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Variants of a lattice of partitions of a countable set

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Abstract. We consider lattice Part_M of partitions of a countable set M, ordered by inclusion. It forms a semigroup with respect to the operation $\wedge$ mapping two elements to their greatest lower bound. We obtain necessary and sufficient conditions for isomorphism of two variants of Part_M.

Introduction

Let S be a semigroup and $a \in S$ be fixed. For arbitrary $x, y \in S$ we define $x *_a y = xay$. This binary operation $*_a$ on S is associative, which is clearly, is called a sandwich-operation, and semigroup $(S, *_a)$ is called a variant of the semigroup S or a sandwich-semigroup of the semigroup S with the sandwich-element a.

The study of variants was initiated in the acclaimed Lyapin's monograph [1]. Although Lyapin formulated this notion for transformation semigroups, later on various authors studied variants of other classes of semigroups (see, for example, [2], [3], [4], [5], chapter 13 of monograph [6], and references therein).

In this paper we consider variants of an ordered by inclusion lattice Part_M of partitions of a countable set M, which is a semigroup under the operation $\wedge$ of taking the greatest lower bound of two elements.

Note that for partitions ρ and τ the relation $\rho \leq \tau$ holds if and only if $\rho \wedge \tau = \rho$.

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The main result of the paper is Theorem 2, which gives an isomorphism criterion for variants of a lattice Part_M.

1. Auxiliary results

Recall that the *height* of a partially ordered set L is defined as the least upper bound of lengths of chains of L. In a partially ordered set with zero 0 the height of interval $[0, a]$ is called a *rank* $\mathrm{rank}\,(a)$ of element a. An element b is called an *atom* if $\mathrm{rank}\,(b) = 1$. For the partition $\rho \in \mathrm{Part}_M$ the number of atoms in the interval $[0, \rho]$ will be denoted by $a(\rho)$. By Part_k we denote a lattice of partition of a set of cardinality k.

The partition $\rho \in \mathrm{Part}_M$ has the *type* $\langle l_2, \ldots, l_k, \ldots, l_\infty\rangle$ if it contains l_2 blocks of cardinality 2, …, l_k blocks of cardinality k, …, l_∞ blocks of countable cardinality; the number of blocks of cardinality 1 is irrelevant. The partition, where one block has cardinality $k > 1$ and the remaining blocks are one-element, has the type $\langle \ldots, 0, 1_k, 0, \ldots, 0\rangle$. Thus, atoms of the lattice Part_M have the type $\langle 1_2, 0, \ldots, 0\rangle$.

The partition σ is a *covering* partition of ρ if $\sigma > \rho$ and there exists no partition χ such that $\sigma > \chi > \rho$.

The following Lemma is obvious.

Lemma 1. *Let ρ be a partition of the set M such that $M = \bigcup\limits_{i\in I} M_i$. Then an interval $[0, \rho]$ of the lattice Part_M is isomorphic to the Cartesian product $\prod\limits_{i\in I} \mathrm{Part}_{M_i}$ of lattices of partitions of blocks M_i, $i \in I$.*

The next lemma follows straightforwardly from Lemma 1 and the fact that for any natural number k, the height of the lattice Part_k is equal to $k - 1$.

Lemma 2. *Let ρ be a partition of n-element set A. Then the height of the interval $[0, \rho]$ of lattice Part_A is equal to $n - m$, where m is a number of blocks in partition ρ.*

Lemma 3. *Let ρ be a partition of rank $k - 1$, which contains more than one block of cardinality greater than 1. Then the interval $[0, \rho]$ and the lattice Part_k have different number of atoms. In particular, the interval $[0, \rho]$ and the lattice Part_k are non-isomorphic.*

