) = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{(u_1, -1)}^+ \cap \cdots \cap H_{(u_s, -1)}^+,$$

Simis cones were introduced in [9] to study symbolic Rees algebras of square-free monomial ideals. If \mathcal{H} is an integral Hilbert basis of $\text{Cn}(I)$, then $R_s(I(\mathcal{C}))$ equals $K[\mathbb{N}\mathcal{H}]$, the semigroup ring of $\mathbb{N}\mathcal{H}$ (see [9, Theorem 3.5]). This result is interesting because it allows us to compute the minimal generators of $R_s(I(\mathcal{C}))$ using Hilbert bases. The program *Normaliz* [3] is suitable for computing Hilbert bases. There is a description of \mathcal{H} valid for perfect graphs [28]. Perfect graphs are defined in Section 2.

If G is a perfect graph, the irreducible b -covers of $\Upsilon(G)$ correspond to cliques of G [28] (cf. Corollary 2.5). In this case, setting $\mathcal{C} = \Upsilon(G)$, it turns out that the image of ψ generates $R_s(I_c(\Upsilon(G)))$. Notice that $I_c(\Upsilon(G))$ is equal to $I(G)$.

In Section 2 we introduce and study the concept of an irreducible graph. A b -cover $a = (a_1, \dots, a_n)$ is called *binary* if $a_i \in \{0, 1\}$ for all i . We present a graph theoretical description of the irreducible binary b -vertex covers of the blocker of G (see Theorem 2.9). It is shown that they are in one to one correspondence with the irreducible induced subgraphs of G . As a byproduct we obtain a method, using Hilbert bases, to obtain all irreducible induced subgraphs of G (see Corollary 2.12). In particular we obtain all induced odd cycles and all induced complements of odd cycles. These cycles are called the *odd holes* and *odd antiholes* of the graph. It was shown recently [10] that \mathfrak{p} is an associated prime of $I_c(G)^2$ if and only if \mathfrak{p} is generated by the vertices of an edge of G or \mathfrak{p} is generated by the vertices of an odd hole of G . The proof of this remarkable result makes use of Theorem 1.7. Odd holes and antiholes play a major role in graph theory. In [4] it is shown that a graph G is perfect if and only if G has no odd holes or antiholes of length at least five. We give a procedure to build irreducible graphs (Proposition 2.18) and a method to construct irreducible b -vertex covers of the blocker of G with high degree relative to the number of vertices of G (see Corollaries 2.24 and 2.25).

Along the paper we introduce most of the notions that are relevant for our purposes. For unexplained terminology we refer to [6, 18, 25].

1. Blowup algebras of ideals of vertex covers

Let G be a simple graph with vertex set $X = \{x_1, \dots, x_n\}$. In what follows we shall always assume that G has no isolated vertices. Here we will give a graph theoretical description of the irreducible b -covers of G , i.e., we will describe the symbolic Rees algebra of $I_c(G)$.

Let S be a set of vertices of G . The *neighbor set* of S , denoted by $N_G(S)$, is the set of vertices of G that are adjacent with at least one vertex of S . The set S is called *independent* if no two vertices of S are adjacent. The empty set is regarded as an independent set whose incidence vector is the zero vector. Notice the following duality: S is a maximal independent set of G (with respect to inclusion) if and only if $X \setminus S$ is a minimal vertex cover of G .

Lemma 1.1. *If $a = (a_i) \in \mathbb{N}^n$ is an irreducible k -cover of G , then $0 \leq k \leq 2$ and $0 \leq a_i \leq 2$ for $i = 1, \dots, n$.*

Proof. Recall that a is a k -cover of G if and only if $a_i + a_j \geq k$ for each edge $\{x_i, x_j\}$ of G . If $k = 0$ or $k = 1$, then by the irreducibility of a it is seen that either $a = e_i$ for some i or $a = e_{i_1} + \dots + e_{i_r}$ for some minimal vertex cover $\{x_{i_1}, \dots, x_{i_r}\}$ of G . Thus we may assume that $k \geq 2$.

Case (I): $a_i \geq 1$ for all i . Clearly $\mathbf{1} = (1, \dots, 1)$ is a 2-cover. If $a - \mathbf{1} \neq 0$, then $a - \mathbf{1}$ is a $k - 2$ cover and $a = \mathbf{1} + (a - \mathbf{1})$, a contradiction to a being an irreducible k -cover. Hence $a = \mathbf{1}$. Pick any edge $\{x_i, x_j\}$ of G . Since a is a k -cover, we get $2 = a_i + a_j \geq k$ and k must be equal to 2.

Case (II): $a_i = 0$ for some i . We may assume $a_i = 0$ for $1 \leq i \leq r$ and $a_i \geq 1$ for $i > r$. Notice that the set $S = \{x_1, \dots, x_r\}$ is independent because if $\{x_i, x_j\}$ is an edge and $1 \leq i < j \leq r$, then $0 = a_i + a_j \geq k$, a contradiction. Consider the neighbor set $N_G(S)$ of S . We may assume that $N_G(S) = \{x_{r+1}, \dots, x_s\}$. Observe that $a_i \geq k \geq 2$ for $i = r+1, \dots, s$, because a is a k -cover. Write

$$a = (0, \dots, 0, a_{r+1} - 2, \dots, a_s - 2, a_{s+1} - 1, \dots, a_n - 1) + \underbrace{(0, \dots, 0)}_r, \underbrace{(2, \dots, 2)}_{s-r}, \underbrace{(1, \dots, 1)}_{n-s} = c + d.$$

Clearly d is a 2-cover. If $c \neq 0$, using that $a_i \geq k \geq 2$ for $r+1 \leq i \leq s$ and $a_i \geq 1$ for $i > s$ it is not hard to see that c is a $(k-2)$ -cover. This gives a contradiction, because $a = c + d$. Hence $c = 0$. Therefore $a_i = 2$ for $r < i \leq s$, $a_i = 1$ for $i > s$, and $k = 2$. \square

The next result complements the fact that the symbolic Rees algebra of $I_c(G)$ is generated by monomials of degree in t at most two [14, Theorem 5.1].

Corollary 1.2. $R_s(I_c(G))$ is generated as a K -algebra by monomials of degree in t at most two and total degree at most $2n$.

