

Dimensions of finite type for representations of partially ordered sets

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ABSTRACT. We consider the *dimensions of finite type* of representations of a partially ordered set, i.e. such that there is only finitely many isomorphism classes of representations of this dimension. We give a criterion for a dimension to be of finite type. We also characterize those dimensions of finite type, for which there is an indecomposable representation of this dimension, and show that there can be at most one indecomposable representation of any dimension of finite type. Moreover, if such a representation exists, it only has scalar endomorphisms. These results (Theorem 1.6, page 25) generalize those of [5, 1, 9].

1. Preliminaries and the Main Theorem

Let (\mathbf{S}, \prec) be a finite partially ordered set (*poset*); we denote by \prec the *strict order* and by $a \preceq b$ the relation “ $a \prec b$ or $a = b$ ”. We usually suppose that $\mathbf{S} = \{1, 2, \dots, n\}$ (certainly, not necessary with the natural order) and denote by $\widehat{\mathbf{S}} = \mathbf{S} \cup \{0\}$. (Note that we do not treat $\widehat{\mathbf{S}}$ as a poset.) A *representation* V of \mathbf{S} over a field \mathbb{k} is an order preserving map of \mathbf{S} into the set of subspaces of a finite dimensional vector space $V(0)$ over \mathbb{k} . A *morphism* $f : V \rightarrow V'$ of such representations is a linear mapping $f : V(0) \rightarrow V'(0)$ such that $f(V(a)) \subseteq V'(a)$ for every $a \in \mathbf{S}$. We denote by $\text{rep } \mathbf{S}$ the category of such representations (supposing the field \mathbb{k} fixed).

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Recall a relation to a bimodule category [1]. Let $\Lambda = \Lambda_{\mathbf{S}}$ be the incidence algebra of the poset \mathbf{S} , i.e. the subalgebra of $\text{Mat}(n, \mathbb{k})$ with the basis $\{e_{ab} \mid a \prec b \text{ in } \mathbf{S}\}$. Let also $\mathbf{U} = \mathbf{U}_{\mathbf{S}}$ be the right Λ -module with the basis v_1, v_2, \dots, v_n and the action $v_a e_{bc} = \delta_{ab} v_c$. We consider \mathbf{U} as \mathbb{k} - Λ -bimodule. Then the category $\text{El}(\mathbf{U})$ of *elements of \mathbf{U}* (or of *matrices with entries from \mathbf{U}*) is defined. Its objects are elements from $\text{Hom}_{\Lambda}(P, L \otimes_{\mathbb{k}} \mathbf{U})$, where L is a finite dimensional vector space and P is a finitely generated (right) projective Λ -module. A *morphism* $\phi : u \rightarrow u'$, where $u : P \rightarrow L \otimes \mathbf{U}$, $u' : P' \rightarrow L' \otimes \mathbf{U}$ is, by definition, a pair ϕ_0, ϕ_1 , where $\phi_0 : L \rightarrow L'$ is a linear map, $\phi_1 : P \rightarrow P'$ is a Λ -homomorphism, such that $u' \phi_1 = (\phi_0 \otimes 1)u$. Given an element $u : P \rightarrow L \otimes \mathbf{U}$, set $V(0) = L$ and $V(a) = \{v \in L \mid v \otimes v_a \in \text{Im } u\}$. We get a representation $V = \rho(u) \in \text{rep } \mathbf{S}$. Obviously, if $\phi = (\phi_0, \phi_1)$ is a morphism $u \rightarrow u'$, then ϕ_0 is a morphism $\rho(\phi) : \rho(u) \rightarrow \rho(u')$. So ρ is a functor $\text{El}(\mathbf{U}) \rightarrow \text{rep } \mathbf{S}$. It is not an equivalence, but one can easily control its defects. Namely, let $\Lambda_a = e_{aa} \Lambda$; they are all indecomposable projective Λ -modules. Consider the so-called *trivial element* T_a , which is the unique element of $\text{Hom}_{\Lambda}(\Lambda_a, 0 \otimes \mathbf{U})$ (it is not zero in the category $\text{El}(\mathbf{U})$). Later on we shall also use the trivial representation $T_0 \in \text{Hom}(0, \mathbb{k} \otimes \mathbf{U})$.

Proposition 1.1. 1. *The functor ρ is dense (i.e. every object from $\text{rep } \mathbf{S}$ is isomorphic to $\rho(u)$ for some u) and full, i.e. all induced maps $\text{Hom}(u, u') \rightarrow \text{Hom}(\rho(u), \rho(u'))$ are surjective.*

2. *$\rho(\phi) = 0$ if and only if ϕ factors through a direct sum $\bigoplus_{a \in \mathbf{S}} m_a T_a$ of trivial elements. In particular, only such direct sums become zero under the functor ρ .*

Proof. 1. Let $V \in \text{rep } \mathbf{S}$, $L = V(0)$. Consider the subspace $M = \sum_{a \in \mathbf{S}} V(a) \otimes v_a \subseteq L \otimes \mathbf{U}$. It is a Λ -submodule. Let $P \rightarrow M$ be a projective cover of M . Considered as a homomorphism $P \rightarrow L \otimes \mathbf{U}$, it defines an element $u \in \text{El}(\mathbf{U})$ and it is obvious that $\rho(u) = V$. If $V' = \rho(u')$, where $u' : P' \rightarrow L' \otimes \mathbf{U}$, and $f : V(0) = L \rightarrow V'(0) = L'$ is a morphism $V \rightarrow V'$, then the inclusions $f(V(a)) \subseteq V'(a)$ for all $a \in \mathbf{S}$ imply that $(f \otimes 1)(\text{Im } u) \subseteq \text{Im } u'$. Hence, there is a homomorphism $g : P \rightarrow P'$ with $(f \otimes 1)u = u'g$, which gives a morphism $\phi = (f, g)$ such that $\rho(\phi) = f$.

