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SURVEY ARTICLE

A survey of results on radicals and torsions in modules

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Abstract. In this work basic results of the author on radicals in module categories are presented in a short form. Principal topics are: types of preradicals and their characterizations; classes of R-modules and sets of left ideals of R; notions and constructions associated to radicals; rings of quotients and localizations; preradicals in adjoint situation; torsions in Morita contexts; duality between localizations and colocalizations; principal functors and preradicals; special classes of modules; preradicals and operations in the lattices of submodules; closure operators and preradicals.

The present review contains the formulations of basic results of the author in the theory of radicals and torsions in modules. We preserve the chronological order, as well as the terminology and notations of surveyed works of *References* [1–66]. For convenience the article is divided into sections, dedicated to cycles of works with close subjects.

1. Radicals in modules: general questions ([1–6])

The theory of radicals and torsions in modules has its source in works of P. Gabriel, S.E. Dickson, J.P. Jans, J.-M. Maranda, K. Morita, J. Lambek, O. Goldman and many other algebraists. Fundamental books in this field were written by L.A. Skornyakov, A.P. Mishina (1969), J.S. Golan (1976), B. Stenström (1975), L. Bican, T. Kepka, P. Nemec (1982).

In the article [1] some general questions on radicals in modules are discussed, the characterizations of hereditary and cohereditary radicals

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are shown by suitable conditions on torsion or torsion free classes of modules. The upper and lower radicals over the given class of modules, as well as special and cospecial radicals are studied. The filters of left ideals corresponding to special radicals are described. In particular, the following theorem is proved.

Theorem 1.1. *A radical filter $\mathcal{F}$ is special if and only if the following condition holds:*

(∗) *If $I \in \mathbb{L}({}_R R)$, $I \subseteq J$, J is an irreducible left ideal and $(J : \lambda) \in \mathcal{F}$ for some $\lambda \notin J$, then $I \in \mathcal{F}$.*

In the paper [2] the axiomatic basis of torsions in R-modules is indicated in the terms of left ideals of R: technique of the work by classes of R-modules is adapted to the sets of left ideals of R. In particular, the approach of S.E. Dickson to torsion theories by two classes of modules is transferred in the terms of ring R by two sets $(\mathcal{F}_1, \mathcal{F}_2)$ of left ideals of R. Properties of sets $\mathcal{F}_1$ and $\mathcal{F}_2$, as well as the relations between them, are shown.

Similar ideas are developed *in the work* [4], where the complete description of relations between the classes of modules and the sets of left ideals of R is obtained. These relations are expressed by mappings Φ and Ψ, where: $\Phi(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \pi(M)$, $\mathcal{M}$ is a class of modules and $\pi(M) = \{(0 : m) \mid m \in M\}$; $\Psi(\mathcal{F}) = \{M \in R\text{-Mod} \mid \pi(M) \subseteq \mathcal{F}\}$, $\mathcal{F} \subseteq \mathbb{L}({}_R R)$.

A class $\mathcal{M} \subseteq R$-Mod is called *closed* if $\mathcal{M} = \Psi\Phi(\mathcal{M})$; the set $\mathcal{F} \subseteq \mathbb{L}({}_R R)$ is called *closed* if $\mathcal{F} = \Phi\Psi(\mathcal{F})$. The descriptions of closed classes and closed sets are obtained. Namely, the class $\mathcal{M}$ is closed if and only if the following condition holds:

(A_1) $M \in \mathcal{M} \Leftrightarrow Rm \in \mathcal{M}$ for every $m \in M$.

The set $\mathcal{F}$ is closed if and only if it satisfies the condition:

(a_1) If $I \in \mathcal{F}$, then $(I : a) \in \mathcal{F}$ for every $a \in R$.

Further, the properties of the class $\mathcal{M}$ are considered to be closed under: (A_2) homomorphic images; (A_3) direct sums; (A_4) direct products; (A_5) extensions; (A_6) essential extensions. In parallel, for arbitrary set $\mathcal{F}$ of left ideals of R special conditions $(a_2), (a_3), \ldots, (a_6)$ are considered.

Theorem 1.2. *The mappings Φ and Ψ define an isotone bijection between closed classes of R-modules and closed sets of left ideals of R.*

Theorem 1.3. *Let $\mathcal{M}$ and $\mathcal{F}$ correspond each to other in the sense of Theorem 1.2. The class $\mathcal{M}$ satisfies the condition (A_n) if and only if the set $\mathcal{F}$ satisfies the condition (a_n), where $n = 2, 3, \ldots, 6$.*

So basic closure conditions on the class $\mathcal{M}$ are "translated" in the terms of left ideals. Combining suitable conditions, this result gives us the descriptions by left ideals of many types of preradicals, such as pretorsions, torsions, cotorsions, special radicals, etc.

In the article [3] some constructions of radicals of special types using the properties of elements are shown. Some concrete examples are indicated, in particular the multiplicative closed systems S of elements of the ring R are considered. It is proved that the set of S-torsion elements of each module M is a submodule of M if and only if S satisfies the left Ore condition. Moreover, the condition is indicated when the associated to S radical r_S is a torsion.

The works [5] *and* [6] represent the thesis for a candidate's degree and its short exposition.

2. Radical closures ([7, 9, 20])

Studying idempotent radicals of R-Mod, their relations with closure operators of special type were observed. If r is an idempotent radical of R-Mod, then for every pair $N \subseteq M$, where $N \in \mathbb{L}({}_RM)$, denoting by $\overline{N}$ such submodule of M that $\overline{N}/N = r(M/N)$, we obtain a closure operator $N \longmapsto \overline{N}$ in $\mathbb{L}({}_RM)$ for every $M \in R$-Mod.

In the paper [7] the notion of *radical closure* of R-Mod is introduced and studied.

Theorem 2.1. *There exists an isotone bijection between idempotent radicals of* R-Mod *and radical closure of this category.*

In continuation it is shown that radical closures of R-Mod can be described both by dense submodules and by closed submodules. For this purpose properties of the functions $\mathcal{F}_1^t$ and $\mathcal{F}_2^t$ are shown, which in every module M distinguish the set of dense submodules $\mathcal{F}_1^t(M)$ and the set of closed submodules $\mathcal{F}_2^t(M)$.

Theorem 2.2. *There exists a bijection between radical closures of* R-Mod *and functions* $\mathcal{F}$ *of the type* $\mathcal{F}_1^t$ *(as well as functions* $\mathcal{F}$ *of the type* $\mathcal{F}_2^t$*).*

The application of this result to torsions is indicated: the *hereditary* radical closures are obtained which uniquely correspond to the torsions of R-Mod. Such a radical closure can be reduced to a closure operator t in $\mathbb{L}({}_RR)$ with the condition: $t(I : a) = \big(t(I) : a\big)$ for every $a \in R$.

Some types of radicals can be described by radical closures.

In the work [9] the characterizations of torsions and of stable radicals are obtained in terms of radical closures.

Theorem 2.3. *For every idempotent radical r the following conditions are equivalent:*

1) *r is a torsion;*
2) $\{\mathcal{F}_1(A)\} \cap B \subseteq \mathcal{F}_1(B)$ *for every pair* $A \subseteq B$*;*
3) $t_r(B \cap C, A) = t_r(B, A) \cap t_r(C, A)$ *for every* $B, C \in \mathbb{L}(A)$*;*
4) *if* $B, C \in \mathcal{F}_1(A)$ *then* $B \cap C \in \mathcal{F}_1(A)$ *for every module* A*.*

Dually the description of stable radicals is obtained.

3. Divisibility, generators, cogenerators ([8, 10, 11, 20])

Some notions and constructions closely related to radicals of modules are studied. In particular, some known notions are generalized with respect to radicals or torsions.

In the article [8] *r-divisible* modules are investigated. They represent a generalization of injectivity with respect to an idempotent radical r. Some characterizations of r-divisible modules are indicated. Moreover, the r-divisible envelope $E_r(A)$ of a module A is constructed. $E_r(A)$ exists for every module A and is unique up to an isomorphism.

Theorem 3.1. *Let* $B \subseteq A$, $A \in R$-Mod *and r be an idempotent radical. The following conditions are equivalent:*

1) *A is a maximal r-essential extension of B;*
2) *A is a minimal r-divisible module containing B;*
3) *A is an r-divisible and r-essential extension of B.*

As an application, using r-divisible envelope $E_r({}_RR)$ of the module ${}_RR$, an analogue of ring of quotients in the sense of Y. Utumi is constructed.

In the paper [11] some modifications of the notions of generator and cogenerator with respect to some preradicals are studied. Any class of modules $\mathcal{M}$ defines in R-Mod preradicals $r^{\mathcal{M}}$ and $r_{\mathcal{M}}$ by the rules:

$$r^{\mathcal{M}}(X) = \sum \{\operatorname{Im} f \mid f\colon M \to X, M \in \mathcal{M}\},$$
$$r_{\mathcal{M}}(X) = \bigcap \{\operatorname{Ker} g \mid g\colon X \to M, M \in \mathcal{M}\}.$$

If $\mathcal{M} = \{M\}$, we have the preradicals r^M and r_M.

The class $\mathcal{R}(r^{\mathcal{M}})$ contains all modules for which $\mathcal{M}$ is a generator class, and similarly $\mathcal{P}(r_{\mathcal{M}})$ contains all modules for which $\mathcal{M}$ is a cogenerator class. A module M is a *generator* (*cogenerator*) of a preradical r if $r = r^M$ ($r = r_M$). Every pretorsion has a generator module and every torsion has an injective cogenerator. Generators or cogenerators for some concrete preradicals are indicated.

Theorem 3.2. 1) *The module M is a cogenerator of the radical r_R if and only if M is faithful and torsion free in the sense of H. Bass.*

2) *The module M is a generator of the preradical $r^{E(R)}$ if and only if it is a faithful, fully divisible and endofinite module.*

3) *The module M is a cogenerator of a Lambek's torsion $r_{E(R)}$ if and only if M is $r_{E(R)}$-torsion free and contains a faithful, fully divisible module.*

For every $M \in R$-Mod the radical r_M, where $r_M(X) = \bigcap\{\operatorname{Ker} f \mid f\colon X \to M\}$, is the greatest between the radicals r such that $r(M) = 0$.

In the article [10] the following question is discussed: for which modules M the radical r_M is idempotent or it is a torsion.

Theorem 3.3. *For every module $M \in R$*-Mod *the following conditions are equivalent:*

1) *r_M is a torsion;*
2) *the class $\mathcal{P}(r_M)$ is closed under extensions and $\mathcal{R}(r_M)$ is hereditary;*
3) *$r_M = r_{E(M)}$, where $E(M)$ is the injective envelope of M;*
4) *M is a pseudo-injective module (i.e. $E(M) \in \mathcal{P}(r_M)$).*

Some applications in the particular case $M = {}_RR$ are considered.

4. Rings of quotients as bicommutators ([12, 13, 35, 36])

A problem of special interest is to determine when the ring of quotients with respect to a torsion has a simple form, in particular when it can be expressed by some known constructions. One of the most convenient forms of representation is the bicommutator of a suitable module.

If $M_R \in$ Mod-R and $E = \operatorname{Hom}_R(M, M)$ is the ring of endomorphisms, then M is a left E-module and the ring $S = \operatorname{Hom}_E({}_EM, {}_EM)$ is called the *bicommutator* of M_R. Then we have the canonical homomorphism $h\colon R \to S$, defined by the rule $(x)[h(r)] = xr$, where $r \in R$, $x \in M_R$ and $h(r)\colon {}_EM \to {}_EM$.

Let M_R be a pseudo-injective module. Then the radical r_M of Mod-R is a torsion, so it defines a localization functor Q_{r_M}. In particular, the ring of quotients $Q_{r_M}(R_R)$ with canonical homomorphism $\sigma\colon R \to Q_{r_M}(R_R)$ is defined.

In the paper [12] the necessary and sufficient conditions for coincidence of ring of quotients $Q_{r_M}(R_R)$ with the bicommutator of M_R are shown. We denote by $\mathcal{F}(r_M)$ the radical filter of the torsion r_M.

Theorem 4.1. *The bicommutator S of the pseudo-injective module M_R is the right ring of quotients of R with respect to r_M if and only if the following conditions hold:*

(A) *for every homomorphism $f\colon S_R \to M_R$ there exists $x \in M$ such that $f(s) = xs$ for every $s \in S$;*

(B) *if $K \in \mathcal{F}(r_M)$, then every homomorphism from $\mathrm{Hom}_R(K_R, M_R)$ of the form φ_x can be extended to $\overline{\varphi}_x\colon R_R \to M_R$, where $\varphi \in \mathrm{Hom}_R(K_R, S_R)$, $x \in M$ and φ_x acts by the rule $\varphi_x(k) = x\varphi(k)$, $k \in K$.*

From this theorem some results of J. Lambek (1971), K. Morita (1971) and H.H. Storrer (1971) follow as particular cases.

