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Torsion-free groups with every proper homomorphic image an $\mathbf{N_1}$-group

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Abstract. In this article it is proved that a torsion-free locally nilpotent groups with non-trivial Fitting subgroup and every proper homomorphic image an $\mathbf{N_1}$-group is an $\mathbf{N_1}$-group(and so it is nilpotent).

Introduction

Let G be a group and let N be a normal subgroup of G. The factor group of G/N is said to be a proper factor-group if $N \neq 1$. Let G be a group, then G is called an $\mathbf{N_1}$-group if all subgroups of G are subnormal. $\mathbf{N_1}$-groups are considered by several authors and obtained remarkable results. If G is an torsion-free $\mathbf{N_1}$-group then G is soluble [4, (7)Satz], and nilpotent [7 or 1].

In this article we consider locally nilpotent torsion-free groups with non-trivial Fitting subgroup and every proper homomorphic image an $\mathbf{N_1}$-group. These groups are certain generalizations of $\mathbf{N_1}$-groups.

Given a subgroup H of a group G, the isolator of H in G is the set

$$I_G(G) = \{x \in G : x^n \in H, 1 \leq n \text{ for some natural number}\}$$

If G is a locally nilpotent group and H is subgroup of G then the isolator of H in G is a subgroup of G. If H is nilpotent of class c, where c is a natural number, then so is $I_G(H)$. If K is a normal subgroup of H,

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then $I_G(K)$ is normal in $I_G(H)$[2]. If H is a subnormal subgroup of G, we denote by $d(G:H)$ the defect in G, i.e. the shortest length of a series

$$H = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_d = G.$$

The notations and definitions are standard and can be found in [5] and [6].

Lemma 1. *Let G be a locally nilpotent torsion-free group whose proper homomorphic images are $\mathbf{N_1}$-group. Then the Fitting subgroup of G is abelian.*

Proof. We show that every normal nilpotent subgroup of G is abelian. Let N be a non-trivial normal nilpotent subgroup of G such that N is non-abelian. Thus $I_G(N')$ is non-trivial and $I_G(N') \triangleleft G$. By hypotehsis, $G/I_G(N')$ is an $\mathbf{N_1}$-group and $G/I_G(N')$ is a torsion-free group. $G/I_G(N')$ is nilpotent by [7 or 1]. Therefore G is nilpotent by Lemma 4.3.1 [3]. This is a contradiction. Thus N is an abelian group. □

Theorem. *Let G be a torison-free locally nilpotent group with non-trivial Fitting subgroup. If all proper homomorphic images of G are $\mathbf{N_1}$-group, then G is an $\mathbf{N_1}$-group (and so it is nilpotent).*

Proof. Assume by contradiction that G is not $\mathbf{N_1}$-group, and let A be its Fitting subgroup. Then A is abelian by Lemma 1, so that also its isolator $I_G(A)$ is abelian. In particular, $I_G(A) \leq C_G(A) = A$, so that $I_G(A) = A$ and G/A is torsion-free. Thus G/A is nilpotent. Let zA be a non-trivial element of $Z(G/A)$. The map

$$\theta : a \in A \longrightarrow [a, z]$$

is a G-endomorphism of A, and so $C = ker\theta = C_A(z)$ is normal subgroup of G. On the other hand, if a is any non-trivial element of A, the subgroup $< a, z >$ is nilpotent, so that $A \cap Z(< a, z >) \neq 1$, and hence $C \neq 1$. Moreover, as $[b^n, z] = [b, z]^n$ for all $b \in A$ and $n \in N$, we have that G/C is torsion-free, and so also nilpotent. Let yC be a non-trivial element of $Z(G/C)$. If g is any element of G, then $y^g = yc$ with $c \in C$, and hence

$$\theta(y)^g = \theta(y^g) = \theta(yc) = \theta(y),$$

so that $\theta(y) \in Z(G) = 1$. Thus $y \in C = ker\theta$, and this contradiction proves the theorem. □

References

[1] C. Casolo. *Torsion-free groups in which every subgroup is subnormal*, Rend. Circ. Mat. Palermo(2) -(2001), 321-324.

[2] P. Hall. *The Edmontes notes on nilpotent groups.* Queen Mary College Mathematics Notes, London,(19690).

[3] W. Möhres. *Gruppen deren untergruppen alle subnormal sind.* Würzburg Ph.D. thesis Aus Karlstadt, (1988).

[4] W. Möhres. *Torsionfreie gruppen, deren untergruppen alle subnormal sind.* Math. Ann. (1989), (284), 245-249.

[5] D. J. S. Robinson, *Finiteness condition and generalized soluble groups.* vols. 1 and vols. 2 (Springer-Verlag, 1972).

[6] D. J. S. Robinson, *A course in the theory of groups.* (Springer-Verlag, Heidelberg-Berlin-Newyork 1982.)

[7] H. Smith, *Torsion-free groups with all subgroups subnormal*, Arch. Math.,(2001), (76),1-6.

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