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RESEARCH ARTICLE

On decompositions of affine Coxeter groups in semi-direct products

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Abstract. In this paper we study affine Coxeter groups. We show how to construct decompositions of the investigated groups into semi-direct products. We use this to give the complete list of irreducible representations of affine Coxeter groups.

1. Introduction

If $S = \{s_1, s_2, ..., s_n\}$ is a set of generators and $m_{s_i,s_k} : S\times S \longrightarrow \mathbb{N}\bigcup\{\infty\}$ are such that

$$m_{s_i,s_i} = 1, \quad i = 1..n,$$
$$m_{s_i,s_j} > 1, \quad \text{for all distinct } i \text{ and } j,$$

then the group $W = G < s_1, s_2, ..., s_n|(s_is_j)^{m_{s_i,s_j}} = e >$ is called a Coxeter group. It is convenient to associate with a Coxeter group a graph called a Coxeter diagram. Its nodes are indexed by S. If $m_{s_i,s_j} \geq 3$ then we connect the nodes s_i and s_j by an edge. If $m_{s_i,s_j} > 3$ we label also the edge between s_i and s_j by m_{s_i,s_j}.

With any Coxeter group one can also associate a matrix $K = \{k_{ij}\}_{i,j=1}^n$

$$k_{ij} = -\cos\frac{\pi}{m_{s_i,s_j}},$$

called a Cartan matrix.

We will use the following classical result (e.g. [2]):

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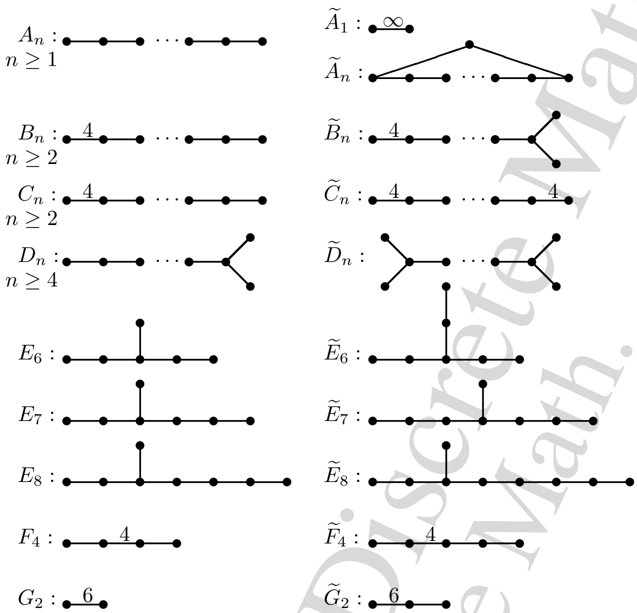

Figure 1:

Statement 1. *1. A Coxeter group W is finite if and only if K is positive definite;*

2. If all principal minors of K are positive and $\det K = 0$*, then W is a semi-direct product of $\mathbb{Z}^{n-1}$ and a finite Coxeter group G_{fin}: $W = \mathbb{Z}^{n-1} \rtimes G_{fin}$, Such W is called an affine Coxeter group.*

3. In the other cases a Coxeter group W contains a free subgroup with two generators.

In Figure 1 we present the complete list of the Coxeter diagrams corresponding to the groups satisfying the first and the second conditions of Statement 1. The first column are the diagrams of the finite Coxeter groups, called basic, and the second one are the diagrams corresponding to the semi-direct products. The diagrams in the second column are called the extensions of the basic ones.

Remark 1. *(to the second condition of Statement 1) In the notations of Statement 1, the diagram of the group W is the extension of the diagram G_{fin}. It will be shown how to build the semi-direct decompositions of such groups efficiently(see Section 3).*

2. Representations of semi-direct products

In this section we recall how to construct all irreducible representations of a semi-direct product, we will use this method in our applications.

Definition 1. *Let A, B be groups and $\varphi : A \to AutB$ be a homomorphism. Then the set of all ordered pairs $\{(b,a)|a \in A, b \in B\}$ with multiplication*

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 \cdot \varphi(a_1)(b_2), a_1 a_2).$$

is called a semi-direct product of the groups B and A (denoted by $B \rtimes A$).

The subgroups (B, e) and (e, A) of the above semi-direct product will be identified with B and A in a natural way.