2. If $\rho(\phi) = 0$, then $\phi = (0, \phi_1)$, so $u'\phi_1 = 0$ and it decomposes as

$$\begin{array}{ccc}
 P & \xrightarrow{u} & L \otimes \mathbf{U} \\
 \parallel & & \downarrow \\
 P & \longrightarrow & 0 \otimes \mathbf{U} \\
 \phi_1 \downarrow & & \downarrow \\
 P' & \xrightarrow{u'} & L' \otimes \mathbf{U}.
 \end{array}$$

Obviously, the second row of this diagram splits in $\text{El}(\mathbf{U})$ into a direct sum of trivial representations. \square

Note that $\text{Hom}_\Lambda(\Lambda_a, \mathbf{U}) \simeq \mathbf{U}e_{aa} = \langle v_a \rangle$. Therefore, a homomorphism $d_a \Lambda_a \rightarrow L \otimes \mathbf{U}$ can be identified with a matrix $M(a)$ of size $d_0 \times d_a$, where $d_0 = \dim L$. Since every projective Λ -module P decomposes uniquely as $\bigoplus_{a \in \mathbf{S}} d_a \Lambda_a$, and

$$\text{Hom}_\Lambda(\Lambda_b, \Lambda_a) = \begin{cases} \mathbb{k} & \text{if } a \prec b, \\ 0 & \text{otherwise,} \end{cases}$$

it gives the original ‘‘matrix’’ definition of [10]. Namely, u is presented as a block matrix

$$M = \left[\begin{array}{|c|c|c|c|} \hline M(1) & M(2) & \dots & M(n) \\ \hline \end{array} \right], \tag{1.1}$$

where $M(a)$ is of size $d_0 \times d_a$. For two matrices of this shape, M and M' , a morphism Φ is given by a set of matrices $\{ \Phi(a) \mid a \in \widehat{\mathbf{S}} \} \cup \{ \Phi(ba) \mid b \prec a \text{ in } \mathbf{S} \}$ such that, for every $a \in \mathbf{S}$,

$$\Phi(0)M(a) = M'(a)\Phi(a) + \sum_{b \prec a} M'(b)\Phi(ba). \tag{1.2}$$

In some respect, this bimodule (or matrix) interpretation has certain advantage, and we shall permanently use it. Especially, it gives rise to a quadratic form useful in many questions.

Definition 1.2. 1. The *dimension* (or *vector dimension*) of an element $u \in \text{Hom}_\Lambda(P, L \otimes \mathbf{U})$, or of the corresponding representation of \mathbf{S} , is the function $\mathbf{d} = \mathbf{dim} u : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ such that $\mathbf{d}(0) = \dim L$ and $P \simeq \bigoplus_{a \in \mathbf{S}} \mathbf{d}(a) \Lambda_a$. We denote by $\text{El}_{\mathbf{d}}(\mathbf{U})$ the set of all elements of dimension \mathbf{d} and by $\text{rep}_{\mathbf{d}}(\mathbf{S})$ the set of the corresponding representations.

If u arises as above from a representation $V \in \text{rep} \mathbf{S}$, then $\mathbf{d}(0) = \dim V(0)$ and $\mathbf{d}(a) = \dim (V(a) / \sum_{b \prec a} V(b))$ for $a \in \mathbf{S}$.

2. The *support* of a dimension $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ is the subset $\text{supp } \mathbf{d} = \{a \in \mathbf{S} \mid \mathbf{d}(a) \neq 0\}$. The dimension \mathbf{d} , as well as the elements from $\text{El}_{\mathbf{d}}(\mathbf{S})$ and the corresponding representations, is called *sincere* if $\text{supp } \mathbf{d} = \widehat{\mathbf{S}}$.

If a dimension \mathbf{d} is not sincere, the representations of this dimension can (and usually will) be treated as representations of a smaller poset, namely its support.

3. The *quadratic form* $Q_{\mathbf{S}}$ associated to a poset \mathbf{S} is, by definition, the quadratic form

$$Q_{\mathbf{S}}(x_0, x_1, \dots, x_n) = \sum_{a \in \widehat{\mathbf{S}}} x_a^2 + \sum_{\substack{a, b \in \mathbf{S} \\ a \prec b}} x_a x_b - \sum_{a \in \mathbf{S}} x_0 x_a.$$

Note that if $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$, then the negative part of $Q_{\mathbf{S}}(\mathbf{d})$ is just the dimension of the vector space $\text{El}_{\mathbf{d}}(\mathbf{U}) = \text{Hom}_{\Lambda}(P, L \otimes \mathbf{U})$ of all elements of dimension \mathbf{d} , while the positive part is the dimension of the algebraic group $\mathbf{G}_{\mathbf{d}} = \text{Aut } L \times \text{Aut } P$ acting on $\text{El}_{\mathbf{d}}(\mathbf{U})$ so that its orbits are the isomorphism classes of elements. From here the following result is evident.

Proposition 1.3. *1. If a dimension $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ is of finite type, i.e. there are only finitely many isomorphism classes in $\text{El}_{\mathbf{d}}(\mathbf{U})$, then $Q_{\mathbf{S}}(\mathbf{d}') > 0$ for each dimension $\mathbf{d}' \leq \mathbf{d}$, i.e. such that $\mathbf{d}'(a) \leq \mathbf{d}(a)$ for all $a \in \widehat{\mathbf{S}}$.*

2. *Epecially, if \mathbf{S} is representation finite, i.e. has only finitely many nonisomorphic indecomposable representations, the quadratic form $Q_{\mathbf{S}}$ is weakly positive, i.e. $Q_{\mathbf{S}}(\mathbf{x}) > 0$ for every nonzero vector \mathbf{x} with non-negative entries.*

In [5, 1] the converse was proved, giving a criterion for \mathbf{S} to be representation finite. We recall this result. A poset \mathbf{S} is called *primitive* if it is a disjoint unit of several chains such that the elements of different chains are noncomparable. We denote such a poset by (n_1, n_2, \dots, n_s) , where n_i are the lengths of the chains. We also denote by \mathfrak{K} the poset $\{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\}$, where the order \prec is defined as follows: $a_2 \prec a_1$, $b_2 \prec b_1$, $b_2 \prec a_1$, $c_1 \prec c_2 \prec c_3 \prec c_4$. The posets $(1, 1, 1, 1)$, $(2, 2, 2)$, $(1, 3, 3)$, $(1, 2, 5)$ and \mathfrak{K} are called *critical*.

Theorem 1.4. *1. The following conditions are equivalent:*

- (a) \mathbf{S} is representation finite.
 (b) $Q_{\mathbf{S}}$ is weakly positive.

- (c) \mathbf{S} contains no critical subset.
2. Let \mathbf{S} is representation finite, $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$. The following conditions are equivalent:
- (a) There is an indecomposable element $u \in \text{El}_{\mathbf{d}}(\mathbf{U})$.
- (b) \mathbf{d} is a root of the form $Q_{\mathbf{S}}$, i.e. $Q_{\mathbf{S}}(\mathbf{d}) = 1$.

Moreover, if the latter condition holds, there is a unique indecomposable element $u \in \text{El}_{\mathbf{d}}(\mathbf{U})$, $\text{End } u = \mathbb{k}$ and the orbit of u is open in the space $\text{El}_{\mathbf{d}}(\mathbf{U})$ (in the Zariski topology).

