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RESEARCH ARTICLE

On dispersing representations of quivers and their connection with representations of bundles of semichains

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Abstract. In the paper we discuss the notion of "dispersing representation of a quiver" and give, in a natural special case, a criterion for the problem of classifying such representations to be tame. In proving the criterion we essentially use representations of bundles of semichains, introduced about fifteen years ago by the author.

1. Introduction

The classical problems of linear algebra on the reduction of the matrix of a linear map (by means of elementary row and column transformations) and the matrix of a linear operator (by means of similarity transformations) to canonical forms can be generalized in the following two ways: by considering a greater number of maps or giving more complicated structure of vector spaces. The first way led finally to the notion of representations of a quiver (P. Gabriel). As examples of a generalization of the second type it may be mentioned the well-known vectorspace problem [1, p. 82], its natural "two-dimensional" analog [2, 3] and a general extension of the classical problem on one linear operator [4, 3]. Clearly one can consider various generalizations of the classical problems combining two indicated ways. In [3] the author consider a common generalization

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of "mixed" type introducing the notion of "dispersing representation of a quiver (with relations)". In terms of these representations one can formulate many classification problems, among them the problems on representations of posets [5], bundles of semichaines [6], tangles [7], etc. (and also all the above mentioned ones). In this paper we study dispersing representations of (finite and infinite) quivers without relations. In considering criterions of tameness we essentially use a main result on representations of bundles of finitely many semichains [8, 6] and his extension to the case of infinitely many ones (see the last section).

2. Main notions and examples

Throughout the paper, we will keep the right-side notation. All vector spaces over a field k will be finite-dimensional; the category of such spaces will be denoted as usual by $\operatorname{mod} k$. Unless otherwise stated, all quivers and posets will be finite. The sign $\coprod$ will denote the direct sum of posets, categories or functors. Singletons will be always identified with the elements themselves.

We first recall the definition of dispersing representations of a quiver [3, Section 10].

Let $\mathcal{A}$ be a Krull-Schmidt category over a field k. By a (right) module over $\mathcal{A}$ we mean as usual a k-linear functor $F : \mathcal{A} \to \operatorname{mod} k$. A collection $M = \{M_i\}$ of modules $M_i : \mathcal{A} \to \operatorname{mod} k$, where i run through a set X, is said to be an *X-bunch of modules over $\mathcal{A}$*. An X-bunch M is said to be *faithful* if $\mathrm{Ann} M = \bigcap_{i \in X} \mathrm{Ann} M_i = 0$ ($\mathrm{Ann} M_i$ being the annihilator of M_i). We call X-bunches M and M' of modules over $\mathcal{A}$ and $\mathcal{A}'$, respectively, *equivalent* if there exists an equivalence $F : \mathcal{A} \to \mathcal{A}'$ such that, for each $i \in X$, the modules M_i and FM'_i are isomorphic; in this case we write $M \cong M'$ or $(\mathcal{A}, M) \cong (\mathcal{A}', M')$.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a (not necessarily finite) quiver with the set of vertices (points) Γ_0 and the set of arrows Γ_1, and k a field. Fix a Krull-Schmidt category $\mathcal{A}$ over k and a Γ_0-bunch M of modules over $\mathcal{A}$. We call *M-dispersing representation* of Γ (or *dispersing representation with respect to M*, or simply *dispersing representation* if M is fixed) a pair $U = (M(X), f)$ formed by the collection of vector spaces $M(X) = \{M_i(X) | \, i \in \Gamma_0\}$ for an object $X \in \mathcal{A}$ and a collection $f = \{f_\alpha | \, \alpha : i \to j \text{ run through } \Gamma_1\}$ of linear maps $f_\alpha : M_i(X) \to M_j(X)$. A morphism from $U = (M(X), f)$ to $U' = (M(X'), f')$ is determined by a morphism $\varphi : X \to X'$ satisfying $f_\alpha M_j(\varphi) = M_i(\varphi) f'_\alpha$ for each arrow $\alpha : i \to j$. The category of M-dispersing representations of Γ is denoted by $\mathrm{rep}_M \Gamma$; by $\mathrm{rep}_M^{\mathrm{inv}} \Gamma$ we denote the full subcategory of $\mathrm{rep}_M \Gamma$ consisting of all objects $U = (M(X), f)$ with invertible linear maps f_α (α runs over Γ_1).

