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Zero-sum subsets of decomposable sets in Abelian groups

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Abstract. A subset D of an abelian group is *decomposable* if $\emptyset \neq D \subset D + D$. In the paper we give partial answers to an open problem asking whether every finite decomposable subset D of an abelian group contains a non-empty subset $Z \subset D$ with $\sum Z = 0$. For every $n \in \mathbb{N}$ we present a decomposable subset D of cardinality $|D| = n$ in the cyclic group of order $2^n - 1$ such that $\sum D = 0$, but $\sum T \neq 0$ for any proper non-empty subset $T \subset D$. On the other hand, we prove that every decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| \leq 7$ contains a non-empty subset $T \subset D$ of cardinality $|Z| \leq \frac{1}{2}|D|$ with $\sum Z = 0$. For every $n \in \mathbb{N}$ we present a subset $D \subset \mathbb{Z}$ of cardinality $|D| = 2n$ such that $\sum Z = 0$ for some subset $Z \subset D$ of cardinality $|Z| = n$ and $\sum T \neq 0$ for any non-empty subset $T \subset D$ of cardinality $|T| < n = \frac{1}{2}|D|$. Also we prove that every finite decomposable subset D of an Abelian group contains two non-empty subsets A, B such that $\sum A + \sum B = 0$.

Introduction

A subset D of an Abelian group G is called *decomposable* if $\emptyset \neq D \subset D+D := \{x+y : x, y \in D\}$. For a finite subset F of an Abelian group put $\sum F = \sum_{x \in F} x$. In this paper we discuss the following open problem, whose special case for the additive group of real numbers was posed by Gjergji Zaimi in 2010 on `MathOverflow` [3].

Problem 1. *Let D be a finite decomposable subset of an Abelian group G. Is $\sum Z = 0$ for some non-empty set $Z \subset D$?*

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Observe that the answer to Problem 1 is affirmative if the group G is *Boolean* (which means that $x + x = 0$ for any $x \in G$). Indeed, take any element $a \in D$ and find elements $b, c \in D$ with $a = b + c$. If $0 \in \{a, b, c\}$, then $T = \{0\} \subset \{a, b, c\} \subset D$ is a subset with $\sum T = 0$. If $0 \notin \{a, b, c\}$, then the elements a, b, c are pairwise distinct and the set $T = \{a, b, c\}$ has $\sum T = a + b + c = 2(b + c) = 0$. Therefore, any decomposable set D in a Boolean group contains a subset $T \subset D$ of cardinality $|T| \le 3$ with $\sum T = 0$.

This simple upper bound does not hold for decomposable sets in arbitrary groups.

Example 2. *Let $n \ge 2$ and G be a cyclic group of order $2^n - 1$ with generator g. The set $D = \{2^k g : 0 \le k < n\}$ is decomposable and has $\sum D = 0$ but $\sum T \ne \emptyset$ for any subset $T \subset D$ of cardinality $0 < |T| < |D| = n$.*

Example 3. *For every $n \in \mathbb{N}$ the subset*

$$D_n := \{2^k : 0 \le k < n\} \cup \{2^k - 2^n + 1 : 0 \le k < n\}$$

of cardinality $|D_n| = 2n$ in the infinite cyclic group $\mathbb{Z}$ is decomposable and the subset

$$Z = \{2^k : 1 \le k < n\} \cup \{2 - 2^n\} \subset D_n$$

of cardinality $|Z| = n$ has $\sum Z = 0$. On the other hand, $\sum T \ne 0$ for every non-empty subset $T \subset D_n$ of cardinality $|T| < n$.

The properties of the set D_n from Example 3 will be established in Section 1.

The above examples suggest to assign to every finite subset D of an Abelian group G the largest number $z(D) \le |D| + 1$ such that $\sum T \ne 0$ for any non-empty subset $T \subset D$ of cardinality $|T| < z(D)$. Therefore, $z(D)$ is equal to the smallest cardinality of a subset $Z \subset D$ with $\sum Z = 0$ if such set Z exists and $z(D) = |D| + 1$ in the opposite case.

In terms of the number $z(D)$, Problem 1 can be reformulated as follows.

Problem 4. *Let D be a finite decomposable subset of an Abelian group. Is $z(D) \le |D|$?*

The decomposable set D_n from Example 2 has $z(D) = |D| = n$ and the decomposable set $D_n \subset \mathbb{Z}$ from Example 3 has $z(D_n) = n = \frac{1}{2}|D_n|$.