We shall generalize this result using the following notions.

Definition 1.5. Let $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$.

1. The dimension \mathbf{d} is called *critical*, if its support \mathbf{C} is a critical subset, $Q_{\mathbf{C}}(\mathbf{d}) = 0$ and the values $\{\mathbf{d}(a) \mid a \in \widehat{\mathbf{S}}\}$ are coprime (equivalently, at least one of these values equals 1).

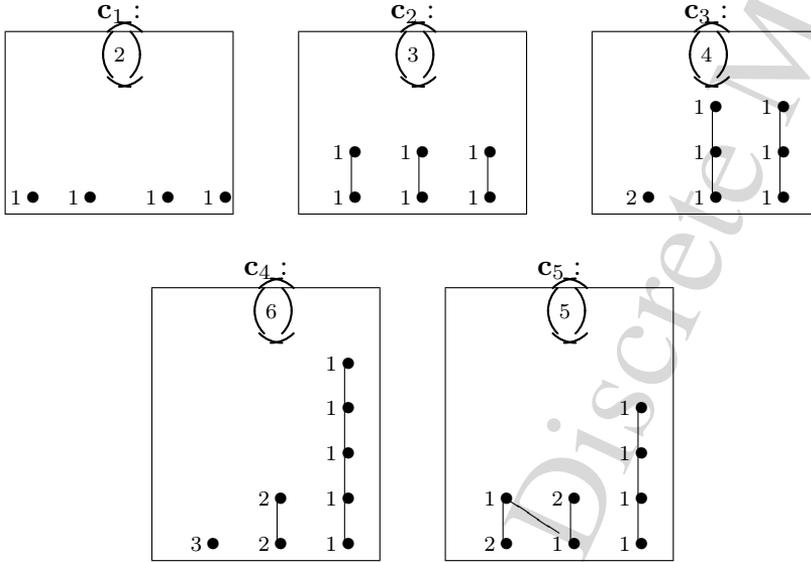
Table 1 below presents all critical dimensions (there are 5 of them, denoted by \mathbf{c}_i , $1 \leq i \leq 5$). In every picture from this table the bullets show the elements $a \in \mathbf{C}$; the numbers nearby are the values $\mathbf{c}_i(a)$. The relations $a \prec b$ are shown by the edges going from a downstairs to b upstairs. The number in a circle above denotes the dimension $\mathbf{c}_i(0)$.

Theorem 1.6 (Main Theorem). 1. The following conditions for a dimension $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ are equivalent:

- (a) \mathbf{d} is a dimension of finite type.
- (b) $Q_{\mathbf{S}}(\mathbf{d}') > 0$ for every nonzero dimension $\mathbf{d}' \leq \mathbf{d}$.
- (c) There is no critical dimension $\mathbf{c} \leq \mathbf{d}$.
2. If a dimension \mathbf{d} is of finite type, the following conditions are equivalent:
- (a) There is an indecomposable element $u \in \text{El}_{\mathbf{d}}(\mathbf{U})$.
- (b) $Q_{\mathbf{S}}(\mathbf{d}) = 1$.

Moreover, if the latter condition holds, there is a unique indecomposable element $u \in \text{El}_{\mathbf{d}}(\mathbf{U})$, $\text{End } u = \mathbb{k}$, and the orbit of u is open and dense in $\text{El}_{\mathbf{d}}(\mathbf{U})$.

Table 3.1: **Critical dimensions**



Note that all claims about indecomposable elements of $\text{El}_d(\mathbf{U})$ obviously remain valid for indecomposable *representations* from $\text{rep}_d(\mathbf{S})$, with the exception of trivial dimensions, which are nonzero on a unique element $a \in \mathbf{S}$.

For primitive posets this theorem was deduced in [9] from the results of Kac [4] about the representations of quivers. Unfortunately, this approach cannot be applied in general case. That is why we have to return to the original technique of *derivations* (or *differentiation*) from [10]. It will be considered in the next section.

2. Derivations and integration

For calculation of representations there is an effective algorithm of *derivations* (or *differentiation*) elaborated in [10, 6]. We recall it; moreover, we show that it can be considered as an equivalence of certain categories. For every element $a \in \mathbf{S}$, denote by $\Delta(a) = \{b \in \mathbf{S} \mid b \preceq a\}$ the *lower cone* of a , $\Delta'(a) = \Delta(a) \setminus \{a\}$, and $\Theta(a)$ the set of elements noncomparable with a . Let also $w(\mathbf{S})$ be the *width* of \mathbf{S} , i.e. the maximal number of pairwise noncomparable elements from \mathbf{S} .

Definition 2.1. Suppose that a is a maximal element of \mathbf{S} . Let $\Pi(a)$ be the set of all pairs $\{b, c\}$ such that $b, c \in \Theta(a)$ and are noncomparable in

\mathbf{S} . Set $\tilde{\mathbf{S}}^a = \mathbf{S} \cup \Pi(a)$ and define a partial order \preccurlyeq on $\tilde{\mathbf{S}}^a$ setting $B \preccurlyeq C$ in $\tilde{\mathbf{S}}^a$ if and only if for each element $b \in B$ there is an element $c \in C$ such that $b \preccurlyeq c$ in \mathbf{S} (we identify elements of \mathbf{S} with one-element sets). We also set $\mathbf{S}^a = \tilde{\mathbf{S}}^a \setminus \{a\}$ and call the poset \mathbf{S}^a the *derivative of \mathbf{S} with respect to a* .

For instance, $b \preccurlyeq \{c, d\}$ means that either $b \preccurlyeq c$ or $b \preccurlyeq d$; $\{b, c\} \preccurlyeq d$ means that both $b \preccurlyeq d$ and $c \preccurlyeq d$, etc.

We fix, for every pair $p \in \Pi(a)$, one element $p' \in p$, and denote by p'' the other element of p .

We also use the following notations.

- For every element $a \in \mathbf{S}$ denote by E_a the representation of \mathbf{S} such that $E_a(0) = \mathbb{k}$, $E_a(b) = \mathbb{k}$ if $a \preccurlyeq b$ and $E_a(b) = 0$ otherwise.
- For every pair of noncomparable elements $p = \{a, b\}$ of \mathbf{S} , denote by E_p the representation of \mathbf{S} such that $E_p(0) = \mathbb{k}$, $E_p(c) = \mathbb{k}$ if $a \preccurlyeq c$ or $b \preccurlyeq c$ and $E_p(c) = 0$ otherwise.

