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On wildness of idempotent generated algebras associated with extended Dynkin diagrams

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Abstract. Let Λ denote an extended Dynkin diagram with vertex set $\Lambda_0 = \{0, 1, \ldots, n\}$. For a vertex i, denote by $S(i)$ the set of vertices j such that there is an edge joining i and j; one assumes the diagram has a unique vertex p, say $p = 0$, with $|S(p)| = 3$. Further, denote by $\Lambda \setminus 0$ the full subgraph of Λ with vertex set $\Lambda_0 \setminus \{0\}$. Let $\Delta = (\delta_i \,|\, i \in \Lambda_0) \in \mathbb{Z}^{|\Lambda_0|}$ be an imaginary root of Λ, and let k be a field of arbitrary characteristic (with unit element 1). We prove that if Λ is an extended Dynkin diagram of type $\tilde{D}_4$, $\tilde{E}_6$ or $\tilde{E}_7$, then the k-algebra $\mathcal{Q}_k(\Lambda, \Delta)$ with generators e_i, $i \in \Lambda_0 \setminus \{0\}$, and relations $e_i^2 = e_i$, $e_i e_j = 0$ if i and $j \neq i$ belong to the same connected component of $\Lambda \setminus 0$, and $\sum_{i=1}^{n} \delta_i \, e_i = \delta_0 1$ has wild representation type.

1. Formulation of the main result

Throughout the paper, we keep the right-side notation. By k we will denote a fixed field of arbitrary characteristic; for a natural number n and $1 \in k$, we identify $n1$ with n.

Let Λ be an nonoriented graph without loops and multiple edges, and let i be a vertex of Λ. Denote by $S(i)$ the set of vertices j such that there is an edge joining i and j. The vertex i is said to be outer if $|S(i)| \leq 1$, inner if $|S(i)| > 1$, weakly inner if $|S(i)| = 2$ and strongly inner if $|S(i)| > 2$.

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Now let Λ be a finite connected tree with vertex set $\Lambda_0 = \{0, 1, \dots, n\}$. We assume that 0 is the unique strongly inner vertex, and denote by $\Lambda \setminus 0$ the full subgraph of Λ with vertex set $\Lambda_0 \setminus \{0\}$. Given a vector $P = (p_0, p_1 \dots, p_n) \in \mathbb{Z}^{1+n}$, we denote by $\mathcal{Q}_k(\Lambda, P)$ the k-algebra with generators e_i, $1 \le i \le n$, and relations

1) $e_i^2 = e_i$ $(1 \le i \le n)$;

2) $e_i e_j = 0$ if i and $j \ne i$ belong to the same connected component of $\Lambda \setminus 0$;

3) $\sum_{i=1}^n p_i\, e_i = p_0$.

In this paper we study finite-dimensional representations of the algebra $\mathcal{Q}_k(\Lambda, P)$ with Λ being an extended Dynkin diagram. What we consider here is concerned with Yu. S. Samoilenko's investigations [1].

Before we formulate the main results of this paper, we recall some definitions.

Let Λ and Γ be algebras over a field k. A matrix representation of Λ over Γ is a homomorphism $\varphi : \Lambda \to \Gamma^{s \times s}$ of algebras, where s is a natural number and $\Gamma^{s\times s}$ the set of all $s \times s$-matrices with entries in Γ) s is called degree of φ and is denoted by $\deg \varphi$. Two representations φ and ψ of Λ over Γ are called equivalent if $\deg \varphi = \deg \psi$ and there exists an invertible matrix α, with entries in Γ, such that $\varphi(\lambda)\alpha = \alpha\psi(\lambda)$ for every $\lambda \in \Lambda$. The indecomposability and direct sum of representations are defined in a natural way.

Let Λ be a k-algebra, and $\Sigma = k\langle x, y\rangle$ be the free associative k-algebra in two noncommuting variables x and y. A representation γ of Λ over Σ is said to be strict if it satisfies the following conditions:

1) the representation $\gamma \otimes \varphi$ of Λ over k is indecomposable if a representation φ of Σ over k is indecomposable;

2) the representations $\gamma \otimes \varphi$ and $\gamma \otimes \varphi'$ of Λ over k are nonequivalent if representations φ and φ' of Σ over k are nonequivalent.

