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On the group of extensions for the bicrossed product construction for a locally compact group

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Abstract. For the cocycle bicrossed product construction applied to a locally compact group and its two subgroups, we give a simple description of the group of the corresponding extensions in terms of the second cohomology group of a certain complex of continuous functions on the group. Using this description, we find pairs of continuous cocycles for two subgroups of the Heisenberg group.

Introduction

A method for constructing nontrivial examples of finite ring groups, now known as finite Kac algebras, was proposed in [1]. It consists in using two subgroups of a group satisfying certain conditions for constructing a commutative algebra of functions on one subgroup and then extending it to a nontrivial Kac algebra, via a pair of cocycles, with the group algebra of the other subgroup. This method is now called the bicrossed product construction and was generalized to bialgebras in [2] and to locally compact quantum groups in [3].

For fixed subgroups, the set of extensions forms a group whose elements are determined by equivalence classes of pairs of cocycles. This group formed by equivalence classes of pairs of measurable functions is

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described in [4] for a locally compact group in terms of the second cohomology group of a certain complex constructed from a bicomplex.

The purpose of this paper is to consider the case where the extensions are obtained using continuous cocycles and to give a simple direct construction of a complex such that its second cohomology group is isomorphic to the group of such extensions. This is done in Section 1 after first making necessary definitions. In Section 2 we use this result and the ideas from [5] to construct pairs of cocycles for two subgroups of the Heisenberg group.

1. Definitions and the main result

Definition 1 ([6]). Let K be a locally compact group, G, H subgroups of K satisfying the conditions

$$G \cdot H = K \quad \text{and} \quad G \cap H = \{e\}.$$

Then (G, H) is called a matched pair of locally compact groups.

Remark 1. For a more general definition of a matched pair of locally compact groups, see [3].

In what follows, g, h, k with, possibly, subscripts denote elements of the groups G, H, K, respectively.

Let (G, H) be a matched pair of locally compact groups. It is known [3] that there are right and left actions, $\triangleleft : H \times G \to H$ and $\triangleright : H \times G \to G$, given by

$$h \cdot g = (h \triangleright g) \cdot (h \triangleleft g)$$

and satisfying

$$\begin{aligned} (h_1 h_2) \triangleleft g &= (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g), \\ h \triangleright (g_1 g_2) &= (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2). \end{aligned} \tag{1}$$

Denote $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and let $\widetilde{\mathbb{T}}$ be the Abelian group $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 2 ([3]). Let (G, H) be a matched pair of locally compact groups. A pair of continuous maps (u, v), $u : H \times G \times G \to \mathbb{T}$, $v : H \times H \times G \to \mathbb{T}$ is called a pair of cocycles for the pair (G, H) if the following identities hold:

$$u(h \triangleleft g_1, g_2, g_3) u(h, g_1, g_2 g_3) = u(h, g_1, g_2) u(h, g_1 g_2, g_3), \tag{2}$$

$$v(h_1, h_2, h_3 \triangleright g) v(h_1 h_2, h_3, g) = v(h_1, h_2 h_3, g) v(h_2, h_3, g), \tag{3}$$

$$\begin{aligned} v(h_1, h_2, g_1 g_2) u(h_1 h_2, g_1, g_2) = {} & v(h_1, h_2, g_1) u(h_2, g_1, g_2) \\ & \cdot v(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1, g_2) \\ & \cdot u(h_1, h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2). \end{aligned} \tag{4}$$

Two pairs of cocycles (u_1, v_1) and (u_2, v_2) for (G, H) are called equivalent if there exists a continuous function $r : H \times G \to \mathbb{T}$ such that

$$\begin{aligned} u_1(h, g_1, g_2)u_2(h, g_1, g_2)^{-1} &= r(h, g_1)r(h \triangleleft g_1, g_2)r(h, g_1g_2)^{-1}, \\ v_1(h_1, h_2, g)v_2(h_1, h_2, g)^{-1} &= r(h_1h_2, g)r(h_1, h_2 \triangleright g)^{-1}r(h_2, g)^{-1}. \end{aligned} \tag{5}$$

We will denote the equivalence class of a pair (u, v) by $[u, v]$. It is known [3] that the set of equivalence classes $[u, v]$ forms a group with respect to the operations $[u_1, v_1] \cdot [u_2, v_2] = [u_1u_2, v_1v_2]$, $[u, v]^{-1} = [u^{-1}, v^{-1}]$, and the identity element $[1, 1]$. We will consider the subgroup of this group formed by the classes $[u, v]$ such that $u(h_1, g_2, g_3) = 1$ if at least one of the elements h_1, g_2, g_3 is the identity of the corresponding group and the same holds for v. This subgroup will be denoted by $H^2_0(\text{m.p.}, \mathbb{T})$.