The similar question on the coincidence of ring of quotients with the bicommutator of suitable module is discussed *in the work [13]*. The situation is studied when by the module M_R the ring $Q_M(R)$ can be constructed as the r_M-closure of R-module R/K in $E(R/K)$, where $K = (0 : M)$. The main result is the following.

Theorem 4.2. *Let M_R be a K-fully divisible module, where $K = (0 : M)$ is a torsion ideal of R. The bicommutator S of M_R coincides with the ring of quotients $Q_M(R)$ if and only if M_R is a module of type F_h (in the sense of K. Morita) and the canonical homomorphism $h\colon R \to S$ is essential.*

We obtain as corollaries the following statements:

1) If M_R is a cofaithful and fully divisible module, then $Q_M(R) \cong S$;
2) If M_R is an injective and endofinite module, then $Q_M(R) \cong S$;
3) If M_R is injective, then $Q_M(R) \cong S$ if and only if M_R is a module of type F_h.

5. Preradicals and adjointness ([14, 15, 20, 35, 36])

Further investigations of radicals in modules require the most intensive utilization of categorical methods, in particular, of adjoint functors and their properties.

In the article [14] preradicals associated to the pair of adjoint functors $R\text{-Mod} \underset{S}{\overset{T}{\rightleftarrows}} \mathfrak{B}$ are studied, where $\mathfrak{B}$ is an abelian category and T is left adjoint to S. Then there exist associated natural transformations $\Phi\colon 1_{R\text{-Mod}} \to ST$ and $\Psi\colon TS \to 1_{\mathfrak{B}}$. Preradicals generated by this situation are studied, the relations between them are elucidated and also criteria of their coincidence are shown.

In particular, the radical r is defined by the rule $r(M) = \operatorname{Ker}\Phi_M$ and if the functor T is exact, then r is a torsion. Therefore r defines a localization functor $\mathbb{L}_r$.

Furthermore, the functor $Q_r\colon R\text{-Mod} \to R\text{-Mod}$ is considered, where $Q_r(M)$ is the r-closure of $\operatorname{Im}\Phi_M$ in $ST(M)$. The question when these functors ($\mathbb{L}_r$ and Q_r) coincide is studied.

Theorem 5.1. *Let T be a selfexact functor and r be a torsion. Then for every module $M \in R$-Mod the module $Q_r(M)$ coincides with the module of quotients of M with respect to r (i.e. $Q_r = \mathbb{L}_r$).*

Theorem 5.2. *Let T be a selfexact functor and r be a torsion. Then the following conditions are equivalent:*

1) $Q_r(M) = \operatorname{Im}\Phi_M$;
2) $\operatorname{Im}\Phi_M$ *is an r_M-injective module.*

These statements generalize some results of K. Morita (1971), J. Lambek (1971) and J.A. Beachy (1974).

In the paper [15] the same adjoint situation (T, S) is considered and the question on the correspondences between preradicals (torsions) of the categories R-Mod and $\mathfrak{B}$ is studied. Some methods of transition from preradicals of R-Mod to preradicals of $\mathfrak{B}$ and inversely are indicated.

In particular, if r is a preradical of R-Mod, then the preradical r^* of $\mathfrak{B}$ is defined by the rule:

$$r^*(B) = \operatorname{Im}\left(\Psi_B \cdot T(i_B)\right),$$

where $B \in \mathfrak{B},\ i_B\colon r\left(S(B)\right) \to S(B)$ is the inclusion and the right part is the image of composition $T\big(r\left(S(B)\right)\big) \xrightarrow{T(i_B)} TS(B) \xrightarrow{\Psi_B} B$.

Similarly the inverse transition $s \longmapsto s^*$ is defined for an arbitrary preradical s of $\mathfrak{B}$.

Theorem 5.3. *The functions r^* and s^* are preradicals. The operators $r \longmapsto r^*$ and $s \longmapsto s^*$ preserve the order of preradicals. Moreover, the following relations hold:*

$$r \leqslant r^{**}, \qquad (r_1 \vee r_2)^* = r_1^* \vee r_2^*, \qquad (r_1 \cdot r_2)^* \leqslant r_1^* \cdot r_2^*,$$
$$s \geqslant s^{**}, \qquad (s_1 \wedge s_2)^* = s_1^* \wedge s_2^*, \qquad (s_1 \cdot s_2)^* \geqslant s_1^* \cdot s_2^*.$$

Theorem 5.4. 1) *The operator $r \longmapsto r^*$ preserves the idempotence of preradicals;*

2) *If T is exact and Ψ is an equivalence, then the operator $r \longmapsto r^*$ preserves the hereditary property;*

3) *If S is exact, then the operator $r \longmapsto r^*$ preserves the cohereditary property.*

Some similar results are obtained for the operator $s \longmapsto s^*$. This situation is analyzed more detailed in the case when T is exact and Ψ is a natural equivalence. The main result is the following.

Theorem 5.5. *Let T be an exact functor and Ψ be a natural equivalence. Then the operators $r \longmapsto r^*$ and $s \longmapsto s^*$ establish an isotone bijection between torsions r of R-Mod such that $r \geqslant r_T$, and all torsions of $\mathfrak{B}$, where $r_T(M) = \operatorname{Ker} \Phi_M$.*

6. Torsions in Morita contexts ([16–18, 27])

Morita context is an important construction with a considerable role in studying the equivalence of module categories (Morita theorems). We use Morita contexts for the investigation of relations between torsions of two module categories. It turned out that in this case there exists a remarkable isomorphism between two parts of the lattices of torsions.

In the article [17] an arbitrary Morita context $(R, {}_RV_S, {}_SW_R, S)$ is considered with bimodule homomorphisms $(\,,\,)\colon V \otimes {}_S W \to R$ and $[\,,\,]\colon W \otimes {}_R V \to S$. The following functors are studied:

$$R\text{-Mod} \underset{H^*=\operatorname{Hom}_S(W,-)}{\overset{H=\operatorname{Hom}_R(V,-)}{\rightleftarrows}} S\text{-Mod}.$$

The trace-ideals $T = \operatorname{Im}(\,,\,)$ and $L = \operatorname{Im}[\,,\,]$ generate torsions r_0 in R-Mod and s_0 in S-Mod such that:

$$\mathcal{P}(r_0) = \{{}_RM \mid Tm = 0 \Rightarrow m = 0\}, \quad \mathcal{P}(s_0) = \{{}_SN \mid Ln = 0 \Rightarrow n = 0\}.$$

We use the notations:

$$\mathcal{L}(R)\ (\mathcal{L}(S)) \text{ is the lattice of torsions of } R\text{-Mod } (S\text{-Mod}),$$
$$\mathcal{L}_0(R) = \{r \in \mathcal{L}(R) \mid r \geqslant r_0\}, \quad \mathcal{L}_0(S) = \{s \in \mathcal{L}(S) \mid s \geqslant s_0\}.$$

Theorem 6.1. *The functors H and H^* determine an isotone bijection between torsions r of R-Mod such that $r \geqslant r_0$ and torsions s of S-Mod such that $s \geqslant s_0$, i.e. $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$.*

This is a key result and it will be repeatedly used in continuation. The particular cases are:

1) If ${}_RV$ is a generator of R-Mod, then $r_0 = 0$, therefore $\mathcal{L}(R) \cong \mathcal{L}_0(S)$;

2) If ${}_RV$ if finitely generated and projective, then $s_0{=}0$, so $\mathcal{L}_0(R){\cong}\mathcal{L}(S)$;

3) If the rings R and S are Morita equivalent, then $\mathcal{L}(R) \cong \mathcal{L}(S)$.

We remark that the bijection of Theorem 6.1 is obtained acting by functors H and H^* on the injective cogenerators of corresponding torsions.

In the papers [16] *and* [18] the question about the preservation of properties of torsions under the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ is investigated. The torsion $r \in \mathcal{L}(R)$ is called *faithful*, if $r({}_RR) = 0$. In [16] it is shown under which conditions on the Morita context the isomorphism of Theorem 6.1 preserves the faithfulness of torsions. In particular the following theorem is true.

Theorem 6.2. *Let V_S and ${}_RV$ be faithful, ${}_RV$ be torsion free in the sense of Bass and $[W, v] = 0$ implies $v = 0$. Then the torsion r is faithful if and only if the torsion s is faithful, where (r, s) is a pair of corresponding torsions.*

Some applications of obtained results to the standard Morita context $(R, {}_RV_S, {}_SV_R^*, S)$ are shown, where $S = \mathrm{End}({}_RV)$ and $V^* = \mathrm{Hom}_R(V, R)$.

In [18] the similar question is studied for two classes of torsions: jansian and ideal torsions. A torsion r is *jansian* if its radical filter $\mathcal{E}_r$ has the smallest element, the ideal I_r of R. A torsion r_I is called *the ideal torsion*, defined by ideal I, if $\mathcal{E}_{r_I}$ is the smallest radical filter containing I.

Theorem 6.3. *A torsion $r \in \mathcal{L}_0(R)$ is jansian if and only if the corresponding torsion $s \in \mathcal{L}_0(S)$ is jansian. In this case the ideals I_r and J_s, defining r and s, are related by the rules:*

$$J_s = [W, I_rV], \qquad I_r = (V, J_sW).$$

Similar methods are used also in the case of ideal torsions.

Theorem 6.4. *Let r and s be the corresponding torsions in the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$. Then r is an ideal torsion if and only if s is an ideal torsion. If $r = r_I$ and $s = r_J$, then the ideals I and J are related by the rules:*

$$J = [W, IV], \qquad I = (V, JW).$$

In the notice [27] one more application of the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ is shown. For the torsion r a submodule $N \subseteq M$ is called *r-closed* in M if $M/N \in \mathcal{P}(r)$. For the Morita context $(R, {}_RU_S, {}_SV_R, S)$ with trace ideals I and J the pair (r, s) of corresponding torsions is considered, where $r \in \mathcal{L}_0(R)$ and $s \in \mathcal{L}_0(S)$. Denote by $\mathbb{L}^r({}_RU)$ the lattice of r-closed submodules of ${}_RU$ and by $\mathbb{L}^s({}_SS)$ the lattice of s-closed left ideals of R.

Theorem 6.5. *The lattices* $\mathbb{L}^r({}_RU)$ *and* $\mathbb{L}^s({}_SS)$ *are isomorphic.*

In this case to every submodule $U' \subseteq U$ the annihilator $\{s' \in S \mid Us' \subseteq U'\}$ corresponds, and to every left ideal $K \subseteq {}_SS$ the submodule $\{u \in U \mid [V, u] \subseteq K\}$ is associated. Some results of B.J. Müller (1974) and S.M. Khuri (1984) follow as particular cases.

7. Adjointness and localizations ([19–22, 24])

Further investigations aim to compare the localizations (colocalizations) of modules with the canonical homomorphisms of adjoint situation, as well as to search for criteria of their coincidence.

In the article [19] the adjointness determined by a bimodule ${}_RU_S$ is considered:

$$S\text{-Mod} \underset{H=\mathrm{Hom}_R(U,-)}{\overset{T=U\otimes_S -}{\rightleftarrows}} R\text{-Mod}$$

with the natural transformations $\Psi\colon 1_{S\text{-Mod}} \to HT$ and $\Phi\colon TH \to 1_{R\text{-Mod}}$. The torsion class $\mathrm{Ker}\, T$ defines in S-Mod the idempotent radical s such that $\mathcal{R}(s) = \mathrm{Ker}\, T$; similarly, the torsion free class $\mathrm{Ker}\, H$ determines in R-Mod the idempotent radical r such that $\mathcal{P}(r) = \mathrm{Ker}\, H$. The following questions are studied:

1) when the homomorphism $\Psi_N : N \to HT(N)$ is the s-localization of N for every $N \in S$-Mod?
2) when the homomorphism $\Phi_M : TH(M) \to M$ is the r-colocalization of M for every $M \in R$-Mod?

To answer these questions the requirements from definitions of localizations and colocalizations are analyzed separately, indicating for each of them some equivalent conditions.

Theorem 7.1. *The following conditions are equivalent:*

1) Ψ_N *is the* s*-localization of* N *for every* $N \in S$-Mod*;*
2) T *is left* Ψ*-exact, full on* $\mathrm{Im}\, H$ *and left selfexact;*
3) HT *is left exact and the pair* (T, H) *is idempotent* (*i.e. the associated triple* $\mathbb{F}$ *is a localization triple*)*;*

4) *the class* $\mathcal{L} = \{{}_S N \mid \Psi_N$ *is an isomorphism*$\}$ *is a Giraud subcategory of* S-Mod*, whose reflector is induced by* HT*.*

The dual result which shows when Φ_M is the r-colocalization of M for every $M \in R$-mod is also proved. Some applications and particular cases are indicated. Some results of T. Kato (1978), R.S. Cunnigham, etc. (1972), K. Morita (1970), R.J. McMaster (1975) are obtained as corollaries.