If we take $\mathcal{A} = \coprod_{i\in\Gamma_0} \mathcal{A}_i$ with $\mathcal{A}_i = \operatorname{mod} k$ for each i, and $M_i = \coprod_{j\in\Gamma_0} M_{ij}$ with $M_{ij} = \delta_{ij}\mathbf{1}_{\mathcal{A}_j} : \mathcal{A}_j \to \operatorname{mod} k$ (δ_{ij} being the Kroneker delta), then the case of usual representations of Γ occurs.

Our notion is naturally generalized to the case of quivers with relations. Moreover, one can take any ring instead of the field k, an arbitrary category instead of the category $\mathcal{A}$ or $\operatorname{mod} k$, etc.

In terms of dispersing representations one can formulate many classification problems.

Example 2.1. Let Γ be the quiver $\overset{1}{\circ}\!\longrightarrow\!\overset{2}{\circ}$ and C a finite poset which is identified with the following category: $\mathrm{Ob}C = C, C(x,y) = \{(x|y)\}$ if $x \le y$ and $C(x,y) = \varnothing$, otherwise; composition is such that $(x|y)(y|z) = (x|z)$. Denote by $\mathcal{C}$ the category $\oplus kC$ (kC being the linearization of C and $\oplus kC$ its additive hull) and by N the module over $\mathcal{C}$ such that $N(x) = k$ for each $x \in C$ and $N(x|y) = \mathbf{1}_k$. Set $\mathcal{A} = \mathcal{B}\coprod\mathcal{C}$ with $\mathcal{B} = \operatorname{mod} k$, and $M_1 = \mathbf{1}_{\mathcal{B}}\coprod \mathbf{0}_{\mathcal{C}}$, $M_2 = \mathbf{0}_{\mathcal{B}}\coprod N$ with the identity module $\mathbf{1}_{\mathcal{B}} : \mathcal{B} \to \operatorname{mod} k$ and the zero ones $\mathbf{0}_{\mathcal{C}} : \mathcal{C} \to \operatorname{mod} k$, $\mathbf{0}_{\mathcal{B}} : \mathcal{B} \to \operatorname{mod} k$. Then the category of $\{M_1, M_2\}$-dispersing representations of Γ is in fact the category of representations of the poset C [5, §4].

A general case of a "decomposable" bunch (as in the example) arise, in other terms, in studying representations of dyadic sets [9, Section 0].

From the point of view of the author, the most interesting cases occur when (in contrast to the previous case) a system M of modules is not "decomposable" or there is a quiver with relations.

Before discussing such examples we give some definitions.

Let $S = (A, *)$ be a (not necessarily finite) poset with involution. By an *S-graded vector space over k* we mean the direct sum $U = \bigoplus_{a\in A} U_a$ of k-vector spaces U_a such that $U_{a^*} = U_a$ for all $a \in A$. A linear map φ of an S-graded space $U = \bigoplus_{a\in A} U_a$ into an S-graded space $U' = \bigoplus_{a\in A} U'_a$ will be called an *S-map* if $\varphi_{a^*a^*} = \varphi_{aa}$ for each $a \in A$ and $\varphi_{bc} = 0$ for each $b, c \in A$ not satisfying $b \le c$, where φ_{xy} denotes the linear map of U_x into U'_y induced by the map φ. The category of S-graded vector spaces over k (with objects the S-graded spaces and with morphisms the S-maps) is denoted by $\operatorname{mod}_S k$ [1]. Because $S = (A, *)$ with trivial involution is naturally identified with A, these definitions involve the case of usual posets. For a poset $A = \coprod_{i=1}^n A_i$, we identify $\operatorname{mod}_A k$ with $\coprod_{i=1}^n \operatorname{mod}_{A_i} k$.

Recall that a *semichain* is by definition a poset A such that every element of A is comparable with all but at most one elements. Obviously, any semichain A can be uniquely represented in the form $A = \bigcup_{i=1}^m A_i$,

[1] When S is infinite and $U \in \operatorname{mod}_S k$, we have $U_a = 0$ for all but finitely many $a \in A$ (because we consider only finite-dimensional vector spaces).

where each A_i (called a *link* of A) consist of either one point or two incomparable points, and $A_1 < A_2 < \cdots < A_m$, where, for subsets X and Y of a poset, $X < Y$ means that $x < y$ for any $x \in X, y \in Y$ (if each A_i consist of one point, the set A is called a *chain*); the number m is called the *length* of A. A semichain A with involution $*$ is called a $*$-*semichain* if $x^* = x$ for every x belonging to the union of all two-point links.