Following [2] a k-algebra Λ is called wild (or of wild representation type) if it has a strict representation over Σ.

Note that the matrix $(\gamma \otimes \varphi)(\lambda)$ is obtained from the matrix $\gamma(\lambda)$ by change x and y, respectively, on the matrices $\varphi(x)$ and $\varphi(y)$ (and $a \in k$ on the scalar matrix aE_s, where E_s is the identity matrix of dimension $s = \deg \varphi$).

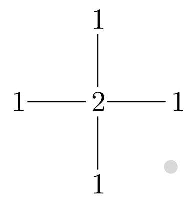
We now formulate the main result of the paper.

Theorem. *Let Λ be an extended Dynkin diagram of type $\tilde{D}_4$, $\tilde{E}_6$ or $\tilde{E}_7$ and $\Delta \in \mathbb{Z}^{|\Lambda_0|}$ an imaginary root of Λ. Then the algebra $\mathcal{Q}_k(\Lambda, \Delta)$ is wild.*

In proving the theorem we can obviously take Δ to be minimal positive, which we denote by Δ_0.

2. Proof of the theorem for $\Lambda = \tilde{D}_4$

In this case the diagram Λ and vector Δ_0 are



By the convention indicated above 0 denotes the strongly inner vertex, and $1, 2, 3$ and 4 the outer vertices. Then the algebra $\mathcal{Q}_k(\Lambda, \Delta_0)$, with generators e_1, e_2, e_3, e_4, has the relations

$1'$) $e_i^2 = e_i$ $(1 \leq i \leq 4)$;

$2'$) $e_1 + e_2 + e_3 + e_4 = 2$.

Consider the following representation γ of $\mathcal{Q}_k(\Lambda, \Delta_0)$ over $\Sigma = k < x, y >$:

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_2) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

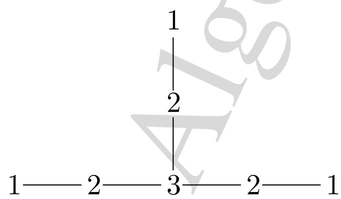
$$\gamma(e_3) = \begin{pmatrix} 1 & 0 & x & y & x^2-x+y & xy-y & 1 \\ 0 & 1 & 1 & 0 & x-1 & y & 0 \\ 0 & 0 & 0 & 0 & -x+1 & -y & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_4) = \begin{pmatrix} 0 & 0 & -x+1 & -y & -x^2+x-y & -xy+y & -1 \\ 0 & 0 & -1 & 1 & -x+1 & -y & 0 \\ 0 & 0 & 1 & 0 & x & y & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In [3] the author has proved that this representation is strict.

3. Proof of the theorem for $\Lambda = \tilde{E}_6$

In this case the diagram Λ and vector Δ_0 are



We assume that the vertices $1, 3, 5$ are outer, the vertices $2, 4, 6$ are weakly inner (the vertex 0 is strongly inner), and the edges join the vertices 1 and 2, 3 and 4, 5 and 6, and consequently 0 with $2, 4, 6$. Then the algebra $\mathcal{Q}_k(\Lambda, \Delta_0)$, with generators $e_1, e_2, \ldots, e_6$, has the relations

1′) $e_i^2 = e_i$ $(1 \le i \le 6)$;

2′) $e_1e_2 = e_2e_1 = 0$, $e_3e_4 = e_4e_3 = 0$, $e_5e_6 = e_6e_5 = 0$;

3′) $e_1 + e_3 + e_5 + 2(e_2 + e_4 + e_6) = 3$.

Consider the following representation γ of $\mathcal{Q}_k(\Lambda, \Delta_0)$ over $\Sigma = k < x, y >$:

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma(e_2) = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x & -y & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_4) = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & x & y & -x & -y \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -x & -y & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_6) = \begin{pmatrix} 1 & 0 & 1 & 1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & x-1 & y \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We will prove that the representation γ is strict.

Let φ and φ' be representations of Σ over k having the same degree: $\deg\varphi = \deg\varphi' = d$. The representation $\gamma\otimes\varphi$ (respectively, $\gamma\otimes\varphi'$) is uniquely defined by the matrices $A_s = (\gamma\otimes\varphi)(e_s)$ (respectively, $A'_s = (\gamma\otimes\varphi')(e_s)$), where $s = 1, 2, \ldots, 6$. It is natural to consider these matrices as block matrices with blocks $(A_s)_{ij}$ and $(A'_s)_{ij}$ of degree d ($i, j = 1, 2, \ldots, 7$). Then $\mathrm{Hom}(\gamma\otimes\varphi, \gamma\otimes\varphi') = \{T \in k^{7d\times 7d} \mid A_sT = TA'_s$ for each $s = 1, 2, \ldots, 6\}$.