The continuation of these investigations *is the article* [21], in which the similar questions are studied for a pair of adjoint *contravariant* functors. For a bimodule ${}_S V_R$ the following functors are considered:

$$S\text{-Mod} \underset{H'=\mathrm{Hom}_R(-,V)}{\overset{H=\mathrm{Hom}_S(-,V)}{\rightleftarrows}} \text{Mod-}R$$

with the natural transformations $\Psi\colon 1_{S\text{-Mod}} \to H'H$ and $\Phi\colon 1_{\text{Mod-}R} \to HH'$. In this situation all the facts which take place in S-Mod have analogous statements in Mod-R, so it is sufficient to study one of these categories, for example S-Mod. We have the idempotent radical s in S-Mod, defined by the class $\mathrm{Ker}\, H = \mathcal{R}(s)$.

Conditions are searched under which Ψ_N is the s-localization of N for every $N \in S$-Mod. The analogue of Theorem 7.1 is proved and some applications are shown.

A slightly different approach to these questions is applied *in the paper* [22]: criteria of coincidence of localizations (colocalizations) of modules with some simple *modifications* of canonical homomorphism of adjointness are searched. For the bimodule ${}_R U_S$ the following functors are considered:

$$R\text{-Mod} \underset{H=\mathrm{Hom}_S(U,-)}{\overset{T=U\otimes_R -}{\rightleftarrows}} S\text{-Mod}$$

with natural transformations $\Phi\colon 1_{R\text{-Mod}} \to HT$ and $\Psi\colon TH \to 1_{S\text{-Mod}}$.

The kernels of the functors T and H define the idempotent radicals r_0 and s_0. For any $M \in R$-Mod we have the canonical homomorphism $\Phi_M\colon M \to HTM$ and we consider its modification Φ'_M, denoting by $Q(M)$ the r_0-closure of $\mathrm{Im}\,\Phi_M$ in $HT(M)$ and representing Φ_M as the decomposition $M \xrightarrow{\Phi'_M} Q(M) \xrightarrow{\subseteq} HT(M)$.

Theorem 7.2. *If the functor* T *is exact, then for every* $M \in R$-Mod *the homomorphism* $\Phi'_M\colon M \to Q(M)$ *is a localization of* M *with respect to the torsion* r_0*.*

The dual result about s_0-colocalizations in S-Mod is also true. As corollaries we obtain the following statements:

1) if T is exact, then Φ_M is the r_0-localization of M if and only if $\Psi_{T(M)}$ is an isomorphism;
2) if H is exact, then Ψ_N is the s_0-colocalization of N if and only if $\Phi_{H(N)}$ is an isomorphism.

To the same cycle of works we can attribute *the article* [24], in which so called *double localizations* are defined and studied. This notion generalizes ordinary localizations and is defined by a pair (r, s), where r is a torsion and s is an idempotent radical of R-Mod (if $r = s$, then the ordinary localization is obtained).

Let $\mathcal{T}_r$ ($\mathcal{T}_s$) be the class of r-torsion (s-torsion) modules and $\mathcal{L}_{rs}$ be the class of r-torsion free and s-injective modules. The homomorphism $\varphi\colon M \to L$ is called (r, s)-*localization* of M if $\operatorname{Ker}\varphi \in \mathcal{T}_r$, $\operatorname{Coker}\varphi \in \mathcal{T}_s$ and $L \in \mathcal{L}_{rs}$.

The uniqueness and the existence of (r, s)-localization are proved for every module M in the case $r \geqslant s$. Then we have the functor of (r, s)-localization $L_{rs}\colon R\text{-Mod} \to R\text{-Mod}$ with the natural transformation $\varphi\colon 1_{R\text{-Mod}} \to L_{rs}$.

Theorem 7.3. *The module $L_{rs}({}_RR)$ can be transformed into a ring and $\varphi_R\colon {}_RR \to {}_RL_{rs}({}_RR)$ can be improved to a ring homomorphism. Every module $H \in \mathcal{L}_{rs}$ is an $L_{rs}(R)$-module, every R-homomorphism $f\colon {}_RH \to {}_RK$, where $H, K \in \mathcal{L}_{rs}$, is an $L_{rs}(R)$-homomorphism.*

The connections between the localization functors L_r, L_s and L_{rs} are indicated. The existence of a close relation between (r, s)-localizations and reflective subcategories of special type is also shown. As a consequence we obtain a bijection between the torsions of R-Mod and the Giraud subcategories of R-Mod. In this way some results of L. Fuchs and K. Messa (1980) are generalized.

In the book [20] both the foundations of radical theory in modules and some special related questions are expounded: localizations and colocalizations; modules and rings of quotients; torsions in diverse situations; Giraud subcategories; torsions and triples (monads); duality between localizations and colocalizations; lattice of torsions of R-Mod, etc. The diversity of possible approaches to radicals and torsions in modules is elucidated.

8. Idempotent radicals and adjointness ([23, 26, 28])

In the article [26] the connection between idempotent radicals of two module categories in the adjoint situation is studied. The adjoint functors defined by a bimodule ${}_SU_R$ are considered:

$$R\text{-Mod} \underset{H=\mathrm{Hom}_S(U,\text{-})}{\overset{T=U\otimes_R\text{-}}{\rightleftarrows}} S\text{-Mod}.$$

Let $\mathcal{I}(R)$ $(\mathcal{I}(S))$ be the class of all idempotent radicals of R-Mod (S-Mod). The following mappings are defined:

$$\mathcal{I}(R) \underset{\alpha}{\overset{\alpha'}{\rightleftarrows}} \mathcal{I}(S),$$

where $\mathcal{R}(\alpha(s)) = T^{-1}(\mathcal{R}(s))$ and $\mathcal{P}(\alpha'(r)) = H^{-1}(\mathcal{P}(r))$ for every $r \in \mathcal{I}(R)$ and $s \in \mathcal{I}(S)$. The operators of orthogonality $(\ )^{\uparrow}$ and $(\ )^{\downarrow}$ on the classes of modules determine the transition from the torsion to the torsion free classes and inversely. By these operators the mappings α and α' can be expressed as follows:

$$\mathcal{P}(\alpha(s)) = [H(\mathcal{P}(s))]^{\uparrow\downarrow}, \qquad \mathcal{R}(\alpha'(r)) = [T(\mathcal{R}(r))]^{\downarrow\uparrow}.$$

Some properties of the mappings α and α' are shown. In particular, α preserves the intersection, while α' preserves the sum of idempotent radicals. Furthermore,

$$\alpha(s) = \alpha\alpha'\alpha(s), \qquad \alpha'(r) = \alpha'\alpha\alpha'(r)$$

for every $s \in \mathcal{I}(S)$ and $r \in \mathcal{I}(R)$. The necessary and sufficient conditions are found for the relations $s = \alpha'\alpha(s)$ and $r = \alpha\alpha'(r)$. Such idempotent radicals are called *U-closed*.

Theorem 8.1. *The mappings α and α' define an isotone bijection between U-closed idempotent radicals of R-*Mod *and U-closed idempotent radicals of S-*Mod.

The conditions under which the mappings α and α' define the isomorphism $\mathcal{I}(R) \cong \mathcal{I}(S)$ are indicated. The dual results are proved for adjoint contravariant functors.

The continuation of these investigations is contained *in the article [28]*, where the action of the mappings α and α' on torsions and cotorsions of given categories is studied. The question is: under which conditions α preserves torsions or α' preserves cotorsions. Furthermore, the question when H preserves localizations or T preserves colocalizations of modules is studied.

Theorem 8.2. *The following conditions are equivalent:*

1) *α preserves torsions;*
2) *H transfers injective modules in up-hereditary modules;*
3) *H preserves up-hereditary property;*
4) *for every monomorphism $i\colon N' \to N$ of S-Mod the relation $\operatorname{Ker} T(i) \in \mathcal{R}(T(N))$ (the smallest torsion class containing $T(N)$) is true.*

Theorem 8.3. *Let T be an exact functor and r be a torsion of R-Mod. The following conditions are equivalent:*

1) *H preserves r-localizations;*
2) *H is r-full and r-exact.*

The dual results are also proved: conditions when α' preserves cotorsions and the functor T preserves colocalizations of modules are shown.

The preprint [23] contains the detailed exposition of all results on the mappings α and α', and also of similar facts on adjoint *contravariant* functors.

9. Classes of modules and localizations in Morita contexts ([29, 30])

If a Morita context $(R, {}_RU_S, {}_SV_R, S)$ with the bimodule homomorphisms $(\, , \,)$ and $[\, , \,]$ is given, then in the categories R-Mod and S-Mod quite a number of classes of modules with diverse closure properties (under homomorphic images, submodules, extensions, etc.) appear in a natural way. Therefore these classes of modules determine preradicals of various types (idempotent radicals, torsions, etc.).

In the article [29] the most important classes of modules determined by a Morita context, as well as the corresponding preradicals are investigated. Properties of these classes and connections between them are shown. Furthermore, criteria of the coincidence of some "near" preradicals are obtained.

For example, the pairs of adjoint functors (T^U, H^U) and (T^V, H^V) lead to the classes $\operatorname{Ker} T^U$ and $\operatorname{Ker} T^V$, which are torsion classes, and also to the classes $\operatorname{Ker} H^U$ and $\operatorname{Ker} H^V$, which are torsion free classes. An important role is played by the classes $\mathrm{Gen}({}_RU)$ and $\mathrm{Gen}({}_SV)$ of modules generated by ${}_RU$ or ${}_SV$, and also by the dual classes $\mathrm{Cog}({}_RV^*)$ and $\mathrm{Cog}({}_SU^*)$.

In the same time, the classes of modules are considered which are defined by trace ideals I and J of a given Morita context. For example, the ideal I determines the classes of modules:

$$ {}_I\mathcal{T} = \{{}_RM \mid IM = M\}, \qquad {}_I\mathcal{F} = \{{}_RM \mid \mathrm{Im} = 0 \Rightarrow m = 0\}, $$
$$ \mathcal{A}(I) = \{{}_RM \mid IM = 0\}, $$

and similarly for ideal J. The classes ${}_I\mathcal{T}$ and ${}_J\mathcal{T}$ are torsion classes, while ${}_I\mathcal{F}$ and ${}_J\mathcal{F}$ are torsion free classes. Various closure properties of the classes $\mathcal{A}(I)$ and $\mathcal{A}(J)$ lead to quite a number of associated preradicals.

Some relations between the studied classes of modules are shown. In particular, is true the following theorem.

Theorem 9.1. 1) $\mathcal{A}(I)^{\uparrow} = {}_I\mathcal{T}$, $\mathcal{A}(J)^{\uparrow} = {}_J\mathcal{T}$; $\mathcal{A}(I)^{\downarrow} = {}_I\mathcal{F}$, $\mathcal{A}(J)^{\downarrow} = {}_J\mathcal{F}$;
2) $\mathrm{Ker}\, T^V \subseteq \mathcal{A}(I)$, $\mathrm{Ker}\, T^U \subseteq \mathcal{A}(J)$; $\mathrm{Ker}\, H^U \subseteq \mathcal{A}(I)$, $\mathrm{Ker}\, H^V \subseteq \mathcal{A}(J)$;
3) $\mathcal{A}(I) \subseteq {}_I\mathcal{T}^{\downarrow}$, $\mathcal{A}(J) \subseteq {}_J\mathcal{T}^{\downarrow}$; $\mathcal{A}(I) \subseteq {}_I\mathcal{F}^{\uparrow}$, $\mathcal{A}(J) \subseteq {}_J\mathcal{F}^{\uparrow}$.

Further, preradicals of diverse types, which are defined by these classes of modules are considered. Some connections between them are indicated and conditions under which some preradicals coincide are found. These results are closely related to the investigations of T. Kato (1978), K. Ohtake (1980, 1982), etc.

The article [30] is devoted to the study of localizations in the Morita context $(R, {}_RU_S, {}_SV_R, S)$, which define the functors:

$$ R\text{-Mod} \underset{H^V = \mathrm{Hom}_S(V,-)}{\overset{H^U = \mathrm{Hom}_R(U,-)}{\rightleftarrows}} S\text{-Mod}. $$

The trace ideal $I = (U, V)$ of R leads to the natural transformation $\varphi\colon 1_{R\text{-Mod}} \to H^V H^U$, where $\varphi_M\colon {}_RM \to {}_RH^V H^U(M)$ acts by the rule $u(v(m\varphi_M)) = (u, v)m$. Furthermore, the ideal I determines in R-Mod the torsion r_I such that $\mathcal{P}(r_I) ={}_I \mathcal{F} = \{{}_RM \mid \mathrm{Im} = 0 \Rightarrow m = 0\}$. The goal of investigation: to find necessary and sufficient conditions under which the homomorphism φ_M is an r_I-localization of M for every $M \in R$-Mod.

Theorem 9.2. *The following conditions are equivalent:*
1) φ_M *is the* r_I*-localization of* M *for every* $M \in R$-Mod*;*
2) $I^2 = I$ *and the module* ${}_R(U \otimes {}_SV)$ *is fully* r_I*-projective;*
3) $I^2 = I, I(U \otimes {}_SV) = U \otimes {}_SV$ *and* ${}_R(U \otimes {}_SV)$ *is projective relative to the epimorphisms* $\pi_M\colon M \to M/_{r_I}(M)$*,* $M \in R$-Mod*.*

In particular, if ${}_R(U \otimes {}_SV)$ is projective with the trace I, then the conditions of this theorem hold. The connection with some results of K. Ohtake (1980, 1982), T. Kato (1978) and B.J. Muller (1974) is indicated.