Proof. Let $x^a t^k$ be a minimal generator of $R_s(I_c(G))$ as a K -algebra. Then $a = (a_1, \dots, a_n)$ is an irreducible k -cover of G . By Lemma 1.1 we obtain that $0 \leq k \leq 2$ and $0 \leq a_i \leq 2$ for all i . If $k = 0$ or $k = 1$, we get that the degree of $x^a t^k$ is at most n . Indeed when $k = 0$ or $k = 1$, one has $a = e_i$ or $a = \sum_{x_i \in C} e_i$ for some minimal vertex cover C of G , respectively. If $k = 2$, by the proof of Lemma 1.1 either $a = \mathbf{1}$ or $a_i = 0$ for some i . Thus $\deg(x^a) \leq 2(n-1)$. \square

Let $I = I(G)$ be the edge ideal of G . For use below consider the vectors ℓ_1, \dots, ℓ_r that occur in the irreducible representation of $\mathbb{R}_+(I)$ given in Eq. (1).

Corollary 1.3. If $\ell_i = (\ell_{i1}, \dots, \ell_{in}, -\ell_{i(n+1)})$, then $0 \leq \ell_{ij} \leq 2$ for $j = 1, \dots, n$ and $1 \leq \ell_{i(n+1)} \leq 2$.

Proof. It suffices to observe that $(\ell_{i1}, \dots, \ell_{in})$ is an irreducible $\ell_{i(n+1)}$ -cover of G and to apply Lemma 1.1. \square

Lemma 1.4. $a = (1, \dots, 1)$ is an irreducible 2-cover of G if and only if G is non bipartite.

Proof. \Rightarrow) We proceed by contradiction assuming that G is a bipartite graph. Then G has a bipartition (V_1, V_2) . Set $a' = \sum_{x_i \in V_1} e_i$ and $a'' = \sum_{x_i \in V_2} e_i$. Since V_1 and V_2 are minimal vertex covers of G , we can decompose a as $a = a' + a''$, where a' and a'' are 1-covers, which is impossible.

\Leftarrow) Notice that a cannot be the sum of a 0-cover and a 2-cover. Indeed if $a = a' + a''$, where a' is a 0-cover and a'' is a 2-cover, then a'' has an entry a_i equal to zero. Pick an edge $\{x_i, x_j\}$ incident with x_i , then $\langle a'', e_i + e_j \rangle \leq 1$, a contradiction. Thus we may assume that $a = c + d$, where c, d are 1-covers. Let C_r be an odd cycle of G of length r . Notice that any vertex cover of C_r must contain a pair of adjacent vertices because r is odd. Clearly a vertex cover of G is also a vertex cover of the subgraph C_r . Hence the vertex covers of G corresponding to c and d must contain a pair of adjacent vertices, a contradiction because c and d are complementary vectors and the complement of a vertex cover is an independent set. \square

Definition 1.5. Let A be the incidence matrix of a clutter \mathcal{C} . A clutter \mathcal{C} has a *cycle* of length r if there is a square sub-matrix of A of order $r \geq 3$ with exactly two 1's in each row and column. A clutter without odd cycles is called *balanced*.

Proposition 1.6 ([12, Proposition 4.11]). *If \mathcal{C} is a balanced clutter, then*

$$R_s(I_c(\mathcal{C})) = R[I_c(\mathcal{C})t].$$

This result was first shown for bipartite graphs in [11, Corollary 2.6] and later generalized to balanced clutters [12] using an algebro combinatorial description of clutters with the max-flow min-cut property [13].

Let S be a set of vertices of a graph G , the *induced subgraph* on S , denoted by $\langle S \rangle$, is the maximal subgraph of G with vertex set S . The next result has been used in [10] to show that any associated prime of $I_c(G)^2$ is generated by the vertices of an edge of G or it is generated by the vertices of an odd hole of G .

We come to the main result of this section.

Theorem 1.7. *Let $0 \neq a = (a_i) \in \mathbb{N}^n$ and let $\Upsilon(G)$ be the family of minimal vertex covers of a graph G .*

- (i) *If G is bipartite, then a is an irreducible b -cover of G if and only if $b = 0$ and $a = e_i$ for some $1 \leq i \leq n$ or $b = 1$ and $a = \sum_{x_i \in C} e_i$ for some $C \in \Upsilon(G)$.*
- (ii) *If G is non-bipartite, then a is an irreducible b -cover if and only if a has one of the following forms:*
 - (a) (0-covers) $b = 0$ and $a = e_i$ for some $1 \leq i \leq n$,
 - (b) (1-covers) $b = 1$ and $a = \sum_{x_i \in C} e_i$ for some $C \in \Upsilon(G)$,
 - (c) (2-covers) $b = 2$ and $a = (1, \dots, 1)$,
 - (d) (2-covers) $b = 2$ and up to permutation of vertices

$$a = (\underbrace{0, \dots, 0}_{|A|}, \underbrace{2, \dots, 2}_{|N_G(A)|}, 1, \dots, 1)$$

for some independent set of vertices $A \neq \emptyset$ of G such that

- (d₁) $N_G(A)$ is not a vertex cover of G and $V \neq A \cup N_G(A)$,
- (d₂) the induced subgraph $\langle V \setminus (A \cup N_G(A)) \rangle$ has no isolated vertices and is not bipartite.

Proof. (i) \Rightarrow) Since G is bipartite, by Proposition 1.6, we have the equality $R_s(J) = R[Jt]$, where $J = I_c(G)$ is the ideal of vertex covers of G . Thus the minimal set of generator of $R_s(J)$ as a K -algebra is the set

$$\{x_1, \dots, x_n, x^{u_1}t, \dots, x^{u_s}t\},$$

where u_1, \dots, u_s are the incidence vectors of the minimal vertex covers of G . By hypothesis a is an irreducible b -cover of G , i.e., $x^a t^b$ is a minimal generator of $R_s(I_C(\mathcal{C}))$. Therefore either $a = e_i$ for some i and $b = 0$ or $a = u_i$ for some i and $b = 1$. The converse follows readily and is valid for any graph or clutter.

(ii) \Rightarrow) By Lemma 1.1 we have $0 \leq b \leq 2$ and $0 \leq a_i \leq 2$ for all i . If $b = 0$ or $b = 1$, then clearly a has the form indicated in (a) or (b) respectively.