This suggests the following refinement of Problems 1 and 4 for the infinite cyclic group $\mathbb{Z}$.

Problem 5. *Is $z(D) \le \frac{1}{2}|D|$ for any finite decomposable subset $D \subset \mathbb{R}$?*

A special case of this problem was posed by Aryabhata in [1].

Problem 6 (Aryabhata)**.** *Is $z(D) \le 7$ for any decomposable subset $D \subset \mathbb{Z}$ of cardinality $|D| = 15$?*

In Section 2 we shall provide an affirmative answer to Problem 5 for decomposable subsets $D \subset \mathbb{Z}$ of cardinality $|D| \le 7$.

Proposition 7. *Any decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| \le 7$ has $z(D) \le \frac{1}{2}|D|$.*

We also observe that for $n \in \{2,3\}$ the decomposable set $D_n = \{2^k, 2^k - 2^n + 1 : 0 \le k < n\}$ form Example 3 is unique in the following sense.

Proposition 8. *Every decomposable set $D \subset \mathbb{R}$ with $n = z(D) = \frac{1}{2}|D| \in \{2,3\}$ is equal to $D_n \cdot r$ for some real number r.*

Problem 9. *Is every finite decomposable set $D \subset \mathbb{R}$ with $z(D) = \frac{1}{2}|D| = n \ge 2$ equal to $D_n \cdot r$ for some real number r?*

The following proposition shows that the problems on fiinite decomposable sets in torsion-free Abelian groups can be reduced to the case of the infinite cyclic group.

Proposition 10. *For any finite decomposable set D is a torsion-free Abelian group G there exists a decomposable set $D' \subset \mathbb{Z}$ such that $|D'| = |D|$ and $z(D') = z(D)$.*

Proof. We lose no generality assuming that G is finitely-generated and hence is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Let $e_1, \dots, e_n$ be the standard generators of the group $\mathbb{Z}^n$. Find $m \in \mathbb{N}$ such that $\{\sum F : F \subset D\} \subset \{\sum_{i=1}^n x_i e_i : (x_i)_{i=1}^n \in [-m,m]^n\} = [-m,m]^n$. Consider the homomorphism $h : \mathbb{Z}^n \to \mathbb{Z}$ such that $h(e_i) = (2m+1)^i$ for all $1 \le i \le n$. It is easy to see that the restriction $h\restriction[-m,m]^n$ is injective. Consequently, a subset $F \subset D$ has $\sum F = 0$ if and only if $\sum h(F) = 0$. This implies that for the set $D' = h(D)$ we have the equalities $|D'| = |D|$ and $z(D') = z(D)$. $\square$

Our final result provides an affirmative answer to a weak version of Problem 1. The following theorem will be proved in Section 3.

Theorem 11. *For any finite decomposable subset D of an Abelian group there are two non-empty sets $A, B \subset D$ such that $\sum A + \sum B = 0$.*

Corollary 12. *For any finite decomposable subset D of an Abelian group, there exists a non-empty subset $T \subset D$ and a function $f : T \to \{1, 2\}$ such that $\sum_{x \in T} f(x) \cdot x = 0$.*

A decomposable subset D of an Abelian group is called *minimal decomposed* if no proper subset of D is decomposed. It is clear that every finite decomposed set contains a minimal decomposed set.

Corollary 12 can be compared with the following result that was essentially proved by Hsien-Chih Chang [2] in his answer to the problem of Zaimi [3].

Proposition 13. *For every finite minimal decomposable subset D of an Abelian group there exists a function $f : D \to \omega$ such that $\sum_{x \in D} f(x) = |D|$ and $\sum_{x \in D} f(x) \cdot x = 0$.*

Proof. By the decomposability of D, there exist functions $\alpha, \beta : D \to D$ such that $x = \alpha(x) + \beta(x)$ for every $x \in X$. For every $x \in D$ let $g(x) = |\alpha^{-1}(x)| + |\beta^{-1}(x)|$ and observe that $\sum_{x \in D} g(x) = |D| + |D|$. The minimal decomposability of D ensures that $D = \alpha(D) \cup \beta(D)$ and hence $f(x) := g(x) - 1 \geq 0$ for every $x \in D$. Then $\sum_{x \in D} f(x) = (|D| + |D|) - |D| = |D|$. It follows that $\sum D = \sum_{x \in D}(\alpha(x) + \beta(x)) = \sum_{y \in D} g(y) \cdot y$ and hence $\sum_{x \in D} f(x) \cdot x = \sum_{x \in D}(g(x) - 1) \cdot x = 0$. □

1. Properties of the decomposable set in Example 3

Given a natural number n consider the subsets $A = \{1, 2, 4, \dots, 2^{n-1}\}$ and

$$D_n := A \cup (A - (2^n - 1))$$

in the group $\mathbb{Z}$ of integers. Then $|D_n| = 2n$.