Lemma 1. *Let $T = (T_{ij})$, $i, j = 1, 2, \ldots, 7$, be a block matrix (over k) with blocks T_{ij} of degree d, belonging to* $\mathrm{Hom}(\gamma\otimes\varphi, \gamma\otimes\varphi')$. *Then $T_{ij} = 0$ if $i \neq j$ and $(i, j) \neq (1, 6), (1, 7)$, and $T_{11} = T_{22} = \ldots = T_{77}$.*

Proof. Denote by $\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}$ the matrix equalities $A_1T = TA'_1$, $A_2T = TA'_2, \ldots, A_6T = TA'_6$, respectively. The (matrix) equality $(A_sT)_{ij} = (TA'_s)_{ij}$, $i, j \in \{1, 2, \ldots, 7\}$, induced by an equality $A_sT = TA'_s$, is denoted by $\mathrm{I}(i, j)$ for $s = 1$, $\mathrm{II}(i, j)$ for $s = 2$, …, $\mathrm{VI}(i, j)$ for $s = 6$.

It is easy to see that $\mathrm{I}(2,1)$ implies $T_{21} = 0$; $\mathrm{I}(3,1)$ implies $T_{31} = 0$; $\mathrm{I}(6,4)$ implies $T_{64} = 0$; $\mathrm{I}(6,5)$ implies $T_{65} = 0$; $\mathrm{I}(7,4)$ implies $T_{74} = 0$; $\mathrm{I}(7,5)$ implies $T_{75} = 0$; $\mathrm{II}(2,4)$ implies $T_{24} = 0$; $\mathrm{II}(2,5)$ implies $T_{25} = 0$; $\mathrm{II}(2,6)$ implies $T_{26} = 0$; $\mathrm{II}(2,7)$ implies $T_{27} = 0$; $\mathrm{II}(3,4)$ implies $T_{34} = 0$; $\mathrm{II}(3,5)$ implies $T_{35} = 0$; $\mathrm{II}(3,6)$ implies $T_{36} = 0$; $\mathrm{II}(3,7)$ implies $T_{37} = 0$; $\mathrm{III}(1,2)$ implies $T_{12} = 0$; $\mathrm{III}(1,3)$ implies $T_{13} = 0$; $\mathrm{III}(4,2)$ implies $T_{42} = 0$; $\mathrm{III}(4,3)$ implies $T_{43} = 0$; $\mathrm{III}(5,2)$ implies $T_{52} = 0$; $\mathrm{III}(5,3)$ implies $T_{53} = 0$; $\mathrm{III}(6,2)$ implies $T_{62} = 0$; $\mathrm{III}(6,3)$ implies $T_{63} = 0$; $\mathrm{III}(7,2)$ implies $T_{72} = 0$; $\mathrm{III}(7,3)$ implies $T_{73} = 0$; $\mathrm{V}(4,6)$ implies $T_{46} = 0$; $\mathrm{V}(4,7)$ implies $T_{47} = 0$; $\mathrm{V}(5,6)$ implies $T_{56} = 0$; $\mathrm{V}(5,7)$ implies $T_{57} = 0$; $\mathrm{I}(1,4)$ and $T_{34} = 0$ imply $T_{14} = 0$; $\mathrm{I}(1,5)$ and $T_{35} = 0$ imply $T_{15} = 0$; $\mathrm{IV}(6,4)$, $T_{62} = 0$, $T_{63} = 0$ and $T_{64} = 0$ imply $T_{61} = 0$; $\mathrm{IV}(7,4)$, $T_{72} = 0$, $T_{73} = 0$ and $T_{74} = 0$ imply $T_{71} = 0$; $\mathrm{IV}(4,1)$ and $T_{61} = 0$ imply $T_{41} = 0$; $\mathrm{IV}(5,1)$ and $T_{71} = 0$ imply $T_{51} = 0$; $\mathrm{VI}(1,2)$, $T_{12} = 0$, $T_{42} = 0$, $T_{52} = 0$, $T_{62} = 0$ and $T_{72} = 0$ imply $T_{32} = 0$; $\mathrm{V}(3,5)$, $T_{31} = 0$, $T_{32} = 0$ and $T_{35} = 0$ imply $T_{45} = 0$; $\mathrm{IV}(3,7)$, $T_{31} = 0$, $T_{32} = 0$, $T_{35} = 0$ and $T_{47} = 0$ imply $T_{67} = 0$; $\mathrm{IV}(1,4)$, $\mathrm{VI}(1,4)$, $T_{12} = 0$, $T_{13} = 0$, $T_{14} = 0$, $T_{34} = 0$, $T_{64} = 0$ and $T_{74} = 0$ imply $T_{54} = 0$; $\mathrm{IV}(5,6)$, $T_{51} = 0$, $T_{52} = 0$, $T_{53} = 0$, $T_{54} = 0$