Assume $b = 2$. If $a_i \geq 1$ for all i , the $a_i = 1$ for all i , otherwise if $a_i = 2$ for some i , then $a - e_i$ is a 2-cover and $a = e_i + (a - e_i)$, a contradiction. Hence $a = \mathbf{1}$. Thus we may assume that a has the form

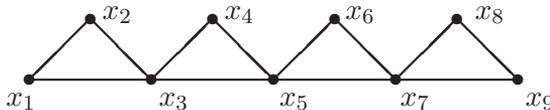
$$a = (0, \dots, 0, 2, \dots, 2, 1, \dots, 1).$$

We set $A = \{x_i \mid a_i = 0\} \neq \emptyset$, $B = \{x_i \mid a_i = 2\}$, and $C = V \setminus (A \cup B)$. Observe that A is an independent set because a is a 2-cover and $B = N_G(A)$ because a is irreducible. Hence it is seen that conditions (d₁) and (d₂) are satisfied. Using Lemma 1.4, the proof of the converse is direct. \square

Lemma 1.8. *Let \mathcal{C} be a clutter and let $I = I(\mathcal{C})$ be its edge ideal. If $\ell_k = (a_k, -d_k)$ is any of the vectors that occur in Eq. (1), where $a_k \in \mathbb{N}^n$, $d_k \in \mathbb{N}$, then a_k is an irreducible d_k -cover of \mathcal{C} .*

Proof. We proceed by contradiction assume there is a d'_k -cover a'_k and a d''_k -cover a''_k such that $a_k = a'_k + a''_k$ and $d_k = d'_k + d''_k$. Set $F' = H_{(a'_k, -d'_k)} \cap \mathbb{R}_+(I)$ and $F'' = H_{(a''_k, -d''_k)} \cap \mathbb{R}_+(I)$. Clearly F', F'' are proper faces of $\mathbb{R}_+(I)$ and $F = \mathbb{R}_+(I) \cap H_{\ell_k} = F' \cap F''$. Recall that any proper face of $\mathbb{R}_+(I)$ is the intersection of those facets that contain it (see [30, Theorem 3.2.1(vii)]). Applying this fact to F' and F'' it is seen that $F' \subset F$ or $F'' \subset F$, i.e., $F = F'$ or $F = F''$. We may assume $F = F'$. Hence $H_{(a'_k, -d'_k)} = H_{\ell_k}$. Taking orthogonal complements we get that $(a'_k, -d'_k) = \lambda(a_k, -d_k)$ for some $\lambda \in \mathbb{Q}_+$, because the orthogonal complement of H_{ℓ_k} is one dimensional and it is generated by ℓ_k . Since the non-zero entries of ℓ_k are relatively prime, we may assume that $\lambda \in \mathbb{N}$. Thus $d'_k = \lambda d_k \geq d_k \geq d'_k$ and λ must be equal to 1. Hence $a_k = a'_k$ and a''_k must be zero, a contradiction. \square

Example 1.9. Consider the following graph G :



Using *Normaliz* [3] it is seen that the vector $a = (1, 1, 2, 0, 2, 1, 1, 1, 1)$ is an irreducible 2-cover of G such that the supporting hyperplane $H_{(a,-2)}$ does not define a facet of the Rees cone of $I(G)$. Thus, in general, the image of ψ described in the introduction does not determine $R_s(I_c(G))$. We may use Lemma 1.4 to construct non-connected graphs with this property.

Edge cones of graphs. Let G be a connected simple graph and let $\mathcal{A} = \{v_1, \dots, v_q\}$ be the set of all vectors $e_i + e_j$ such that $\{x_i, x_j\}$ is an edge of G . The *edge cone* of G , denoted by $\mathbb{R}_+\mathcal{A}$, is defined as the cone generated by \mathcal{A} . Below we give an explicit combinatorial description of the edge cone.

Let A be an *independent set* of vertices of G . The supporting hyperplane of the edge cone of G defined by

$$\sum_{x_i \in N_G(A)} x_i - \sum_{x_i \in A} x_i = 0$$

will be denoted by H_A .

Edge cones and their representations by closed halfspaces are a useful tool to study a -invariants of edge subrings [23, 26]. The following result is a prototype of these representations. As an application we give a direct proof of the next result using Rees cones.

Corollary 1.10 ([24, Corollary 2.8]). *A vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ is in $\mathbb{R}_+\mathcal{A}$ if and only if a satisfies the following system of linear inequalities*

$$a_i \geq 0, \quad i = 1, \dots, n;$$

$$\sum_{x_i \in N_G(A)} a_i - \sum_{x_i \in A} a_i \geq 0, \quad \text{for all independent sets } A \subset V(G).$$

Proof. Set $\mathcal{B} = \{(v_1, 1), \dots, (v_q, 1)\}$ and $I = I(G)$. Notice the equality

$$\mathbb{R}_+(I) \cap \mathbb{R}\mathcal{B} = \mathbb{R}_+\mathcal{B}, \tag{2}$$

where $\mathbb{R}\mathcal{B}$ is \mathbb{R} -vector space spanned by \mathcal{B} . Consider the irreducible representation of $\mathbb{R}_+(I)$ given in Eq. (1) and write $\ell_i = (a_i, -d_i)$, where $0 \neq a_i \in \mathbb{N}^n$, $0 \neq d_i \in \mathbb{N}$. Next we show the equality:

$$\mathbb{R}_+\mathcal{A} = \mathbb{R}\mathcal{A} \cap \mathbb{R}_+^n \cap H_{(2a_1/d_1-1)}^+ \cap \dots \cap H_{(2a_r/d_r-1)}^+, \tag{3}$$

where $\mathbf{1} = (1, \dots, 1)$. Take $\alpha \in \mathbb{R}_+\mathcal{A}$. Clearly $\alpha \in \mathbb{R}\mathcal{A} \cap \mathbb{R}_+^n$. We can write

$$\alpha = \lambda_1 v_1 + \dots + \lambda_q v_q \Rightarrow |\alpha| = 2(\lambda_1 + \dots + \lambda_q) = 2b.$$

Thus $\langle \alpha, b \rangle = \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1)$, i.e., $(\alpha, b) \in \mathbb{R}_+\mathcal{B}$. Hence from Eq. (2) we get $(\alpha, b) \in \mathbb{R}_+(I)$ and

$$\langle (\alpha, b), (a_i, -d_i) \rangle \geq 0 \Rightarrow \langle \alpha, a_i \rangle \geq bd_i = (|\alpha|/2)d_i = |\alpha|(d_i/2).$$

Writing $\alpha = (\alpha_1, \dots, \alpha_n)$ and $a_i = (a_{i1}, \dots, a_{in})$, the last inequality gives:

$$\alpha_1 a_{i1} + \cdots + \alpha_n a_{in} \geq (\alpha_1 + \cdots + \alpha_n)(d_i/2) \Rightarrow \langle \alpha, a_i - (d_i/2)\mathbf{1} \rangle \geq 0.$$