All irreducible representations of groups, realized as semi-direct products, can be obtained using the Mackey algorithm [1].

Let us sketch this method. For detailed constructions see [1]. Assume that $G = B \rtimes A$, where B is a commutative group and A is a subgroup of $Aut(B)$. Every element of G is of the form (b, a), where $a \in A$, $b \in B$. Denote by $\mathcal{X}$ the group of all characters of B with the standard group operation.

For any automorphism a of the group B define $\widehat{a} \in Aut(\mathcal{X})$ by the following rule:

$$\widehat{a}(\chi(b)) = \chi(a^{-1}b).$$

Denote by $\widehat{A} := \{\widehat{a} \quad | \quad a \in A\}$. Since $\widehat{a_1}\widehat{a_2} = \widehat{a_1 a_2}$, the correspondence $a \to \widehat{a}$ determines an isomorphism $A \simeq \widehat{A}$. Let us decompose $\mathcal{X}$ into classes of transitivity with respect to the action of $\widehat{A}$. Then we get representation T_g, $g \in G$, in the space of the functions defined on a certain class of transitivity via

$$T_g(f(\chi)) = \chi(b) f(\widehat{a}^{-1}\chi).$$

Note that the maximum of the dimensions of irreducible representations of $B \rtimes A$ is the order of the group A.

3. Coxeter groups realized as semi-direct products

Let an affine Coxeter group, W, be defined by its Coxeter diagram, Γ. In this section we show how to recover the decomposition of W into semi-direct product from the structure of Γ.

In the notations of Section 1 the generators of G_{fin} corresponds to the vertices of the basic subdiagram of the extended one. Now it is remained

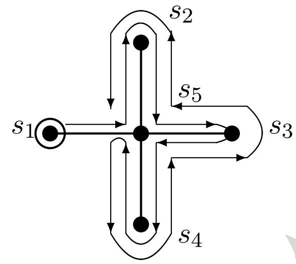

Figure 2: Path-trace of $\widetilde{D}_4$.

to construct generators of $\mathbb{Z}^{n-1}$. It is sufficient to construct one of them. Then the others can be obtained acting by G_{fin} on the known one.

Let us consider first the groups with $m_{s_i,s_j} \leq 4$.

We call a vertex x *a node* if and only if one of the following conditions holds:

1. There are more than two edges coming out of x;

2. there exists an edge starting from x labeled by 4.

If the graph Γ does not contain any node we mark by circle one arbitrary chosen vertex. Otherwise, consider all branches of Γ starting from a fixed node x and pick up the longest one, say B. Then we mark by circle the vertex on B neighboring with x.

Let us construct our first generator, T_1, of $\mathbb{Z}^{n-1}$. First of all we introduce a certain path-tracing of the graph Γ. Let us imagine that the edges of Γ form a wall and we are walking along this wall. We start from the vertex labeled by circle and move in the direction to the fixed node x or in an arbitrary direction if there is no node in the diagram. Evidently, in such way we fix our orientation with respect to the wall. Further, we a continue our walking along with the same orientation until we reach the last not traversed vertex. Then we go back by the inverse path and stop one step before the starting vertex. Then T_1 is the product of all s_i corresponding to the passed vertices (started from the one labeled by circle) taking in such an order that s_j whose vertex appears one step later then the one corresponding to s_i stay to the right of s_i in the product. (see Figure [2] for illustration, the fixed node is $x{=}s_5$, $T_1 = s_1s_5s_2s_5s_3s_5s_4s_5s_3s_5s_2s_5$).

There exists only one affine group, $\widetilde{G_2}$, which has an edge with multiplicity greater than 4. The path-tracing method can be used also in this case.We modify first the corresponding graph. Let us split the vertex s_2

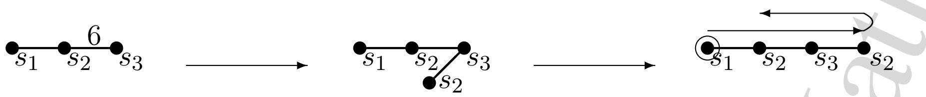

Figure 3: $\widetilde{G}_2$.

and the edge, joining s_2 and s_3, into two nodes and two edges and denote the new vertex also by s_2 (see Figure 3). Now the path-tracing described above can be applied to the modified diagram. We have the following

Lemma 1. *Generators of the lattice $\mathbb{Z}^2$ in the decomposition $\widetilde{G_2} \simeq \mathbb{Z}^2 \rtimes G_2$, where $G_2 =< s_2, s_3 | (s_2 s_3)^6 = e, s_2^2 = e, s_3^2 = e >$, can be obtained by application of the path-tracing method to the modified diagram.*

Proof. We construct T_1 by path-tracing, see Figure 3, $T_1 = s_1 s_2 s_3 s_2 s_3 s_2$. Then put T_2 to be $s_2 T_1 s_2 = s_2 s_1 s_2 s_3 s_2 s_3$.