and $T_{56} = 0$ imply $T_{76} = 0$; IV$(2,5)$,IV$(5,7)$, VI$(2,7)$, $T_{21} = 0$, $T_{25} = 0$, $T_{27} = 0$, $T_{45} = 0$, $T_{51} = 0$, $T_{52} = 0$, $T_{57} = 0$, $T_{65} = 0$, $T_{67} = 0$ and $T_{75} = 0$ imply $T_{23} = 0$.

So $T_{ij} = 0$ when $i \neq j$ and $(i,j) \neq (1,6),(1,7)$. Then it follows from IV$(1,4)$,IV$(1,5)$, IV$(1,6)$, IV$(1,7)$, III$(3,4)$, III$(2,4)$ and VI$(2,6)$ that $T_{11} = T_{22} = \ldots = T_{77}$. □

It follows from the lemma that a matrix $T = (T_{ij})$ belonging to $\mathrm{Hom}(\gamma \otimes \varphi, \gamma \otimes \varphi')$ satisfies the following conditions:

a) T is invertible if and only if $T_0 = T_{11} = T_{22} = \ldots = T_{77}$ is invertible;

b) $\varphi(x)T_0 = T_0\varphi'(x)$ and $\varphi(y)T_0 = T_0\varphi'(y)$.

(In fact it follows from the lemma that the equalities I-VI are equivalent to the equalities *b*)).

Therefore the representation γ satisfies the condition 2) (of the definition of a strict representation).

It remains to prove that γ satisfies the condition 1) or, in other words, φ is decomposable if so is $\gamma \otimes \varphi$. We will denote by 0_s and E_s the $s \times s$ zero and identity matrices, respectively.

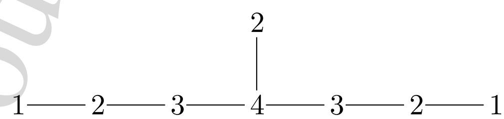
Denote by $\mathrm{Hom}(\varphi, \varphi)$ the algebra of endomorphisms of φ, i.e.

$$\mathrm{Hom}(\varphi,\varphi) = \{S \in k^{d\times d} \mid \varphi(x)S = S\varphi(x), \varphi(y)S = S\varphi(y)\}.$$

Decomposability of $\gamma\otimes\varphi$ implies that the k-algebra $\mathrm{Hom}(\gamma\otimes\varphi, \gamma\otimes\varphi)$ (of endomorphisms of $\gamma \otimes \varphi$) contains an idempotent $T \neq 0_{7d}, E_{7d}$ (see, for example, [4, ch.V]). Then, by the lemma, the matrix $T_0 = T_{11} = T_{22} = \ldots = T_{77}$ is an idempotent; moreover, $T_0 \neq 0_d, E_d$, because otherwise it would follow from the equality $T^2 = T$ that $T = T_0 \oplus T_0 \oplus \ldots \oplus T_0$, where T_0 occurs 7 times, or in other words $T = 0_{7d}$ or $T = E_{7d}$, respectively. Since T_0 belong to the algebra $\mathrm{Hom}(\varphi,\varphi) = \{S \in k^{d\times d} \mid \varphi(x)S = S\varphi(x), \varphi(y)S = S\varphi(y)\}$ (see the condition b)), the representation φ is decomposable (see again [4, ch.V]).