Then $\langle \alpha, 2a_i/d_i - \mathbf{1} \rangle \geq 0$ and $\alpha \in H_{(2a_i/d_i - \mathbf{1})}^+$ for all i , as required. This proves that $\mathbb{R}_+\mathcal{A}$ is contained in the right hand side of Eq. (3). The other inclusion follows similarly. Now by Lemma 1.8 we obtain that a_i is an irreducible d_i -cover of G . Therefore, using the explicit description of the irreducible b -covers of G given in Theorem 1.7, we get the equality

$$\mathbb{R}_+\mathcal{A} = \left(\bigcap_{A \in \mathcal{F}} H_A^+ \right) \cap \left(\bigcap_{i=1}^n H_{e_i}^+ \right),$$

where \mathcal{F} is the collection of all the independent sets of vertices of G . From this equality the assertion follows at once. \square

The edge cone of G encodes information about both the Hilbert function of the edge subring $K[G]$ (see [23]) and the graph G itself. As a simple illustration, we recover the following version of the marriage theorem for bipartite graphs, see [2]. Recall that a pairing by an independent set of edges of all the vertices of a graph G is called a *perfect matching* or a *1-factor*.

Corollary 1.11. *If G is a bipartite connected graph, then G has a perfect matching if and only if $|A| \leq |N_G(A)|$ for every independent set of vertices A of G .*

Proof. Notice that the graph G has a perfect matching if and only if the vector $\beta = (1, 1, \dots, 1)$ is in $\mathbb{N}\mathcal{A}$. By [23, Lemma 2.9] we have the equality $\mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\mathcal{A}$. Hence β is in $\mathbb{N}\mathcal{A}$ if and only if $\beta \in \mathbb{R}_+\mathcal{A}$. Thus the result follows from Corollary 1.10. \square

2. Symbolic Rees algebras of edge ideals

Let G be a graph with vertex set $X = \{x_1, \dots, x_n\}$ and let $I = I(G)$ be its edge ideal. As before we denote the clutter of minimal vertex covers of G by $\Upsilon(G)$. The clutter $\Upsilon(G)$ is called the *blocker* of G . Recall that the symbolic Rees algebra of $I(G)$ is given by

$$R_s(I(G)) = K[x^a t^b \mid a \text{ is an irreducible } b\text{-cover of } \Upsilon(G)], \quad (4)$$

where the set $\{x^{at^b} | a \text{ is an irreducible } b\text{-cover of } \Upsilon(G)\}$ is the minimal set of generators of $R_s(I(G))$ as a K -algebra. The main purpose of this section is to study the symbolic Rees algebra of $I(G)$ via graph theory. We are interested in finding combinatorial representations for the minimal set of generators of this algebra.

Lemma 2.1. *Let $0 \neq a = (a_1, \dots, a_m, 0, \dots, 0) \in \mathbb{N}^n$ and let $a' = (a_1, \dots, a_m)$. If $0 \neq b \in \mathbb{N}$, then a is an irreducible b -cover of $\Upsilon(G)$ if and only if a' is an irreducible b -cover of $\Upsilon(\langle S \rangle)$, where $S = \{x_1, \dots, x_m\}$.*

Proof. It suffices to prove that a is a b -cover of the blocker of G if and only if a' is a b -cover of the blocker of $\langle S \rangle$.

\Rightarrow) The induced subgraph $\langle S \rangle$ is not a discrete graph. Take a minimal vertex cover C' of $\langle S \rangle$. Set $C = C' \cup (V(G) \setminus S)$. Since C is a vertex cover of G such that $C \setminus \{x_i\}$ is not a vertex cover of G for every $x_i \in C'$, there is a minimal vertex cover C_ℓ of G such that $C' \subset C_\ell \subset C$ and $C' = C_\ell \cap S$. Notice that

$$\sum_{x_i \in C'} a_i = \sum_{x_i \in C_\ell \cap S} a_i = \langle a, u_\ell \rangle \geq b,$$

where u_ℓ is the incidence vector of C_ℓ . Hence $\sum_{x_i \in C'} a_i \geq b$, as required.

\Leftarrow) Take a minimal vertex cover C_ℓ of G . Then $C_\ell \cap S$ contains a minimal vertex cover C'_ℓ of $\langle S \rangle$. Let u_ℓ (resp. u'_ℓ) be the incidence vector of C_ℓ (resp. C'_ℓ). Notice that

$$\langle a, u_\ell \rangle = \sum_{x_i \in C_\ell \cap S} a_i \geq \sum_{x_i \in C'_\ell} a_i = \langle a', u'_\ell \rangle \geq b.$$

Hence $\langle a, u_\ell \rangle \geq b$, as required. \square

We denote a complete subgraph of G with r vertices by \mathcal{K}_r . If v is a vertex of G , we denote its neighbor set by $N_G(v)$.

Lemma 2.2. *Let G be a graph and let $a = (a_1, \dots, a_n)$ be an irreducible b -cover of $\Upsilon(G)$ such that $a_i \geq 1$ for all i . If $\langle N_G(x_n) \rangle = \mathcal{K}_r$, then $a_i = 1$ for all i , $b = r$, $n = r + 1$, and $G = \mathcal{K}_n$.*

Proof. We may assume that $N_G(x_n) = \{x_1, \dots, x_r\}$. We set

$$c = e_1 + \dots + e_r + e_n; \quad d = (a_1 - 1, \dots, a_r - 1, a_{r+1}, \dots, a_{n-1}, a_n - 1).$$

Notice that $\langle x_1, \dots, x_r, x_n \rangle = \mathcal{K}_{r+1}$. Thus c is an r -cover of $\Upsilon(G)$ because any minimal vertex cover of G must intersect all edges of \mathcal{K}_{r+1} . By the irreducibility of a , there exists a minimal vertex cover C_ℓ of G such that

$\sum_{x_i \in C_\ell} a_i = b$. Clearly we have $b \geq g \geq r$, where g is the height of $I(G)$. Let C_k be an arbitrary minimal vertex cover of G . Since C_k contains exactly r vertices of \mathcal{K}_{r+1} , from the inequality $\sum_{x_i \in C_k} a_i \geq b$ we get $\sum_{x_i \in C_k} d_i \geq b - r$, where d_1, \dots, d_n are the entries of d . Therefore $d = 0$; otherwise if $d \neq 0$, then d is a $(b - r)$ -cover of $\Upsilon(G)$ and $a = c + d$, a contradiction to the irreducibility of a . It follows that $g = r$, $n = r + 1$, $a_i = 1$ for $1 \leq i \leq r$, $a_n = 1$, and $G = \mathcal{K}_n$. \square

Notation We regard \mathcal{K}_0 as the empty set with zero elements. A sum over an empty set is defined to be 0.