Let T be a group generated by T_1 and T_2. Since $s_i T s_i^{-1} = T$ for any s_i, $i = 1, 2, 3$, T is normal subgroup of $\widetilde{G_2}$. Let us show that T_1 and T_2 form a basis of $\mathbb{Z}^2$. Note that $T \cap G_2 =< e >$. It is easy to verify that $< T_1 >= \mathbb{Z}$, $< T_2 >= \mathbb{Z}$. Further, note that

$$T_1 T_2^{-1} = s_1 s_2 s_3 s_2 s_3 s_2 s_3 s_2 s_3 s_2 s_1 s_2 = s_1 s_3 s_2 s_3 s_1 s_2,$$
$$T_2^{-1} T_1 = s_3 s_2 s_3 s_2 s_1 s_2 s_1 s_2 s_3 s_2 s_3 s_2 = s_3 s_2 s_3 s_1 s_3 s_2 s_3 s_2 = s_1 s_3 s_2 s_3 s_1 s_2.$$

Thus, T_1 and T_2 commute. The independence of T_1 and T_2 over $\mathbb{Z}$ follows from the action of G_2 on T. Indeed, let $T_1^{x_1} T_2^{x_2} = e$. Then

$$s_3 (T_1^{x_1} T_2^{x_2}) s_3 = (T_1^{x_1} T_2^{x_1 - x_2}) = e, \quad s_2 (T_1^{x_1} T_2^{x_2}) s_2 = T_1^{x_2} T_2^{x_1}$$
$$\text{hence} \quad T_2^{x_1 - 2x_2} = e, \text{ and } T_2^{x_2 - 2x_1} = e$$
$$x_1 = 2x_2, \quad 2x_1 = x_2 \Longrightarrow x_1 = x_2 = 0$$

So T_1 and T_2 are independent over $\mathbb{Z}$ and $T \simeq \mathbb{Z}^2$. In the following we identify $T_1^{x_1} T_2^{x_2} \in T$ with $(x_1, x_2) \in \mathbb{Z}^2$

Note that G_{fin} acts on T as follows: $\widehat{S}_2(x_1, x_2) = (x_2, x_1)$,

$$\widehat{S}_3(x_1, x_2) = (x_1, x_1 - x_2).$$

To show that $\widetilde{G_2} \simeq \mathbb{Z}^2 \rtimes G_2$ it remains only to prove that $\widetilde{G_2} = T \cdot G_2$. Obviously it is sufficient to decompose the generators of $\widetilde{G_2}$:

$$s_1 = T_1 \cdot s_2 s_3 s_2 s_3 s_2 \to ((1, 0), \widehat{S}_2 \widehat{S}_3 \widehat{S}_2 \widehat{S}_3 \widehat{S}_2) \in \mathbb{Z}^2 \rtimes G_2,$$
$$s_2 = e \cdot s_2 \to ((0, 0), \widehat{S}_2) \in \mathbb{Z}^2 \rtimes G_2,$$
$$s_3 = e \cdot s_3 \to ((0, 0), \widehat{S}_3) \in \mathbb{Z}^2 \rtimes G_2.$$

$\square$

Now we are able to formulate our general result.

Theorem 1. *The decomposition of an affine Coxeter group in a semi-direct product $\mathbb{Z}^{n-1} \rtimes G_{fin}$ can be obtained using the path-tracing method for corresponding Coxeter diagram or the modified Coxeter graph in the case of $\tilde{G}_2$.*

Proof. As above G_{fin} is generated by s_i corresponding to the basic subgraph. Then we construct T_1 by the path-tracing method. The other T_i are obtained by the action of the generators of G_{fin} on T_1. Let $T =< T_1, T_2, ...T_{n-1} >$.