Example 2.2. Let Γ be the quiver with one vertex, one loop φ and one relation $f(\varphi) = 0$, where $f(t) = t^2$, and let $S = (A, *)$ be a poset with involution. Set $\mathcal{A} = \operatorname{mod}_S k$ and denote by $M : \operatorname{mod}_S k \to \operatorname{mod} k$ the natural imbedding module. In the case when S is a $*$-semichain, M-dispersing representations of Γ was classified in [10, §2] (in connection with classifying the modular representations of quasidihedral groups); the case of a chain with involution was considered earlier in [11, §1]. The case, when $\mathcal{A}$ is an arbitrary Krull-Schmidt subcategory in $\operatorname{mod} k$ and $f(t)$ an arbitrary polynomial, is considered in [4, 3].

Finally we consider an example which plays a central role in our consideration.

Example 2.3. Let $S = \{A_1, \dots, A_n, B_1, \dots, B_n\}$ be a family of pairwise disjoint semichains; set $A = \coprod_{i=1}^n A_i$ and $B = \coprod_{i=1}^n B_i$. A *bundle of semichains* $A_1, \dots, A_n, B_1, \dots, B_n$ is a pair $\overline{S} = (S, *)$, where $*$ is an involution on the set $S_0 = A \coprod B$ such that $x^* = x$ for each x from the union of all two-point links (of the given semichains).

Let $\overline{S} = (S, *)$ be a bundle of semichaines $A_1, \dots, A_n, B_1, \dots, B_n$. A representation of the bundle $\overline{S} = (S, *)$ over a field k is a triple (U, V, φ), where

(1) $U = \{U_1, \dots, U_n\}$ and $V = \{V_1, \dots, V_n\}$ are collections of k-spaces such that $U_i \in \operatorname{mod}_{A_i} k$, $V_i \in \operatorname{mod}_{B_i} k$ $(i = 1, \dots, n)$, and the $A \coprod B$-graded space $(\bigoplus_{i=1}^n U_i) \oplus (\bigoplus_{i=1}^n V_i)$ belong to the subcategory $\operatorname{mod}_{(A \coprod B, *)} k$ of $\operatorname{mod}_{A \coprod B} k$;

(2) $\varphi = \{\varphi_1, \dots, \varphi_n\}$ is a collection of linear maps $\varphi_i \in \operatorname{Hom}_k(U_i, V_i)$, $i = 1, \dots, n$.

A morphism from

$$(U, V, \varphi) = (\{U_1, \dots, U_n\}, \{V_1, \dots, V_n\}, \{\varphi_1, \dots, \varphi_n\})$$

to

$$(U', V', \varphi') = (\{U'_1, \dots, U'_n\}, \{V'_1, \dots, V'_n\}, \{\varphi'_1, \dots, \varphi'_n\})$$

is determined by a pair (α, β) formed by a collection $\alpha = \{\alpha_1, \dots, \alpha_n\}$ of A_i-maps $\alpha_i \colon U_i \to U'_i$ and a collection $\beta = \{\beta_1, \dots, \beta_n\}$ of B_i-maps $\beta_i \colon V_i \to V'_i$ $(i = 1, \dots, n)$ such that

(3) the $A\coprod B$-map $(\bigoplus_{i=1}^n \alpha_i) \oplus (\bigoplus_{i=1}^n \beta_i)$ of $(\bigoplus_{i=1}^n U_i) \oplus (\bigoplus_{i=1}^n V_i)$ into $(\bigoplus_{i=1}^n U_i') \oplus (\bigoplus_{i=1}^n V_i')$ belong to the subcategory $\mathrm{mod}_{(A\coprod B,*)}\, k$;

(4) $\varphi_i\beta_i = \alpha_i\varphi_i'$ for each $i = 1, \ldots, n$.

The category of representations of the bundle of semichains $\overline{S} = (S, *)$ is denoted by $\mathcal{B}_k\,(\overline{S}) = \mathcal{B}_k\,(S, *) = \mathcal{B}_k\,(A_1, \ldots, A_n, B_1, \ldots, B_n, *)$.

The definition of representations of bundles of semichaines can be easily rewrited in terms of dispersing representations.

