Algebra and Discrete Mathematics
Number 2. **(2003).** pp. 87–92

RESEARCH ARTICLE

Principal quasi-ideals of cohomological dimension 1

B. V. Novikov

Abstract. We prove that a principal quasi-ideal of a non-commutative free semigroup has cohomological dimension 1 if and only if it is free.

In this note we continue to study semigroups of cohomological dimension 1 (c.d. 1). In [4] an analog of the Stallings—Swan theorem [1] was proved: a cancellative semigroup of c.d. 1 can be embedded into a free group. However, we have not complete description even for a free semigroup, because its subsemigroups can both have and not have c.d. 1.

In [4] following results about ideals of free semigroups were obtained:

– c.d. of every proper two-side ideal is greater than 1;

– c.d. of every left ideal equals 1 if and only if it is free;

– c.d. of every principal right ideal equals 1 if and only if it is free; this is not true for non-principal right ideals.

In [4] a problem was proposed: describe principal quasi-ideals of the free semigroup having c.d. 1. We solve this problem below. The answer is the same as for ideals: a principal quasi-ideal has c.d. 1 if and only if it is free. Nevertheless the proof of this assertion is carried out in different ways for two kinds of quasi-ideals.

In what follows we shall denote by S a semigroup with an adjoined identity; by F a free non-commutative semigroup; by $|a|$ the length of a word $a \in F$; by $\langle a \rangle$ (resp. $\langle a \rangle_q$) the subsemigroup (resp. quasi-ideal) generated by element a.

We recall that a subset Q of a semigroup S is called a *quasi-ideal* [6] if $QS \cap SQ \subset Q$. A *principal quasi-ideal* generated by element $w \in S$ is a subset $\langle w \rangle_q = S^1 w \cap w S^1$. We need to separate the case when w is

2001 Mathematics Subject Classification: *20M05, 20M50.*

Key words and phrases: *cohomology of semigroups, cohomological dimension, quasi-ideal, free semigroup.*

not a power of any element different from w; such element w is called *primitive*.

The next properties of free semigroups will be used mostly without reference to them:

1) If $ab = cd$ for some $a, b, c, d \in F$ and $|a| \leq |c|$ then $c \in aF^1$.

2) A subsemigroup $S \subset F$ is not free if and only if $\alpha S \cap S \neq \emptyset \neq S\alpha \cap S$ for some $\alpha \in F \setminus S$ ([3], Prop. 5.2.2).

3) If $uw = wv$ for some $u, v \in F$, $w \in F^1$ then there are $x, y \in F^1$ and an integer $k \geq 0$ such that $u = xy$, $v = yx$, $w = (xy)^k x$ ([3], Lemma 11.5.1).

In particular:

4) If $uv = uv$ $(u, v \in F^1$ then $u = x^m$, $v = x^n$ for some $x \in F$ and integers $m, n \geq 0$.

First we shall study the structure of principal quasi-ideals.

Lemma 1. *Let $x, w \in F$, $S = \langle w \rangle_q$. If $xS \cap S \neq \emptyset$ then $xS \subset S$; if $Sx \cap S \neq \emptyset$ then $Sx \subset S$.*

Proof. Let $y \in xS \cap S \subset xwF^1 \cap wF^1$, i.e. $y = xwf = wg$ for some $f, g \in F^1$. Then $|g| \geq |f|$ whence $g = hf$ $(h \in F^1)$ and $xw = wh$. So

$$xS = xF^1 w \cap xwF^1 = xF^1 w \cap whF^1 \subset F^1 w \cap wF^1 = S.$$

The second part of Lemma is proved analogously. □

Theorem 1. *A quasi-ideal $\langle w \rangle_q$ $(w \in F)$ is free if and only if w is primitive.*

Proof. 1) Let $w = x^n$, $n \geq 2$. Since $|x| < |w|$, $x \notin \langle w \rangle_q$. On another hand

$$xw \in x\langle w \rangle_q \cap \langle w \rangle_q, \qquad wx \in \langle w \rangle_q x \cap \langle w \rangle_q,$$

i.e. $\langle w \rangle_q$ $(w \in F)$ is not free.

