Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **20** (2015). Number 1, pp. 55–68

Quivers of 3×3-exponent matrices

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Dedicated to 60-th anniversary of E. Zelmanov

Abstract. We show how to use generating exponent matrices to study the quivers of exponent matrices. We also describe the admissible quivers of 3×3 exponent matrices.

Introduction

Exponent matrices were introduced in the study of semi-maximal rings (see [10]), as important ingredients of tiled orders. Recall that a *semi-maximal ring* is a semiperfect semiprime right Noetherian ring A such that for any local idempotent $e \in A$ the endomorphism ring eAe is a (non-necessarily commutative) discrete valuation ring, i.e. all principal endomorphism rings of A are discrete valuation rings (see also [3, pp. 349-350]). A square $n \times n$ matrix $A = (\alpha_{ps})$ is called an *exponent matrix* if its diagonal entries are equal to zero and for all possible indices i, j, k, one has

$$\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}. \tag{1}$$

Throughout this paper, unless otherwise stated, n will denote the size of the matrix under consideration. We shall refer to (1) as *ring inequalities*, and this term is explained by the following fact.

Theorem 1 ([10], [3, Th. 14.5.2]). *An arbitrary semi-maximal ring is isomorphic to a direct product of rings of the form*

$$\Lambda = \sum_{i,j=1}^{n} e_{ij}(\pi^{\alpha_{ij}}\mathcal{O}) \subseteq M_n(\mathcal{O}), \tag{2}$$

2010 MSC: 16H99, 16Z05.
Key words and phrases: quiver, tiled Order, exponent matrix.

where $n \geqslant 1$, $\mathcal{O}$ *is a discrete valuation ring with prime element* π, (α_{ij}) *is an exponent matrix,* $e_{ij}(\pi^{\alpha_{ij}}\mathcal{O}) = \{e_{ij}(a), a \in \pi^{\alpha_{ij}}\mathcal{O}\}$ *and* $e_{ij}(a)$ *is the* $n \times n$*-matrix whose unique non-zero entry* a *is placed in the* (i,j)*-position.*

The ring $\mathcal{O}$ can be embedded into classical division ring $\mathcal{D}$ and (2) is the set of all matrices $(\alpha_{ij}) \in M_n(\mathcal{D})$ such that

$$\alpha_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}\Lambda e_{jj},$$

where $e_{11}, \ldots, e_{nn}$ are matrix units in $M_n(\mathcal{D})$.

Clearly, $Q = M_n(\mathcal{D})$ is a classical division ring of Λ. Obviously, Λ is left and right Noetherian.

We recall next some additional definitions and facts.

Definition 1. A module M is called distributive, if so is its lattice of submodules, i.e.

$$K \cup (L + N) = L \cup L + K \cup N$$

for all submodules K, L and N.

Clearly, that every submodule of a distributive lattice is also distributive.

A direct sum of distributive modules is called a semidistributive module. A ring A is called right (left) semidistributive, if it is semidistributive as a right (left) module over itself. We say that a ring is semidistributive if it is right and left semidistributive (see. [8]).

Theorem 2 ([7])**.** *The following conditions are equivalent for a semiprime right Noetherian ring* A*:*

1) A *is semidistributive;*
2) A *is a direct product of a semiprime Artinian ring and a semimaximal ring.*

A tiled order Λ over a discrete valuation ring $\mathcal{O}$ is a Noetherian prime semiperfect semidistributive ring with zero Jacobson radical. In this case, $\mathcal{O} = e\Lambda e$ where $e \in \Lambda$ is a primitive idempotent. We shall write

$$\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\},$$

where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of Λ, i.e. Λ is of the form (2).

A tiled order is called reduced, if $\Lambda/R(\Lambda)$ is a direct product of division rings. In this case $\alpha_{ij}+\alpha_{ji}>0$ for all $i\neq j$. Exponent matrices with this property are called reduced.

Denote by $\mathcal{M}(\Lambda)$ the partially ordered set (with respect to inclusion) of all projective right Λ-modules, which are contained in some fixed Q-module W. All simple Q-modules are isomorphic to each other, whence we may take any of them. Notice, that the partially ordered sets $\mathcal{M}_l(R)$ and $\mathcal{M}_r(R)$, which correspond to left and right modules are anti-isomorphic.