The set D_n is decomposable, because, clearly, each element of the set $A \setminus \{1\}$ is decomposable, $1 = 2^{n-1} + (2^{n-1} - (2^n - 1))$, $2^k - (2^n - 1) = 2^{k-1} + (2^{k-1} - (2^n - 1))$ for every positive $k < n$, and $1 - (2^n - 1) = (2^{n-1} - (2^n - 1)) + (2^{n-1} - (2^n - 1))$.

It is clear that the set $Z = \{2^k : 1 \leq k < n\} \cup \{2^n - 2\}$ has cardinality $|Z| = n = \frac{1}{2}|D_n|$ and $\sum Z = 0$.

Next, we prove that every subset $T \subset D_n$ of cardinality $|T| < n$ has $\sum T \neq 0$. Assuming that $\sum T = 0$, we conclude that $n \geq 3$ and T contains at most $n - 2$ positive elements. Since all of them are distinct elements of A, their sum is at most $2^n - 4$. On the other hand, the largest negative element of the set $A - (2^n - 1)$ is $-2^{n-1} + 1 = -(2^n - 2)/2$. Thus if T contains at least two negative elements then $\sum T < 0$. If T contains exactly one negative element $2^k - 2^n + 1$ then $\sum T = 0$ implies that we have a representation of $2^n - 1$ as a sum of at most $n-1$ powers of 2 with at most one power used twice. This representation collapses to a sum of at most $n-1$ distinct powers of 2. If the representation contains a power 2^l with $l \geq n$ then it is bigger than $2^n - 1$. Otherwise the sum is at most $2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 2 < 2^k - 1$. Thus $z(D_n) \geq n$.

2. Proof of Propositions 7 and 8

We divide the proof of Propositions 7 and 8 into five lemmas. In fact, Proposition 8 follows from Lemmas 15 and 18 proved below.

Lemma 14. *Every decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| \leq 3$ contains zero and hence has $z(D) = 1 \leq \frac{1}{2}|D|$.*

Proof. To derive a contradiction, assume that $0 \notin D$. Replacing D by $-D$, if necessary, we can assume that D contains a unique positive element p. Since $p = a + b$ for some elements $a, b \in D$, one of the numbers a or b is positive, so it equals p and the other summand is zero. □

The following lemma was proved by Ingdas [3]. We present a short proof for convenience of the reader.

Lemma 15 (Ingdas)**.** *Any decomposable subset $D \subset \mathbb{R}$ with $|D| = 4$ and $z(D) \geq 2$ is equal to the set $\{-2, -1, 1, 2\}\cdot r = D_2 \cdot r$ for some positive real number r.*

Proof. Since $z(D) \geq 2$, the set D does not contain zero. Then D should contains at least two positive numbers and at least two negative numbers (otherwise D will be not decomposable). Since $|D| = 4$, the set D contains exactly two positive and two negatice numbers. Let c be the largest positive element of D. Since D is decomposable, $c = a + b$ for some $a, b \in D$. Since D does not contain zero, the maximality of c ensures that the elements a, b are strictly positive and hence coincide with the unique positive element r of the set $D \setminus \{c\}$. Therefore, $c = a + b = 2r$. By the same reason, the subset $\{x \in D : x < 0\}$ of negative numbers is equal to $\{2n, n\}$ for some negative real number n. Write the element r as $r = x + y$ for some $x, y \in D$ with $x \leq y$. Taking into account that D contains no zero and $r, 2r$ are unique positive elements of D, we conclude that $y = 2r$ and then $x = r - y = r - 2r = -r \in \{2n, n\}$. If $-r = 2n$, then $D = \{-r, -\frac{1}{2}r, r, 2r\}$ is not decomposable as $-\frac{1}{2}r \notin D + D$. So, $r = -n$ and hence $D = \{-2r, -r, r, 2r\} = \{-2, -1, 1, 2\}\cdot r$. □

Lemma 16. *Any decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| \in \{4, 5\}$ contains a subset $T \subset D$ of cardinality $|Z| \in \{1, 2\}$ with $\sum Z = 0$. Consequently, $z(D) \leq 2 \leq \frac{1}{2}|D|$.*

Proof. If $0 \in D$, then $Z = \{0\}$ has $\sum Z = 0$ and witnesses that $z(D) = 0 \le \frac{1}{2}|D|$.