Proof. Let M_1, M_2, …, M_n be all non-singleton blocks of a partition ρ, of cardinalities m_1, m_2, …, m_n respectively. Then by Lemma 2

$$m_1 + m_2 + \cdots + m_n = n + k - 1. \tag{1}$$

Assume that the number of atoms of the interval $[0, \rho]$ and the lattice Part_k is the same. Then

$$\binom{m_1}{2} + \binom{m_2}{2} + \cdots + \binom{m_n}{2} = \binom{k}{2}. \tag{2}$$

Using equation (1), we can rewrite equation (2) as follows:

$$\sum_{i=1}^{n} m_i^2 = k^2 - k + \sum_{i=1}^{n} m_i = k^2 + n - 1. \tag{3}$$

On the other hand, squaring both sides of (1), we get

$$\sum_{i=1}^{n} m_i^2 + \sum_{i \neq j} m_i m_j = n^2 + k^2 + 1 + 2nk - 2n - 2k. \tag{4}$$

It follows from (3) and (4) that

$$\sum_{i \neq j} m_i m_j = n^2 + 2nk - 3n - 2k + 2. \tag{5}$$

Assume that $m_i = 1 + a_i$. Then

$$\sum_{i \neq j} m_i m_j = \sum_{i \neq j} (1 + a_i)(1 + a_j) = n^2 - n + 2(n-1) \sum_{i=1}^{n} a_i + \sum_{i \neq j} a_i a_j =$$
$$= n^2 - n + 2(n-1)(k-1) + \sum_{i \neq j} a_i a_j.$$

Hence, considering (5) we obtain:

$$\sum_{i \neq j} a_i a_j = (n^2 + 2nk - 3n - 2k + 2) - (n^2 - n + 2(n-1)(k-1)) = 0.$$

However, the sum $\sum_{i \neq j} a_i a_j$ must be positive by assumption. This contradiction proves the lemma. □

Lemma 4. *Let partitions μ and σ be covering partitions of a partition ρ, which has the type $\langle \ldots, 0, 1_k, 0, \ldots, 0 \rangle$. If μ and σ have different types, then intervals $[0, \mu]$ and $[0, \sigma]$ contain different number of partitions of rank k.*

Proof. The type of a covering partition of ρ is either $\langle \ldots, 0, 1_{k+1}, 0, \ldots, 0 \rangle$ or $\langle 1_2, 0, \ldots, 0, 1_k, 0, \ldots, 0 \rangle$. Without loss of generality we may assume that μ is a partition of the first type and σ is a partition of the second type.

The partition μ has a unique non-singleton block A of cardinality $k+2$.

Every partition of the interval $[0, \mu]$ of rank k can be obtained by partitioning the block A into two smaller blocks. Obviously, this can be done in $2^{k+1}-1$ ways.

Partition σ has two non-singleton blocks, namely a two-element block B and a block C of cardinality $k+1$. Partition of rank k of the interval $[0, \sigma]$ can be obtained by partitioning one of these blocks into two smaller blocks. The block B can be partitioned uniquely, and the block C can be partitioned in $2^k - 1$ ways. Hence, in this case we get 2^k partitions of rank k.

Since for every natural number k the inequality $2^{k+1}-1 > 2^k$ holds, the lemma is proved. □

Lemma 5. *Let partitions μ and σ be the partitions of types $\langle l_2, l_3, \ldots, l_\infty \rangle$ and $\langle t_2, t_3, \ldots, t_\infty \rangle$ respectively. If the intervals $[0, \mu]$ and $[0, \sigma]$ are isomorphic, then $l_k = t_k$ for each $k \in \mathbb{N}$.*

Proof. Let the partition μ have the form $M = \bigcup\limits_{i \in I} A_i$, and the partition σ, the form $M = \bigcup\limits_{j \in J} B_j$. Let $\varphi : [0, \mu] \to [0, \sigma]$ be an isomorphism.

Note that φ maps atoms to atoms, a covering of partition $\tau \in [0, \mu]$ to a covering of partition $\varphi(\tau)$. Also φ preserves ranks of elements.