The set $\mathcal{M}(\Lambda)$ is completely determined by $\mathcal{E}(\Lambda)=(\alpha_{ij})$. More precisely, if Λ is a reduced, then

$$\mathcal{M}(\Lambda)=\{P_i^z : i=1,\ldots,n,\, z\in\mathbb{Z}\},$$

where

$$P_i^z \leqslant P_j^{z'} \Longleftrightarrow \begin{cases} z-z'\geqslant \alpha_{ij}, & \text{if} \quad \mathcal{M}(\Lambda)=\mathcal{M}_l(\Lambda)\\ z-z'\geqslant \alpha_{jj}, & \text{if} \quad \mathcal{M}(\Lambda)=\mathcal{M}_r(\Lambda).\end{cases}$$

Evidently, $\mathcal{M}(\Lambda)$ is an infinite periodical set.

Let Λ and Γ be tiled orders over discrete valuated rings $\mathcal{O}$ and Δ.

Definition 2 ([10])**.** An isomorphism $\varphi:\mathcal{M}(\Lambda)\simeq\mathcal{M}(\Gamma)$ is called coordinated, if

$$B\simeq C \Longleftrightarrow \varphi(B)\simeq\varphi(C)$$

for all B, $C\in\mathcal{M}(\Lambda)$.

Theorem 3 ([10, Prop. 2.9])**.** *The tiled orders Λ and Γ are Morita equivalent if and only if the following hold:*

1) *The discrete valuated rings $\mathcal{O}$ and Δ are isomorphic;*
2) *There is coordinated isomorphism between the partially ordered sets $\mathcal{M}(\Lambda)$ and $\mathcal{M}(\Gamma)$.*

Let I be a two sided ideal of the tiled order Λ. Evidently,

$$I=\sum_{i,j} e_{ij}\pi^{\beta_{ij}}\mathcal{O},$$

where e_{ij} are matrix units. Denote by $\mathcal{E}(I)=(\beta_{ij})$ the exponent matrix of the ideal I.

For twosided ideals I and J with exponent matrices $\mathcal{E}(I)=(\beta_{ij})$ and $\mathcal{E}(J)=(\gamma_{ij})$ we have $\mathcal{E}(IJ)=(\delta_{ij})$, where $\delta_{ij}=\min\limits_k(\beta_{ik}+\gamma_{kj})$.

Assume that Λ is reduced and write $\mathcal{E}(\Lambda) = (\alpha_{ij})$. Then the exponent matrix $\mathcal{E}(R) = (\beta_{ij})$ of the Jacobson radical R of Λ can be found as follows: $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for all i.

Let Q be the quiver of the reduced tiled order Λ and let $[Q(\Lambda)]$ be its adjacency matrix. By [3, Theor. 14.6.2], $[Q(\Lambda)]$ is a (0, 1)-matrix, more precisely, $[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R)$.

For the $n \times n$-exponent matrix $\mathcal{E} = (\alpha_{ij})$ define the following matrices:

$$\mathcal{E}^{(1)} = (\beta_{ij}) = \mathcal{E} + E,$$

where E is the identity matrix.

$$\mathcal{E}^{(2)} = (\gamma_{ij}),\ \gamma_{ij} = \min_k(\beta_{ik} + \beta_{kj}). \tag{3}$$

Evidently, $[Q(\Lambda)] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$.

Theorem 4 ([5]). *The matrix $[Q(\Lambda)]$ is the adjacency matrix of a strongly connected simply laced quiver.*

Definition 3. A quiver is called admissible, if it is the quiver of some exponent matrix.

Theorem 5 ([6]). *Let Q be a strongly connected quiver which has a loop at each vertex. Then Q is admissible.*

Theorem 6 ([1, Teor. 5.3]). *For every natural m, ($1 \leqslant m \leqslant n$, $m \neq n-1$), there exists an admissible quiver with n vertices and exactly m loops.*

Theorem 7 ([1]). *Let Q be a strongly connected quiver with n vertices which has exactly $n-1$ loop. Then Q is not admissible.*

Definition 4. Two exponent matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are called equivalent if they can be obtained from each other by transformations of the following two types:

(1) subtraction of an integer from the i-th row with simultaneous addition of the same integer to the i-th column;

(2) simultaneous interchanging of two rows and of the equally numbered columns.