10. Principal functors and preradicals ([25, 35, 36, 43])

The investigation of general questions on connections between preradicals of two module categories is continued (see Section 5).

In the preprint [25] (see also [35, 36, 43]) the pair of adjoint functors defined by the bimodule ${}_RU_S$ is considered:

$$R\text{-Mod} \underset{T=U\otimes_S -}{\overset{H=\mathrm{Hom}_R(U,-)}{\rightleftarrows}} S\text{-Mod}.$$

Let $\Phi\colon TH \to 1_{R\text{-Mod}}$ and $\Psi\colon 1_{S\text{-Mod}} \to HT$ be the associated natural transformations. These functors permit to define some mappings between preradicals of diverse types of the categories R-Mod and S-Mod on different "levels":

1) for *preradicals*, with the help of Φ and Ψ;
2) for *radicals* and *idempotent preradicals*, by functors T and H;
3) for *idempotent radicals*, applying T and H to torsion or torsion free classes (see Section 8).

In general case of preradicals the "star" mappings $r \longmapsto r^*$ and $s \longmapsto s^*$ are considered (see Section 5). For radicals other method is used: acting by T and H on the generating or cogenerating classes, $r_{\mathcal{K}} \longmapsto r_{H(\mathcal{K})}$, $r^{\mathcal{K}} \longmapsto r^{T(\mathcal{K})}$. Properties of these mappings are indicated and also their relation with the "star" mappings is shown. On the next "level" of idempotent radicals the mappings α and α' are used (see Section 8). It is proved that $\alpha(r)$ is the greatest idempotent radical contained in r^* and similarly for α'. If the functors H or T are exact, then α and α' coincide with the "star" mappings and in this case they preserve torsions or cotorsions.

The next step consists in the comparison of localizations of modules with special *modifications* of canonical homomorphisms for every torsion r of R-Mod (the particular case $r = r_0$ is considered in [22]).

By functors (T, H) and a torsion r of R-Mod for every $N \in S$-Mod we consider the homomorphism:

$$\Psi_N^r\colon N \xrightarrow{\Psi_N} HT(N) \xrightarrow{H(\pi_{T(N)})} (H \cdot 1/r \cdot T)(N),$$

where $\pi_{T(N)}$ is the natural homomorphism. In such a way we obtain the natural transformation $\Psi^r \colon 1_{S\text{-Mod}} \to H \cdot 1/r \cdot T$. Furthermore, in S-Mod we have the idempotent radical $s = \alpha(r)$, where $\mathcal{R}(s) = T^{-1}(\mathcal{R}(s))$. The problem is to find necessary and sufficient conditions under which Ψ^r_N is the s-localization of N for every $N \in S$-Mod. The torsion r defines a new pair of adjoint functors:

$$\mathcal{P}(r) \underset{T^r}{\overset{H^r}{\rightleftarrows}} S\text{-Mod}$$

(closely related to (H, T)) with the natural transformations Ψ^r and Φ^r. To this pair the triple $\mathbb{F}^r$ is associated.

Theorem 10.1. *The following conditions are equivalent:*
1) Ψ^r_N *is the s-localization of N for every $N \in S$-*Mod*;*
2) T^r *is left Ψ^r-exact, full on* $\operatorname{Im} H^r$ *and left selfexact;*
3) $\mathbb{F}^r$ *is a localization triple;*
4) $\mathcal{P}(s) = \mathcal{P}(r^*)$ *and* $\operatorname{Im} H^r = \operatorname{Fix} \Psi^r = \mathcal{L}_s$*;*
5) $\operatorname{Fix} \Psi^r$ *is a Giraud subcategory of S-*Mod*.*

In the same situation the colocalizations of modules with canonical homomorphisms are compared for some cotorsion s of S-Mod and $r = \alpha'(s)$. The modification Φ^s_M of Φ_M is considered and conditions under whichΦ^s_M is the r-colocalization of M for any $M \in R$-Mod (the analogue of Theorem 10.1) are shown. In addition, the work [25] contains the exposition of the case of *contravariant* functors with dual results.

11. Principal functors and lattices of submodules ([31–37, 39, 43])

The problem of the influence of principal functors of module categories on the lattices of submodules is of considerable interest. More exactly, if $F \colon \mathcal{M}_1 \to \mathcal{M}_2$ is a functor between module categories, then the question on the relation between the lattices of submodules $\mathbb{L}(X)$ and $\mathbb{L}(F(X))$ is considered, where $X \in \mathcal{M}_1$ and $F(X) \in \mathcal{M}_2$. This problem is studied for the principal functors:

$$H = \operatorname{Hom}_R(U, \text{-}), \quad T = U \otimes_S \text{-}, \qquad H_1 = \operatorname{Hom}_R(\text{-}, U)$$

for a bimodule ${}_RU_S$. The mainly used method is the transition from the lattices of all submodules to the lattices of special submodules determined by associated preradicals r and s. To these questions the cycle of works [31–34] is dedicated (see also [35, 36, 43]).

Further we expose shortly the basic results for every principal functor.

Functor $H = \mathrm{Hom}_R(U, \text{-})\colon R$-Mod $\to S$-Mod

The bimodule ${}_RU_S$ defines the pair (T, H) of adjoint functors with the natural transformations $\Phi\colon TH \to 1_{R\text{-Mod}}$ and $\Psi\colon 1_{S\text{-Mod}} \to HT$. Then we have preradicals r of R-Mod and s of S-Mod such that:

$$r({}_RM) = \mathrm{Im}\,\Phi_M, \qquad s({}_SN) = \mathrm{Ker}\,\Psi_N.$$

In the lattices of submodules $\mathbb{L}({}_RX)$ and $\mathbb{L}({}_SY)$ the following sublattices are defined:

$$\mathcal{L}^r({}_RX) = \{X' \subseteq X \mid r(X') = X'\}, \qquad \mathcal{L}_s({}_SY) = \{Y' \subseteq Y \mid s(Y/Y') = 0\}.$$

For every $X \in R$-Mod the following mappings are defined:

$$\mathbb{L}({}_RX) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}({}_SH(X)),$$

where

$$\alpha(X') = \{f\colon {}_RU \to {}_RX \mid \mathrm{Im}\, f \subseteq X'\}, \quad X' \subseteq X,$$
$$\beta(Y') = \sum\{\mathrm{Im}\, f' \mid f' \in Y'\}\ (= UY'), \quad Y' \subseteq H(X).$$

Some properties of the mappings α and β are shown. In particular, $\alpha(X') \in \mathcal{L}_s\big(H(X)\big)$ and $\beta(Y') \in \mathcal{L}^r(X)$. The lattices $\mathbb{L}({}_RX)$ and $\mathbb{L}\big({}_SH(X)\big)$ are called *canonically isomorphic* if α and β determine the isomorphism of these lattices, which is equivalent to the conditions:

I) $\mathbb{L}({}_RX) = \mathcal{L}^r({}_RX)$;
II) $\mathbb{L}\big({}_SH(X)\big) = \mathcal{L}_s\big({}_SH(X)\big)$;
III) $\mathcal{L}^r({}_RX) \cong \mathcal{L}_s\big({}_SH(X)\big)$.

Further, every of these conditions is investigated in detail, the third being the most nontrivial. For its fulfilment the key question is when the relation $Y' = \alpha\beta(Y')$ is true for $Y' \subseteq H(X)$. Necessary and sufficient conditions for this relation being satisfied are shown. We mention some results.

Theorem 11.1. *If ${}_RU$ is a projective module and $S = \mathrm{End}({}_RU)$, then the lattices $\mathcal{L}^r({}_RX)$ and $\mathcal{L}_s\big({}_SH(X)\big)$ are canonically isomorphic for every $X \in R$-Mod.*

Theorem 11.2. *The following conditions are equivalent:*
1) *the lattices $\mathcal{L}^r({}_RX)$ and $\mathcal{L}\big({}_SH(X)\big)$ are canonically isomorphic;*
2) *the module ${}_RU$ is inner X-projective and X-compact.*

Some results of F.L. Sandomierski (1972), G.M. Brodskii (1983), A.K. Gupta, K. Varadarajan (1980) are obtained as corollaries.

Functor $T = U \otimes_S$ -: S-Mod $\to$ R-Mod

Similar questions for the functor of tensor multiplication T are investigated. For the adjoint situation generated by ${}_RU_S$ and for every module $Y \in S$-Mod the following mappings are considered:

$$\mathbb{L}\big({}_RT(Y)\big) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \mathbb{L}({}_SY),$$

where

$$\alpha'(Y') = \operatorname{Im} T(j), \quad j\colon Y' \xrightarrow{\subseteq} Y,$$
$$\beta'(X') = \{y \in Y \mid U \otimes {}_Sy \subseteq X'\}, \quad X' \subseteq T(Y).$$

Then $\alpha'(Y') \in \mathcal{L}^r\big({}_RT(Y)\big)$ and $\beta'(X') \in \mathcal{L}_s({}_SY)$. Therefore the canonical isomorphism $\mathbb{L}({}_SY) \cong \mathbb{L}\big({}_RT(Y)\big)$ holds if and only if:

I) $\mathbb{L}({}_SY) = \mathcal{L}_s({}_SY)$;

II) $\mathbb{L}\big({}_RT(Y)\big) = \mathcal{L}^r\big({}_RT(Y)\big)$;

III) $\mathcal{L}_s({}_SY) \cong \mathcal{L}^r\big({}_RT(Y)\big)$.

The analysis of these conditions (the condition III) is basic) elucidates the situation when the required isomorphism takes place. The main question is when the equality $X' = \alpha'\beta'(X')$ holds for $X' \subseteq {}_RT(Y)$. In particular, is proved the following theorem.

Theorem 11.3. *If U_S is flat and the pair (T, H) is idempotent, then $\mathcal{L}^r\big({}_R(T(Y)\big) \cong \mathcal{L}_s({}_SY)$ for every $Y \in S$-Mod.*

One of basic results in this case is the following.

Theorem 11.4. *Let ${}_RC$ be a cogenerator of R-Mod which is finitely generated and injective. For every $AB5^*$-module $Y \in S$-Mod the following conditions are equivalent:*

1) *the lattices $\mathbb{L}\big({}_RT(Y)\big)$ and $\mathbb{L}({}_SY)$ are canonically isomorphic;*
2) *the module ${}_SH(C)$ is inner Y-injective, Y-cofinitely cogenerated and Y-cogenerating.*

Functor $H_1 = \mathrm{Hom}_R(\text{-}, U)$: R-Mod $\to$ Mod-S

The question of the impact of functors on the lattices of submodules is investigated by similar methods for *contravariant* functor H_1 ([31, 34–36, 43]). The bimodule ${}_RU_S$ determines the adjoint functors:

$$R\text{-Mod} \underset{H_2=\mathrm{Hom}_S(\text{-},U)}{\overset{H_1=\mathrm{Hom}_R(\text{-},U)}{\rightleftarrows}} \text{Mod-}S$$

with the natural transformations $\Phi\colon 1_{R\text{-Mod}} \to H_2H_1$ and $\Psi\colon 1_{\text{Mod-}S} \to H_1H_2$. In view of the full symmetry, it is sufficient to study one of these functors, for example H_1.

For every $X \in R$-Mod the following mappings are defined:

$$\mathbb{L}({}_RX) \underset{\beta^*}{\overset{\alpha^*}{\rightleftarrows}} \mathbb{L}(H_1(X)_S),$$

where

$$\alpha^*(X') = \{f \in H_1(X) \mid \operatorname{Ker} f \supseteq X'\}, \quad X' \subseteq {}_RX;$$
$$\beta^*(Y') = \cap\{\operatorname{Ker} f' \mid f' \in Y'\}, \quad Y' \subseteq H_1(X)_S.$$

Further, preradicals r of R-Mod and s of Mod-S are used, where $r({}_RX) = \operatorname{Ker} \Phi_X$ and $s(Y_S) = \operatorname{Ker} \Psi_Y$. For every ${}_RX \in R$-Mod and $Y_S \in$ Mod-S the following lattices of submodules are considered:

$$\mathcal{L}_r({}_RX) = \{X' \subseteq X \mid r(X/X') = 0\}, \qquad \mathcal{L}_s(Y_S) = \{Y' \subseteq Y \mid s(Y/Y') = 0\}.$$

The problem is to find conditions under which α^* and β^* determine an antiisomorphism of lattices $\mathbb{L}({}_RX)$ and $\mathbb{L}(H_1(X)_S)$, which is equivalent to the conditions:

I) $\mathbb{L}({}_RX) = \mathcal{L}_r({}_RX)$;
II) $\mathbb{L}(H_1(X)_S) = \mathcal{L}_s(H_1(X)_S)$;
III) $\mathcal{L}_r({}_RX) \mathrel{\overline{\approx}} \mathcal{L}_s(H_1(X)_S)$.