2) Let w is primitive and $\langle w \rangle_q$ $(w \in F)$ is not free. Then

$$x\langle w \rangle_q \cap \langle w \rangle_q \neq \emptyset \neq \langle w \rangle_q x \cap \langle w \rangle_q$$

for some $x \in F \setminus \langle w \rangle_q$. By Lemma 1 $x\langle w \rangle_q \cup \langle w \rangle_q x \subset \langle w \rangle_q$. In particular

$$xw \in wF^1, \qquad wx \in F^1 w, \tag{1}$$

so if $|x| \geq |w|$ then $x \in \langle w \rangle_q$, what is impossible.

Therefore $|x| < |w|$. It follows from (1) that $w = ux = xv$ for some $u, v \in F$. Then there are $a, b \in F^1$ and $k \geq 0$ such that $u = ab$, $v = ba$, $x = (ab)^k a$ whence $w = (ab)^{k+1} a$.

The inclusions (1) imply too that $xw = wt$ for some $t \in F^1$. Substituting the values of x and w in this equation and cancelling, we obtain: $(ab)^{k+1}a = bat$. Then $ab = ba$ because $|ab| = |ba|$. Therefore $a = c^p$, $b = c^q$ for some $c \in F$ and $p, q \geq 0$. Then $w = c^{(p+q)(k+1)+p}$. The primitivity of w implies $(p+q)(k+1) + p = 0$ whence $p = q = 0$ in contradiction with $w \in F$. □

Now we express arbitrary principal quasi-ideals by means of the free ones.

Lemma 2. *Let $a, b, w \in F$, w is primitive, $n \geq 2$ and $aw^n = w^n b$. Then either $a = b \in \langle w \rangle$ or $a = w^{n-1}x$, $b = yw^{n-1}$ for such $x, y \in F^1$ that $xw = wy$.*

Proof uses induction on n. Suppose that $|a| < |w|$. Then $w = aw_1$ and $(aw_1)^n = w_1(aw_1)^{n-1}b$. Since $|aw_1| = |w_1 a|$, the last equation implies $aw_1 = w_1 a$. Hence $a = t^p$, $w_1 = t^q$ ($t \in F$, $p, q \geq 0$). But $a \neq 1$, so $w_1 = 1$ and $w = a$ in contradiction with $|a| < |w|$.

Thus $|a| = |b| \geq |w|$ whence $a = wa_1$, $b = b_1 w$, $a_1 w^{n-1} = w^{n-1} b_1$. If $a_1 = 1$ then $b_1 = 1$ and $a = b = w$. Otherwise we get by induction either $a_1 = b_1 \in \langle w \rangle$ (and then $a = b \in \langle w \rangle$) or $a_1 = w^{n-2}x$, $b_1 = yw^{n-2}$ (and then $a = w^{n-1}x$, $b = yw^{n-1}$). □

Corollary 1. *Let w is primitive, $n \geq 2$. Then*

$$\langle w^n \rangle_q = w^{n-1} \langle w \rangle_q w^{n-1} \cup \{w^k \mid k \geq n\}. \quad \square$$

Now we pass to studying of cohomological dimension.

Recall that the nth cohomology group of semigroup S with values in a left S-module A is defined as $H^n(S, A) = \mathrm{Ext}^n_{\mathbf{Z}S}(\mathbf{Z}, A)$; another definition of semigroup cohomology in terms of cochains see, e. g. in [2] or [5]. The *cohomological dimension* of S (c.d.(S)) is the smallest integer n such that $H^k(S, A) = 0$ for every S-module A and $k > n$.