Let (G, H) be a matched pair of locally compact groups. Denote by ${}_G\mathrm{C}^n_H(K^{n+1}, \widetilde{\mathbb{T}})$, or simply by ${}_G\mathrm{C}^n_H(K)$, $n = 0, 1, \dots$, the set of continuous functions $c^n \colon K^{n+1} \to \widetilde{\mathbb{T}}$ such that

$$c^n(gk_1, k_2, \dots, k_n, k_{n+1}h) = c^n(k_1, k_2, \dots, k_n, k_{n+1}), \tag{6}$$

$$c^n(k_1, \dots, k_{j-1}, e_K, k_{j+1}, \dots, k_{n+1}) = 0 \tag{7}$$

for all $g \in G$, $h \in H$, $k_j \in K$, $j = 1, \dots, n + 1$. Here e_K denotes the identity element in K. ${}_G\mathrm{C}^n_H$ is an Abelian group with respect to the pointwise addition. As usual [7], define

$$\begin{aligned} (d^nc^n)(k_1, \dots, k_{n+2}) = c^n(k_2, \dots, k_{n+2}) \\ + \sum_{j=1}^{n+1}(-1)^j c^n(k_1, \dots, k_jk_{j+1}, \dots, k_{n+2}) \\ + (-1)^{n+2}c^n(k_1, \dots, k_{n+1}) \end{aligned} \tag{8}$$

and thus obtain the following complex:

$${}_G\mathrm{C}^0_H(K) \xrightarrow{d^0} {}_G\mathrm{C}^1_H(K) \xrightarrow{d^1} {}_G\mathrm{C}^2_H(K) \xrightarrow{d^2} {}_G\mathrm{C}^3_H(K) \xrightarrow{d^3} \dots, \tag{9}$$

where, as easily seen, ${}_G\mathrm{C}^0_H(K)$ can be identified with the group that has the only element 0 and ${}_G\mathrm{C}^1_H(K)$ with the group of all continuous functions c^1 on K satisfying $c^1 \restriction_G = c^1 \restriction_H = 0$.

Theorem. *Let ${}_GH^2_H(K) = \operatorname{Ker} d^2/\operatorname{Im} d^1$ denote the second cohomology group of complex (9). Then the map $\theta \colon H^2_0(\text{m.p.}, \mathbb{T}) \to {}_GH^2_H(K)$ defined by*

$$\theta([u, v])(k_1, k_2, k_3) = \frac{1}{i} \ln u(h_1, g_2, h_2 \triangleright g_3) + \frac{1}{i} \ln v(h_1 \triangleleft g_2, h_2, g_3), \tag{10}$$

is a group isomorphism. Here $\ln\colon \mathbb{T} \to i\,\widetilde{\mathbb{T}}$ *is the principle branch of the logarithm,* $k_j = g_j h_j$, $j = 1, 2, 3$, *and* $i = \sqrt{-1}$.

The proof of the theorem will be divided into several lemmas. But before, let us denote $\tilde{u} = \frac{1}{i}\ln u$, $\tilde{v} = \frac{1}{i}\ln v$, and $\tilde{r} = \frac{1}{i}\ln r$. With these notations, the map (10) becomes

$$\theta([u,v])(k_1,k_2,k_3) = \tilde{u}(h_1, g_2, h_2 \triangleright g_3) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3), \tag{11}$$

the defining relations (2), (3), (4) will read

$$\tilde{u}(h \triangleleft g_1, g_2, g_3) - \tilde{u}(h, g_1 g_2, g_3) + \tilde{u}(h, g_1, g_2 g_3) - \tilde{u}(h, g_1, g_2) = 0, \tag{12}$$
$$\tilde{v}(h_2, h_3, g) - \tilde{v}(h_1 h_2, h_3, g) + \tilde{v}(h_1, h_2 h_3, g) - \tilde{v}(h_1, h_2, h_3 \triangleright g) = 0, \tag{13}$$