So, assume that $0 \notin D$. Replacing D by $-D$ we can assume that D has at most two positive elements. Let c be the largest positive element of D. Since D is decomposable, $c = a + b$ for some $a, b \in D$. Since D does not contain zero, the maximality of c ensures that the elements a, b are strictly positive and hence coincide with the unique positive element of the set $D \setminus \{c\}$. Therefore, $c = a + b = 2a$. Write the element a as $a = x + y$ for some $x, y \in D$ with $x \le y$. Taking into account that D contains no zero and a, c are unique positive elements of D, we conclude that $y = c$ and then $x = a - y = a - c = a - 2a = -a$. Then the set $Z = \{-a, a\}$ has $\sum Z = 0$ and witnesses that $z(D) \le 2 \le \frac{1}{2}|D|$. □

Remark 17. It can be shown that each decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| = 5$ with $0 \notin D$ is equal to one of the sets

$$\{-3, -2, -1, 1, 2\} \cdot r, \quad \{-4, -2, -1, 1, 2\} \cdot r, \quad \{-3, -2, -1, 2, 4\} \cdot r$$

for some non-zero number r.

Lemma 18. *Any decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| = 6$ with $z(D) \ge 3$ coincides with $\{1, 2, 4, -3, -5, -6\} \cdot r$ for some real number r and hence has $z(D) = 3 = \frac{1}{2}|D|$.*

Proof. Let D be a decomposable set consisting of six real numbers. If $0 \in D$ then $z(D) = 1$. If D contains exactly one (resp. two) positive (or negative) elements then similarly to the case $n \le 3$ (resp. $4 \le n \le 5$) we can show that $z(D) = 1$ (resp. $z(D) \le 2$).

So it remains to consider the case when D consists of three positive and three negative numbers. Let $p_{\max}$ (resp. $n_{\min}$) be the largest positive (resp. the smallest negative) element of D. Write $p_{\max}$ as $p_{\max} = p + p'$ for some numbers $p, p' \in D$ with $p \le p'$. By the maximality of $p_{\max}$, both numbers p and p' are positive. Assume that $p < p'$ and write $p = \bar{p} + a$, $p' = \bar{p}' + a'$ for some elements of D with positive $\bar{p}$ and $\bar{p}'$. If $\bar{p} = p_{\max}$ (resp. $\bar{p}' = p_{\max}$) then $p' + a = 0$ (resp. $p + a' = 0$), so $z(D) = 2$. Therefore, we can assume that $\bar{p} \ne p_{\max} \ne \bar{p}'$. In this case $\bar{p} = p'$ and $\bar{p}' = p$. Consequently, $a = p - \bar{p} = p - p' = \bar{p}' - p' = -a'$ and $Z = \{a, a'\}$ is a set with $\sum Z = 0$, witnessing that $z(D) \le 2$.

Thus it remains to consider the case when $p_{\max} = 2p'$ for some $p' \in D$ and, by the symmetry, $n_{\min} = 2n'$ for some $n' \in D$. Let p',$2p' = p_{\max}, p''$ be positive elements of the set D and n', $2n' = n_{\min}, n''$ be its negative elements. Then n' is a sum of two elements of D at least one of which is negative. If $n' = 2n' + x$ for some $x \in D$ then $n' + x = 0$ and $z(D) = 2$.