Proposition 2.3. *Let G be a graph and let $J = I_c(G)$ be its ideal of vertex covers. Then the set*

$$F = \{(a_i) \in \mathbb{R}^{n+1} \mid \sum_{x_i \in \mathcal{K}_r} a_i = (r - 1)a_{n+1}\} \cap \mathbb{R}_+(J)$$

is a facet of the Rees cone $\mathbb{R}_+(J)$.

Proof. If $\mathcal{K}_r = \emptyset$, then $r = 0$ and $F = H_{e_{n+1}} \cap \mathbb{R}_+(J)$, which is clearly a facet because $e_1, \dots, e_n \in F$. If $r = 1$, then $F = H_{e_i} \cap \mathbb{R}_+(J)$ for some $1 \leq i \leq n$, which is a facet because $e_j \in F$ for $j \notin \{i, n + 1\}$ and there is at least one minimal vertex cover of G not containing x_i . We may assume that $X' = \{x_1, \dots, x_r\}$ is the vertex set of \mathcal{K}_r and $r \geq 2$. For each $1 \leq i \leq r$ there is a minimal vertex cover C_i of G not containing x_i . Notice that C_i contains $X' \setminus \{x_i\}$. Let u_i be the incidence vector of C_i . Since the rank of u_1, \dots, u_r is r , it follows that the set

$$\{(u_1, 1), \dots, (u_r, 1), e_{r+1}, \dots, e_n\}$$

is a linearly independent set contained in F , i.e., $\dim(F) = n$. Hence F is a facet of $\mathbb{R}_+(J)$ because the hyperplane that defines F is a supporting hyperplane. \square

Proposition 2.4. *Let G be a graph and let $0 \neq a = (a_i) \in \mathbb{N}^n$. If*

(a) $a_i \in \{0, 1\}$ for all i , and

(b) $\langle \{x_i \mid a_i > 0\} \rangle = \mathcal{K}_{r+1}$,

then a is an irreducible r -cover of $\Upsilon(G)$.

Proof. By Proposition 2.3, the closed halfspace $H_{(a, -r)}^+$ occurs in the irreducible representation of the Rees cone $\mathbb{R}_+(J)$, where $J = I_c(G)$. Hence a is an irreducible r -cover by Lemma 1.8. \square

A *clique* of a graph G is a set of vertices that induces a complete subgraph. We will also call a complete subgraph of G a clique. Symbolic Rees algebras are related to perfect graphs as is seen below. Let us recall the notion of perfect graph. A *colouring* of the vertices of G is an assignment of colours to the vertices of G in such a way that adjacent vertices have distinct colours. The *chromatic number* of G is the minimal number of colours in a colouring of G . A graph is *perfect* if for every induced subgraph H , the chromatic number of H equals the size of the largest complete subgraph of H . We refer to [5, 6, 20] and the references there for the theory of perfect graphs.

Notation The *support* of $x^a = x_1^{a_1} \cdots x_n^{a_n}$ is $\text{supp}(x^a) = \{x_i \mid a_i > 0\}$.

Corollary 2.5 ([28]). *If G is a graph, then*

$$K[x^{at^r} \mid x^a \text{ square-free}; \langle \text{supp}(x^a) \rangle = \mathcal{K}_{r+1}; 0 \leq r < n] \subset R_s(I(G))$$

with equality if and only if G is a perfect graph.

Proof. The inclusion follows from Proposition 2.4. If G is a perfect graph, then by [28, Corollary 3.3] the equality holds. Conversely if the equality holds, then by Lemma 1.8 and Proposition 2.3 we have

$$\mathbb{R}_+(I_c(G)) = \{(a_i) \in \mathbb{R}^{n+1} \mid \sum_{x_i \in \mathcal{K}_r} a_i \geq (r-1)a_{n+1}; \forall \mathcal{K}_r \subset G\}. \quad (5)$$

Hence a direct application of [28, Proposition 2.2] gives that G is a perfect graph. \square

The *vertex covering number* of G , denoted by $\alpha_0(G)$, is the number of vertices in a minimum vertex cover of G (the cardinality of any smallest vertex cover in G). Notice that $\alpha_0(G)$ equals the height of $I(G)$. If H is a discrete graph, i.e., all the vertices of H are isolated, we set $I(H) = 0$ and $\alpha_0(H) = 0$.

Lemma 2.6. *Let G be a graph. If $a = e_1 + \cdots + e_r$ is an irreducible b -cover of $\Upsilon(G)$, then $b = \alpha_0(H)$, where $H = \langle x_1, \dots, x_r \rangle$.*

Proof. The case $b = 0$ is clear. Assume $b \geq 1$. Let C_1, \dots, C_s be the minimal vertex covers of G and let u_1, \dots, u_s be their incidence vectors. Notice that $\langle a, u_i \rangle = b$ for some i . Indeed if $\langle a, u_i \rangle > b$ for all i , then $a - e_1$ is a b -cover of $\Upsilon(G)$ and $a = (a - e_1) + e_1$, a contradiction. Hence

$$b = \langle a, u_i \rangle = |\{x_1, \dots, x_r\} \cap C_i| \geq \alpha_0(H).$$

This proves that $b \geq \alpha_0(H)$. Notice that H is not a discrete graph. Then we can pick a minimal vertex cover A of H such that $|A| = \alpha_0(H)$. The set

$$C = A \cup (V(G) \setminus \{x_1, \dots, x_r\})$$

is a vertex cover of G . Hence there is a minimal vertex cover C_ℓ of G such that $A \subset C_\ell \subset C$. Observe that $C_\ell \cap \{x_1, \dots, x_r\} = A$. Thus we get $\langle a, u_\ell \rangle = |A| \geq b$, i.e., $\alpha_0(H) \geq b$. Altogether we have $b = \alpha_0(H)$. \square

This result has been recently extended to clutters using the notion of parallelization [7]. Let \mathcal{C} be a clutter on the vertex set $X = \{x_1, \dots, x_n\}$ and let $x_i \in X$. Then *duplicating* x_i means extending X by a new vertex x'_i and replacing $E(\mathcal{C})$ by

$$E(\mathcal{C}) \cup \{(e \setminus \{x_i\}) \cup \{x'_i\} \mid x_i \in e \in E(\mathcal{C})\}.$$

The *deletion* of x_i , denoted by $\mathcal{C} \setminus \{x_i\}$, is the clutter formed from \mathcal{C} by deleting the vertex x_i and all edges containing x_i . A clutter obtained from \mathcal{C} by a sequence of deletions and duplications of vertices is called a *parallelization*. If $w = (w_i)$ is a vector in \mathbb{N}^n , we denote by \mathcal{C}^w the clutter obtained from \mathcal{C} by deleting any vertex x_i with $w_i = 0$ and duplicating $w_i - 1$ times any vertex x_i if $w_i \geq 1$. The map $w \mapsto \mathcal{C}^w$ gives a one to one correspondence between \mathbb{N}^n and the parallelizations of \mathcal{C} .