$$\begin{aligned}\tilde{v}(h_1, h_2, g_1 g_2) + \tilde{u}(h_1 h_2, g_1, g_2) = \tilde{v}(h_1, h_2, g_1) + \tilde{v}(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1, g_2)\\ + \tilde{u}(h_2, g_1, g_2) + \tilde{u}(h_1, h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2),\end{aligned} \tag{14}$$

and the equivalence relations (5) are

$$\begin{aligned}\tilde{u}_1(h, g_1, g_2) - \tilde{u}_2(h, g_1, g_2) &= \tilde{r}(h, g_1) + \tilde{r}(h \triangleleft g_1, g_2) - \tilde{r}(h, g_1 g_2),\\ \tilde{v}_1(h_1, h_2, g) - \tilde{v}_2(h_1, h_2, g) &= \tilde{r}(h_1 h_2, g) - \tilde{r}(h_1, h_2 \triangleright g) - \tilde{r}(h_2, g).\end{aligned} \tag{15}$$

Lemma 1. *Let a pair* $(\tilde{u}, \tilde{v})$ *satisfy* (12), (13), (14), *and*

$$f(k_1, k_2, k_3) = \tilde{u}(h_1, g_2, h_2 \triangleright g_3) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3), \tag{16}$$

$k_j = g_j h_j$, $j = 1, 2, 3$. *Then* f *is a* 2*-cocycle for the complex* (9).

Proof. Indeed, by the definition of f, for $k_j = g_j h_j$, $j = 1, \dots, 4$, we have

$$\begin{aligned}
&(d^2 f)(k_1, k_2, k_3, k_4)\\
&\quad= f(k_2, k_3, k_4) - f(k_1 k_2, k_3, k_4) + f(k_1, k_2 k_3, k_4)\\
&\qquad - f(k_1, k_2, k_3 k_4) + f(k_1, k_2, k_3)\\
&\quad= f(g_2 h_2, g_3 h_3, g_4 h_4) - f(g_1 (h_1 \triangleright g_2)(h_1 \triangleleft g_2) h_2, g_3 h_3, g_4 h_4)\\
&\qquad + f(g_1 h_1, g_2 (h_2 \triangleright g_3)(h_2 \triangleleft g_3) h_3, g_4 h_4)\\
&\qquad - f(g_1 h_1, g_2 h_2, g_3 (h_3 \triangleright g_4)(h_3 \triangleleft g_4) h_4) + f(g_1 h_1, g_2 h_2, g_3 h_3)\\
&\quad= \tilde{u}(h_2, g_3, h_3 \triangleright g_4) + \tilde{v}(h_2 \triangleleft g_3, h_3, g_4) - \tilde{u}((h_1 \triangleleft g_2) h_2, g_3, h_3 \triangleright g_4)\\
&\qquad - \tilde{v}\big(((h_1 \triangleleft g_2) h_2) \triangleleft g_3, h_3, g_4\big) + \tilde{u}(h_1, g_2 (h_2 \triangleright g_3), ((h_2 \triangleleft g_3) h_3) \triangleright g_4)\\
&\qquad + \tilde{v}(h_1 \triangleleft (g_2 (h_2 \triangleright g_3)), (h_2 \triangleleft g_3) h_3, g_4) - \tilde{u}\big(h_1, g_2, h_2 \triangleright (g_3 (h_3 \triangleright g_4))\big)\\
&\qquad - \tilde{v}(h_1 \triangleleft g_2, h_2, g_3 (h_3 \triangleright g_4)) + \tilde{u}(h_1, g_2, h_2 \triangleright g_3) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3).
\end{aligned}$$

Replacing h_1, g_1, and g_2 in (14) with $h_1 \triangleleft g_2$, g_3, and $h_3 \triangleright g_4$, respectively, we get that

$$\begin{aligned}
&\tilde{u}((h_1 \triangleleft g_2)h_2, g_3, h_3 \triangleright g_4) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3(h_3 \triangleright g_4)) \\
&\quad = \tilde{u}(h_2, g_3, h_3 \triangleright g_4) + \tilde{u}(h_1 \triangleleft g_2, h_2 \triangleright g_3, (h_2 \triangleleft g_3) \triangleright (h_3 \triangleright g_4)) \\
&\qquad + \tilde{v}((h_1 \triangleleft g_2) \triangleleft (h_2 \triangleright g_3), h_2 \triangleleft g_3, h_3 \triangleright g_4) + \tilde{v}(h_1 \triangleleft g_2, h_2, g_3).
\end{aligned}$$