Denote by $\Lambda(n)$ the quiver with the set of vertices

$$\Lambda_0(n) = \{1^-, \ldots, n^-, 1^+, \ldots, n^+\}$$

and the arrows $(i^-, i^+) : i^- \to i^+$ for $i = 1, \ldots, n$. In our new terms, a representation of the bundle $\overline{S} = (S, *)$ is a P-dispersing representation of $\Lambda(n)$ with the category $\mathcal{S} = \mathcal{K}(\overline{S}) = \mathrm{mod}_{(A\coprod B,*)}k$ (as $\mathcal{A}$) and the modules $P_i = P_i(\overline{S}) : \mathcal{S} \to \mathrm{mod}\, k$ (i run through $\Lambda_0(n)$) to be the composition of the natural embedding of $\mathcal{S}$ in $\mathcal{S}_0 = \mathrm{mod}_{A\coprod B}k$ and the projection of $\mathcal{S}_0$ onto $\mathrm{mod}_{A_i}k$ (resp. $\mathrm{mod}_{B_i}k$) for $i = j^-$ (resp. $i = j^+$). Obviously, the category $\mathcal{B}_k\,(\overline{S})$ is isomorphic to the category $\mathrm{rep}_P\Lambda(n)$ with $P = \{P_i | i \in \Lambda_0(n)\}$.

The representations of a bundle of semichains (and the notion of "bundle of semichains" itself) were introduced in [6, §1] (for the first time, in [8]). In these papers the author give (in terms of matrices) a complete classification of the indecomposable representations of an arbitrary bundle of semichains; the classifying is obtained in the explicit and invariant (without "trace" of the method of solution) form.

In special case, when there is only two semichaines, representations of bundles arose under consideration a problem of I. M. Gelfand [12][2], in the classification of the modular representations of quasidihedral groups [13, 10] (see also [6, §2]) and in studying numerous other problems: in studying representations of different classes of quivers with relations and algebras (see e.g. [14, 15, 16, 17, 18]), in the classification of faithful posets of infinite (non-polynomial, in other terminology) growth [19], under consideration representations of posets with involution [20] and equivalence relation [21]. In studying representations of posets with non-singularity conditions [22, 23] there arose representations of bundles of four semichaines. For an arbitrary (even) number of semichaines, representations of bundles arose first in solving the Gelfand problem and its generalizations [6, §3]. Recently the main classification theorem of [6, §1] is used in solving various classification problems of representation theory,

[2]The tameness of the problem under consideration also follows from properties of an algorithm described in [12, §2]), but an inductive answer indicated there (for two semichaines, if one use our terminology) is false.

topology and algebraic geometry (see e.g. [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

The main reason of wide application of representations of bundles of semichains is that, for many classification problems, "most" of tame cases are reduced to them (such a reduction is today the only method of solving many problems of infinite growth, as for representations of quasidihedral groups [10] or partially ordered sets [19]). As will be seen below, this is also true for dispersing representations of quiver.

3. Main result

We assume from now on that k is an algebraic closed field. For a Krull-Schmidt category $\mathcal{A}$ (over k), we denote by $\mathcal{A}_0$ a fixed full subcategory of $\mathcal{A}$ formed by chosen representatives of all isomorphism classes of indecomposables; we will assume throughout this section that $|\mathrm{Ob}\mathcal{A}_0| < \infty$ (for the case $|\mathrm{Ob}\mathcal{A}_0| = \infty$ see the next section).

Let N be a module over $\mathcal{A}$, and define

$$\mathrm{supp}_0 N = \{X \in \mathrm{Ob}\mathcal{A}_0 |\, N(X) \neq 0\};$$

for $X, Y \in \mathrm{Ob}\mathcal{A}_0$, set $N(X,Y) = N(\mathcal{A}_0(X,Y))$. We call N *saturated* if $\dim_k N(X,X) = \dim_k N(X)(\dim_k N(X) - 1)/2 + 1$ for any $X \in \mathrm{supp}_0 N$ and $\dim_k N(X,Y) + \dim_k N(Y,X)$ is equal to 0 or to $\dim_k N(X) \dim_k N(Y)$ for any distinct $X, Y \in \mathrm{supp}_0 N$ (i.e. nonzero $\dim_k N(X,X)$ and $\dim_k N(X,Y) + \dim_k N(Y,X)$ take the greatest possible values). For $X \in \mathrm{Ob}\mathcal{A}_0$, we will denote by N^X the submodule of N generated by $N(X)$. Let $\mathcal{L}(N)$ denote the lattice of submodules in N ordered by inclusion. We call N *lattice-chained* (resp. *lattice-semichained*) if $\mathcal{L}(N)$ is a chain (resp. a semichaine), and *chained* (resp. *semichained*) if in addition it is saturated. Finally, we say that a submodule N' of N is *singular* if it is comparable (in $\mathcal{L}(N)$) to each submodule of N.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver. For an arrow α, denote by $s(\alpha)$ and $e(\alpha)$ its starting point and its endpoint, respectively. By $w^-(i)$ (resp. $w^+(i)$), where $i \in \Gamma_0$, denote the number of arrows α with $s(\alpha) = i$ (resp. $e(\alpha) = i$); put $w(i) = w^-(i) + w^+(i)$. A vertex i is said to be *trivial* if $w(i) = 0$, *outer* if $w(i) = 1$ and *inner* if $w(i) > 1$. The sets of all trivial, outer and inner vertices are denoted by Γ_0^0, Γ_0^1 and Γ_0^2, respectively. Let $M = \{M_i\}$ be a fixed Γ_0-bunch of $\mathcal{A}$-modules. We call M_i *isolated* if $\mathrm{supp}_0 M_i \cap \mathrm{supp}_0 M_j = \varnothing$ for any $j \neq i$. An isolated chained module $M_i \neq 0$ with $\dim_k M_i(X) \leq 1$ for any object $X \in \mathcal{A}_0$ is said to be *elementary*.