We use the same notations for the objects of $\text{El}(\mathbf{U})$ corresponding to these representations. In the matrix form, $E_a(a) = (1)$, $E_a(b) = \emptyset$ if $b \neq a$; $E_p(a) = E_p(b) = (1)$, $E_p(c) = \emptyset$ if $c \neq a, c \neq b$.

If V is a representation of \mathbf{S} , define the *derived representation* $D_a V$ of \mathbf{S}^a as follows:

- $D_a V(0) = V(a)$;
- $D_a V(b) = V(b) \cap V(a)$ for $b \in \mathbf{S} \setminus \{a\}$;
- $D_a V(p) = (V(p') + V(p'')) \cap V(a)$ for $p \in \Pi(a)$.

Obviously, every morphism $f : V \rightarrow W$ induces a morphism $D_a f : D_a V \rightarrow D_a W$. So we obtain a functor $D_a : \text{rep}(\mathbf{S}) \rightarrow \text{rep}(\mathbf{S}^a)$.

On the contrary, let V be a representation of \mathbf{S}^a . For every $p \in \Pi(a)$, let $\tilde{V}(p) = V(p)/(V(p') + V(p''))$ and $\pi_p : V(p) \rightarrow \tilde{V}(p)$ be the natural surjection. We can choose sections $\iota_p : \tilde{V}(p) \rightarrow V(p)$ such that $\pi_p \iota_p = \text{Id}$ and $\iota_p|_{V(q)} = \iota_q$ if $p, q \in \Pi(a)$, $q \prec p$. Set $\tilde{V}(0) = \bigoplus_{p \in \Pi(a)} \tilde{V}(p)$ and define, for $b \in \Theta(a)$, a map $\tilde{\iota}_b : \tilde{V}(0) \rightarrow V(0) \oplus \tilde{V}(0)$ by the rule

$$\tilde{\iota}_b(v) = \begin{cases} (0, 0) & \text{if } b \notin p, \\ (0, v) & \text{if } b = p', \\ (\iota_p(v), v) & \text{if } b = p'', \end{cases}$$

where $v \in \tilde{V}(p)$. We construct the *integrated representation* $\int_a^\iota V$ as follows:

- $\int_a^\iota V(0) = V(0) \oplus \tilde{V}(0)$;
- $\int_a^\iota V(a) = V(0)$;
- $\int_a^\iota V(b) = V(b)$ for $b \in \Delta(a)$;
- $\int_a^\iota V(b) = V(b) + \text{Im } \tilde{\iota}_b$ for $b \in \Theta(a)$.

We have included the choice $\iota = \{\iota_p\}$ of sections $\iota_p : \tilde{V}(p) \rightarrow V(p)$ into this notation. Nevertheless, if $\iota' = \{\iota'_p\}$ is another choice of such sections, $\text{Im}(\iota'_p - \iota_p) \in V(p') + V(p'')$ for each $p \in \Pi(a)$. Thus we can find maps $\delta_p : \tilde{V}(p) \rightarrow V(p')$ such that $\text{Im}(\iota'_p - \iota_p - \delta_p) \subseteq V(p'')$. Moreover, we can again suppose that $\delta_p|_{\tilde{V}(q)} = \delta_q$ if $p, q \in \Pi(a)$, $q \prec p$. It defines a map $\delta : \tilde{V}(0) \rightarrow V(0)$ such that the map $V(0) \oplus \tilde{V}(0) \rightarrow V(0) \oplus \tilde{V}(0)$ given by the matrix

$$\begin{pmatrix} \text{Id} & \delta \\ 0 & \text{Id} \end{pmatrix}$$

is indeed a morphism (hence, an isomorphism) $\int_a^\iota V \rightarrow \int_a^{\iota'} V$. So we can use the notation $\int_a V$ without mentioning ι . Note that we have only defined the operation \int_a on representations, not on their morphisms, so it is not a functor. Nevertheless, Proposition 2.2 below shows that it can be considered as a functor from $\text{rep } \mathbf{S}^a$ to a factorcategory of $\text{rep } \mathbf{S}$.

This integration is easier in the matrix language. Namely, let a set of matrices $\{M(x) \mid x \in \mathbf{S}^a\}$ define an object $u \in \text{El}(\mathbf{U}_{\mathbf{S}^a})$, like in (1.1), and $\mathbf{d} = \mathbf{dim } u$. Let also $\Delta(a) = \{a = a_1, a_2, \dots, a_k\}$. We choose a matrix $M(a)$ with $\mathbf{d}(0)$ rows so that its columns are linear independent and

$$\text{rank} \begin{array}{|c|c|c|c|} \hline M(a_1) & M(a_2) & \dots & M(a_n) \\ \hline \end{array} = \mathbf{d}(0).$$

We denote by $\mathbf{d}(a)$ the number of columns of $M(a)$. Define $\mathbf{d}^* : \hat{\mathbf{S}} \rightarrow \mathbb{N}$ as follows:

$$\mathbf{d}^*(b) = \begin{cases} \mathbf{d}(0) + \sum_{p \in \Pi(a)} \mathbf{d}(p) & \text{if } b = 0, \\ \mathbf{d}(b) & \text{if } b \in \Delta(a), \\ \mathbf{d}(b) + \sum_{\substack{p \in \Pi(a) \\ b \in p}} \mathbf{d}(p) & \text{if } b \in \Theta(a). \end{cases}$$

The integrated element $\int_a u$ is of dimension \mathbf{d}^* and is given by the set of matrices $M^*(b)$ defined as follows. We consider the element $z \in \text{El}(\mathbf{U})$, which is the direct sum

$$z = \left(\bigoplus_{b \in \Theta(a)} \mathbf{d}(b) T_b \right) \oplus \left(\bigoplus_{p \in \Pi(a)} E_p \right).$$

In the block matrix Z defining this element only blocks $Z(b)$, $b \in \Theta(a)$, are nonzero; let

$$Z(b) = \begin{array}{|c|c|c|c|} \hline Z_b(0) & Z_b(p_1) & \dots & Z_b(p_s) \\ \hline \end{array},$$

$$Y(b) = \begin{array}{|c|c|c|c|} \hline M(b) & M_b(p_1) & \dots & M_b(p_s) \\ \hline \end{array},$$

where

- p_1, p_2, \dots, p_s are all pairs from $\Pi(a)$ containing b ; $Z_b(p_i)$ denotes the part of $Z(b)$ corresponding to the direct summand E_{p_i} of Z , and $Z_b(0)$ is the part of $Z(b)$ corresponding to the direct summand T_b (it is the zero matrix with $\mathbf{d}(b)$ columns);
- the vertical stripes of the matrix $Y(b)$ are of the same size as the corresponding stripes of the matrix $Z(b)$;
- $M_b(p) = M(p)$ if $b = p''$, and $M_b(p) = 0$ if $b = p'$.