For each k-element block A_i of the partition μ, there is a partition $\mu_{A_i} \in [0, \mu]$ of the type $\langle \ldots, 0, 1_{k-1}, 0, \ldots, 0 \rangle$; its blocks are A_i and singletons. Similarly, we define partitions σ_{B_j} for blocks of partition σ. Note that all covering partitions for μ_{A_i} have the type $\langle 1_2, 0, ..., 0, 1_{k-1}, 0, ..., 0 \rangle$.

By Lemma 3, the partition $\varphi(\mu_{A_i})$ has the type $\langle \ldots, 0, 1_{k-1}, 0, \ldots, 0 \rangle$; by Lemma 4, all covering partitions for $\varphi(\mu_{A_i})$ have the type $\langle 1_2, 0, \ldots, 0, 1_{k-1}, 0, \ldots, 0 \rangle$. Hence the image $\varphi(\mu_{A_i})$ of partition μ_{A_i} is a partition σ_{B_j} for some block B_j of cardinality k.

Since the inverse mapping φ^{-1} is an isomorphism, there is a one-to-one between k-element blocks of partition μ and those of partition σ.

□

Lemma 6. *A partition μ contains an infinite block if and only if the interval $[0, \mu]$ is an infinite increasing chain*

$$0 < \tau_1 < \tau_2 < \tau_3 < \cdots , \tag{6}$$

where for every k the partition τ_k has rank k and the interval $[0, \tau]$ contains $\binom{k+1}{2}$ atoms.

Proof. Let A be an infinite block of partition μ. We consider an infinite increasing chain

$$\{a_0, a_1\} \subset \{a_0, a_1, a_2\} \subset \{a_0, a_1, a_2, a_3\} \subset$$

of subsets $A_k = \{a_0, a_1, a_2, \ldots, a_k\}$ of the set A. For each subset A_k consider a partition $\mu_{A_k} \in [0, \mu]$ such that its blocks are A_k and singletons. Then the chain of partitions

$$0 < \mu_{A_1} < \mu_{A_2} < \mu_{A_3} < \cdots$$

is as required.

Conversely, let the interval $[0, \mu]$ be an infinite increasing chain (6) satisfying the statement of lemma. By Lemma 3, every partition τ_k has only one non-singleton block A_k. Since the partition τ_k is a chain, then the blocks A_k form a chain by inclusion. Hence the set $A = \bigcup_{k \geq 1} A_k$ is an infinite block of partition μ. □

Lemma 7. *Let the types of partitions μ and σ be $\langle l_2, l_3, \ldots, l_\infty \rangle$ and $\langle t_2, t_3, \ldots, t_\infty \rangle$ respectively. If the intervals $[0, \mu]$ and $[0, \sigma]$ are isomorphic, then $l_\infty = t_\infty$.*

Proof. It follows from Lemma 3 that the partial order relation in the lattice Part_M is sufficient to identify the partitions of finite rank, which have unique non-singleton block. Hence it is enough to show that $l_\infty = k \in \mathbb{N}$ if and only if in $[0, \mu]$ there are k chains $L_1, \ldots, L_k$ such that

(i) every chain L_i satisfies the conditions of Lemma 6;

(ii) if partitions $\tau' \neq 0$ and $\tau'' \neq 0$ belong to different chains and $\nu > \tau'$, $\nu > \tau''$, then ν contains at least two non-singleton blocks;

but there is no $k + 1$ chains satisfying these conditions.

Let $l_\infty = k$ and $N_1, \ldots, N_k$ be all infinite blocks of partition μ. According to the proof of Lemma 6, for every infinite block N_i we can construct a chain L_i in $[0, \mu]$ satisfying the conditions of Lemma 6. Moreover, the chains corresponding to different blocks satisfy condition (ii). Therefore, if $l_\infty = k$, then in $[0, \mu]$ there exist k chains satisfying conditions (i) and (ii).