Example 2.7. Let G be the graph whose only edge is $\{x_1, x_2\}$ and let $w = (3, 3)$. Then $G^w = \mathcal{K}_{3,3}$ is the complete bipartite graph with bipartition $V_1 = \{x_1, x_1^2, x_1^3\}$ and $V_2 = \{x_2, x_2^2, x_2^3\}$. Notice that x_i^k is a vertex, i.e., k is an index not an exponent.

Proposition 2.8 ([7]). *Let \mathcal{C} be a clutter and let $\Upsilon(\mathcal{C})$ be the blocker of \mathcal{C} . If $w = (w_i)$ is an irreducible b -cover of $\Upsilon(\mathcal{C})$, then*

$$b = \min \left\{ \sum_{x_i \in C} w_i \mid C \in \Upsilon(\mathcal{C}) \right\} = \alpha_0(\mathcal{C}^w).$$

The next result gives a nice graph theoretical description for the irreducible binary b -vertex covers of the blocker of G .

Theorem 2.9. *Let G be a graph and let $a = (1, \dots, 1)$. Then a is a reducible $\alpha_0(G)$ -cover of $\Upsilon(G)$ if and only if there are H_1 and H_2 induced subgraphs of G such that*

- (i) $V(G)$ is the disjoint union of $V(H_1)$ and $V(H_2)$, and

(ii) $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$.

Proof. \Rightarrow) We may assume that $a_1 = e_1 + \cdots + e_r$, $a_2 = a - a_1$, a_i is a b_i -cover of $\Upsilon(G)$, $b_i \geq 1$ for $i = 1, 2$, and $\alpha_0(G) = b_1 + b_2$. Consider the subgraphs $H_1 = \langle x_1, \dots, x_r \rangle$ and $H_2 = \langle x_{r+1}, \dots, x_n \rangle$. Let A be a minimal vertex cover of H_1 with $\alpha_0(H_1)$ vertices. Since

$$C = A \cup (V(G) \setminus \{x_1, \dots, x_r\})$$

is a vertex cover G , there is a minimal vertex cover C_k of G such that $A \subset C_k \subset C$. Hence

$$|A| = |C_k \cap \{x_1, \dots, x_r\}| = \langle a_1, u_k \rangle \geq b_1,$$

and $\alpha_0(H_1) \geq b_1$. Using a similar argument we get that $\alpha_0(H_2) \geq b_2$. If C_ℓ is a minimal vertex cover of G with $\alpha_0(G)$ vertices, then $C_\ell \cap V(H_i)$ is a vertex cover of H_i . Therefore

$$b_1 + b_2 = \alpha_0(G) = |C_\ell| = \sum_{i=1}^2 |C_\ell \cap V(H_i)| \geq \sum_{i=1}^2 \alpha_0(H_i) \geq b_1 + b_2,$$

and consequently $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$.

\Leftarrow) We may assume that $V(H_1) = \{x_1, \dots, x_r\}$ and $V(H_2) = V(G) \setminus V(H_1)$. Set $a_1 = e_1 + \cdots + e_r$ and $a_2 = a - a_1$. For any minimal vertex cover C_k of G , we have that $C_k \cap V(H_i)$ is a vertex cover of H_i . Hence

$$\langle a_1, u_k \rangle = |C_k \cap \{x_1, \dots, x_r\}| \geq \alpha_0(H_1),$$

where u_k is the incidence vector of C_k . Consequently a_1 is an $\alpha_0(H_1)$ -cover of $\Upsilon(G)$. Similarly we obtain that a_2 is an $\alpha_0(H_2)$ -cover of $\Upsilon(G)$. Therefore a is a reducible $\alpha_0(G)$ -cover of $\Upsilon(G)$. \square

Definition 2.10. A graph satisfying conditions (i) and (ii) is called a *reducible* graph. If G is not reducible, it is called *irreducible*.

These notions appear in [8]. As far as we know there is no structure theorem for irreducible graphs. Examples of irreducible graphs include complete graphs, odd cycles, and complements of odd cycles. Below we give a method, using Hilbert bases, to obtain all irreducible induced subgraphs of G .

By [16, Lemma 5.4] there exists a finite set $\mathcal{H} \subset \mathbb{N}^{n+1}$ such that

(a) $\text{Cn}(I(G)) = \mathbb{R}_+ \mathcal{H}$, and

$$(b) \mathbb{Z}^{n+1} \cap \mathbb{R}_+ \mathcal{H} = \mathbb{N} \mathcal{H},$$

where $\mathbb{N} \mathcal{H}$ is the additive subsemigroup of \mathbb{N}^{n+1} generated by \mathcal{H} .

Definition 2.11. The set \mathcal{H} is called a *Hilbert basis* of $\text{Cn}(I(G))$.

If we require \mathcal{H} to be minimal (with respect inclusion), then \mathcal{H} is unique [19].

Corollary 2.12. *Let G be a graph and let $\alpha = (a_1, \dots, a_n, b)$ be a vector in $\{0, 1\}^n \times \mathbb{N}$. Then α is an element of the minimal integral Hilbert basis of $\text{Cn}(I(G))$ if and only if the induced subgraph $H = \langle \{x_i \mid a_i = 1\} \rangle$ is irreducible with $b = \alpha_0(H)$.*

Proof. The map $(a_1, \dots, a_n, b) \mapsto x_1^{a_1} \cdots x_n^{a_n} t^b$ establishes a one to one correspondence between the minimal integral Hilbert basis of $\text{Cn}(I(G))$ and the minimal generators of $R_s(I(G))$ as a K -algebra. Thus the result follows from Lemma 2.1 and Theorem 2.9. \square

The next result shows that irreducible graphs occur naturally in the theory of perfect graphs.