The main question here is when the relation $Y' = \alpha^*\beta^*(Y')$ holds for $Y' \subseteq H_1(X)_S$. A series of equivalent conditions, which ensures this relation, is found. From the basic results we mention the following.

Theorem 11.5. *If ${}_RU$ is injective and $H_1(\Phi_X)$ is a monomorphism, then $\mathcal{L}_r({}_R(X)) \mathrel{\overline{\approx}} \mathcal{L}_s(H_1(X)_S)$.*

Theorem 11.6. *For every $X \in R$-Mod the following conditions are equivalent:*

1) $\mathcal{L}_r({}_RX) \mathrel{\overline{\approx}} \mathbb{L}(H_1(X)_S)$;
2) *${}_RU$ is inner X-injective and X-cocompact.*

As a consequence it is obtained that the antiisomorphism $\mathbb{L}({}_RX) \mathrel{\overline{\approx}} \mathbb{L}(H_1(X)_S)$ is equivalent to the conditions that ${}_RU$ is inner X-injective, X-cocompact and X-cogenerator.

From these results all the earlier known facts on the antiisomorphism of lattices of submodules follow, in particular those of K.R. Fuller (1974), F.L. Sandomierski (1972), C. Năstăsescu (1979), A.K. Gupta, K. Varadarajan (1980), G.M. Brodskii (1983).

A combined investigation of mappings between the lattices of submodules for all principal functors is realized *in the paper* [37] (see also [39]). For a bimodule ${}_RU_S$ the following functors are considered:

$$S\text{-Mod} \underset{T=U\otimes_S -}{\overset{H=\mathrm{Hom}_R(U,-)}{\leftrightarrows}} R\text{-Mod} \underset{H_2=\mathrm{Hom}_S(-,U)}{\overset{H_1=\mathrm{Hom}_R(-,U)}{\rightleftarrows}} \text{Mod-}S$$

Every module $M \in R$-Mod defines the mappings:

$$\mathbb{L}({}_SH(M)) \underset{\beta_M}{\overset{\alpha_M}{\leftrightarrows}} \mathbb{L}({}_RM) \underset{\beta_M^*}{\overset{\alpha_M^*}{\rightleftarrows}} \mathbb{L}(H_1(M)_S),$$

$$\mathbb{L}({}_SH(M)) \underset{\mathcal{L}_M}{\overset{\mathcal{R}_M}{\rightleftarrows}} \mathbb{L}(H_1(M)_S),$$

where

$$\alpha_M(M') = \{f \in H(M) \mid \operatorname{Im} f \subseteq M'\}, \quad \beta_M(N') = \sum\{\operatorname{Im} f' \mid f' \in N'\};$$
$$\alpha_M^*(M') = \{f \in H_1(M) \mid \operatorname{Ker} f \supseteq M'\}, \quad \beta_M^*(N') = \bigcap\{\operatorname{Ker} f' \mid f' \in N'\};$$
$$\mathcal{R}_M(N') = \{g\colon {}_RM \to {}_RU \mid fg = 0 \ \forall f \in N'\},$$
$$\mathcal{L}_M(L') = \{f\colon {}_RU \to {}_RM \mid fg = 0 \ \forall g \in L'\}.$$

These pairs of mappings constitute the "triangular Galois theory". They are combined with the pairs of mappings defined by the natural transformations $\Psi\colon 1_{R\text{-Mod}} \to HT$ and $\Phi\colon TH \to 1_{R\text{-Mod}}$. The commutativity of the resulting diagram is studied (it contains 12 pairs of mappings) and for the obtained Galois connections the accompanying projectivities (isotone bijections) or dualities (antiisotone bijections) are indicated. For example, is true the next theorem.

Theorem 11.7. *For any module* $M \in R$-Mod *the restrictions of the mappings* α_M^* *and* β_M^* *define a duality* $\operatorname{Im}\mathcal{R}_M \underset{\beta_M^*}{\overset{\alpha_M^*}{\leftrightarrows}} \mathcal{S}_1$, *where* $\mathcal{S}_1 = \{X \subseteq M \mid \beta_M^*\alpha_M^*\beta_M\alpha_M(X) = X\}$. *The restrictions of the mappings* α_M *and* $\beta_M^*\mathcal{R}_M$ *define the projectivity* $\operatorname{Im}\mathcal{L}_M \underset{\beta_M^*\mathcal{R}_M}{\overset{\alpha_M}{\leftrightarrows}} \mathcal{S}_1$. *The symmetrical statement is also true.*

These facts generalize some results of G.M. Brodskii (1983) and S.M. Khuri (1989).

12. Morita contexts and lattices of submodules ([38, 40, 42–45, 50])

Ample opportunities for the investigation of relations between the lattices of submodules are provided by Morita contexts. Some mappings between the lattices of submodules in this case are considered *in the work* [38]. A Morita context $(R, {}_RM_S, {}_SN_R, S)$ with the bimodule homomorphisms $(\,,\,)\colon M \otimes {}_SN \to R$ and $[\,,\,]\colon N \otimes {}_RM \to S$ and with trace ideals $I_0 = (M, N)$ and $J_0 = [N, M]$ determines some pairs of mappings between the lattices of submodules, in particular:

$$\mathbb{L}({}_RM) \underset{l'}{\overset{r'}{\rightleftarrows}} \mathbb{L}(N_R), \qquad \mathbb{L}({}_RM) \underset{f_M}{\overset{P_M}{\rightleftarrows}} \mathbb{L}({}_SS),$$

$$\mathbb{L}({}_SS) \underset{Q_N}{\overset{G_N}{\leftrightarrows}} \mathbb{L}(N_R),$$

where

$$\begin{aligned} r'(K) &= \{n \in N \mid (K, n) = 0\}, & l'(L) &= \{m \in M \mid (m, L) = 0\}; \\ p_M(J) &= N^{-1}J, & f_M(K) &= (N, K); \\ G_N(L) &= \operatorname{ann}_S(L), & Q_N(J) &= \operatorname{ann}_N(J). \end{aligned}$$

Properties of these mappings, as well as connections between them are shown. Conditions under which the restrictions of these mappings determine projectivities or dualities are obtained. In particular, is proved the following theorem.

Theorem 12.1. *Let N_R be a faithful module and $[N, M] = S$. Then the pair (p_M, f_M) defines a projectivity between $\mathcal{C} = \{J \subseteq {}_SS \mid J = G_NQ_N(J)\}$ and $\mathcal{L}' = \{K \subseteq {}_RM \mid K = l'r'(K)\}$; also, the pair (G_N, Q_N) determines the duality between $\mathcal{C}$ and $\mathcal{R}' = \{L \subseteq N_R \mid L = l'r'(L)\}$.*

Together with dual results the "quadrangular Galois theory" is obtained, which consists in five bijections (projectivities and dualities). These facts generalize some results of S. Kyuno, M.-S.B. Smith (1989), J.J. Hutchinson (1987), G.M. Brodskii (1983).

A rather full picture of the relation between the lattices of submodules in Morita contexts is exposed *in the article* [40], using preradicals determined by trace ideals $I = (M, N)$ and $J = [N, M]$ of the Morita context $(R, {}_RM_S, {}_SN_R, S)$. Two types of mappings between the lattices of submodules are distinguished.

Mappings of the first type are defined using the idempotent radicals r^I in R-Mod and r^J in S-Mod, where $\mathcal{R}(r^I) = \{_R X \mid IX = X\}$ and $\mathcal{R}(r^J) = \{_S Y \mid JY = Y\}$. In the lattices of submodules $\mathbb{L}(_R X)$ and $\mathbb{L}(_S S)$ the following sublattices are considered:

$$\mathcal{L}^{r^I}(_R X) = \{X' \subseteq {_R X} \mid IX' = X'\}, \qquad \mathcal{L}^{r^J}(_S S) = \{A \subseteq {_S S} \mid A = JA\}.$$

The following mappings are studied:

$$\mathbb{L}(_R M) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \mathbb{L}(_S S), \quad \alpha_M(_R K) = [N, K], \quad \beta_M(_S A) = MA;$$

$$\mathbb{L}(_S N) \underset{\beta_N}{\overset{\alpha_N}{\rightleftarrows}} \mathbb{L}(_R R), \quad \alpha_N(_S L) = (M, L), \quad \beta_N(_R B) = NB.$$

Theorem 12.2. *The pair of mappings (α_M, β_M) defines the lattice isomorphism $\mathcal{L}^{r^I}(_R M) \cong \mathcal{L}^{r^J}(_S S)$ and the pair (α_N, β_N) defines the lattice isomorphism $\mathcal{L}^{r^J}(_S N) \cong \mathcal{L}^{r^I}(_R R)$. Right variants of these statements also hold: $\mathcal{L}^{r^J}(M_S) \cong \mathcal{L}^{r^I}(R_R)$, $\mathcal{L}^{r^I}(N_R) \cong \mathcal{L}^{r^J}(S_S)$.*

It is interesting that all considered mappings can be restricted to *subbimodules* and then isomorphisms of lattices of subbimodules are obtained. In particular, we have the mappings

$$\mathbb{L}(_R M_S) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}(_S N_R), \qquad \mathbb{L}(_S S_S) \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \mathbb{L}(_R R_R),$$

whose restrictions lead to lattice isomorphisms.

Mappings of the second type are defined by the torsions r_I of R-Mod and r_J of S-Mod (the ideal torsions determined by I and J). For every module $_R X$ and every torsion r of R-Mod the lattice of r-closed submodules $\{X' \subseteq X \mid r(X/X') = 0\}$ is denoted by $\mathcal{L}_r(_R X)$.

The following mappings are considered:

$$\mathbb{L}(_R M) \underset{\delta_M}{\overset{\gamma_M}{\rightleftarrows}} \mathbb{L}(_S S), \qquad \mathbb{L}(_S N) \underset{\delta_N}{\overset{\gamma_N}{\rightleftarrows}} \mathbb{L}(_R R),$$

where $\gamma_M(_R K) = \{s \in S \mid Ms \subseteq K\}$, $\delta_M(_S A) = \{m \in M \mid [N, m] \subseteq A\}$, and similarly for γ_N and δ_N.

Theorem 12.3. *The mappings γ_M and δ_M define the lattice isomorphism $\mathcal{L}_{r_I}(_R M) \cong \mathcal{L}_{r_J}(_S S)$; the mappings γ_N and δ_N define the lattice isomorphism $\mathcal{L}_{r_J}(_S N) \cong \mathcal{L}_{r_I}(_R R)$. Right variants of these statements also hold.*

As in the first case, these mappings can be restricted to the *subbimodules* and isomorphisms of lattices of subbimodules are obtained.

These results are closely related to the investigations of S. Kyuno, M.-S.B. Smith (1989), S.M. Khuri (1986), B.J. Müller (1974).

In the paper [42] these investigations are continued: the pair of torsions (r_I, r_J) of the Theorem 12.3 is substituted by an arbitrary pair (r, s) of torsions which correspond each to other in the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ (see Theorem 6.1). For a Morita context $(R, {}_RU_S, {}_SV_R, S)$ with the trace ideals I and J, the following mappings are considered:

$$\mathbb{L}({}_RU) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbb{L}({}_SS) \underset{L_S}{\overset{R_S}{\rightleftarrows}} \mathbb{L}(S_S),$$

$$\mathbb{L}({}_RU) \underset{Q}{\overset{G}{\rightleftarrows}} \mathbb{L}(S_S),$$

where

$$\begin{aligned} \alpha(U') &= \{s' \in S \mid Us' \subseteq U'\}, & \beta({}_SA) &= \{u \in U \mid [V, u] \subseteq A\}; \\ R_S({}_SX) &= \{s' \in S \mid s'X \subseteq s({}_SS)\}, & L_S(X) &= \{s' \in S \mid s'X \subseteq s({}_SS)\}; \\ G(U') &= \{s' \in S \mid U's' \subseteq r({}_RU)\}, & Q(X) &= \{u \in U \mid uX \subseteq r({}_RU)\}. \end{aligned}$$

Theorem 12.4. *The restrictions of the mappings α and β on the* $\operatorname{Im} Q$ *and* $\operatorname{Im} L_S$ *define a projectivity between these lattices, which is a part of "triangular Galois theory":*

$$\operatorname{Im} Q \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \operatorname{Im} L_S \underset{L_S}{\overset{R_S}{\rightleftarrows}} \operatorname{Im} G = \operatorname{Im} R_S,$$

$$\operatorname{Im} Q \underset{Q}{\overset{G}{\rightleftarrows}} \operatorname{Im} G.$$

Symmetrically the other triangular Galois theory is constructed, which supplements the previous and in common the square is obtained, consisting of four pairs of co-ordinated mappings with the diagonal (G, Q): two projectivities and three dualities. Some known results of Zhou Zhengping (1983), J.J. Hutchinson (1987), S. Kyuno, M.-S.B. Smith (1989) are obtained as particular cases.