It is easy to see that the group T is invariant under the action of inner automorphisms of G_{fin}. So we can consider the group $T \cdot G_{fin}$. We show that $W = T \cdot G_{fin} = \mathbb{Z}^{n-1} \rtimes G_{fin}$. In order to see that $T_1, T_2, ..., T_{n-1}$ form the lattice $\mathbb{Z}^{n-1}$ it is sufficient to verify they commutate and are independent over $\mathbb{Z}$. In the general situation the independence follows from the action of G_{fin} on T in a manner presented in the proof of Lemma 1. To make sure that W coincides with $T \cdot G_{fin}$ one has only to decompose the unique generator of W which is not a generator of G_{fin} (the decompositions of the generators of G_{fin} are trivial).

Below we realize the plan described above for the group $\widetilde{D}_4$ and present the formulas for generators of the lattice and action of G_{fin} for the groups corresponding to $\tilde{A}_n$ and $\tilde{C}_n$. For the remained groups the arguments are similar.

So, let $W = \widetilde{D}_4$, then $\mathbb{Z}^{n-1} = \mathbb{Z}^4$, $G_{fin} = D_4$. Using the path-tracing described in Figure [2] we set $T_1 = s_1s_5s_2s_5s_3s_5s_4s_5s_3s_5s_2s_5$. Acting by s_5, s_2 and s_3 we obtain other generators:

$$
\begin{aligned}
T_2 &= s_5s_1s_5s_2s_5s_3s_5s_4s_5s_3s_5s_2, \\
T_3 &= s_2s_5s_1s_5s_2s_5s_3s_5s_4s_5s_3s_5, \\
T_4 &= s_3s_2s_5s_1s_5s_2s_5s_3s_5s_4s_3s_4.
\end{aligned}
$$

Let us show that T_i, $i = \overline{1,4}$ generate $\mathbb{Z}^4$. By direct verification one gets $T_iT_j = T_jT_i$, $i \neq j$. The independence of T_i over $\mathbb{Z}$ follows easily from the action of G_{fin} on $T_1^{x_1}T_2^{x_2}...T_{n-1}^{x_{n-1}}$, see Lemma 1. In the following we will write $\widehat{S}_i(x_1, x_2, x_3, x_4)$ instead of $s_iT_1^{x_1}T_2^{x_2}T_3^{x_3}T_4^{x_4}s_i$. Then one has

$$
\begin{aligned}
\widehat{S}_1(x_1, x_2, x_3, x_4) &= (-x_1, x_2 - x_1, x_3 - x_1, x_4 - x_1), \\
\widehat{S}_2(x_1, x_2, x_3, x_4) &= (x_1, x_3, x_2, x_2 - x_3 + x_4), \\
\widehat{S}_3(x_1, x_2, x_3, x_4) &= (x_1, x_2 - x_3 + x_4, x_4, x_3), \\
\widehat{S}_5(x_1, x_2, x_3, x_4) &= (x_2, x_1, x_3, x_4 - x_1 + x_2),
\end{aligned}
$$

where $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$.

The generators s_1, s_2, s_3, $s_5 \in G_{fin}$ and for s_4 has $T \cdot D_4 \ni s_4 = (T_1^{-1}T_2T_3) \cdot s_5s_3s_2s_5s_1s_5s_2s_3s_5 \rightarrow ((-1,1,0,1), \widehat{S}_5\widehat{S}_3\widehat{S}_2\widehat{S}_5\widehat{S}_1\widehat{S}_5\widehat{S}_2\widehat{S}_3\widehat{S}_5)$. Thus $\widetilde{D}_4 = T \cdot D_4$ and our method gives the decomposition $\widetilde{D}_4 = \mathbb{Z}^4 \rtimes D_4$.

Below we present a description of generators of the lattice and the action of G_{fin} for some other series of the Coxeter groups.