4. Proof of the theorem for $\Lambda = \tilde{E}_7$

In this case the diagram Λ and vector Δ_0 are



We assume that the vertices $1, 4, 7$ are outer, the vertices $2, 3, 5, 6$ are weakly inner (the vertex 0 is strongly inner), and the edges join the vertices 1 and 2, 2 and 3, 4 and 5, 5 and 6, and consequently 0 with $3, 6, 7$. Then the algebra $\mathcal{Q}_k(\Lambda, \Delta_0)$, with generators $e_1, e_2, \ldots, e_7$, has the relations

1′) $e_i^2 = e_i$ $(1 \leq i \leq 7)$;

2′) $e_1e_2 = e_2e_1 = 0$, $e_2e_3 = e_3e_2 = 0$, $e_1e_3 = e_3e_1 = 0$, $e_4e_5 = e_5e_4 = 0$, $e_5e_6 = e_6e_5 = 0$, $e_4e_6 = e_6e_4 = 0$;

3′) $e_1 + e_4 + 2(e_2 + e_5 + e_7) + 3(e_3 + e_6) = 4$.

Consider the following representation γ of $\mathcal{Q}_k(\Lambda, \Delta_0)$ over $\Sigma = k < x, y >$:

$$\gamma(e_1) = \begin{pmatrix} 0 & 0 & 1 & 0 & -3 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 & 0 & x & y \\ 0 & 0 & 1 & 0 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 3 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_5) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -3 & -3 & -12 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma(e_6) = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 1 & 3 & 3 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & -x-3 & -y \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma(e_7) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We will prove that the representation γ is strict.

Let φ and φ' be representations of Σ over k having the same degree: $\deg\varphi = \deg\varphi' = d$. The representation $\gamma\otimes\varphi$ (respectively, $\gamma\otimes\varphi'$) is uniquely defined by the matrices $A_s = (\gamma\otimes\varphi)(e_s)$ (respectively, $A'_s = (\gamma\otimes\varphi')(e_s)$), where $s = 1, 2, \ldots, 7$. It is natural to consider these matrices as block matrices with blocks $(A_s)_{ij}$ and $(A'_s)_{ij}$ of degree d $(i, j = 1, 2, \ldots, 9)$. Then $\mathrm{Hom}(\gamma\otimes\varphi, \gamma\otimes\varphi') = \{T \in k^{9d\times 9d} \mid A_sT = TA'_s$ for each $s = 1, 2, \ldots, 7\}$.

Lemma 2. *Let $T = (T_{ij})$, $i, j = 1, 2, \ldots, 9$, be a block matrix (over k) with blocks T_{ij} of degree d, belonging to $\mathrm{Hom}(\gamma\otimes\varphi, \gamma\otimes\varphi')$. Then $T_{ij} = 0$ if $i \neq j$ and $(i, j) \neq (1, 8), (1, 9)$, and $T_{11} = T_{22} = \ldots = T_{99}$.*

Proof. Denote by I, II, ..., VII the matrix equalities $A_1T = TA'_1$, $A_2T = TA'_2, \ldots, A_7T = TA'_7$, respectively. The (matrix) equality $(A_sT)_{ij} = (TA'_s)_{ij}$, $i, j \in \{1, 2, \ldots, 9\}$, induced by an equality $A_sT = TA'_s$, is denoted by $\mathrm{I}(i, j)$ for $s = 1$, $\mathrm{II}(i, j)$ for $s = 2$, ..., $\mathrm{VII}(i, j)$ for $s = 7$.