The next assertion is a start point for the further consideration:

Lemma 3. ([5], Prop. 3.2) *Let c.d.$(S) = 1$ and $\alpha S \cap S \neq \emptyset \neq S\alpha \cap S$ for some $\alpha \in F \setminus S$ (so S is not free). There exists $x \in \alpha S \cap S$ such that for every $u \in \alpha S \cap S$ one can choose $\lambda_1, \ldots, \lambda_n \in F^1$ satisfying the next conditions:*

(i) $x\lambda_i \in \alpha S \cap S \qquad (1 \leq i \leq n)$,

(ii) $S^1 \cap \lambda_1 S^1 \neq \emptyset, \quad \lambda_i S^1 \cap \lambda_{i+1} S^1 \neq \emptyset \qquad (1 \leq i < n)$,

(iii) $u = x\lambda_n$. □

For quasi-ideals this lemma is modified as follows:

Lemma 4. *Let a quasi-ideal $S = \langle w \rangle_q \subset F$ is not free and c.d.$(S) = 1$. Then for every $\lambda \in F^1$ from $w\lambda \in F^1 w$ it follows $\lambda S \subset S$.*

Proof. First note that in situation when $S = \langle w \rangle_q$, one can set $n = 1$ in Lemma 3. Indeed, $\lambda_1 S \subset S$ (see Lemma 1), so it follows from $\lambda_1 S \cap \lambda_2 S \neq \emptyset$ that $\lambda_2 S \cap S \neq \emptyset$, i. e. $\lambda_2 S \subset S$. Repeating this reasoning we get at last $\lambda_n S \subset S$. But then the sequence $\lambda_1, \dots, \lambda_n$ can be replaced by the single element $\lambda = \lambda_n$ with the conditions (i) – (iii) be preserved (the condition (ii) turns into $\lambda S \subset S$).

Further, let $\alpha S \cap S \neq \emptyset \neq S\alpha \cap S$. Then an element x from Lemma 3 has the least length in $\alpha S \cap S = \alpha S = \alpha F^1 w \cap \alpha w F^1$ accordingly to (iii). Hence $x = \alpha w$. Now setting $u = \alpha t$ ($t \in S$), we can rewrite the conclusion of Lemma 3 in the form:

$$\text{for every } t \in S \text{ there is } \lambda \in F^1 \text{ such that} \quad \begin{array}{lrcl} \text{(a)} & \lambda S & \subset & S, \\ \text{(b)} & t & = & w\lambda. \end{array} \tag{2}$$

Evidently, here λ is defined uniquely by given t.

Now we can finish the proof of Lemma. Let $w\lambda \in F^1 w$. Then $w\lambda \in F^1 w \cap w F^1 = S$. Applying (2) to $t = w\lambda$, we obtain $\lambda S \subset S$. □

Every principal quasi-ideal can be written in the form $S = \langle w^n \rangle_q$ where w is primitive and $n \geq 1$. If $n = 1$, S is free (Theorem 1) and hence c.d.$(S) = 1$ (see, e. g. [2]). Therefore we suppose further on that $n \geq 2$. We shall show that c.d.$(S) > 1$, but the proof depends on if the word w can be presented in the form aba or not.

Theorem 2. *Let $S = \langle w^n \rangle_q \subset F$, $n \geq 2$, w is primitive and $w = aba$ for some $a, b \in F$. Then c.d.$(S) > 1$.*

Proof. Set $\lambda = baw^{n-1}$. Then

$$w^n \lambda = w^{n-1} ababaw^{n-1} = w^{n-1} abw^n \in F^1 w^n$$

Show that $\lambda S \not\subset S$. Indeed, let $t \in S$ and $\lambda t \in S$. Then $\lambda t = w^n f$ for some $f \in F^1$, i. e. $baw^{n-1} t = abaw^{n-1} f$. From here $ba = ab$, so $a = c^p, b = c^q, w = c^{2p+q}$ ($c \in F$) in contradiction with primitivity of w.

Therefore the conclusion of Lemma 4 is not valid and c.d.$(S) > 1$. □

Now consider the second kind of quasi-ideals.