We call Γ *M-tame* (resp. *M-wild*) if so is the problem of classifying the objects of the category $\mathrm{rep}_M \Gamma$ [35]; a quiver of *M-finite* (*M-infinite*) *type* is defined similarly. Further, we call Γ M-inv-*wild* if the problem of classifying the object of the category $\mathrm{rep}_M^{\mathrm{inv}}\Gamma$ is wild. In considering these problems, it is obviously sufficient to confine oneself to quivers without trivial vertices.

Our main result is the following theorem.

Theorem 3.1. *Let Γ be a finite (not necessarily connected) quiver without trivial vertices and $M = \{M_i\}$ a Γ_0-bunch of nonzero $\mathcal{A}$-modules without elementary ones for outer vertices. Then Γ is M-tame if and only if the following conditions hold*:

(1) *$w(i) \le 2$ for any $i \in \Gamma_0$;*

(2) *the module M_i is semichained for each $i \in \Gamma_0^1$ and is simple and isolated for each $i \in \Gamma_0^2$;*

(3) *$\sum_{i\in\Gamma_0^1} \dim_k M_i(X) \le 2$ for each object $X \in \mathcal{A}_0$; moreover, when $\dim_k M_j(X) = \dim_k M_s(X) = 1$ for $j \ne s$, the submodules $M_j^X \subseteq M_j$ and $M_s^X \subseteq M_s$ are both singular.*

*Otherwise, the quiver Γ is M-*inv-*wild.*

Note that in all cases Γ is of M-infinite type.

Sketch of proof. We may assume $\Gamma_0^1 = \Gamma_0$, because otherwise one can take the new quiver $\vec{\Gamma}$ with $\vec{\Gamma}_0 = \{\alpha^-, \alpha^+ \mid \alpha \in \Gamma_1\}$, $\vec{\Gamma}_1 = \{\overline{\alpha} : \alpha^- \to \alpha^+ \mid \alpha \in \Gamma_1\}$ and the $\vec{\Gamma}_0$-bunch of $\mathcal{A}$-modules $\vec{M}$ with $\vec{M}_{\alpha^-} = M_{s(\alpha)}$, $\vec{M}_{\alpha^+} = M_{e(\alpha)}$ (taking into account that Γ is M-tame iff $\vec{\Gamma}$ is $\vec{M}$-tame). Then (1)–(3) imply that $(\mathcal{A}/\mathrm{Ann}M, M) \cong (\mathcal{K}(\overline{S}), P(\overline{S}))$ for a bundle of semichaines $\overline{S} = (S, *)$ with $S = \{A_\alpha, B_\alpha \mid \alpha \in \Gamma_1\}$, and it follows from [6, §1] that Γ is M-tame (of M-infinite type). The proof of the fact that Γ is M-wild if the condition (1), (2) or (3) does not hold is divided into several steps.

Step 1. Let $S = \{A_1, \ldots, A_n, B_1, \ldots, B_n\}$ be a family of pairwise disjoint posets. We call $*$-*bundle* (or *involution bundle*) *of* these *posets* a pair $\overline{S} = (S, *)$, where $*$ is an involution on $S_0 = A \coprod B$ ($A = \coprod_{i=1}^n A_i$, $B = \coprod_{i=1}^n B_i$). $\overline{S}$ is said to be *nodal* if $x^* \ne x$ implies that x is comparable to any element of his poset. Nonempty A_i or B_i is said to be *elementary* if it is a chain with all elements being involutory to themselves. We say "bundle of semichaines" instead "nodal $*$-bundle of semichaines". *Representations of a $*$-bundle $\overline{S}$* are defined in the same way as those of a bundle of semichains.

We have the following statement: a $*$-bundle $\overline{S}$ of nonempty and nonelementary posets is wild if (a) there is a poset A_i or B_i which is not semichained, or (b) the bundle is not nodal.