Use the above to replace the sum of the 3rd and 8th terms in the previous expression, collect the terms with $\tilde{u}$ and $\tilde{v}$, and apply (12) and (13) to get 0. □

Lemma 2. *Let a pair* $(\tilde{u}_1, \tilde{v}_1)$ *satisfy* (15) *with* $\tilde{u}_2 = \tilde{v}_2 = 0$. *Then* f_1 *defined by* (16) *for* $(\tilde{u}_1, \tilde{v}_1)$ *is a* 1*-coboundary.*

Proof. With the notations as before, we have

$$\begin{aligned}
f_1(k_1, k_2, k_3) &= \tilde{u}_1(h_1, g_2, h_2 \triangleright g_3) + \tilde{v}_1(h_1 \triangleleft g_2, h_2, g_3) \\
&= \tilde{r}(h_1, g_2) + \tilde{r}(h_1 \triangleleft g_2, h_2 \triangleright g_3) - \tilde{r}(h_1, g_2(h_2 \triangleright g_3)) \\
&\quad + \tilde{r}((h_1 \triangleleft g_2)h_2, g_3) - \tilde{r}(h_1 \triangleleft g_2, h_2 \triangleright g_3) - \tilde{r}(h_2, g_3).
\end{aligned}$$

On the other hand, extending $\tilde{r}$ from $H \times G \to \widetilde{\mathbb{T}}$ to $K \times K \to \widetilde{\mathbb{T}}$ by setting $\tilde{r}(g_1 h_1, g_2 h_2) = \tilde{r}(h_1, g_2)$, we have

$$\begin{aligned}
(d^1 \tilde{r})(k_1, k_2, k_3) &= \tilde{r}(k_2, k_3) - \tilde{r}(k_1 k_2, k_3) + \tilde{r}(k_1, k_2 k_3) - \tilde{r}(k_1, k_2) \\
&= \tilde{r}(h_2, g_3) - \tilde{r}((h_1 \triangleleft g_2)h_2, g_3) + \tilde{r}(h_1, g_2(h_2 \triangleright g_3)) - \tilde{r}(h_1, g_2).
\end{aligned}$$

Comparing the above, we see that $f_1 = d^1(-r)$. □

Corollary 1. *The map* θ *is well-defined.*

Lemma 3. *Let* $f \in {}_G H^2_H$. *Then* f *satisfies the following:*

$$f(k_1, k_2, k_3) = f(h_1, g_2, h_2 \triangleright g_3) + f(h_1 \triangleleft g_2, h_2, g_3).$$

Proof. Since f is a 2-cocycle,

$$\begin{aligned}
0 = (d^2 f)(h_1, g_2, h_2, g_3) &= f(g_2, h_2, g_3) - f(h_1 g_2, h_2, g_3) + f(h_1, g_2 h_2, g_3) \\
&\quad - f(h_1, g_2, h_2 g_3) + f(h_1, g_2, h_2).
\end{aligned}$$

But $f(g_2, h_2, g_3) = f(e_K, h_2, g_3) = 0$, $f(h_1 g_2, h_2, g_3) = f(h_1 \triangleleft g_2, h_2, g_3)$, $f(h_1, g_2, h_2 g_3) = f(h_1, g_2, h_2 \triangleright g_3)$, and $f(h_1, g_2, h_2) = f(h_1, g_2, e_K) = 0$. □

Corollary 2. *The map* θ *is a surjection.*

Proof. For $f \in {}_G\mathrm{H}^2_H$, define $\tilde{u}(h_1, g_2, g_3) = f(h_1, g_2, g_3)$ and $\tilde{v}(h_1, h_2, g_3) = f(h_1, h_2, g_3)$. Then $\tilde{u}$ and $\tilde{v}$ satisfy (12), (13), and (14). □