Lemma 5. *Let a primitive word w cannot be written in the form $w = aba$, $a, b \in F$. Then*

$$\langle w \rangle_q = wF^1 w \cup \{w\}.$$

Proof. Let $t \in \langle w\rangle_q \setminus (wF^1w \cup \{w\})$. Then $t = uw = wv$ and $u \neq 1 \neq v$ since $t \neq w$. Hence $u = xy$, $v = yx$, $w = (xy)^k x$ $(x, y \in F^1)$. Consider various values of k.

1) $\underline{k = 0}$. Then $w = x$ and $t = uw = wyw \in wF^1w$, what is impossible.

2) $\underline{k = 1}$. Then $w = xyx$ and $x = 1$ because of primitivity. Hence $t = uw = w^2 \in wF^1w$; contradiction.

3) $\underline{k > 1}$. Then $w = x \cdot y(xy)^{k-1} \cdot x$. Again $x = 1$ and $w = y^k$ contrary to primitivity. □

Remark. The converse is true too: if $w = aba$ then $ababa \in \langle w\rangle_q \setminus (wF^1w \cup \{w\})$ whence $\langle w\rangle_q \neq wF^1w \cup \{w\}$.

Lemma 6. *Let w is the same as in Lemma 5. Then the semigroup $T_n = \langle w^n\rangle_q \cup \langle w\rangle$ is free for all $n \geq 1$.*

Proof is fulfilled by induction on n. For $n = 1$ the assertion follows from Theorem 1 since $T_1 = \langle w\rangle_q$.

Let T_n is free. Accordingly to Corollary 1

$$\begin{aligned} T_{n+1} &= w^n\langle w\rangle_q w^n \cup \langle w\rangle = w(w^{n-1}\langle w\rangle_q w^{n-1} \cup \langle w\rangle)w \cup \{w, w^2\} \\ &= wT_nw \cup \{w, w^2\} = wT_n^1 w \cup \{w\}. \end{aligned}$$

Since T_n is free and $T_{n+1} \subset T_n$, this equality and Lemma 5 imply T_{n+1} be coinciding with the quasi-ideal generating by w in T_n. By Theorem 1 T_{n+1} is free. □

Theorem 3. *Let a primitive word w cannot be written in the form $w = aba$, $a, b \in F$. Then c.d.$\langle w^n\rangle_q > 1$ for $n \geq 2$.*

Proof. We use the fact that every proper subsemigroup $S \subset F$ of finite defect (i. e. $|F \setminus S| < \infty$) has c.d. > 1 ([5], Example 3.5). It follows immediately from here that c.d.$\langle w^n\rangle_q > 1$ $(n \geq 2)$ since $1 \leq |T_n \setminus \langle w^n\rangle_q| < n$. □

Joining Theorems 2 and 3 we obtain finally:

Theorem 4. *A principal quasi-ideal of a free non-commutative semigroup has c.d. 1 if and only if it is free.* □

References

[1] K. S. Brown. *"Cohomology of Groups"*. Springer-Verlag, 1982.

[2] H. Cartan, S. Eilenberg. *"Homological Algebra"*. Princeton, 1956.

[3] G. Lallement. *"Semigroups and Combinatorial Applications"*. Wiley & Sons, NY, 1979.

[4] B. V. Novikov. *Semigroups of cohomological dimension one.* J. Algebra **204**(1998), no. 2, 386–393

[5] B. V. Novikov. *Ideals of free semigroups having cohomological dimension 1.* Semigroup Forum, **65**(2002), N2, 225–232.

[6] O. Steinfeld. *“Quasi-ideals in rings and semigroups”.* Akadémiai Kiado, Budapest, 1978.

Contact information

B. V. Novikov
Saltovskoye shosse 258, apt. 20, 61178 Kharkov, Ukraine
E-Mail: boris.v.novikov@univer.kharkov.ua

Received by the editors: 10.02.2003.