Present the idea of the proof. For $x, y \in S_0$, we write $x \sim_* y$ iff $x = y$ or $x^* = y$, and $x - y$ iff, for some i, $x \in A_i, y \in B_i$ or $x \in B_i, y \in A_i$;

put $r_*(x) = |\{y|y \sim_* x\}|$, and, for $X \subseteq S_0$, $r_*(X) = \max_{x\in X} r_*(x)$. The notation $X — Y$ for subsets X, Y of S_0 means that $x — y$ for any $x \in X$, $y \in Y$. A chain $\{1 < 2 < \ldots < p\}$ is denoted by $\langle p\rangle$ and a poset $\langle i\rangle \coprod \ldots \coprod\langle j\rangle$ by $\langle i, \ldots, j\rangle$.

It is proved that (a) or (b) holds iff there is an "alternating" chain $f = \{C — x_1 \sim_* x_2 — \cdots — x_{2m-1} \sim_* x_{2m} — D\}\,(m \geq 0)$ with $C, D \subset S_0$ such that (c) $C \cong \langle 1, 1\rangle$ and $r_*(C) = 2$, or $C \cong \langle 1, 2\rangle$ and $r_*(C) = 1$, or $C \cong \langle 1, 1, 1\rangle$ and $r_*(C_1) = 1$; (d) $D \cong \langle 1, 1\rangle$ and $r_*(D) = 1$, or $D = \{x_i\}$ with $1 \leq i < 2m$; (e) $x_i \neq x_j$ for any $i \neq j$ (for $m = 0$, $f = \{C — D\}$ with D to be of the first form). The main stage of the proof is to describe all minimal $*$-bundles with a chain f of the above type and construct for each such $*$-bundle a $k\langle x, y\rangle$-representation from the definition of wildness.

Step 2. One can introduce an $\sim$-*bundle of posets* and its *representations* (and define *elementary* posets, etc.) in the same way as those in the case of an involution $*$, replacing everywhere (in particular, in the definition of $\mathrm{mod}_S k$) $*$, or equivalently the equivalence relation $\sim_*$, by an arbitrary equivalence relation $\sim$. It is proved that an $\sim$-bundle $\overline{S}$ of nonempty and nonelementary posets is wild if $r_\sim(S_0) > 2$. The idea of the proof is similar to that in Step 1. The differences are only (besides the taking $\sim$ instead of $\sim_*$) that, in (c), C is only of the form $\{y\}$ with $r_\sim(y) > 2$, that, in (d), D can be (in addition) of this form, and that, in (e), in addition $r_\sim(x_i) = 2$ for any i.

Step 3. Keeping the notation of Step 1, we call $(*, \circ)$-*bundle* (or *biinvolution bundle*) *of* the given *posets* a triple $\overline{\overline{S}} = (S, *, \circ)$, where $*$ and $\circ$ are, respectively, involutions on S_0 and S_0^2 satisfying $(x, y)^\circ = (y, x)^\circ$ for any x, y, $(x, y)^\circ = (x, y)$ for incomparable x, y and the natural conditions 1)–4) of [5, 4.11] if $x \leq y$. Its representations are defined similar to that for a $*$-bundle (for a poset A, $(A, *, \circ)$-graded spaces are $(A, *)$-graded ones; by $(A, *, \circ)$-maps one must mean $(A, *)$-maps φ such that $\varphi_{ab} = \varphi_{cd}$ whenever $(a, b)^\circ = (c, d)$). It is proved that an $(*, \circ)$-bundle of nonempty and nonelementary (respect to $*$) posets is wild if $\circ$ is nontrivial. The idea of the proof is similar to that in Step 2. The difference is only that the role of x with $r_\sim(x) = 1, 2$ is played by x with $r_\circ(x) = 1, 2$, where $r_\circ(x) = 1$ if $(x, y)^\circ = (x, y)$ for any y and $r_\circ(x) = 2$ otherwise.

Step 4. Identifying the modules $M_i\,(i \in \Gamma_0)$ with their images in $\mathrm{mod}\, k$, it is proved (with the help of not very complicated arguments) that the general case is reduced to the cases of Step 1–3.

It follows from the above that our main result can be reformulated in the following way.

Theorem 3.2. *Let* Γ *and* M *be as in Theorem* 3.1. *Then* Γ *is* M*-tame* *if and only if there is a bundle of semichaines* $\overline{S} = (S, *)$ *with* $S = \{A_\alpha, B_\alpha | \alpha \in \Gamma_1\}$ *such that* $(\mathcal{A}/\mathrm{Ann}M, \overrightarrow{M}) \cong (\mathcal{K}(\overline{S}), P(\overline{S}))$. *Otherwise,* Γ *is* M-inv-*wild.*

For $\sim$-bundles of posets (which include $*$-bundles), we classify tame cases in the general situation.