Proposition 1 ([1]). *Suppose, that $\mathcal{E}$ and Θ are exponent matrices and Θ can be obtained from $\mathcal{E}$ by transformations of type (1). Then $Q(\mathcal{E}) = Q(\Theta)$.*

For an $n \times n$-matrix A and a permutation σ of $\{1, \ldots, n\}$ denote by $\sigma \circ A$ the matrix, which is obtained from A by simultaneous permutation of rows and columns, defined by σ.

Proposition 2 ([1]). *Let τ be an arbitrary permutation of $\{1, \ldots, n\}$. Suppose that $\mathcal{E}$ and Θ are exponent matrices such that Θ can be obtained applying τ to the rows and columns of $\mathcal{E}$. Then $[Q(\Theta)] = \tau \circ [Q(\mathcal{E})]$.*

Since any permutation is a product of transpositions, the above fact explains how does an adjacency matrix changes under transformations of the second type.

1. Generating exponent matrices in the study of quivers of exponent matrices

A non-negative exponent matrix is called **generating**, if it can not be represented as a sum of non-negative non-zero exponent matrices. Denote by $\mathcal{G}_n$ the set of all generating $n \times n$ exponent matrices. By [9] cardinality of $\mathcal{G}_n$ is finite.

For a quiver Q denote by Q^* the quiver, which is obtained from Q by deleating all loops.

Lemma 1. *Let $A_1, \ldots, A_s$ be exponent matrices and Q be the quiver of $A = \sum\limits_{t=1}^{s} \alpha_t A_t$, where all α_s are positive integers, such that A is reduced.*

1) *Let $\widetilde{\alpha}_t = \min\{2, \alpha_s\}$ for all s. Then Q is also the quiver of $\widetilde{A} = \sum\limits_{t=1}^{s} \widetilde{\alpha}_t A_t$.*
2) *Let $\alpha_t^* = \min\{1, \alpha_s\}$ for all s. Then Q^* coincides with $(Q(A^*))^*$, where $A^* = \sum\limits_{t=1}^{s} \alpha_t^* A_t$.*

Proof. Write $A = (\alpha_{pq})$, $A + E = B = (\beta_{pq})$ and $C = (\gamma_{pq})$, where

$$\gamma_{ij} = \min_k \{\beta_{ik} + \beta_{kj}\} - \beta_{ij}.$$

Write also $\beta_{ijk} = \beta_{ik} + \beta_{kj} - \beta_{ij}$ and $\alpha_{ijk} = \alpha_{ik} + \alpha_{kj} - \alpha_{ij}$.

Notice, that if $k = i$, or $k = j$, then $\beta_{ijk} = 1$. Indeed, if $i \neq j$, then $\beta_{iji} = (\alpha_{ii} + 1) + \alpha_{ij} - \alpha_{ij} = 1$ and $\beta_{ijj} = (\alpha_{ij} + 1) + \alpha_{jj} - \alpha_{ij} = 1$. Also if $i = j$, then $\beta_{iii} = (\alpha_{ii} + 1) + (\alpha_{ii} + 1) - (\alpha_{ii} + 1) = 1$.

We conclude that $\gamma_{ij} = \min\{1, \min\limits_{k \notin \{i, j\}} \beta_{ijk}\}$.

Notice, that C is the adjacency matrix of the quiver of A. We do next some transformations of formulas for entries of C, which will prove the Lemma.