Thus $n' = n'' + a$ for some $a \in D$. If $n'' = n' + b$ for some $b \in D$ then $a + b = 0$ and $z(D) = 2$. Thus $n'' = 2n' + \bar{p}$ for some positive $\bar{p} \in D$. Similarly, $p' = p'' + a'$ for some $a' \in D$ and $p'' = 2p' + \bar{n}$ for some negative $\bar{n} \in D$. If $a > 0$ and $a' < 0$ then $\max\{\bar{n}, a'\} \le n'$ and $p' \le \min\{\bar{p}, a\}$. So

$$0 = n' + a + \bar{p} \ge n' + p' + p' > p' + n' > p' + n' + n' \ge p' + a' + \bar{n} = 0,$$

a contradiction. Thus $a < 0$ or $a' > 0$. If $a < 0$, then since $n' = n'' + a$ we have $n' = 2n''$. Similarly, if $a' > 0$, then $p' = 2p''$. Reverting the signs of elements of D, if needed, we can suppose that $p' = 2p''$. Then $\{p'', 2p'', 4p''\} \in D$, so $\bar{n} = -3p''$ and $n'' = \bar{p} + 2n'$. The following cases are possible:

1. $n' = \bar{n} = -3p''$, so $2n' = -6p''$.

1.1. If $\bar{p} = p''$ then $n'' = -5p''$ so the proposition claim holds.

1.2. If $\bar{p} = 2p''$ then $n'' = -4p''$, so $n'' + 4p'' = 0$ and $z(D) = 2$.

1.3. If $\bar{p} = 4p''$ then $n'' = -2p''$, so $n'' + 2p'' = 0$ and $z(D) = 2$.

2. $2n' = \bar{n} = -3p''$, so $n' = -1.5p''$.

2.1. If $\bar{p} = p''$ then $n'' = -2p''$, so $n'' + 2p'' = 0$ and $z(D) = 2$.

2.2. If $\bar{p} = 2p''$ then $n'' = -p''$, so $n'' + p'' = 0$ and $z(D) = 2$.

2.3. If $\bar{p} = 4p''$ then $n'' = p'' > 0$, a contradiction.

3. $n'' = \bar{n} = -3p''$. Then $\bar{p} + 2n' + 3p'' = 0$.

3.1. If $\bar{p} = p''$ then $n' = -2p''$, so $n' + 2p'' = 0$ and $z(D) = 2$.

3.2. If $\bar{p} = 2p''$ then $n' = -2.5p''$, so $2n'' = -5p''$ and $n' \notin D + D$, a contradiction.

3.3. If $\bar{p} = 4p''$ then $n' = -3.5p''$, so $2n'' = -7p''$ and $n' \notin D + D$, a contradiction.

□

Lemma 19. *Every decomposable subset $D \subset \mathbb{R}$ of cardinality $|D| = 7$ has $z(D) \le 3 \le \frac{1}{2}|D|$.*

Proof. Let D be a decomposable set consisting of seven real numbers. If $0 \in D$ then $z(D) = 1$. If D contains exactly one (resp. two) positive (or negative) elements then similarly to the case $n \le 3$ (resp. $4 \le n \le 5$) we can show that $z(D) = 1$ (resp. $z(D) \le 2$). So, reverting the signs of elements of D, if needed, it remains to consider the case when D consists of four positive and three negative numbers. Let $p_{\max}$ (resp. $n_{\min}$) be the largest positive (resp. the smallest negative) element of D. Similarly to the proof of Lemma 18 we can show that $n_{\min} = 2n'$ for some $n' \in D$.

Let n', $2n' = n_{\min}, n''$ be negative elements of the set D. Similarly to the proof of Lemma 18 we can show that $n' = n'' + a$ for some $a \in D$ and $n'' = 2n' + \bar{p}$ for some positive $\bar{p} \in D$. Then $n' + a + \bar{p} = 0$. If these numbers are distinct then $z(D) \le 3$. So we assume the converse. Since $n' = n'' + a$, $a \ne n'$, so $a = \bar{p} = n' - n'' = n'' - 2n'$ and $3n' = 2n''$. Divide all elements of the set D by $\frac{1}{2}|n'|$. Then it will have elements -2, -3, -4, and 1.

Depending on the representation of 1 as a sum of elements of D, the following cases are possible:

1. $1 = 3 - 2$, and $3 \in D$. Since $-3 \in D$, we have $z(D) \le 2$.

2. $1 = 4 - 3$, and $4 \in D$. Since $-4 \in D$, we have $z(D) \le 2$.

3. $1 = 5 - 4$, and $5 \in D$. Since $-2, -3 \in D$, we have $z(D) \le 3$.

4. 1 is a sum of distinct positive elements of D. Then $p_{\max} < 2$ so a sum of each positive and negative elements of D is negative. Then the smallest positive element of D does not belong to $D + D$, a contradiction.