4. Extensions of the main result

4.1. The main result for $|\mathrm{Ob}\mathcal{A}_0| = \infty$.

Let $\mathcal{A}$ be a Krull-Schmidt category over a field k, with $|\mathrm{Ob}\mathcal{A}_0| = \infty$. The definitions of various types of $\mathcal{A}$-modules, which we gave for $|\mathrm{Ob}\mathcal{A}_0| < \infty$ (at the beginning of Section 2), can be directly transferred to this case. It is easy to see that a module N is chained (resp. semichained) if and only if so is $N|_{\oplus\mathcal{B}}$ (the restriction of N on $\oplus\mathcal{B}$), for any $\mathcal{B}$ to be a full subcategory of $\mathcal{A}_0$ with finite many objects (because an infinite poset is a chaine if and only if all its finite subposets are chaines, and the same is true for semichaines). Using these facts, one can easily prove that the main result of this section is also true for $|\mathrm{Ob}\mathcal{A}_0| = \infty$.

4.2. The main result for infinite quivers.

Our main result remains also true for an infinite quiver Γ, and the proof of this fact can be carried out in the same way as that for finite quivers; moreover, in view of what we said in the preceding section, it suffices to consider the case when $|\mathrm{Ob}\mathcal{A}_0| < \infty$. But here we already need to know that the problem of classifying the representations of a bundle of semichains is tame when the number of ones is infinite (because $|\mathrm{Ob}\mathcal{A}_0| < \infty$, all the semichaines can be assumed to be finite). The intuition tell us that this fact is true and that the representations of such bundle can be classified analogously to that for finitely many semichaines. In this subsection we clarify an explicit solution of this problem.

Let $\overline{S}$ be $S = \{A_i, B_i | \, i \in \mathcal{I}\}$ be a family of pairwise disjoint (finite) semichains, where $\mathcal{I}$ is some set. Put $A = \coprod_{i \in \mathcal{I}} A_i$, $B = \coprod_{i \in \mathcal{I}} B_i$, $S_0 = A \coprod B$. A bundle of semichains A_i, B_i, where i runs through $\in \mathbb{I}$, is defined similar to that for finitely many semichaines: it is a pair $\overline{S} = (S, *)$ with $*$ to be is an involution on S_0 such that $x^* = x$ for each x belonging to the union of all two-point links. In the new situation, the category $\mathcal{B}_k\,(\overline{S})$ of representations of the bundle $\overline{S}$ are defined in the same way as that for finitely many semichaines, and it is a Krull-Schmidt category too.

It is easy to see that a faithful bundle of infinitely many semichaines has only countable many ones, and hence we can confine oneself to the countable case[3]. As usual, $\mathbb{Z}$ denotes the integer numbers and $\mathbb{N}$ the natural ones.

Thus, let $S = \{A_i, B_i | \, i \in \mathbb{N}\}$ be a family of pairwise disjoint (finite) semichains and $\overline{S} = (S, *)$ a bundle of semichains $A_1, A_2, \ldots, B_1, B_2, \ldots$; recall that $A = \coprod_{i\in\mathbb{N}} A_i$, $B = \coprod_{i\in\mathbb{N}} B_i$ and $S_0 = A \coprod B$. If $R = (U, V, \varphi)$ is a representation of $\overline{S}$ with the dimension-function $d : S_0 \to \mathbb{N} \cup 0$ (sending $x \in A_i$ to $\dim_k(U_i)_x$ and $y \in B_i$ to $\dim_k(V_i)_y$), then the set of all elements $x \in S_0$ such that $d(x) \neq 0$ will be called the *support* of R.

The indecomposable representations with finite supports (or equivalently, with finitely many semichains) were classified in [8, 6]. Here we classify the indecomposable representations (of a bundle of countable many semichains) with infinite supports.

Let, for a semichaine X, $L(X)$ denotes the set of its links (which is ordered in a natural way). Put $L(A) = \cup_{i\in\mathbb{N}} L(A_i)$, $L(B) = \cup_{i\in\mathbb{N}} L(B_i)$, and denote by $L(S)$, or simply L, the union of the sets $L(A)$ and $L(B)$. It is convenient for us to denote elements of L by lower case letters and to identify the one-points links with the points themselves. The number of points of a link $x \in L$ is denoted by $l(x)$.