Compositions of dualities in a nondegenerated Morita context $(R, {}_RM_S, {}_SN_R, S)$ are studied *in the article* [45]. Mappings between the lattices of submodules $\mathbb{L}({}_RM)$, $\mathbb{L}(N_S)$, $\mathbb{L}({}_SS)$ and $\mathbb{L}(S_S)$, which are defined as annihilators are studied. In particular, the following mappings are considered:

$$\mathbb{L}({}_R M) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \mathbb{L}({}_S S) \underset{l}{\overset{r}{\rightleftarrows}} \mathbb{L}(S_S),$$

$$\mathbb{L}({}_R M) \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \mathbb{L}(S_S),$$

where

$$\begin{aligned} \alpha_M({}_R K) &= \{s \in S \mid Ms \subseteq K\}, & \beta_M({}_S J) &= \{m \in M \mid [N, m] \subseteq J\}; \\ r({}_S J) &= \{s \in S \mid Js = 0\}, & l(J_S) &= \{s \in S \mid sJ = 0\}; \\ G_M({}_R K) &= \{s \in S \mid Ks = 0\}, & Q_M({}_S J) &= \{m \in M | mJ = 0\}. \end{aligned}$$

The pairs (r, l) and (G_M, Q_M) form Galois connections, so they define the dualities:

$$\operatorname{Im} l \underset{l}{\overset{r}{\rightleftarrows}} \operatorname{Im} r, \qquad \operatorname{Im} Q_M \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \operatorname{Im} G_M.$$

To obtain their composition the equality Im $r =$ Im G_M is necessary. Some weak conditions under which this relation holds are indicated. Then the composition of these dualities generates a projectivity, which coincides with the pair (α_M, β_M):

$$\operatorname{Im} Q_M \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \operatorname{Im} G_M = \operatorname{Im} r \underset{r}{\overset{l}{\rightleftarrows}} \operatorname{Im} l,$$

$$\operatorname{Im} Q_M \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} \operatorname{Im} l.$$

For a given context, four such triangles are obtained and under suitable conditions the compositions of dualities can be formed. If the context is *nondegenerated*, then all these conditions are satisfied and so the compositions of dualities can be considered.

Theorem 12.5. *If the Morita context* $(R, {}_R M_S, {}_S N_R, S)$ *is nondegenerated, then we obtain four co-ordinated dualities* (r, l), (G_M, Q_M), (r', l'), (G_N, Q_N) *and two projectivities* (α_M, β_M), (α_N, β_N), *which are compositions of corresponding dualities.*

To the same cycle of works can be attributed *the article* [50], in which the equivalence of special subcategories is proved for the Morita context $(R, {}_R V_S, {}_S W_R, S)$, using the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ of Theorem 6.1.

For a fixed pair (r, s) of corresponding torsions some modifications of the functors

$$R\text{-Mod} \underset{T^V = V \otimes_S -}{\overset{T^W = W \otimes_R -}{\rightleftarrows}} S\text{-Mod}$$

are defined such that they are of the form $\mathcal{T}_r \underset{\overline{T}^V}{\overset{\overline{T}^W}{\rightleftarrows}} \mathcal{F}_s$. The subcategories $\mathcal{A} = \mathcal{F}_r \cap \mathcal{F}_I \subseteq R$-Mod and $\mathcal{B} = \mathcal{F}_s \cap \mathcal{F}_J \subseteq S$-Mod are considered.

Theorem 12.6. *For every pair of corresponding torsions (r, s) the functors $\overline{T}^W$ and $\overline{T}^V$ define an equivalence between the subcategories of torsion free accessible modules: $\mathcal{A} \approx \mathcal{B}$.*

For the smallest pair (r_I, r_J) of corresponding torsions this result was proved by W.K. Nicholson, J.F. Watters (1988) and F.C. Iglesias, J.G. Torrecillas (1995).

The article [44] is a survey of some results on radicals in modules.

The book [43] is devoted to torsions in modules and contains some new results on the following subjects:

Chapter 1. Adjoint functors and radicals;

Chapter 2. Morita contexts and torsions;

Chapter 3. Principal functors and lattices of submodules.

13. Divisible and reduced modules ([41, 46])

In the paper [41] the investigations of torsions and accompanying constructions in Morita contexts are continued. For a torsion r of R-Mod with radical filter (Gabriel topology) $\mathcal{F}(r)$ the *right* R-module M_R is called *r-divisible* if $MK = M$ for every $K \in \mathcal{F}(r)$. The class $\mathcal{D}_r$ of all r-divisible right R-modules is a torsion class Mod-R. For every torsion r of R-Mod there exists the greatest torsion s of R-Mod such that $\mathcal{D}_r = \mathcal{D}_s$.

For the Morita context $(R, {}_RU_S, {}_SV_R, S)$ with the trace ideals I and J the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$ (Theorem 6.1) of the lattices of torsions for *left* R-modules implies a close relation between the corresponding classes of divisible *right* R-modules.

Theorem 13.1. *For the given Morita context there exists an isotone bijection between the subcategories of divisible modules $\mathcal{D}_r$ $(r \geqslant r_I)$ of* Mod-R *and subcategories of divisible modules $\mathcal{D}_{r'}$ $(r' \geqslant r_J)$ of* Mod-S, *where (r, r') is a pair of corresponding torsions in the isomorphism $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$.*

A similar result is obtained for r-reduced modules.

In the article [46] the notions of r-divisible and r-reduced modules are generalized for idempotent radicals. If r is an idempotent radical of R-Mod, then the *right* R-module D_R is called *r-divisible* if $D \otimes {}_R X = 0$ for every ${}_R X \in \mathcal{T}_r$. A module Y_R is called *r-reduced* if it has no nontrivial r-divisible submodule. Let $\mathcal{D}_r$ ($\mathcal{R}_r$) be the class of all r-divisible (r-reduced) right R-modules. Then the pair $(\mathcal{D}_r, \mathcal{R}_r)$ defines a torsion theory in Mod-R (i.e. it determines an idempotent radical r^* of Mod-R).

The connections between the classes $\mathcal{T}_r$, $\mathcal{F}_r$ of R-Mod and corresponding classes $\mathcal{D}_r$, $\mathcal{R}_r$ of Mod-R are shown. In particular, is proved the following.

Theorem 13.2. $\mathcal{D}_r = H^{-1}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{T}_r)$, $\mathcal{D}_r = [H_{\mathbb{Q}/\mathbb{Z}}(\mathcal{T}_r)]^{\uparrow}$, $H^{-1}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{R}_r) = \mathcal{D}_r^{\downarrow}$.

The inverse transition from the class of modules $\mathcal{L} \subseteq$ Mod-R to the idempotent radical $r_{(\mathcal{L})}$ of R-Mod is defined by the rule: $\mathcal{T}_{r_{(\mathcal{L})}} = [H_{\mathbb{Q}/\mathbb{Z}}(\mathcal{L})]^{\uparrow}$. The class $\mathcal{T}_{r_{(\mathcal{L})}}$ (as well as $r_{(\mathcal{L})}$) is described by the relations $\mathcal{T}_{r_{(\mathcal{L})}} = \mathcal{L}^{\perp}$ and $\mathcal{T}_{r_{(\mathcal{L})}} = H^{-1}_{\mathbb{Q}/\mathbb{Z}}(\mathcal{L}^{\downarrow})$. The mappings $r \mapsto \mathcal{D}_r$ and $\mathcal{L} \mapsto r_{(\mathcal{L})}$ define a Galois connection. Closed elements of this connection are characterized. Furthermore, is true the following theorem.

Theorem 13.3. $\mathcal{D}_{\bigvee\limits_{\alpha \in \mathfrak{A}} r_\alpha} = \bigcap\limits_{\alpha \in \mathfrak{A}} \mathcal{D}_{r_\alpha}$.

Some curious applications of these results are indicated in the case of Morita contexts: for the corresponding torsions r and s the equivalences of accompanying subcategories are shown:

$$\mathbb{L}_r \approx \mathbb{L}_s, \qquad \mathbb{K}_r \approx \mathbb{K}_s,$$

where $\mathbb{L}_r$ is the class of r-torsion free r-injective modules, and $\mathbb{K}_r$ is the class of r-divisible r-flat modules.

These results are closely related to the investigations of B. Stenström (1975), B.J. Müller (1974), Zhou Zhengping (1991), T. Kato, T. Ohtake (1979).

14. Methods of construction of preradicals and their approximations ([47–49])

In the paper [47] the three-sided relation between the ideals of the ring R, classes of left R-modules and preradicals of R-Mod is studied.

Every ideal I of R defines the classes of modules $\mathcal{T}_I$, $\mathcal{A}(I)$, $\mathcal{F}_I$ and also the preradicals $r^{(I)}$, $r_{(J)}$, where:

$$\mathcal{T}_I = \{{}_RM \mid IM = M\}, \qquad \mathcal{A}(I) = \{{}_RM \mid IM = 0\},$$
$$\mathcal{F}_I = \{{}_RM \mid \mathrm{Im} = 0 \Rightarrow m = 0\};$$
$$r^{(I)}(M) = IM, \qquad r_{(I)}(M) = \{m \in M \mid \mathrm{Im} = 0\}.$$

Various restrictions on the ideal I imply some special properties of associated classes of modules and preradicals. In this way some bijections between ideals, classes of modules and preradicals of diverse types are obtained. This method is used in the five cases: 1) I is an arbitrary ideal; 2) $I = I^2$; 3) I is a "still" ideal, i.e. with the condition (a): $a \in Ia \ \forall\, a \in I$; 4) I is a left direct summand of R; 5) I is a ring direct summand of R. As an example we expose the case when $I = I^2$.

Theorem 14.1. *There exists a bijection between:*
1) *idempotent ideals of* R (I)*;*
2) *cotorsions of* R-Mod $(r^{(I)})$*;*
3) *jansian torsions of* R-Mod $(r_{(I)})$*;*
4) *TTF-classes of* R-Mod $\big(\mathcal{A}(I)\big)$*;*
5) *three-fold torsion theories of* R $\big((\mathcal{T}_I, \mathcal{A}(I), \mathcal{F}_I)\big)$*;*
6) *radical filters of* R *closed under intersection* $(\mathcal{E}_{r_{(I)}})$.

These results generalize and supplement some known facts on the construction of preradicals by classes of modules and by ideals: L. Bican, T. Kepka, P. Nemec (1982), Y. Kurata (1972), G. Azumaya (1973), J.S. Golan (1986), J.P. Jans (1965), etc.

Diverse methods of "improvement" of preradicals are known, i.e. of the construction of nearest preradicals with the required properties. *In the paper* [48] these methods are systematized and supplemented, using various means: functors, classes of modules, ideals, filters, etc. For example, for every preradical r it is possible to show the upper and the lower approximations by torsions or by cotorsions. In particular, is true the following.

Theorem 14.2. *If* r *is an idempotent radical of* R-Mod *and* $\mathcal{E} = \{I \in \mathbb{L}({}_RR) \mid R/I \in \mathcal{T}_r\}$*, then the set of left ideals* $\overline{\mathcal{E}} = \{K \in \mathbb{L}({}_RR) \mid (K : a) \in \mathcal{E} \ \forall\, a \in R\}$ *is a radical filter and the corresponding torsion* $\overline{r}$ *is the greatest torsion contained in* r.

The work [49] is a text-book on the theory of modules and contains an introduction in this theory, using methods of the theory of categories.

The basic subjects: principal functors; main classes of modules (projective, injective, flat); generators and cogenerators of R-Mod; homological classification of rings.

15. Natural classes of modules ([51–54])

In connection with diverse problems of module theory a group of authors (J. Dauns, 1997, 1999; Y. Zhou, 1996; A.A. Garcia, H. Rincon, J.R. Montes, 2001, etc.) introduced and studied so-called *natural* (or saturated) classes of modules, i.e. classes closed under submodules, direct sums and injective envelopes. A series of articles on this subject was completed by the book: J. Dauns, Y. Zhou, “Classes of modules”, Chapman and Hall, 2006.

In the article [51] it is proved that natural classes are closed (see Section 1), i.e. they possess the description by sets of left ideals of the ring R (Theorem 1.2). The inner characterization of natural sets of left ideals (i.e. of the form $\Gamma(\mathcal{K})$, where $\mathcal{K}$ is a natural class) is obtained. Using the mappings Γ and Δ, defined by the rules:

$$\Gamma(\mathcal{K}) = \{(0:m) \mid m \in M, M \in \mathcal{K}\},$$
$$\Delta(\mathcal{E}) = \{{}_RM \mid (a:m) \in \mathcal{E}\ \forall\, m \in M\}$$

is proved the next theorem.

Theorem 15.1. *The operators* Γ *and* Δ *define an isotone bijection between natural classes* $\mathcal{K}$ *of* R-Mod *and natural sets* $\mathcal{E}$ *of left ideals of* R. *In this case the following relations hold:*

$$\Gamma(\mathcal{K}^{\perp}) = \mathcal{E}, \qquad \Delta(\mathcal{E}^{\perp}) = \mathcal{K}^{\perp}.$$

In this way the operators Γ and Δ determine the isomorphism of boolean lattices, which consist in natural classes of R-modules and natural sets of left ideals of R. Moreover, in this case the operators of complementation are concordant with Γ and Δ.