1. The path-trace for $\widetilde{A}_n$ is described in Figure 4: Then

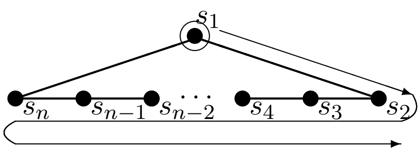

Figure 4: Path-trace of $\widetilde{A}_n$.

$$
\begin{aligned}
T_1 &= s_1s_2...s_{n-1}s_ns_{n-1}...s_3s_2\\
T_2 &= s_2s_1s_2...s_{n-1}s_ns_{n-1}...s_3,\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
T_{n-1} &= s_{n-1}s_{n-2}s_{n-3}...s_1s_2s_1...s_{n-1}s_n
\end{aligned}
$$

The action of $G_{fin} = S_n$ has the following form:

$$
\begin{aligned}
\widehat{S}_1(x_1, x_2, ..., x_{n-1}) &= (-x_1, x_2 - x_1, x_3 - x_1, ..., x_{n-1} - x_1),\\
\widehat{S}_i(x_1, x_2, ..., x_{n-1}) &= (x_1, x_2, ..., x_{i-2}, x_i, x_{i-1}, x_{i+1}, ...x_{n-1}),\\
&\qquad i = 2..n-1.
\end{aligned}
$$

The generators of $\widetilde{A}_n$ have the following form:

$$
\begin{aligned}
s_k &:= ((0, 0, ..., 0), \widehat{S}_k), \quad k = 1..n-1,\\
s_n &:= (T_{n-1}^{-1}, \widehat{S}_{n-1}...\widehat{S}_2\widehat{S}_1\widehat{S}_2...\widehat{S}_{n-3}\widehat{S}_{n-2}\widehat{S}_{n-1}),
\end{aligned}
$$

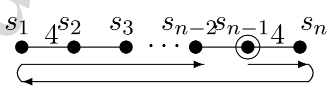

Figure 5: $\widetilde{C}_n$.

2. Decomposition of $\widetilde{C}_n$ (figure[5]).

$$T_1 = s_{n-1}s_{n-2}s_{n-3}...s_1s_2s_1...s_{n-1}s_n,$$
$$T_2 = s_ns_{n-1}s_{n-2}s_{n-3}...s_1s_2s_1...s_{n-1},$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$
$$T_{n-1} = s_1s_2...s_{n-1}s_ns_{n-1}...s_3s_2$$

The action of G_{fin} on the lattice has the form:

$$\widehat{S}_1(x_1, x_2, ..., x_{n-1}) = (-x_1, x_2, x_3, ..., x_{n-1}),$$
$$\widehat{S}_i(x_1, x_2, ..., x_{n-1}) = (x_1, x_2, ..., x_{i-2}, x_i, x_{i-1}, x_{i+1}, ...x_{n-1}),$$
$$i = 2..n-1.$$

Generators of $\widetilde{C}_n$ are identified with $s_k := ((0,0,...,0), \widehat{S}_k)$, $k = 1..n-1$, $s_n := (T_{n-1}, \widehat{S}_{n-1}...\widehat{S}_2\widehat{S}_1\widehat{S}_2...\widehat{S}_{n-3}\widehat{S}_{n-2}\widehat{S}_{n-1})$.

□

4. Examples

In this section we apply the results of Section 3 to describe irreducible representations of $\widetilde{A}_1$ and $\widetilde{C}_2$.

Let $c_i = e^{i\varphi_i}$, $i = 1..n-1$ be an irreducible representation of $\mathbb{Z}^{n-1}$. Now, using the Mackey algorithm we obtain all irreducible representations of the groups under consideration.

Example 1. $\widetilde{A_1} \simeq \mathbb{Z} \rtimes \mathbb{Z}_2$. *Generators of* $\mathbb{Z}$ *and* $\mathbb{Z}_2$ *are* s_2s_1 *and* s_2 *respectively. Obviously, the generators of* $\widetilde{A}_1$ *have the following form in the semi-direct product:* $s_1 := ((s_2s_1)^{-1}, s_2)$, $s_2 = (e, s_2)$. *Evidently, any representation,* π, *of* $\mathbb{Z}$ *is determined by some* $c \in \mathbb{C}, |c| = 1$. *So let* $\chi(s_2s_1) = c$. *We write* χ_c *for such* χ.

Then since

$$s_2(s_2s_1)^xs_2 = (s_2s_1)^{-x},$$

the class of transitivity of χ *with respect to this action on one-dimensional representations of* $\mathbb{Z}$ *has the form:* $\{c, c^{-1}\}$. *Further, using Mackey's algorithm we obtain all irreducible representation* π, *of* $\widetilde{A}_1$: $\pi(s_i) = S_i$

1. *If* $c = 1$ *then* $S_1 = 1$, $S_2 = 1$ *or* $S_1 = -1$, $S_2 = -1$.
   If $c = -1$ *then* $S_1 = 1$, $S_2 = -1$ *or* $S_1 = -1$, $S_2 = 1$.