Lemma 4. *The map θ is an injection.*

Proof. Indeed, let $\theta([u, v])(k_1, k_2, k_3) = (d^1\tilde{r})(k_1, k_2, k_3)$, that is,

$$\begin{aligned}\tilde{u}(h_1, g_2, h_2 \triangleright g_3) &+ \tilde{v}(h_1 \triangleleft g_2, h_2, g_3)\\ &= \tilde{r}(k_2, k_3) - \tilde{r}(k_1k_2, k_3) + \tilde{r}(k_1, k_2k_3) - \tilde{r}(k_1, k_2)\\ &= \tilde{r}(h_2, g_3) - \tilde{r}((h_1 \triangleleft g_2)h_2, g_3) + \tilde{r}(h_1, g_2(h_2 \triangleright g_3)) - \tilde{r}(h_1, g_2).\end{aligned}$$

By setting $h_2 = e_H$ and then $g_2 = e_G$, we obtain

$$\begin{aligned}\tilde{u}(h_1, g_2, g_3) &= -\tilde{r}(h_1 \triangleleft g_2, g_3) + \tilde{r}(h_1, g_2g_3) - \tilde{r}(h_1, g_2)\\ \tilde{v}(h_1, h_2, g_3) &= \tilde{r}(h_2, g_3) - \tilde{r}(h_1h_2, g_3) + \tilde{r}(h_1, h_2 \triangleright g_3),\end{aligned}$$

that is $[u, v] = [1, 1]$. □

Now, Corollaries 1, 2 and Lemma 4 prove the theorem.

2. An example for the Heisenberg group

As follows from (16), to find the functions $\tilde{u}$, $\tilde{v}$, we need to construct a 2-cocycle f of the complex (9). Note that f is, actually, a 3-cocycle of the group K [7] satisfying the additional relations (6), (7).

Thus we construct a 3-cocycle F for the corresponding matched pair of the Lie algebras, $(\mathfrak{g}, \mathfrak{h})$, by using Proposition 1 in [5], find nonequivalent 3-cocycles in terms of Proposition 2 in [5], and consider the corresponding left-invariant form ω_F on K. Following the procedure of finding a 3-cocycle on a Lie group from a 3-cocycle on the Lie algebra [7], we consider the 3-simplex σ given by

$$\begin{aligned}\sigma(h_1, g_2h_2, g_3) = \Big(h_1 \triangleright \big(g_2 \cdot (h_2 \triangleright g_3^{p_3})\big)^{p_2}\Big)&\\ \cdot \Big(\big(h_1 \triangleleft \big(g_2(h_2 \triangleright g_3^{p_3})\big)\big)^{p_2}(h_2 \triangleleft g_3^{p_3})^{q_2}\Big)^{q_1},&\end{aligned} \tag{17}$$

where $p_j, q_j : \Delta_3 \to \mathbb{R}$ are some differentiable functions on the standard 3-simplex $\Delta_3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1, t_2, t_3 \geq 0,\ t_1 + t_2 + t_3 \leq 1\}$. Then the 2-cocycle f can be found in the form

$$f(h_1, g_2h_2, g_3) = \int_{\sigma(h_1, g_2h_2, g_3)} \omega_F. \tag{18}$$

Using the above procedure, we now construct pairs of continuous cocycles $(\tilde{u}, \tilde{v})$ for the matched pair of Lie groups (G, H) associated with the Heisenberg group. Denote

$$g(\vec{a}) = \begin{pmatrix} 1 & \vec{0}^{\,t} & 0 \\ \vec{0} & 1_n & \vec{a} \\ 0 & \vec{0}^{\,t} & 1 \end{pmatrix}, \qquad h(\vec{x}, y) = \begin{pmatrix} 1 & \vec{x}^{\,t} & y \\ \vec{0} & 1_n & \vec{0} \\ 0 & \vec{0}^{\,t} & 1 \end{pmatrix},$$

where $\vec{a}, \vec{x} \in \mathbb{R}^n$, $y \in \mathbb{R}$, $\vec{x}^{\,t}$ is the transpose of $\vec{x}$, and 1_n denotes the unit matrix on $\mathbb{R}^n$. The groups $G = \{g(\vec{a}) : \vec{a} \in \mathbb{R}^n\}$ and $H = \{h(\vec{x}, y) : \vec{x} \in \mathbb{R}^n, y \in \mathbb{R}\}$ form a matched pair of Abelian Lie groups with the mutual actions

$$h(\vec{x}, y) \triangleright g(\vec{a}) = g(\vec{a}), \qquad h(\vec{x}, y) \triangleleft g(\vec{a}) = h(\vec{x}, y + \vec{a} \cdot \vec{x}),$$

where $\vec{a} \cdot \vec{x}$ is the scalar product of $\vec{a}, \vec{x} \in \mathbb{R}^n$.