We also set $Z(b) = 0$ if $b \in \Delta(a)$. Then

$$M^*(b) = \begin{array}{|c|} \hline Y(b) \\ \hline Z(b) \\ \hline \end{array}.$$

Since $w(\Theta(a)) \leq 2$, every object of $\text{rep } \mathbf{S}$ with support in $\Theta(a)$ is a direct sum of the trivial representation T_0 , the representations E_b and E_p . Set $\mathcal{O}(a) = \{T_0, E_b, E_p \mid b \in \Theta(a), p \in \Pi(a)\}$. They are all indecomposable representations V such that $D_a V = 0$. Straightforward matrix calculations immediately imply the following result (cf. also [6] and, for paragraphs 3 and 4, the proof of Lemma 4.4 below).

Proposition 2.2. 1. If $V \in \text{rep}(\mathbf{S}^a)$, then $D_a \int_a V \simeq V$.

2. If $V \in \text{rep}(\mathbf{S})$, then $\int_a D_a V \simeq V$ if and only if V has no direct summands from $\mathcal{O}(a)$.
3. For every morphism $\phi : V \rightarrow W$ of representations of \mathbf{S}^a , there is a morphism $f : \int_a V \rightarrow \int_a W$ such that $\phi = D_a f$. If, moreover, ϕ is an isomorphism, so is f .
4. The operations D_a and \int_a induce an equivalence between the categories $\text{rep } \mathbf{S} / \mathcal{J}_a$ and $\text{rep } \mathbf{S}^a$, where \mathcal{J}_a is the ideal generated by the identity morphisms of all representations from $\mathcal{O}(a)$.

We shall call a dimension $\mathbf{d}' : \widehat{\mathbf{S}}^a \rightarrow \mathbb{N}$ *subordinate* to a dimension $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ if $\mathbf{dim} \int_a V = \mathbf{d}$ for some representation $V \in \text{rep}_{\mathbf{d}'}(\mathbf{S}^a)$. Obviously, for every dimension $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ there is only a finite set of subordinate dimensions $\mathbf{d}' : \widehat{\mathbf{S}}^a \rightarrow \mathbb{N}$. Proposition 2.2 immediately implies the following corollary.

Corollary 2.3. *If a dimension \mathbf{d}' is subordinate to a dimension \mathbf{d} , which is of finite type, then \mathbf{d}' is of finite type as well.*

3. Dimensions of finite type

In this section we shall prove paragraph 1 of the Main Theorem 1.6. In fact, $1(a) \Rightarrow 1(b)$ is the claim of Proposition 1.3.1, and $1(b) \Rightarrow 1(c)$ is obvious. So we only have to prove that $1(c) \Rightarrow 1(a)$.

Definition 3.1. We call a representation V *quite sincere* if it is indecomposable and the following conditions hold:

- $V(a) \neq V(0)$ for every $a \in \mathbf{S}$.
- $V(a) \neq \sum_{b \prec a} V(b)$ for every $a \in \mathbf{S}$.

In particular, since V is indecomposable, $\sum_{a \in \mathbf{S}} V(a) = V(0)$. If there is a quite sincere representation of dimension \mathbf{d} , we call this dimension *quite sincere* as well.

Obvious necessary conditions for a dimension \mathbf{d} to be quite sincere are:

- $\mathbf{d}(a) \neq 0$ for every $a \in \mathbf{S}$;
- $\sum_{i=1}^k \mathbf{d}(a_i) < \mathbf{d}(0)$ for every chain $a_1 \prec a_2 \prec \dots \prec a_k$ from \mathbf{S} .

Note also that if there is a quite sincere dimension of representations of \mathbf{S} , \mathbf{S} must have at least 2 maximal elements.

We shall deduce the implication $1(c) \Rightarrow 1(a)$ from the following result.

Lemma 3.2. *Suppose that $w(\mathbf{S}) \leq 3$ and a quite sincere dimension \mathbf{d} satisfies condition 1(c) of Theorem 1.6. There is a maximal element $a \in \mathbf{S}$ such that every dimension \mathbf{d}' of representations of the derived poset \mathbf{S}^a , which is subordinate to \mathbf{d} , satisfies this condition too.*

Proof. Note, first of all, that $\mathbf{d}'|_{\mathbf{S} \setminus \{a\}} \leq \mathbf{d}|_{\mathbf{S} \setminus \{a\}}$. Hence, if $\mathbf{d}' \geq \mathbf{c}$ for a critical dimension \mathbf{c} , the support of this \mathbf{c} must contain at least one element from $\Pi(a)$: otherwise also $\mathbf{d} \geq \mathbf{c}$. If there is a maximal element $a \in \mathbf{S}$ such that $w(\Theta(a)) \leq 1$, then $\mathbf{S}^a = \mathbf{S} \setminus \{a\}$, so there is

nothing to prove. Hence, we may suppose that $w(\Theta(a)) = 2$ for every maximal element $a \in \mathbf{S}$. We show that then \mathbf{S} must have 3 maximal elements. Indeed, suppose that \mathbf{S} has only 2 maximal elements, a and b . If both a and b can be included in non-comparable triples, respectively, $\{a, a', a''\}$ and $\{b, b', b''\}$, the quadruple $\{a', a'', b', b''\}$ is non-comparable too, which contradicts the condition. So, for one of these elements, say for a , $w(\Theta(a)) = 1$, the case already excluded.

We also recall the following (rather easy) lemma from [10].

Lemma 3.3 ([10]). *Suppose that $\mathbf{S} = \mathbf{S}_1 \sqcup \mathbf{S}_2 \sqcup \mathbf{S}_3$, where \mathbf{S}_3 is a chain (maybe empty), $b \prec a$ for every $a \in \mathbf{S}_1$, $b \in \mathbf{S}_2$ and $\mathbf{S}_1 \neq \emptyset$, $\mathbf{S}_2 \neq \emptyset$. If $V \in \text{rep } \mathbf{S}$ is indecomposable and $\mathbf{d} = \mathbf{dim } V$, then either $\mathbf{d}|_{\mathbf{S}_1} = 0$ or $\mathbf{d}|_{\mathbf{S}_2} = 0$. Especially, if neither \mathbf{S}_1 nor \mathbf{S}_2 are empty, \mathbf{S} has no sincere indecomposable representations.*

In this case the poset \mathbf{S} is called *semidecomposable*. Thus in what follows we may suppose that \mathbf{S} is not semidecomposable.