Similar questions are discussed *in the paper* [52], where the lattice R-cl of all *closed* classes of R-Mod is studied (i.e. the classes $\mathcal{K} \subseteq R$-Mod such that $\mathcal{K} = \Delta\Gamma(\mathcal{K})$).

Theorem 15.2. *The lattice* R*-cl of closed classes of* R-Mod *is a frame.*

For every class $\mathcal{K} \in R$-cl it is proved that its pseudocomplement in R-cl coincides with the class

$$\mathcal{K}^{\perp} = \{{}_RM \mid M \text{ has no nontrivial submodules from } \mathcal{K}\}.$$

An interesting relation between closed and natural classes of modules is elucidated.

Theorem 15.3. *The skeleton of the lattice* R-cl *coincides with the lattice of natural classes* R-nat *of the category* R-Mod*, where*

$$\mathrm{Sk}(R\text{-cl}) = \{\mathcal{K}^{\perp} \mid \mathcal{K} \in R\text{-cl}\}.$$

Similar results for closed and natural sets of left ideals of R are obtained.

The investigations of natural classes of modules are continued *in the article* [53], where the relation between the torsion free classes and natural classes of R-Mod is studied. Every torsion free class (i.e. of the form $\mathcal{F}_r$ for a torsion r) is natural, so we have the inclusion $i\colon \mathbb{P} \to R$-nat, where $\mathbb{P}$ is the family of all torsion free classes of R-Mod. The inverse mapping $\phi\colon R\text{-nat} \to \mathbb{P}$ is defined, where $\phi(\mathcal{K})$ is the smallest torsion free class containing $\mathcal{K}$. Various forms of presentation of the class $\phi(\mathcal{K})$ are shown, in particular: $\phi(\mathcal{K}) = \mathcal{K}^{\uparrow\downarrow}$, $\phi(\mathcal{K}) = \mathrm{Cog}(\mathcal{K})$.

For every set of natural classes $\{\mathcal{K}_\alpha \mid \alpha \in \mathfrak{A}\}$ of R-Mod the relation $(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha)^{\uparrow} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow})$ is proved, from which follows the next theorem.

Theorem 15.4. *The mapping* ϕ *preserves the join of natural classes of modules:*

$$(\bigvee_{\alpha\in\mathfrak{A}} \mathcal{K}_\alpha)^{\uparrow\downarrow} = \bigvee_{\alpha\in\mathfrak{A}} (\mathcal{K}_\alpha^{\uparrow\downarrow}).$$

In the paper [54] some results on natural and conatural classes of R-Mod are translated in the terms of left ideals of R. If $\mathcal{K} \in R$-nat, then $\mathcal{E} = \Gamma(\mathcal{K}) = \{I \in \mathbb{L}({}_RR \mid R/I \in \mathcal{K}\}$ is the corresponding natural set of left ideals of R. Diverse descriptions of natural sets of left ideals are obtained. The lattice R-Nat of natural sets of left ideals of R coincides with the skeleton of the lattice R-Cl of closed sets. All these facts are dualized, proving similar results for conatural sets of left ideals of R.

16. Preradicals associated to principal functors ([55–57])

The next cycle of works is devoted to the investigation of preradicals accompanying principal functors of module categories, i.e. functors of the form $H = H^U = \mathrm{Hom}_R({}_RU, \text{-})$, $H' = H_U = \mathrm{Hom}_R(\text{-}, {}_RU)$ and $T = T^U = U \otimes {}_S\text{-}$, where ${}_RU_S$ is an arbitrary bimodule.

In the article [55] from this point of view the functor

$$H = \mathrm{Hom}_R({}_RU, \text{-}) \colon R\text{-Mod} \to \mathcal{A}b$$

is studied for an arbitrary module ${}_RU \in R$-Mod. Associated preradicals and their properties, as well as the conditions of coincidence of some preradicals are shown. The module ${}_RU$ defines the trace ideal $I = \sum\{\mathrm{Im}\, f \mid f \colon {}_RU \to {}_RR\}$, which in its turn determines the classes of modules ${}_I\mathcal{T}$, ${}_I\mathcal{F}$ and $\mathcal{A}(I)$. Properties of these classes lead to preradicals of diverse types: r^I, r_I, $r^{(I)}$, $r_{(I)}$, where:

$$\mathcal{R}(r^I) = {}_I\mathcal{T}, \qquad \mathcal{P}(r_I) = {}_I\mathcal{F}, \qquad \mathcal{P}(r^{(I)}) = \mathcal{A}(I), \qquad \mathcal{R}(r_{(I)}) = \mathcal{A}(I).$$

Numerous relations between the studied classes of modules (and so between the corresponding preradicals) are elucidated. Some simple conditions, under which the "near" preradicals coincide are found. In particular, the following conditions are equivalent:

$$1)\ r^U = r^I; \quad 2)\ \bar{r}^U = r^I; \quad 3)\ \bar{r}^U = r^{(I)}; \quad 4)\ r^U = r^{(I)}; \quad 5)\ IU = U.$$

In the paper [56] similar questions are studied for the functor

$$T = U \otimes {}_S\text{-} \colon S\text{-Mod} \to \mathcal{A}b,$$

where $U_S \in$ Mod-S. Classes of modules accompanying this functor are shown. All constructions indicated earlier for the functor H, have their analogues for the functor T. In particular, there exist the radical t_U in S-Mod and the nearest idempotent radical $\bar{t}_U \leqslant t_U$, where $t_U({}_SM) = \{m \in M \mid U \otimes {}_Sm = 0\}$ and $\mathcal{R}(\bar{t}_U) = \mathrm{Ker}\, T$. Conditions under which $t_U = \bar{t}_U$ or t_U is a torsion are shown.

Further, the ideal $J = (0 : U_S)$ of S and the corresponding classes of left S-modules: ${}_J\mathcal{T}$, ${}_J\mathcal{F}$, $\mathcal{A}(J)$ as well as the preradicals defined by them are considered. Properties and connections between these classes and preradicals are studied. Also the close relation with the case of functor H, which follows from the adjointness of the functors T and H, is mentioned.

The case of *contravariant* functor

$$H' = \mathrm{Hom}_R(\text{-}\,, {}_RU) \colon R\text{-Mod} \to \mathcal{A}b,$$

where ${}_RU \in R$-Mod, is studied *in the article* [57]. The module ${}_RU$ defines the radical r_U and the nearest idempotent radical $\bar{r}_U \leqslant r_U$. Some conditions are indicated when $\bar{r}_U = r_U$ or r_U is a torsion. Further, the ideal $I = (0 : {}_RU)$ of R, which defines preradicals r^I, r_I, $r^{(I)}$ and $r_{(I)}$ is considered. Numerous relations between the classes of modules and preradicals appearing in this situation are shown.

An important remark: there exists a rather full analogy between the situations for the functors H' and T. The reason of this fact is shown: the functor $T = U \otimes {}_S$- determines the same classes of modules as the functor $H' = \mathrm{Hom}_S(-, U^*)$, where ${}_SU^* = \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}U_S, \mathbb{Q}/\mathbb{Z})$ and $\mathbb{Q}/\mathbb{Z}$ is the injective cogenerator of the category of abelian groups $\mathcal{A}b$.

17. Preradicals and new operations in the lattices of submodules ([58–61])

The works [58–61] are related with some new operations in the lattices of submodules, defined by standard preradicals. We remind that standard preradicals α_N^M and ω_N^M are determined by a pair $N \subseteq M$, where $N \in \mathbb{L}({}_RM)$, as follows:

$$\alpha_N^M(X) = \sum_{f:\, M \to X} f(N), \qquad \omega_N^M(X) = \bigcap_{f:\, X \to M} f^{-1}(N), \qquad X \in R\text{-Mod}.$$

Denote by $\mathbb{L}^{\mathrm{ch}}({}_RM)$ the lattice of *characteristic* (fully invariant) submodules of M (i.e. such $N \subseteq M$ that $f(N) \subseteq N$ for every $f\colon M \to M$). *In the article* [58] the relations between the lattice $\mathbb{L}^{\mathrm{ch}}({}_RM)$ and some sublattices of the lattice R-pr of all preradicals of R-Mod are studied. The mappings

$$\alpha^M : \mathbb{L}^{\mathrm{ch}}({}_RM) \to R\text{-pr}, \qquad \omega^M : \mathbb{L}^{\mathrm{ch}}({}_RM) \to R\text{-pr},$$

which transfer N into α_N^M and N into ω_N^M, are injective, so we obtain the isomorphisms:

$$\mathbb{L}^{\mathrm{ch}}({}_RM) \cong \mathrm{Im}\,\alpha^M \subseteq R\text{-pr}, \qquad \mathbb{L}^{\mathrm{ch}}({}_RM) \cong \mathrm{Im}\,\omega^M \subseteq R\text{-pr}.$$

In other form the relation between $\mathbb{L}^{\mathrm{ch}}({}_RM)$ and R-pr can be expressed defining on R-pr the equivalence relation:

$$r \cong_M s \Leftrightarrow r(M) = s(M).$$

Then R-pr is divided into the classes of equivalence, which have the form $[\alpha_N^M, \omega_N^M]$ and the isomorphism between $\mathbb{L}^{\mathrm{ch}}({}_RM)$ and $R\text{-pr}/\cong_M$ holds.

Using the product and coproduct of preradicals and the standard preradicals α_N^M and ω_N^M, for $K, N \in \mathbb{L}^{\mathrm{ch}}({}_RM)$ in the lattice $\mathbb{L}^{\mathrm{ch}}({}_RM)$ four operations are defined:

1) *α-product* $K \cdot N$ is $\alpha_K^M \alpha_N^M(M) = \sum_{f:\, M \to N} f(K)$;

2) *ω-product* $K \odot N$ is $\omega_K^M \omega_N^M(M) = \bigcap_{f:\, N \to M} f^{-1}(K)$;

3) *α-coproduct* $N : K$ is $(\alpha_N^M : \alpha_K^M)(M)$;

4) *ω-coproduct* $N \odot K$ is $(\omega_N^M : \omega_K^M)(M)$.

Basic properties of these operations are shown, in particular the distributivity of diverse types. For example, is true the following theorem.

Theorem 17.1. *The following relations hold:*

$$(K_1 + K_2) \cdot N = (K_1 \cdot N) + (K_2 \cdot N);$$
$$(K_1 \cap K_2) \odot N = (K_1 \odot N) \cap (K_2 \odot N);$$
$$N : (K_1 + K_2) = (N : K_1) + (N : K_2);$$
$$N \odot (K_1 \cap K_2) = (N \odot K_1) \cap (N \odot K_2).$$

If ${}_R M = {}_R R$ (i.e. $\mathbb{L}^{\text{ch}}({}_R M)$ is the lattice of ideals of R), then two operations coincide with multiplication and addition of ideals.

The foregoing ideas are developed *in the paper* [59], where the standard preradicals α_N^M and ω_N^M are used to define four operations in the lattice $\mathbb{L}({}_R M)$ of *all* submodules of M. Namely, *α-product* of $K, N \in \mathbb{L}({}_R M)$ is

$$K \cdot N = \alpha_K^M(N) = \sum_{f:\, M \to N} f(K),$$

and *ω-product* is

$$K \odot N = \omega_K^M(N) = \bigcap_{f:\, N \to M} f^{-1}(K).$$

A series of properties of these operations is shown (as associativity and distributivity of diverse types).

In a dual form the other two operations are defined, using the coproduct of preradicals. Namely, *α-coproduct* $N : K$ is defined by the relation $(N : K)/N = \alpha_K^M(M/N)$, and the *ω-coproduct* $N \odot K$ is defined similarly: $(N \odot K)/N = \omega_K^M(M/N)$. Various forms of presentation of these operations, as well as a series of their properties are exposed. In the case $M =_R R$ we have:

$$I \cdot J = IJ, \qquad I \odot J = \bigcap_{f:\, {}_R I \to {}_R R} f^{-1}(I),$$
$$I : J = JR + I, \qquad I \odot J = \big(J : (0 : I)_r\big)_l.$$

A natural continuation of the previous investigations *is the article* [60], in which the *inverse operations* with respect to α-product and ω-coproduct are introduced and studied. Namely, for $N, K \in \mathbb{L}({}_RM)$ the *left quotient relative to α-product* is defined as

$$N \,/_{\!\bullet}\, K = \sum\{L_\alpha \subseteq M \mid L_\alpha \cdot K \subseteq N\},$$

and the *right quotient relative to ω-coproduct* is defined by the rule:

$$N \,{}_{\odot}\!\backslash\, K = \bigcap\{L_\alpha \subseteq M \mid N \odot L_\alpha \supseteq K\}.$$

The distributivity of the operations of α-product and ω-coproduct ensures the existence of these quotients for all submodules.

Various possibilities of representation of these quotients, as well as basic properties of the considered operations are shown. For example, is true the next theorem.