Thus the group $K = GH$ is the Heisenberg group, that is, the group of matrices of the form

$$\begin{pmatrix} 1 & \vec{x}^{\,t} & y \\ \vec{0} & 1_n & \vec{a} \\ 0 & \vec{0}^{\,t} & 1 \end{pmatrix}.$$

Consider the corresponding matched pair of the Abelian Lie algebras $(\mathfrak{g}, \mathfrak{h})$. The Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ consist of the matrices

$$\begin{pmatrix} 0 & \vec{0}^{\,t} & 0 \\ \vec{0} & 0_n & \vec{a} \\ 0 & \vec{0}^{\,t} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \vec{x}^{\,t} & y \\ \vec{0} & 0_n & \vec{0} \\ 0 & \vec{0}^{\,t} & 0 \end{pmatrix},$$

respectively. Let $A_j \in \mathfrak{g}$, $j = 1, \ldots, n$, denote the matrix with $\vec{a}$ having 1 at the jth place and the rest 0. The matrices X_j and Y are defined similarly. Then A_j, X_j, Y form a basis in the Lie algebra $\mathfrak{k}$ of the Lie group K. The mutual actions are given by

$$X_j \triangleleft A_k = \delta_{jk} Y, \qquad Y \triangleleft A_j = X_j \triangleright A_k = Y \triangleright A_k = 0,$$

where δ_{jk} denotes Kronecker's symbol.

Using Propositions 1, 2 in [5] we find that the functionals F^1_{jkl} and F^2_{jkl}, defined on the basis of the Lie algebra $\mathfrak{k}$ by

$$F^1_{jkl}(X_j, X_k, A_l) = 1, \quad 1 \le j < k \le n,\ l = 1, \ldots, n, \tag{19}$$

and zero on other basis elements, and

$$F^2_{jkl}(X_j, A_k, A_l) = 1, \quad j = 1, \ldots, n,\ 1 \le k < l \le n,\ j \ne k,\ j \ne l, \tag{20}$$

and zero otherwise, make a basis in the space of equivalence classes of 3-cocycles for the matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$. Thus the dimension of the corresponding cohomology group of the matched pair of the Lie algebras is $n(n-1)^2$.

The left-invariant 3-forms on K that correspond to F^1_{jkl} and F^2_{jkl} are

$$\omega^1_{jkl} = dx^j \wedge dx^k \wedge da^l, \qquad \omega^2_{jkl} = dx^j \wedge da^k \wedge da^l,$$

where x^j denotes the j-th coordinates of $\vec{x} \in \mathbb{R}^n$.

Using (16), (17) and (18), we obtain the corresponding pairs of cocycles $\tilde{u}$, $\tilde{v}$ for the matched pair (G, H),

$$\begin{aligned} &\tilde{u}^1_{jkl}\big(h(\vec{x}, y), g(\vec{a}_1), g(\vec{a}_2)\big) = 0, \\ &\tilde{v}^1_{jkl}\big(h(\vec{x}_1, y_1), h(\vec{x}_2, y_2), g(\vec{a})\big) = a^l \begin{vmatrix} x^j_1 & x^j_2 \\ x^k_1 & x^k_2 \end{vmatrix}, \end{aligned} \tag{21}$$

where $j < k$, $l = 1, \ldots, n$, and

$$\begin{aligned} &\tilde{u}^2_{jkl}\big(h(\vec{x}, y), g(\vec{a}_1), g(\vec{a}_2)\big) = x^j \begin{vmatrix} a^k_1 & a^k_2 \\ a^l_1 & a^l_2 \end{vmatrix}, \\ &\tilde{v}^2_{jkl}\big(h(\vec{x}_1, y_1), h(\vec{x}_2, y_2), g(\vec{a})\big) = 0, \end{aligned} \tag{22}$$

for $j < k$, $j \neq k$, $j \neq l$.

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