Let the maximal elements of \mathbf{S} be a, b, c . Using the Dilworth theorem [11, Theorem 10.2.3], we consider \mathbf{S} as a union of three chains

$$\begin{aligned} \mathbf{A} &= \{a = a_1 \succ a_2 \succ \cdots \succ a_r\}, \\ \mathbf{B} &= \{b = b_1 \succ b_2 \succ \cdots \succ b_s\}, \\ \mathbf{C} &= \{c = c_1 \succ c_2 \succ \cdots \succ c_t\}, \end{aligned}$$

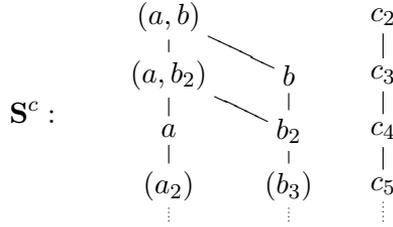
Since \mathbf{d} is quite sincere, \mathbf{S} contains no primitive subset of type $(2, 2, 2)$. Consider the *top* of \mathbf{S} , i.e. the maximal primitive subset $\mathbf{T} \subseteq \mathbf{S}$ containing $\{a, b, c\}$. Then $\mathbf{T} \simeq (1, m, n)$, namely, we may suppose that $\mathbf{T} = \{a, b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_n\}$ ($m \leq s, m \leq n \leq t$). If $m = 1$, the derived poset \mathbf{S}^c only has one new point $p = (a, b)$, such that $p \succ a_i$ and $p \succ b_j$ for all i, j . So p cannot occur in any critical subset. If $m \geq 3$ and $n \geq 3$, then $\mathbf{d}(a) = 1$. Hence, in any subordinate dimension \mathbf{d}' of the poset \mathbf{S}^c , which consists of $\mathbf{S} \setminus \{c\}$ and the points $p_j = (a, b_j)$ ($1 \leq j \leq m$), only one of the points p_j can occur with $\mathbf{d}'(p_j) = 1$, and then $\mathbf{d}'(a) = 0$. Thus, replacing this p_j by a , we get the same dimension for \mathbf{S} , so $\mathbf{d}' \geq \mathbf{c}_i$ is impossible. Therefore, we may suppose that $m = 2, n \geq 2$.

We distinguish the following cases.

Case 1. Either $r = 1$ or $a_2 \prec c$.

Then $\mathbf{S}^c = (\mathbf{S} \setminus \{c\}) \cup \{(a, b), (a, b_2)\}$ and (a, b) cannot occur in any

critical subset of \mathbf{S}^c :



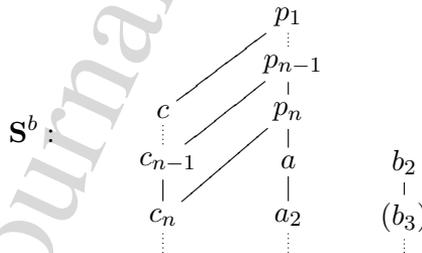
(a_2 and b_3 are definitely not in the top of \mathbf{S}). Set $p = (a, b_2)$ and suppose that $\mathbf{d}' \geq \mathbf{c}_i$ for some i , $\mathbf{S}_0 = \text{supp } \mathbf{c}_i$. It is easy to see that $\mathbf{S}_0 \neq (2, 2, 2)$. If it is $(1, 3, 3)$, it can only be $\{b, p \succ a \succ a_2, c_j \succ c_{j+1} \succ c_{j+2}\}$. Then $\{a \succ a_2, b \succ b_2, c_j \succ c_{j+1}\}$ is a subset of \mathbf{S} of type $(2, 2, 2)$, which is impossible. Suppose that \mathbf{S}_0 is $(1, 2, 5)$, so $\mathbf{d}' = \mathbf{c}_4$. It can only occur as $\{b, p \succ a, c_j \succ c_{j+1} \succ \dots \succ c_{j+4}\}$ with $\mathbf{d}'(p) \geq 2$, $\mathbf{d}'(a) \geq 2$, $\mathbf{d}'(b) \geq 3$. Then $\{a, b \succ b_2, c_j \succ \dots \succ c_{j+4}\}$ is also of type $(1, 2, 5)$ and $\mathbf{d}(a) \geq 4$, $\mathbf{d}(b) \geq 3$, $\mathbf{d}(b_2) \geq 2$, so $\mathbf{d} \geq \mathbf{c}_4$, which is impossible. Just in the same way, if $\mathbf{S}_0 \simeq \mathfrak{K}$, it can only be $\{a \prec p \succ b_2 \prec b, c_j \succ \dots \succ c_{j+3}\}$ with $\mathbf{d}'(a) \geq 2$, $\mathbf{d}'(b) \geq 2$. Then $\mathbf{d}(a) \geq 3$, $\mathbf{d}(b_2) \geq 2$, hence $\mathbf{d} \geq \mathbf{c}_4$ with $\text{supp } \mathbf{c}_4 = \{a, b \succ b_2, c \succ c_j \succ \dots \succ c_{j+3}\}$. It is also impossible, which accomplishes the consideration of Case 1.

Case 2. $a_2 \prec b_2$.

Then, if $b_3 \prec a$, \mathbf{S} is semidecomposable with $\mathbf{S}_1 = \{a, b, b_2\}$, $\mathbf{S}_2 = \{a_i, b_j \mid i > 1, j > 2\}$, both nonempty, so there are no quite sincere dimensions at all. Hence either $s = 2$ or $b_3 \leq c$. In both cases \mathbf{S}^c is as in Case 1 and analogous considerations prove the lemma.

Case 3. $a_2 \prec b$, $a_2 \not\prec b_2$, $a_2 \not\prec c$.