Proposition 2.13. *A graph G is perfect if and only if the only irreducible induced subgraphs of G are the complete subgraphs.*

Proof. \Rightarrow) Let H be an irreducible induced subgraph of G . We may assume that $V(H) = \{x_1, \dots, x_r\}$. Set $a' = (1, \dots, 1) \in \mathbb{N}^r$ and $a = (a', 0, \dots, 0) \in \mathbb{N}^n$. By Theorem 2.9, a' is an irreducible $\alpha_0(H)$ cover of $\Upsilon(H)$. Then by Lemma 2.1, a' is an irreducible $\alpha_0(H)$ cover of $\Upsilon(G)$. Since $x_1 \cdots x_r t^{\alpha_0(H)}$ is a minimal generator of $R_s(I(G))$, using Corollary 2.5 we obtain that $\alpha_0(H) = r - 1$ and that H is a complete subgraph of G on r vertices.

\Leftarrow) In [4] it is shown that G is a perfect graph if and only if no induced subgraph of G is an odd cycle of length at least five or the complement of one. Since odd cycles and their complements are irreducible subgraphs. It follows that G is perfect. \square

Definition 2.14. A graph G is called *vertex critical* if $\alpha_0(G \setminus \{x_i\}) < \alpha_0(G)$ for all $x_i \in V(G)$.

Remark 2.15. If x_i is any vertex of a graph G and $\alpha_0(G \setminus \{x_i\}) < \alpha_0(G)$, then $\alpha_0(G \setminus \{x_i\}) = \alpha_0(G) - 1$

Lemma 2.16. *If the graph G is irreducible, then it is connected and vertex critical*

Proof. Let G_1, \dots, G_r be the connected components of G . Since $\alpha_0(G)$ is equal to $\sum_i \alpha_0(G_i)$, we get $r = 1$. Thus G is connected. To complete the proof it suffices to prove that $\alpha_0(G \setminus \{x_i\}) < \alpha_0(G)$ for all i (see Remark 2.15). If $\alpha_0(G \setminus \{x_i\}) = \alpha_0(G)$, then $G = H_1 \cup H_2$, where $H_1 = G \setminus \{x_i\}$ and $V(H_2) = \{x_i\}$, a contradiction. \square

Definition 2.17. The *cone* $C(G)$, over a graph G , is obtained by adding a new vertex v to G and joining every vertex of G to v .

The next result can be used to build irreducible graphs. In particular it follows that cones over irreducible graphs are irreducible.

Proposition 2.18. *Let G be a graph with n vertices and let H be a graph obtained from G by adding a new vertex v and some new edges joining v with $V(G)$. If $a = (1, \dots, 1) \in \mathbb{N}^n$ is an irreducible $\alpha_0(G)$ -cover of $\Upsilon(G)$ such that $\alpha_0(H) = \alpha_0(G) + 1$, then $a' = (a, 1)$ is an irreducible $\alpha_0(H)$ -cover of $\Upsilon(H)$.*

Proof. Clearly a' is an $\alpha_0(H)$ -cover of $\Upsilon(H)$. Assume that $a' = a'_1 + a'_2$, where $0 \neq a'_i$ is a b'_i -cover of $\Upsilon(H)$ and $b'_1 + b'_2 = \alpha_0(H)$. We may assume that $a'_1 = (1, \dots, 1, 0, \dots, 0)$ and $a'_2 = (0, \dots, 0, 1, \dots, 1)$. Let a_i be the vector in \mathbb{N}^n obtained from a'_i by removing its last entry. Set $v = x_{n+1}$. Take a minimal vertex cover C_k of G and consider $C'_k = C_k \cup \{x_{n+1}\}$. Let u'_k (resp. u_k) be the incidence vector of C'_k (resp. C_k). Then

$$\langle a_1, u_k \rangle = \langle a'_1, u'_k \rangle \geq b'_1 \text{ and } \langle a_2, u_k \rangle + 1 = \langle a'_2, u'_k \rangle \geq b'_2,$$

consequently a_1 is a b'_1 -cover of $\Upsilon(G)$. If $b'_2 = 0$, then a_1 is an $\alpha_0(H)$ -cover of $\Upsilon(G)$, a contradiction; because if u is the incidence vector of a minimal vertex cover of G with $\alpha_0(G)$ elements, then we would obtain $\alpha_0(G) \geq \langle u, a_1 \rangle \geq \alpha_0(H)$, which is impossible. Thus $b'_2 \geq 1$, and a_2 is a $(b'_2 - 1)$ -cover of $\Upsilon(G)$ if $a_2 \neq 0$. Hence $a_2 = 0$, because $a = a_1 + a_2$ and a is irreducible. This means that $a'_2 = e_{n+1}$ is a b'_2 -cover of $\Upsilon(H)$, a contradiction. Therefore a' is an irreducible $\alpha_0(H)$ -cover of $\Upsilon(H)$, as required. \square

Definition 2.19. A graph G is called *edge critical* if $\alpha_0(G \setminus e) < \alpha_0(G)$ for all $e \in E(G)$.

Proposition 2.20. *If G is a connected edge critical graph, then G is irreducible.*

Proof. Assume that G is reducible. Then there are induced subgraphs H_1, H_2 of G such that $V(H_1), V(H_2)$ is a partition of $V(G)$ and $\alpha_0(G) = \alpha_0(H_1) + \alpha_0(H_2)$. Since G is connected there is an edge $e = \{x_i, x_j\}$ with x_i a vertex of H_1 and x_j a vertex of H_2 . Pick a minimal vertex cover C of $G \setminus e$ with $\alpha_0(G) - 1$ vertices. As $E(H_i)$ is a subset of $E(G \setminus e) = E(G) \setminus \{e\}$ for $i = 1, 2$, we get that C covers all edges of H_i for $i = 1, 2$. Hence C must have at least $\alpha_0(G)$ elements, a contradiction. \square

Corollary 2.21. *The following hold for any connected graph:*

$$\text{edge critical} \implies \text{irreducible} \implies \text{vertex critical}.$$

Finding generators of symbolic Rees algebras using cones The cone $C(G)$, over the graph G , is obtained by adding a new vertex t to G and joining every vertex of G to t .

Example 2.22. A pentagon and its cone:



In [1] Bahiano showed that if $H = C(G)$ is the graph obtained by taking a cone over a pentagon G with vertices x_1, \dots, x_5 , then

$$R_s(I(H)) = R[I(H)t][x_1 \cdots x_5 t^3, x_1 \cdots x_6 t^4, x_1 \cdots x_5 x_6 t^5].$$

This simple example shows that taking a cone over an irreducible graph tends to increase the degree in t of the generators of the symbolic Rees algebra. Other examples using this “cone process” have been shown in [14, Example 5.5].

Let G be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$. The aim here is to give a general procedure—based on the irreducible representation of the Rees cone of $I_c(G)$ —to construct generators of $R_s(I(H))$ of high degree in t , where H is a graph constructed from G by recursively taking cones over graphs already constructed.