Define two symmetric binary relations, α and β, on the set L by putting $x\alpha y$ if and only if $x \neq y$, $l(x) = l(y) = 1$ and $x^* = y$, or $x = y$ and $l(x) = 2$; $x\beta y$ if and only if either $x \in L(A_i)$, $y \in L(B_i)$ or $x \in L(B_i)$, $y \in L(A_i)$ for some $i \in \mathbb{N}$.

We now introduce the notion of L-chains of type $(0, +\infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$.

Throughout, all graphs are nonoriented. For a graph C, we denote by C_0 and C_1 the sets of its vertices and edges, respectively. Let $C^{+\infty}$ be the graph with $C_0^{+\infty} = \mathbb{N}$ and $C_1^{+\infty} = \{(i, i+1) | \, i \in \mathbb{N}\}$, $C^{-\infty}$ be the graph with $C_0^{-\infty} = \{-n | \, n \in \mathbb{N}\}$ and $C_1^{-\infty} = \{(-i-1, -i) | \, i \in \mathbb{N}\}$, and C^{∞} be the graph with $C_0^{\infty} = \mathbb{Z}$ and $C_1^{\infty} = \{(i, i+1) | \, i \in \mathbb{Z}\}$. A *countable L-chain* is a function g, defined on a graph $C \in \{C^{+\infty}, C^{-\infty}, C^{\infty}\}$, that associates to each $j \in C_0$ an element $g(j) \in L$ and to each edge $(j, j+1) \in C_1$ a relation $g(j, j+1) \in \{\alpha, \beta\}$ subject to the following conditions: (a) $g(j)$ and $g(j+1)$ satisfy the relation $g(j, j+1)$; (b) $g(j-1, j) \neq g(j, j+1)$; (c) for each $x \in L$, the set $g^{-1}(x) = \{j \in C_0 | \, g(j) = x\}$ is finite. An *isomorphism* of L-chains g and g', defined on C and C', respectively, is an isomorphism $\tau : C \to C'$ such that $g = \tau g'$.

[3]A representation (U, V, φ) of a bundle $\overline{S}$ is called faithful if $(U_i)_x, (V_i)_y \neq 0$ for any $i \in \mathcal{I}$, $x \in A_i$ and $y \in B_i$; the bundle $\overline{S}$ is called faithful if it has a faithful indecomposable representation.

A countable L-chain defined on $C = C^{+\infty}$, $C^{-\infty}$, C^{∞} will be called an *L-chain of type* $(0, +\infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$, respectively.

A countable L-chain g is called *admissible* if $x\alpha y$ for distinct elements $x, y \in L$ and $g(j) = x$ imply the existence of an edge ρ containing the vertex j and satisfying $g(\rho) = \alpha$ (an L-chain of type $(-\infty, +\infty)$ is always admissible), and *symmetric* if there exist a vertex i such that $g(i - s) = g(i + s)$ for any $s \in \mathbb{N}$ (an L-chain of type $(0, +\infty)$ or $(-\infty, 0)$ is always nonsymmetric). The vertex 1 (resp. -1) of an L-chain of type $(0, +\infty)$ (resp. $(-\infty, 0)$) is called *double* if $g(1, 2) = \beta$ and $g(1)\alpha g(1)$ in L (resp. $g(-2, -1) = \beta$ and $g(-1)\alpha g(-1)$ in L). We write $d(g) = 1$ if the vertex 1 (resp. -1) is double and $d(g) = 0$, otherwise; for an L-chain of type $(-\infty, +\infty)$, we put $d(g) = 0$.

Denote by $G_1(L)$ the set of admissible nonsymmetric (countable) L-chains. To an $g \in G_1(L)$, we associate the representation $U_1(g)$ if $d(g) = 0$, and the representations $U_1(g), U_2(g)$ if $d(g) = 1$. These representations are defined in the same way as those in [8, 6] for finite many semichains (in these paper we used the language of matrices, but all the results and proofs can be easily rewrited in terms of vector spaces and linear maps).

The representations $U_i(g)$ of the bundle $\overline{S} = (S, *)$ are all indecomposable. Moreover, the following statement holds.

Theorem 4.1. *Let $\overline{S} = (S, *)$ be a bundle of countable many semichains. Choose one representative in each isomorphism class of L-chains of type $(0, +\infty)$, $(-\infty, 0)$ and $(-\infty, +\infty)$ belonging to $G_1(L)$. Then the set of representations of the form $U_i(g)$ associated to the chosen L-chains is a complete set of pairwise nonisomorphic indecomposable representations with infinite support.*

The idea of the proof is similar to that in [8] for finite many semichaines.

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