It is easy to see that $\mathrm{VII}(i, j)$ implies $T_{ij} = 0$ for each $(i, j) \in \{1, 4, 5, 8, 9\} \times \{2, 3, 6, 7\}$ and each $(i, j) \in \{2, 3, 6, 7\} \times \{1, 4, 5, 8, 9\}$; $\mathrm{II}(1, 4)$ and $T_{12} = 0$ imply $T_{14} = 0$; $\mathrm{II}(1, 5)$ and $T_{13} = 0$ imply $T_{15} = 0$; $\mathrm{II}(4, 1)$ and $T_{61} = 0$ imply $T_{41} = 0$; $\mathrm{II}(4, 8)$ and $T_{68} = 0$ imply $T_{48} = 0$; $\mathrm{II}(4, 9)$ and $T_{69} = 0$ imply $T_{49} = 0$; $\mathrm{II}(5, 1)$ and $T_{71} = 0$ imply $T_{51} = 0$; $\mathrm{II}(5, 8)$ and $T_{78} = 0$ imply $T_{58} = 0$; $\mathrm{II}(5, 9)$ and $T_{79} = 0$ imply $T_{59} = 0$; $\mathrm{II}(8, 4)$ and $T_{82} = 0$ imply $T_{84} = 0$; $\mathrm{II}(8, 5)$ and $T_{83} = 0$ imply $T_{85} = 0$; $\mathrm{II}(9, 4)$ and $T_{92} = 0$ imply $T_{94} = 0$; $\mathrm{II}(9, 5)$ and $T_{93} = 0$ imply $T_{95} = 0$; $\mathrm{III}(6, 2)$ and $T_{82} = 0$ imply $T_{62} = 0$; $\mathrm{III}(6, 3)$ and $T_{83} = 0$ imply $T_{63} = 0$; $\mathrm{III}(7, 2)$ and $T_{92} = 0$ imply $T_{72} = 0$; $\mathrm{III}(7, 3)$ and $T_{93} = 0$ imply $T_{73} = 0$; $\mathrm{III}(6, 1)$ and $T_{61} = 0$ imply $T_{81} = 0$; $\mathrm{III}(1, 9)$, $T_{13} = 0$, $T_{15} = 0$, $T_{17} = 0$ and $T_{69} = 0$ imply $T_{89} = 0$; $\mathrm{III}(6, 9)$, $T_{63} = 0$, $T_{65} = 0$, $T_{69} = 0$ and $T_{89} = 0$ imply $T_{67} = 0$; $\mathrm{III}(7, 1)$ and $T_{71} = 0$ imply $T_{91} = 0$; $\mathrm{IV}(2, 6)$, $T_{21} = 0$ and $T_{24} = 0$ imply $T_{26} = 0$; $\mathrm{IV}(2, 7)$, $T_{21} = 0$ and $T_{25} = 0$ imply $T_{27} = 0$; $\mathrm{IV}(3, 6)$, $T_{31} = 0$ and $T_{34} = 0$ imply $T_{36} = 0$; $\mathrm{IV}(3, 7)$, $T_{31} = 0$ and $T_{35} = 0$ imply $T_{37} = 0$; $\mathrm{VI}(1, 2)$, $T_{12} = 0$, $T_{52} = 0$, $T_{72} = 0$, $T_{82} = 0$ and $T_{92} = 0$ imply $T_{32} = 0$; $\mathrm{VI}(2, 7)$, $T_{21} = 0$, $T_{27} = 0$, $T_{47} = 0$, $T_{67} = 0$, $T_{87} = 0$ and $T_{97} = 0$ imply $T_{23} = 0$; $\mathrm{VI}(2, 5)$, $T_{21} = 0$, $T_{23} = 0$, $T_{25} = 0$, $T_{65} = 0$, $T_{85} = 0$ and $T_{95} = 0$ imply $T_{45} = 0$; $\mathrm{VI}(1, 4)$, $T_{12} = 0$, $T_{34} = 0$, $T_{74} = 0$, $T_{84} = 0$ and $T_{94} = 0$ imply $T_{54} = 0$; $\mathrm{II}(5, 6)$, $T_{52} = 0$, $T_{54} = 0$ and $T_{56} = 0$ imply $T_{76} = 0$; $\mathrm{III}(5, 8)$, $T_{51} = 0$, $T_{52} = 0$, $T_{54} = 0$, $T_{56} = 0$ and $T_{78} = 0$ imply $T_{98} = 0$.

So $T_{ij} = 0$ when $i \neq j$ and $(i, j) \neq (1, 8), (1, 9)$. Then it follows from $\mathrm{III}(1, 6)$, $\mathrm{III}(1, 8)$, $\mathrm{III}(5, 7)$, $\mathrm{III}(5, 9)$, $\mathrm{VI}(1, 3)$, $\mathrm{VI}(1, 5)$ $\mathrm{VI}(2, 4)$ and $\mathrm{VI}(2, 6)$. that $T_{11} = T_{22} = \ldots = T_{99}$. □

The final part of the proof is analogous to that in the case $\Lambda = \tilde{E}_6$ (see Section 3).

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