Then the new elements in \mathbf{S}^b are $p_i = (a, c_i)$, ($1 \leq i \leq n$) and p_1 cannot occur in any critical subset:



(a_2 is in the top of \mathbf{S}^b , while b_3 is not). Note that either $b_3 \prec a$ or $b_3 \prec c_{n-1}$: otherwise \mathbf{S} contains a subset $(2, 2, 2)$. It implies that b_3 cannot occur in a critical subset $\mathbf{S}_0 \subseteq \mathbf{S}^b$ containing a new element p_i . Hence,

$\mathbf{S}_0 \neq (2, 2, 2)$ and $\mathbf{S}_0 \neq \mathfrak{R}$. Suppose that $\mathbf{d}' \geq \mathbf{c}_i$ with $\text{supp } \mathbf{c}_i = \mathbf{S}_0$. If $\mathbf{S}_0 = (1, 3, 3)$, then $\mathbf{S}_0 = \{b_2, c_j \succ c_{j+1} \succ c_{j+2}\} \cup \mathbf{A}'$, where $\mathbf{A}' \subset \mathbf{A} \cup \Pi(b)$. Note that \mathbf{A}' contains at least two elements from $\Pi(b) \cup \{a\}$. Then $\mathbf{d}(b_2) = \mathbf{d}'(b_2) \geq 2$, $\mathbf{d}(a) \geq 2$. Hence, $\mathbf{d} \geq \mathbf{c}_5$ with $\text{supp } \mathbf{c}_5 = \{a \succ a_2 \prec b \succ b_2, c \succ c_j \succ c_{j+1} \succ c_{j+2}\}$, which is impossible. Analogously, $\mathbf{S}_0 = (1, 2, 5)$ is impossible too, which accomplishes the proof of the lemma. \square

Now the implication $1(c) \Rightarrow 1(a)$ of Theorem 1.6 is easy. Namely, let a dimension \mathbf{d} satisfy $1(c)$. Without loss of generality, we may suppose \mathbf{d} quite sincere. Then either $w(\mathbf{S}) \leq 3$ or $\mathbf{d}(0) = 1$. In the latter case \mathbf{d} is obviously of finite type. In the former case choose a maximal element $a \in \mathbf{S}$ as stated in Lemma 3.2. Every representation V of dimension \mathbf{d} without direct summands from $\mathcal{O}(a)$ is isomorphic to $\int_a W$ for a representation W of \mathbf{S}^a . The dimension \mathbf{d}' of W is subordinate to \mathbf{d} . Especially it satisfies $1(c)$ too; moreover, $\mathbf{d}'(0) = \dim V(a) < \mathbf{d}(0)$. Thus, using induction by $\mathbf{d}(0)$, we get that there are finitely many nonisomorphic representations of dimension \mathbf{d}' . Since there are finitely many subordinate dimension, we obtain the same for the dimension \mathbf{d} . \square

4. Indecomposable representations

Now we shall prove paragraph 2 of the Main Theorem 1.6. To do it, we combine derivations with analogues of some results of [1, 2] about posets of finite type. Namely, we use induction by $|\mathbf{d}| = \sum_{a \in \widehat{\mathbf{S}}} \mathbf{d}(a)$. The case $|\mathbf{d}| = 1$ is obvious. Thus, from now on, we suppose that $\mathbf{d} : \widehat{\mathbf{S}} \rightarrow \mathbb{N}$ is a dimension of finite type, Theorem 1.6 holds for every dimension of finite type \mathbf{d}' of representations of any poset \mathbf{S}' such that $|\mathbf{d}'| < |\mathbf{d}|$, and, moreover, \mathbf{d} is sincere. Let $u \in \text{El}_{\mathbf{d}}(u)$ be an indecomposable element, V be the corresponding representation of \mathbf{S} and M be the block matrix of the form (1.1) describing u . Fix an element $a \in \mathbf{S}$ and denote by M_a the part of M consisting of the blocks $M(b)$ with $b \preceq a$.

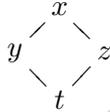
Lemma 4.1. *The columns of the matrix M_a are linear independent.*

Proof. Obviously, we may suppose that the element a is maximal. Consider the part \overline{M} of M consisting of all blocks $M(b)$ with $b \neq a$. It also describes an object $\overline{u} \in \text{El}(\mathbf{U})$ (certainly, non-sincere) and $|\mathbf{dim } \overline{u}| < |\mathbf{d}|$. Hence, Theorem 1.6 holds for every indecomposable direct summand v of \overline{u} . Especially, the orbit of v is open dense in the space of all objects of the same dimension. Let N be the block matrix describing v , $\overline{\mathbf{d}} = \mathbf{dim } v$, $m = \overline{\mathbf{d}}(0)$ and $n = \sum_{b \prec a} \overline{\mathbf{d}}(b)$. For every object $v' \in \text{El}_{\overline{\mathbf{d}}}(\mathbf{U})$, denote by N' the corresponding block matrix and by N'_a its part consisting of the

blocks $N'(b)$ with $b \prec a$. If $m \leq n$, the objects v' such that the rows of N'_a are linear independent form an open subset in $\text{El}_{\bar{\mathbf{d}}}(\mathbf{U})$. Hence, v belongs to this subset, i.e. $\text{rk } N_a = \bar{\mathbf{d}}(0)$. Then, using automorphisms of u , one can make zero the part of the matrix $M(a)$ consisting of the rows that occur in v . Therefore, v is a direct summand of u , which is impossible. Thus $m > n$. Then the same argument shows that the columns of v are linear independent. Since it is so for every direct summand of \bar{u} , it holds for \bar{u} too. If, nevertheless, the columns of M_a are linear dependent, then, using an automorphism of u , one can make a zero column in $M(a)$, which is also impossible. \square

Corollary 4.2. *For any $a \in \mathbf{S}$, $\text{Hom}(u, T_a) = 0$. Especially, neither nonzero endomorphism of u factors through a direct sum of trivial elements; thus $\text{End } u = \text{End } V$, where $V = \rho(u)$ is the corresponding representation of \mathbf{S} .*

Lemma 4.3. *Let \mathbf{S} contain a subset*



and V be an indecomposable representation of \mathbf{S} such that its dimension \mathbf{d} is of finite type. Then either $\mathbf{d}(x) = 0$ or $\mathbf{d}(t) = 0$.

Proof. We may suppose that $w(\mathbf{S}) = 3$. Again we use the induction by $|\mathbf{d}|$; for $|\mathbf{d}| = 1$ the claim is trivial. If $a \in \mathbf{S}$ is maximal and $\mathbf{d}' = \dim D_a V$, then \mathbf{d}' is also of finite type and $|\mathbf{d}'| < |\mathbf{d}|$. If the element x is not maximal, choose a such that $a \succ x$; then $\mathbf{d}(x) = \mathbf{d}'(x)$ and $\mathbf{d}(t) = \mathbf{d}'(t)$, so one of them is 0. Suppose that x is maximal and there is another maximal element a such that $t \prec b$. If $p = \{x, b\} \in \Pi(a)$, then $y \prec p$ and $z \prec p$, hence either $\mathbf{d}'(t) = \mathbf{d}(t) = 0$ or $\mathbf{d}'(x) = 0$ and $\mathbf{d}'(p) = 0$ for each pair $p = \{x, b\} \in \Pi(a)$, wherefrom $\mathbf{d}(x) = 0$. At last, suppose that $t \not\prec a$ for any maximal $a \neq x$. If such an element a exists, then $w(\Theta(a)) = 2$, hence $w(\Theta(x)) = 1$ and \mathbf{S} is semidecomposable as $\{x\} \cup \Delta'(x) \cup \Theta(x)$. Therefore either $\mathbf{d}(x) = 0$ or $\mathbf{d}|_{\Delta'(x)} = 0$, especially $\mathbf{d}(t) = 0$. \square

The following result is crucial for the proof.