2. *If $c \neq \pm 1$ one has:*

$$T_{(e,s_2)} f(\chi_c) = f(\chi_{c^{-1}}), \quad T_{(e,s_2)} f(\chi_{c^{-1}}) = f(\chi_c);$$

$$T_{((s_2 s_1)^{-1}, s_2)} f(\chi_c) = c^{-1} f(\chi_{c^{-1}}), \quad T_{((s_2 s_1)^{-1}, s_2)} f(\chi_{c^{-1}}) = c f(\chi_c).$$

and in the matrix form the generators are:

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & c \\ c^{-1} & 0 \end{pmatrix}.$$

Example 2. *Representations of $\widetilde{C}_2 = \mathbb{Z}^2 \rtimes C_2$: As above we identify the one-dimensional representation of $\mathbb{Z}^2$ with the pair (φ_1, φ_2), $\varphi_k \in [-\pi, \pi]$ and $c_j := e^{i\varphi_j}$, $j = 1, 2$.*

1. *8-dimensional representations. The corresponding classes of transitivity of the action of C_2 on the characters of $\mathbb{Z}^2$ have the form:*

$$M_{\varphi_1,\varphi_2} = \{\big((-1)^{k_1}\varphi_1, (-1)^{k_2}\varphi_2\big), \big((-1)^{k_2}\varphi_2, (-1)^{k_1}\varphi_1\big) \mid k_1, k_2 = 0, 1\},$$

where $\varphi_1 \neq \varphi_2$, $\varphi_j \in (0, \pi)$,

Then the induced representation defined on the space of the functions on M_{φ_1,φ_2} have the following form

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & c_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2^{-1} \\ 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & c_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1^{-1} & 0 & 0 & 0 & 0 \\ c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. *4-dimensional representations. We have the following representations:*

   (a) $\varphi_1 = 0$, $\varphi_2 \in (0, \pi)$. *The corresponding classes of transitivity have the form*

   $$M_{0,\varphi_2} = \{(0, \varphi_2), \quad (-\varphi_2, 0), \quad (\varphi_2, 0), \quad (0, -\varphi_2)\},$$

   and the irreducible representation, π, *corresponding to the class* M_{0,φ_2} *is the following*

   $$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

   $$S_3 = \begin{pmatrix} 0 & 0 & 0 & c_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ {c_2}^{-1} & 0 & 0 & 0 \end{pmatrix};$$

   (b) $\varphi_1 = \pi$, $\varphi_2 \in (0, \pi)$. *In this case the classes of transitivity are*

   $$M_{\pi,\varphi_2} = \{(\pi, \varphi_2), \ (-\varphi_2, \pi), \ (\varphi_2, \pi), \ (\pi, -\varphi_2)\},$$

   and for the representations defined by M_{π,φ_2} *one has*

   $$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

   $$S_3 = \begin{pmatrix} 0 & 0 & 0 & c_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ {c_2}^{-1} & 0 & 0 & 0 \end{pmatrix};$$

(c) $\varphi_1 = \varphi_2 \in (0, \pi)$. *Then*

$$M_{\varphi_2} = \{(\varphi_2, \varphi_2),\ (-\varphi_2, \varphi_2),\ (\varphi_2, -\varphi_2),\ (-\varphi_2, -\varphi_2)\}.$$

The representations corresponding to M_{φ_2} *have the form*

$$S_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_2 \\ {c_2}^{-1} & 0 & 0 & 0 \\ 0 & {c_2}^{-1} & 0 & 0 \end{pmatrix}.$$

3. *1 and 2-dimensional representations. One-dimensional representations correspond to the classes* $M_{0,0} = \{(0,0)\}$, *and* $M_{\pi,\pi} = \{(\pi, \pi)\}$. *Namely, one has* $S_1 = 1, S_2 = 1, S_3 = 1$ *and* $S_1 = -1, S_2 = -1, S_3 = -1$ *respectively. The two-dimensional representation corresponds to* $M_{0,\pi} = \{(0, \pi), (\pi, 0)\}$*:*

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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