Theorem 17.2. *The following relations hold:*

$$(\bigcap_{\alpha\in\mathfrak{A}} N_\alpha) \,/_{\!\bullet}\, K = \bigcap_{\alpha\in\mathfrak{A}} (N_\alpha \,/_{\!\bullet}\, K), \quad N \,{}_{\odot}\!\backslash\, (\sum_{\alpha\in\mathfrak{A}} K_\alpha) = \sum_{\alpha\in\mathfrak{A}} (N \,{}_{\odot}\!\backslash\, K_\alpha).$$

In the particular case when $M =_R R$ we have:

$$I \,/_{\!\bullet}\, J = (I : J)_l = \{a \in R \mid aJ \subseteq I\},$$
$$I \,{}_{\odot}\!\backslash\, J = J(0 : I)_r, \text{ where } (0 : I)_r = \{b \in R \mid Ib = 0\}.$$

In *the article* [61] the inverse operations for ω-product and α-coproduct are introduced and investigated. In contrast to the previous cases, these operations are *partial*.

The *left quotient of N by K relative to ω-product $N \,/_{\odot}\, K$* is defined as the smallest submodule $L \subseteq M$ with the property $L \odot K \supseteq N$. Similarly, the *right quotient $N \,{}_{:}\!\backslash\, K$ relative to α-coproduct* is the greatest submodule $L \subseteq M$ such that $N : L \subseteq K$.

The existence of the quotients $N \,/_{\odot}\, K$ and $N \,{}_{:}\!\backslash\, K$ is equivalent to the relation $N \subseteq K$. Diverse forms of representation of these quotients are shown. Also properties of these operations and relations with the lattice operations of $\mathbb{L}({}_RM)$ are obtained.

Theorem 17.3. *The following relations hold:*

$$(\sum_{\alpha\in\mathfrak{A}} N_\alpha) \,/_{\odot}\, K = \sum_{\alpha\in\mathfrak{A}} (N_\alpha \,/_{\odot}\, K), \quad N_\alpha \subseteq K,\ \alpha \in \mathfrak{A};$$
$$N \,{}_{:}\!\backslash\, (\bigcap_{\alpha\in\mathfrak{A}} K_\alpha) = \bigcap_{\alpha\in\mathfrak{A}} (N \,{}_{:}\!\backslash\, K_\alpha), \quad N \subseteq K_\alpha,\ \alpha \in \mathfrak{A}.$$

18. Closure operators and preradicals ([62–66])

The cycle of works [62–66] is devoted to closure operators in module categories and their relations with preradicals of these categories. Basic types of closure operators of R-Mod, their properties and connections, as well as operations with closure operators are investigated. The question of the interrelations between closure operators and preradicals of R-Mod has a special interest.

Earlier (in the works [7, 9] and [20]) the fact that every idempotent radical of R-Mod defines a special closure operator of this category was remarked and used. The notion of *radical closure* of R-Mod was defined and studied.

A more general notion of closure operator of R-Mod is studied in works of D. Dikranjan, E. Giuli, W. Tholen, etc. A result of these investigations is the book: D. Dikranjan, W. Tholen "Categorical structures of closure operators", Kluwer Acad. Publ., 1995.

In the article [62] the main types of closure operators (weakly hereditary and idempotent) are described by dense and closed submodules. Let $\mathbb{CO}$ be the class of all closure operators of R-Mod. For an operator $C \in \mathbb{CO}$ a submodule $N \in \mathbb{L}({}_RM)$ is called *C-dense* (*C-closed*) in M if $C_M(N) = M$ $(C_M(N) = M)$. We denote:

$$\mathcal{F}_1^C(M) = \{N \subseteq M \mid C_M(N) = M\},$$
$$\mathcal{F}_2^C(M) = \{N \subseteq M \mid C_M(N) = N\}.$$

In this way every operator $C \in \mathbb{CO}$ defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, distinguishing in each $M \in R$-Mod the sets of C-dense or C-closed submodules. Basic properties of these functions are shown. Also, the possibilities of restoration of C by the functions $\mathcal{F}_1^C$ or $\mathcal{F}_2^C$ are indicated.

For an abstract function $\mathcal{F}$ the definitions of a function *of type* $\mathcal{F}_1$ and a function *of type* $\mathcal{F}_2$ are given, using properties of the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$.

Theorem 18.1. *There exists an isotone bijection between weakly hereditary closure operators of $C \in \mathbb{CO}$ and abstract functions of type $\mathcal{F}_1$. So weakly hereditary closure operators are described by dense submodules.*

Dual result for idempotent closure operators is obtained.

Theorem 18.2. *There exists an antiisotone bijection between idempotent closure operators of $\mathbb{CO}$ and abstract functions of the type $\mathcal{F}_2$, so idempotent closure operators are described by closed submodules.*

Further the case, when an operator $C \in \mathbb{CO}$ is simultaneously weakly hereditary and idempotent, is considered. Then the operator C can be reestablished both by $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$. Properties of the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ are indicated in this case and analogues of previous theorems are proved. Namely, bijections are obtained between weakly hereditary idempotent closure operators and functions of type $\mathcal{F}_1$ or of type $\mathcal{F}_2$ with the property of transitivity:

($**$) If $N \subseteq P \subseteq M$, $N \in \mathcal{F}(P)$ and $P \in \mathcal{F}(M)$, then $N \in \mathcal{F}(M)$.

On the basis of previous results *in the paper [63]* the characterizations by the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ are obtained for some other important classes of closure operators: hereditary, weakly hereditary maximal, hereditary maximal, minimal and cohereditary. In each of these cases conditions on the functions $\mathcal{F}_1^C$ or $\mathcal{F}_2^C$, which are necessary and sufficient for the restoration of the operator C are indicated. As an example, we consider the case of *hereditary* closure operators, i.e. such $C \in \mathbb{CO}$ that in the situation $L \subseteq N \subseteq M$ the relation $C_N(L) = C_M(L) \cap N$ holds. Such an operator is weakly hereditary, so it is described by $\mathcal{F}_1^C$ (Theorem 18.1). For an abstract function $\mathcal{F}$ the following condition is considered:

($***$) If $N \subseteq P \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(P)$.

Theorem 18.3. *There exists an isotone bijection between hereditary closure operators $C \in \mathbb{CO}$ and abstract functions of type $\mathcal{F}_1$ with the condition ($***$).*

A similar method is used for the characterization of the rest of named types of closure operators.

The article [64] is a continuation of these researches and contains the study of the basic four operations in the class $\mathbb{CO}$ of closure operators of R-Mod: meet ($\wedge$), join ($\vee$), multiplication ($\cdot$), comultiplication ($\#$). Principal properties of these operations and relations between them are shown. In particular, some properties of distributivity are proved.

Theorem 18.4. *The following relations hold:*

$$(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha)\cdot D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D), \qquad (\bigvee_{\alpha\in\mathfrak{A}} C_\alpha)\cdot D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D);$$

$$(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha)\# D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \# D), \qquad (\bigvee_{\alpha\in\mathfrak{A}} C_\alpha)\# D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \# D).$$

Some other distributivity relations are true under additional conditions. For example, is proved the following theorem.

Theorem 18.5. a) *If the operator* $C \in \mathbb{CO}$ *is* hereditary, *then*

$$C \# (\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha) = \bigwedge_{\alpha \in \mathfrak{A}} (C \# D_\alpha)$$

for arbitrary $D_\alpha \in \mathbb{CO}$, $\alpha \in \mathfrak{A}$.

b) *If the operator* $C \in \mathbb{CO}$ *is* minimal, *then*

$$C \cdot (\bigvee_{\alpha \in \mathfrak{A}} D_\alpha) = \bigvee_{\alpha \in \mathfrak{A}} (C \cdot D_\alpha)$$

for arbitrary $D_\alpha \in \mathbb{CO}$, $\alpha \in \mathfrak{A}$.

The question on the preservation of properties of closure operators by the indicated operations in $\mathbb{CO}$ is studied. Some results on this subject are:

1) if C_α, $\alpha \in \mathfrak{A}$, are weakly hereditary, then $\bigvee_{\alpha \in \mathfrak{A}} C_\alpha$ is weakly hereditary;
2) if C_α, $\alpha \in \mathfrak{A}$, are maximal (minimal), then $\bigvee_{\alpha \in \mathfrak{A}} C_\alpha$ is maximal (minimal);
3) if C_α, $\alpha \in \mathfrak{A}$, are hereditary, then $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is hereditary;
4) if C_α, $\alpha \in \mathfrak{A}$, are maximal, then $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is maximal.

Some relations between closure operators and preradicals of R-Mod are studied *in the article* [66]. Three mappings are defined between the classes $\mathbb{CO}$ of closure operators and $\mathbb{PR}$ of preradicals of R-Mod:

$$\Phi \colon \mathbb{CO} \to \mathbb{PR}, \qquad \Psi_1 \colon \mathbb{PR} \to \mathbb{CO}, \qquad \Psi_2 \colon \mathbb{PR} \to \mathbb{CO},$$

where

$$\begin{aligned} \Phi(C) &= r_{C}, & r_{C}(M) &= C_M(0); \\ \Psi_1(r) &= C^r, & [(C^r)_M(N)]/N &= r(M/N); \\ \Psi_2(r) &= C_r, & (C_r)_M(N) &= N + r(M). \end{aligned}$$

Denote by $\mathbb{M}\mathrm{ax}(\mathbb{CO})$ ($\mathbb{M}\mathrm{in}(\mathbb{CO})$) the class of all maximal (minimal) closure operators of R-Mod. The pair of mappings (Φ, Ψ_1) determines an isomorphism $\mathbb{M}\mathrm{ax}(\mathbb{CO}) \cong \mathbb{PR}$, while the pair (Φ, Ψ_2) defines an isomorphism $\mathbb{M}\mathrm{in}(\mathbb{CO}) \cong \mathbb{PR}$. Using these isomorphisms, in continuation some bijections are obtained between preradicals of diverse types (idempotent, radical, hereditary etc.) and closure operators with special properties. In particular, is proved the following.

Theorem 18.6. 1) *The pair of mappings* (Φ, Ψ_1) *determines isotone bijections between:*

- *idempotent preradicals of* $\mathbb{PR}$ *and maximal weakly hereditary closure operators of* $\mathbb{CO}$*;*
- *pretorsions of* $\mathbb{PR}$ *and maximal hereditary closure operators of* $\mathbb{CO}$*.*

2) *The pair of mappings* (Φ, Ψ_2) *defines isotone bijections between:*

- *idempotent preradicals of* $\mathbb{PR}$ *and minimal weakly hereditary closure operators of* $\mathbb{CO}$*;*
- *pretorsions of* $\mathbb{PR}$ *and minimal hereditary closure operators of* $\mathbb{CO}$*;*
- *cotorsions of* $\mathbb{PR}$ *and weakly hereditary and cohereditary closure operators of* $\mathbb{CO}$*.*

The influence of the mappings Φ, Ψ_1 and Ψ_2 on operations in the classes $\mathbb{CO}$ and $\mathbb{PR}$ is studied *in the paper* [65]. In particular, it is proved that the mapping Φ preserves the meets and joins:

$$\Phi(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha) = \bigwedge_{\alpha\in\mathfrak{A}} [\Phi(C_\alpha)], \qquad \Phi(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha) = \bigvee_{\alpha\in\mathfrak{A}} [\Phi(C_\alpha)].$$

Furthermore, the mapping Φ transforms the coproducts of $\mathbb{CO}$ into the products of $\mathbb{PR}$: $\Phi(C \# D) = \Phi(C) \cdot \Phi(D)$.

The mapping Ψ_1 preserves the meets and joins:

$$\Psi_1(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha) = \bigwedge_{\alpha\in\mathfrak{A}} [\Psi_1(r_\alpha)], \qquad \Psi(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha) = \bigvee_{\alpha\in\mathfrak{A}} [\Psi_1(r_\alpha)].$$

Moreover, Ψ_1 transforms the products of $\mathbb{PR}$ into the coproducts of $\mathbb{CO}$: $\Psi_1(r \cdot s) = \Psi_1(r) \# \Psi_1(s)$, and the coproducts of $\mathbb{PR}$ into the products of $\mathbb{CO}$: $\Psi_1(r : s) = \Psi_1(r) \cdot \Psi_1(s)$.

Some similar results are obtained for the mapping Ψ_2.

On the whole the works of this cycle show the expediency and usefulness of the combined investigations of preradicals and closure operators of R-Mod.

19. Instead of conclusion

The present work reflects the aspiration to embrace by one survey the majority of results of author on radicals and torsions in modules. The requirement of conciseness made impossible the minimal full exposition of results, and so we limit ourselves by formulation of questions and of small part of results. Furthermore, it is impossible to give the definitions and preliminary facts, which strongly restrict the possibilities to show the "entourage" in which the exposed results were obtained. The references in the main text are reduced to the minimum, otherwise the bibliography sharply increases.

The effort to overcome these difficulties led us to the presented form of the review, which can give a general idea of the direction of investigations and of the type of results.

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