Suppose that in $[0, \mu]$ there exist $k + 1$ chains $L_1, \ldots, L_{k+1}$, which satisfy conditions (i) and (ii). The proof of Lemma 6 implies that every chain of this kind

$$0 < \tau_1^i < \tau_2^i < \tau_3^i < \cdots$$

is formed by partitions of finite rank having only one non-singleton block, and all the non-singleton blocks are subsets of some infinite block. Since there are only k infinite blocks and there are $k+1$ chains, then there exist two chains which correspond to the same infinite block. Without loss of generality we assume that chains L_1 and L_2 correspond to the same block N_1. We can choose arbitrary non-zero elements $\tau_i^1 \in L_1$ and $\tau_j^2 \in L_2$.

Let C_1 and C_2 be non-singleton blocks of partitions τ_i^1 and τ_j^2 respectively. Since $C_1 \subset N_1$ and $C_2 \subset N_1$, then $C_1 \cup C_2 \subset N_1$. Hence the interval $[0, \mu]$ contains the partition ν with only one non-singleton block $C_1 \cup C_2$. But $\nu > \tau_i^1$ and $\nu > \tau_j^2$, which contradicts to (ii). □

Let a be a fixed element of a commutative band S with zero. Let $S_{[a,b]} = \{x \in S | x \cdot a = a,\ b \cdot x = x\}$. For every element $x \in S_{[a,b]}$ we define the set $\Omega(x) = \{y \in S | a \cdot y = x\}$, and the weight $\omega(x)$ of the element x by $\omega(x) = |\Omega(x)|$.

Theorem 1 ([7]). *Two variants $(S, *_a)$ and $(S, *_b)$ of a commutative band S with zero are isomorphic if and only if there exists a weight-preserving isomorphism of intervals $S_{[0,a]}$ and $S_{[0,b]}$.*

2. Main result

Proposition 1. *Intervals $[0, \mu]$ and $[0, \sigma]$ of a lattice Part_M of partitions of a countable set M are isomorphic if and only if partitions μ and σ have the same type.*

Proof. It follows from Lemmas 5, 7 and 1. □

Lemma 8. *If partitions μ and τ have the same type and the same number of one-element blocks, then there exists an isomorphism of intervals $[0, \mu]$ and $[0, \sigma]$ induced by the permutation of the set M.*

Proof. Let partitions μ and τ be of the same type and have the same number of one-element blocks. In this case blocks of both partitions can be indexed by the same set of indexes I ($(A_i)_{i \in I}$ for partition μ and $(B_i)_{i \in I}$ for partition τ). Moreover, for every $i \in I$ blocks A_i and B_i have the same cardinality. Let φ_i be a bijection from A_i to B_i. Since $M = \bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i$, then $\varphi = \bigcup_{i \in I} \varphi_i$ is a permutation of the set M, which maps the blocks of partition μ to the blocks of partition τ. Since by Lemma 1 intervals $[0, \mu]$ and $[0, \tau]$ are isomorphic to the Cartesian products $\prod\limits_{i \in I} \mathrm{Part}_{A_i}$ and $\prod\limits_{i \in I} \mathrm{Part}_{B_i}$ respectively, then permutation φ induces an isomorphism of this intervals. □

Note that semigroup Part_M is a commutative band with zero. Thus, we can consider the weight $\omega(\chi)$ of the element $\chi \in [0, \rho]$. By definition it equals the cardinality of the set $\Omega(\chi) = \{\xi \in \mathrm{Part}_M \mid \rho \ \wedge \xi = \chi\}$.

Proposition 2 (On the weight of partitions)**.** *a) If the partition ρ contains an infinite number of blocks, then the weight $\omega(\chi)$ of every element $\chi \in [0, \rho]$ is continuum.*

b) If the partition ρ contains a finite number n of blocks, then in the interval $[0, \rho]$ the weight $\omega(\rho)$ of the element ρ is equal to the n-th Bell number B_n.