5. $1 = 0.5 + 0.5$ and $0.5 \in D$. If the smallest positive element $p_{\min}$ of D is less than 0.5 then $p_{\max} \le 1 + 1 = 2$. Then a sum of each positive and negative elements of D is non-positive and $p_{\min} \notin D + D$, a contradiction. Thus $p_{\min} = 0.5$ and one of numbers 2.5, 3.5, and 4.5 belongs to D. But $2.5 + 0.5 + (-3) = 0$ and $3.5 + 0.5 + (-4) = 0$, so either $z(D) \le 3$ or 4.5 belongs to D. We assume the last case. Then $D' = \{-4, -3, -2, 0.5, 1, 4.5\} \subset D$. Let p be a unique element of $D \setminus D'$. Since $D' + D' \not\ni 4.5$, $4.5 = p + s$ for some $s \in D$. Since $p \ge 4.5/2 = 2.25$, $p \notin D' + D'$ and hence $p = 4.5 + s'$ for some $s' \in D$. Then $s + s' = 0$ and so $z(D) \le 2$.

□

3. Proof of Theorem 11

First we introduce some notation. For a function $f : X \to Y$ and subset $A \subset X$ put $f[A] = \{f(a) : a \in A\}$.

By a *tree* we understand any non-empty finite partially ordered set $(T, \leq)$ such that for every $x \in T$ the set ${\downarrow}x = \{t \in T : t \leq x\}$ is linearly ordered.

Let T be a tree. By $\min T$ we denote the smallest element of T and by $\max T$ the set of all maximal elements of T. A *branch* in a tree is a maximal linearly ordered subset $B \subset T$, which can be identified with the largest element of B.

For an element x of a tree T let ${\uparrow}x := \{y \in T : x \leq y\}$ and $\mathrm{succ}_T(x) := \min({\uparrow}x \setminus \{x\})$ be the set of immediate successors of x in the tree T. For any $x \in \max T$ we have $\mathrm{succ}_T(x) = \emptyset$. For any element $x \neq \min T$ let $\mathrm{prec}_T(x)$ be the unique element $y \in T$ such that $x \in \mathrm{succ}_T(y)$.

A tree T is called *binary* if for each $x \in T \setminus \max T$ the set $\mathrm{succ}_T(x)$ has cardinality 2.

For a branch B in a binary tree, let $\perp_B$: $B \setminus \max B \to T$ be the function assigning to each element $x \in B \setminus \max B$ the unique element of the set $\mathrm{succ}_T(x) \setminus B$.

A function $f : \max T \to T$ is called *regressive* if $f(x) < x$ for each $x \in \max T$.

Lemma 20. *For any regressive function $f : \max T \to T$ on a binary tree T there are distinct elements $x, y \in \max T$ such that $f(x) = f(y) = \max({\downarrow}x \cap {\downarrow}y)$.*

Proof. The proof if by induction on the height $\hbar(T) := \max\{|{\downarrow}x| : x \in T\}$ of the binary tree T. If $\hbar(T) = 1$, then no regressive function $f : \max T \to T$ exists, so the statement of the lemma holds.

Assume that the lemma has been proved for all binary trees of height $< n$. Take a binary tree T of height $n > 1$. Let $\min T$ be the root of T and x_1, x_2 be two immediate successors of $\min T$ in T. Then $T_1 := {\uparrow}x_1 \setminus \{\min T\}$ and $T_2 := {\uparrow}x_2 \setminus \{\min T\}$ are trees of height $< n$. Two cases are possible.

1. $f(\max T_i) \subset T_i$ for some $i \in \{1, 2\}$. In this case we can apply the inductive assumption and find two elements $x, y \in \max T_i$ such that $f(x) = f(y) = \max({\downarrow}x \cap {\downarrow}y)$.

2. For every $i \in \{1, 2\}$ there exists an element $t_i \in \max T_i$ such that $f(t_i) = \min T$. Then $x = t_1$ and $y = t_2$ are two elements with $f(x) = f(y) = \max({\downarrow}x \cap {\downarrow}y)$. □

Now we can present the *proof of Theorem 11*. Given a finite decomposable subset D of an Abelian group, we should find two non-empty subsets $A, B \subset D$ with $\sum A + \sum B = 0$.