By the finite basis theorem [30, Theorem 4.11] there is a unique irreducible representation

$$\mathbb{R}_+(I_c(G)) = H_{e_1}^+ \cap H_{e_2}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\alpha_1}^+ \cap H_{\alpha_2}^+ \cap \cdots \cap H_{\alpha_p}^+ \quad (6)$$

such that each α_k is in \mathbb{Z}^{n+1} , the non-zero entries of each α_k are relatively prime, and none of the closed halfspaces $H_{e_1}^+, \dots, H_{e_{n+1}}^+, H_{\alpha_1}^+, \dots, H_{\alpha_p}^+$ can be omitted from the intersection. For use below we assume that α is any of the vectors $\alpha_1, \dots, \alpha_p$ that occur in the irreducible representation. Thus we can write $\alpha = (a_1, \dots, a_n, -b)$ for some $a_i \in \mathbb{N}$ and for some $b \in \mathbb{N}$.

Lemma 2.23. *Let H be the cone over G . If*

$$\beta = (a_1, \dots, a_n, (\sum_{i=1}^n a_i) - b, -\sum_{i=1}^n a_i) = (\beta_1, \dots, \beta_{n+1}, -\beta_{n+2})$$

and $a_i \geq 1$ for all i , then $F = H_\beta \cap \mathbb{R}_+(I_c(H))$ is a facet of $\mathbb{R}_+(I_c(H))$.

Proof. First we prove that $\mathbb{R}_+(I_c(H)) \subset H_\beta^+$, i.e., H_β is a supporting hyperplane of the Rees cone. By Lemma 1.8, (a_1, \dots, a_n) is an irreducible b -cover of $\Upsilon(G)$. Hence there is $C \in \Upsilon(G)$ such that $\sum_{x_i \in C} a_i = b$. Therefore β_{n+1} is greater or equal than 1. This proves that e_1, \dots, e_{n+1} are in H_β^+ . Let C be any minimal vertex cover of H and let $u = \sum_{x_i \in C} e_i$ be its characteristic vector. Case (i): If $x_{n+1} \notin C$, then $C = \{x_1, \dots, x_n\}$ and

$$\sum_{x_i \in C} \beta_i = \sum_{i=1}^n a_i = \beta_{n+2},$$

that is, $(u, 1) \in H_\beta^+$. Case (ii): If $x_{n+1} \in C$, then $C_1 = C \setminus \{x_{n+1}\}$ is a minimal vertex cover of G . Hence

$$\sum_{x_i \in C} \beta_i = \sum_{x_i \in C_1} \beta_i + \beta_{n+1} \geq b + \beta_{n+1} = \beta_{n+2},$$

that is, $(u, 1) \in H_\beta^+$. Therefore $\mathbb{R}_+(I_c(H)) \subset H_\beta^+$. To prove that F is a facet we must show it has dimension $n + 1$ because the dimension of $\mathbb{R}_+(I_c(H))$ is $n + 2$. We denote the characteristic vector of a minimal vertex cover C_k of G by u_k . By hypothesis there are minimal vertex covers C_1, \dots, C_n of G such that the vectors $(u_1, 1), \dots, (u_n, 1)$ are linearly independent and

$$\langle (a, -b), (u_k, 1) \rangle = 0 \iff \langle a, u_k \rangle = b, \tag{7}$$

for $k = 1, \dots, n$. Therefore

$$\begin{aligned} \langle (\beta_1, \dots, \beta_{n+1}), (u_k, 1) \rangle &= \beta_{n+2} \quad \text{and} \\ \langle (\beta_1, \dots, \beta_{n+1}), (1, \dots, 1, 0) \rangle &= \beta_{n+2}, \end{aligned}$$

i.e., the set $\mathcal{B} = \{(u_1, 1), \dots, (u_n, 1), (1, \dots, 1, 0)\}$ is contained in H_β . Since

$$C_1 \cup \{x_{n+1}\}, \dots, C_n \cup \{x_{n+1}\}, \{x_1, \dots, x_n\}$$

are minimal vertex covers of H , the set \mathcal{B} is also contained in $\mathbb{R}_+(I_c(H))$ and consequently in F . Thus it suffices to prove that \mathcal{B} is linearly independent. If $(1, \dots, 1, 0)$ is a linear combination of $(u_1, 1), \dots, (u_n, 1)$, then we can write

$$(1, \dots, 1) = \lambda_1 u_1 + \dots + \lambda_n u_n$$

for some scalars $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 0$. Hence from Eq. (7) we get

$$|a| = \langle (1, \dots, 1), a \rangle = \lambda_1 \langle u_1, a \rangle + \dots + \lambda_n \langle u_n, a \rangle = (\lambda_1 + \dots + \lambda_n)b = 0,$$

a contradiction. \square

Corollary 2.24. *If $a_i \geq 1$ for all i , then $x_1^{\beta_1} \dots x_{n+1}^{\beta_{n+1}} t^{\beta_{n+2}}$ is a minimal generator of $R_s(I(H))$.*

Proof. By Lemma 2.23, $F = H_\beta \cap \mathbb{R}_+(I_c(H))$ is a facet of $\mathbb{R}_+(I_c(H))$. Therefore using Lemma 1.8, the vector $(\beta_1, \dots, \beta_{n+1})$ is an irreducible β_{n+2} -cover of $\Upsilon(H)$, i.e., $x_1^{\beta_1} \dots x_{n+1}^{\beta_{n+1}} t^{\beta_{n+2}}$ is a minimal generator of $R_s(I(H))$. \square

Corollary 2.25. *Let $G_0 = G$ and let G_r be the cone over G_{r-1} for $r \geq 1$. If $\alpha = (1, \dots, 1, -g)$, then*

$$\underbrace{(1, \dots, 1)}_n, \underbrace{(n-g, \dots, n-g)}_r$$

is an irreducible $n + (r-1)(n-g)$ cover of G_r . In particular $R_s(I(G_r))$ has a generator of degree in t equal to $n + (r-1)(n-g)$.

As a very particular example of our construction consider:

Example 2.26. Let $G = C_s$ be an odd cycle of length $s = 2k + 1$. Note that $\alpha_0(C_s) = (s+1)/2 = k+1$. Then by Corollary 2.25

$$x_1 \dots x_s x_{s+1}^k \dots x_{s+r}^k t^{rk+k+1}$$

is a minimal generator of $R_s(I(G_r))$. This illustrates that the degree in t of the minimal generators of $R_s(I(G_r))$ is much larger than the number of vertices of the graph G_r [14].

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Received by the editors: 01.03.2009
and in final form 26.02.2011.