Lemma 4.4. *A maximal element $a \in \mathbf{S}$ can be so chosen that $\text{End } v \simeq \text{End } u$ whenever $u = \int_a v$ for an object $v \in \text{El}(\mathbf{U}^a)$.*

Proof. By the matrix description of $\int_a v$, the matrix M , up to a permutation of columns, has the form

$$M = \begin{array}{|c|c|c|c|c|} \hline X_0 & X & 0 & Y & Z \\ \hline 0 & 0 & I & I & 0 \\ \hline \end{array},$$

where I denotes an identity matrix, the part X_0 is in the matrix $M(a)$ and the remaining part of the first row describes the element v . Namely, the part X is in the matrices $M(b)$, $b \prec a$; the part Y corresponds to the part of $M(b)$, $b \in \Theta(a)$, arising from the elements $p \in \Pi(a)$ of S^a , such that $b = p''$, while the zero matrix in the first row arises from those p with $b = p'$. At last, the part Z arises from the elements $b \in \Theta(a)$ considered as the elements of \mathbf{S}^a . An endomorphism of u is given by a pair of matrices

$$\Phi_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} S_0 & 0 & 0 & 0 & 0 \\ S_1 & S_2 & S_3 & S_4 & S_5 \\ 0 & 0 & T_{11} & T_{12} & T_{13} \\ 0 & 0 & T_{21} & T_{22} & T_{23} \\ 0 & 0 & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

such that $\Phi_0 M = M \Phi_1$. Here the diagonal blocks are square, the division of Φ_0 reflects the horizontal division of M , while the division of Φ_1 reflects the vertical division of M . Note that, by construction, the rows of the matrix $(X_0 \ X)$ are linear independent, and, by Lemma 4.1, its columns are linear independent too; thus this matrix is invertible. It immediately gives that $C = 0$. The other equalities for the elements of $\Phi_0 M$ and $M \Phi_1$ are

$$\begin{aligned} AX_0 &= X_0 S_0 + X S_1, \\ AX &= X S_2, \\ B &= X S_3 + Y T_{21} + Z T_{31}, \\ B + AY &= X S_4 + Y T_{22} + Z T_{32}, \\ AZ &= X S_5 + Y T_{23} + Z T_{33}, \\ D &= T_{11} + T_{21} = T_{12} + T_{22}, \\ 0 &= T_{13} + T_{23}. \end{aligned}$$

Equivalently, the matrices

$$\Psi_0 = A, \quad \Psi_1 = \begin{pmatrix} S_2 & S_3 - S_4 & S_5 \\ 0 & T_{22} - T_{21} & T_{23} \\ 0 & T_{32} - T_{31} & T_{33} \end{pmatrix}$$

define an endomorphism of the representation v (then the matrices S_0, S_1 can be uniquely calculated from the first equality). We have to show that if $\Psi_0 = \Psi_1 = 0$, also $\Phi_0 = \Phi_1 = 0$. From the proof of Lemma 3.2 we have got to know that the element a can be so chosen that every pair from $\Pi(a)$ is of the sort $p = \{b, c\}$, where b is a maximal element. If $\{b, c'\}$ is another pair, then neither $b \preccurlyeq c'$ nor $c \preccurlyeq b$. If an element at some position in the matrix T_{21} is nonzero, it corresponds either to the relation $c \prec c'$ or to $b \preccurlyeq b$; thus the element in the same position of the matrix T_{22} is zero. Consequently, the equality $T_{22} - T_{21} = 0$ implies that $T_{22} = T_{21} = 0$. Lemma 4.3 implies that if $t \prec p$, i.e. $t \prec b$ and $t \prec c$, then either t -part or p -part in the element v is empty. It implies, just as above, that if $S_3 - S_4 = 0$, then $S_3 = S_4 = 0$, and if $T_{32} - T_{31} = 0$, then $T_{31} = T_{32} = 0$. Thus we get the necessary assertion. \square

Now the induction is obvious (just as in [1, 2]). Namely, if u is an indecomposable element such that \mathbf{d} is of finite type and $u \notin O(a)$, then $u = \int D_a u$, $v = D_a u$ is indecomposable, its dimension $\mathbf{d}' = \mathbf{dim} v$ is of finite type and $|\mathbf{d}'| < |\mathbf{d}|$. Therefore, $\text{End} u = \text{End} v = \mathbb{k}$, so the stabilizer of the element u in the group $\mathbf{G}_{\mathbf{d}}$ is 1-dimensional. Recall that there always is an open subset $U \subseteq \text{El}_{\mathbf{d}}(\mathbf{U})$ such that the stabilizers of the elements of U are of minimal dimension (see e.g. [13]). Thus $u \in U$. Since any orbit is open in its closure, we get that the orbit of u is open (hence dense), so its dimension, which is $\dim \mathbf{G}_{\mathbf{d}} - 1$, equals $\dim \text{El}_{\mathbf{d}}(\mathbf{U})$. Therefore $Q_{\mathbf{S}}(\mathbf{d}) = \dim \mathbf{G}_{\mathbf{d}} - \dim \text{El}_{\mathbf{d}}(\mathbf{U}) = 1$. Moreover, if u' is another indecomposable element of the same dimension \mathbf{d} , its orbit is also open, hence coincide with that of u , i.e. u is the unique indecomposable element of this dimension.

On the other hand, if a dimension is of finite type and $Q_{\mathbf{S}}(\mathbf{d}) = 1$, the number of orbits of the group $\mathbf{G}_{\mathbf{d}}$ in $\text{El}_{\mathbf{d}}(\mathbf{U})$ is finite. Therefore, there is an open orbit and its dimension equals $\dim \text{El}_{\mathbf{d}}(\mathbf{U})$. If u is an element of this orbit, $\dim \text{End} u = \dim \mathbf{G}_{\mathbf{d}} - \dim_{\mathbf{d}}(\mathbf{U}) = Q_{\mathbf{S}}(\mathbf{d}) = 1$, so $\text{End} u = \mathbb{k}$ and u is indecomposable. It accomplishes the proof of the Main Theorem.

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