Proof. a) Let partition ρ of the set M be of the form $M = \bigcup\limits_{i \in I} M_i$, where I is infinite, and partition χ from the interval $[0, \rho]$ be of the form $M = \bigcup\limits_{i \in I} \bigcup\limits_{j \in J_i} N_{ij}$, where $M_i = \bigcup\limits_{j \in J_i} N_{ij}$ is a partition of the block M_i. Consider a partition ξ such that its blocks have the form of union of blocks N_{ij} and none block of a partition ξ contains two blocks N_{ij} with the same first index. Since the set I is countable, the number of such partitions is continuum. On the other hand, the intersection of a block of partition ξ and a block of partition ρ is either empty or a block of the form N_{ij}, i.e. it is a block of a partition χ. Hence $\rho \wedge \xi = \chi$. Thus, the set $\Omega(\chi)$ contains continuum elements, hence the weight $\omega(\chi)$ is continuum.

b) In this case the set $\Omega(\rho)$ is a set of $\xi \in \mathrm{Part}_M$ such that the condition $\rho \wedge \xi = \rho$ holds, in other words, of such ξ which belongs to the interval $[\rho, 1]$. Since the partition ρ consists of n blocks, then the interval $[\rho, 1]$ is isomorphic to the lattice Part_n and its cardinality is equal to B_n. □

We get an immediate corollary from the Proposition 1 and 2.

Corollary 1. *Let intervals $[0, \mu]$ and $[0, \sigma]$ be isomorphic. If partitions μ and σ have the same weight, then one of the following two statements holds:*

(1) both partitions μ and σ have countable number of blocks;

(2) partitions μ and σ have the same finite number of blocks and the same type. In particular, they have the same number of one-element blocks.

Theorem 2 (Isomorphism criterion for variants of the lattice of partitions)**.** *Let Part_M be the lattice of partitions of a countable set M. Variants $(\mathrm{Part}_M, *_\mu)$ and $(\mathrm{Part}_M, *_\sigma)$ of Part_M are isomorphic if and only if partitions μ and σ have the same type and the same number of blocks.*

Proof. By Theorem 1, variants $(\mathrm{Part}_M, *_\mu)$ and $(\mathrm{Part}_M, *_\sigma)$ are isomorphic if and only if there exists a weight-preserving isomorphism between $[0, \mu]$ and $[0, \sigma]$. Since partitions μ and σ are the greatest elements in intervals $[0, \mu]$ and $[0, \sigma]$ respectively, then the isomorphism maps μ to σ. In particular, μ and σ have the same weight.

By Proposition 2, the variants $(\mathrm{Part}_M, *_\mu)$ and $(\mathrm{Part}_M, *_\sigma)$ are isomorphic if both partitions μ and σ have either a countable number of blocks or the same finite number of blocks. Consider each of these cases.

(1) Partitions μ and σ have the countable number of blocks. By Proposition 1, the intervals $[0, \mu]$ and $[0, \sigma]$ are isomorphic if and only if partition μ and σ have the same type. Furthermore, by Proposition 2, all elements in this intervals have continual weights. Hence any isomorphism from $[0, \mu]$ on $[0, \sigma]$ is weight-preserving. Consequently, the variants $(\mathrm{Part}_M, *_\mu)$ and $(\mathrm{Part}_M, *_\sigma)$ are isomorphic if and only if partitions μ and σ have the same type.

(2) Partitions μ and σ have the same finite number of blocks. Since these partitions have the same type, they have the same number of one-element blocks. Hence, by Lemma 8, there exists an isomorphism between the intervals $[0, \mu]$ and $[0, \sigma]$, induced by a permutation of the set M. Obviously, this isomorphism is weight-preserving. Therefore, the variants $(\mathrm{Part}_M, *_\mu)$ and $(\mathrm{Part}_M, *_\sigma)$ are isomorphic if and only if partitions μ and σ have the same type and the same number of blocks. $\square$

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