Let D be a finite subset of an abelian group with $D \subset D + D$. By a *binary D-tree* we understand a pair (T, d) consisting of a binary tree T and a function $d : T \to D$ such that each non-maximal element $x \in T$ we have $d(x) = d(y) + d(z)$, where $\{y, z\} = \mathrm{succ}_T(x)$. A binary D-tree is called *$\perp$-injective* if for each branch $L \subset T$ the restriction $d{\restriction}\perp_L[L \setminus \max L]$ is injective. This implies that $\big|\perp_L[L \setminus \max L]\big| \le |D|$ and hence $|L| \le |D| + 1$. Consequently, each $\perp$-injective binary D-tree is finite, so we can choose a maximal $\perp$-injective binary D-tree T. Since $D \subset D + D$, the tree T is a subtree of a binary D-tree $\tilde{T} = T \cup \max \tilde{T}$ such that $T \cap \max \tilde{T} = \emptyset$. The maximality of the tree T ensures that for any $x \in \max T$, there exists an element $x' \in \mathrm{succ}_{\tilde{T}}(x)$ such that $d(x') \in d[\perp_{\downarrow x}[{\downarrow} x \setminus \{x\}]$. Let $M_2 = \{x \in \max T : d[\mathrm{succ}_{\tilde{T}}(x)] \subset d[\perp_{\downarrow x}[{\downarrow} x \setminus \{x\}]\}$ and $M_1 = \max T \setminus M_2$.

For every $x \in M_1$ let x_1 be the unique immediate successor of x such that

$$d(x_1) \in d\big[\perp_{\downarrow x}[{\downarrow} x \setminus \{x\}]\big]$$

and $g(x) \in {\downarrow} x$ be a (unique) point such that $d(x_1) = d(\perp_{\downarrow x}(g(x)))$. Let $f(x) := g(x)$.

For every $x \in M_2$ and every point $x' \in \mathrm{succ}_{\tilde{T}}(x)$ there is a point $g(x') \in {\downarrow} x$ such that $d(x') = d(\perp_{\downarrow x}(g(x')))$. Let $f(x) := \max\{g(x') : x' \in \mathrm{succ}_{\tilde{T}}(x)\}$.

Now observe that we have defined a regressive function $f : \max T \to T$. By Lemma 20, there are two maximal elements $x, y \in \max T$ such that $f(x) = f(y) = \max({\downarrow} x \cap {\downarrow} y)$.

By the definition of $f(x)$ the set $\mathrm{succ}_{\tilde{T}}(x)$ contains a point x_1 such that $f(x) = g(x_1)$. Let x_2 be the unique point of $\mathrm{succ}_{\tilde{T}}(x) \setminus \{x_1\}$. The definition of the function f guarantees that $d(x_2) \notin \{d(\perp_{\downarrow x}(t)) : f(x) < t < x\}$. By analogy the set $\mathrm{succ}_{\tilde{T}}(y)$ can be written as $\{y_1, y_2\}$ such that and $f(y) = g(y_1)$ and $d(y_2) \notin \{d(\perp_{\downarrow y}(t)) : f(y) < t < y\}$.

Let $g'(x_1) = \perp_{\downarrow x}(g(x_1)) = \perp_{\downarrow x}(f(x))$ and $g'(y_1) = \perp_{\downarrow y}(g(y_1)) = \perp_{\downarrow y}(f(y))$. Observe that $\{g'(x_1), g'(y_1)\} = \mathrm{succ}_T(f(x))$, $g'(x_1) \in {\downarrow} y$ and $g'(y_1) \in {\downarrow} x$.

Let

$$A_T = \{\perp_{\downarrow x}(t) : f(x) < t < x\} \text{ and } B_T := \{\perp_{\downarrow y}(t) : f(y) < t < y\}.$$

It follows from the definition of $f(x) = f(y)$ that $d(x_2) \notin d[A_T]$ and $d(y_2) \notin d[B_T]$.

By induction it can be shown that

$$d(y_1) = d(g'(y_1)) = d(x_1) + d(x_2) + \sum_{t \in A_T} d(t)$$

and

$$d(x_1) = d(g'(x_1)) = d(y_1) + d(y_2) + \sum_{t \in B_T} d(t).$$

Then

$$d(y_1) - d(x_1) = \sum_{t \in A_T \cup \{x_2\}} d(t) = - \sum_{t \in B_T \cup \{y_2\}} d(t)$$

and finally $\sum A = -\sum B$ for the sets $A = d[A_T] \cup \{d(x_2)\}$ and $B = d[B_T] \cup \{d(y_2)\}$.

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