For $i \neq j$ we can simplify γ_{ij} as follows

$$\begin{aligned}\gamma_{ij} &= \min\{1, \min_{k\notin\{i,j\}}(\beta_{ik}+\beta_{kj}-\beta_{ij})\}\\ &= \min\{1, \min_{k\notin\{i,j\}}(\alpha_{ik}+\alpha_{kj}-\alpha_{ij})\}\\ &= \min\{1, \min_{k\notin\{i,j\}}\alpha_{ijk}\}\\ &= \min\{1, \min_{k\notin\{i,j\}}\sum_{t=1}^{s}\alpha_t\alpha_{ijk}^t\}.\end{aligned}$$

Since $\alpha_{ijk}^t \geqslant 0$ for all i, j, k, t, the conditions $\sum\limits_{t=1}^{s}\alpha_t\alpha_{ijk}^t = 0$ and $\sum\limits_{t=1}^{s}\alpha_t^*\alpha_{ijk}^t = 0$ are equivalent. This proves the first part of Lemma.

For $i = j$ the formulas for γ_{ij} can be transformed as follows

$$\begin{aligned}\gamma_{ij} &= \min\{1, \min_{k\notin\{i,j\}}(\beta_{ik}+\beta_{ki}-1)\}\\ &= \min\{1, \min_{k\notin\{i,j\}}(\alpha_{ijk}-1)\}\\ &= \min\{1, \min_{k\notin\{i,j\}}\sum_{t=1}^{s}\alpha_t\alpha_{ijk}^t-1\}.\end{aligned}$$

Since A is reduced, then $\sum\limits_{t=1}^{s}\alpha_t\alpha_{ijk}^t \geqslant 1$. Nevertheless, the conditions $\sum\limits_{t=1}^{s}\alpha_t\alpha_{ijk}^t = 1$ and $\sum\limits_{t=1}^{s}\widetilde{\alpha}_t\alpha_{ijk}^t = 1$ are equivalent. This proves the second part of Lemma. □

The following two theorems follow from Lemma 1.

Theorem 8. *Let Q be an admissible quiver with n vertices and let $\mathcal{G}_n = \{A_1, \ldots, A_s\}$. Then there exist $\alpha_i \in \{0, 1, 2\}$, $1 \leqslant i \leqslant s$, such that Q is the quiver of $\sum\limits_{i=1}^{s}\alpha_i A_i$.*

Theorem 9. *Let Q be an admissible quiver with n vertices, which has no loops and let $\mathcal{G}_n = \{A_1, \ldots, A_s\}$. Then there exist $\alpha_i \in \{0, 1\}$, $1 \leqslant i \leqslant s$, such that Q is the quiver of $\sum\limits_{i=1}^{s}\alpha_i A_i$.*

2. Quivers of reduced exponent 3 × 3-matrices

The main result of this article is the following theorem.

Theorem 10. *The following* 10 *matrices are the adjacency matrices of the quivers of all* 3×3*-reduced exponent matrices, up to isomorphism of quivers:*

1) *The quivers with a loop at each vertex*

$$N_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, N_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, N_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$N_4 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, N_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

2) *The quivers without loops:*

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

3) *The quivers with exactly one loop:*

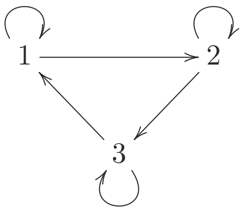
$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Theorem will be proved in Section 3.

Remark 1. Notice, that the quivers with adjacency matrices $N_1, \dots, N_5$ form the complete list of the strongly connected simply laced quivers on 3 vertices up to isomorphism.

Proof of the Remark. Consider an arbitrary strongly connected quiver Q with 3 vertices which has a loop at each vertex.

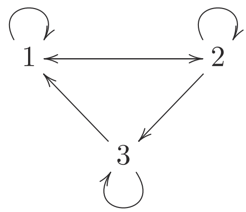
Assume, that Q has exactly 3 arrows, which are not loops. In this case Q is isomorphic to



(4)

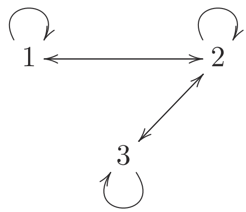
and $[Q] = N_5$.

If Q has more then 3 arrows, which are not loops, then there are vertices i and j, with arrows $i \to j$, $j \to i$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Assume, that Q has exactly 4 arrows, which are not loops. In this case Q is isomorphic to either



(5)

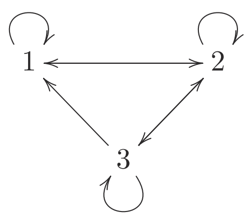
or



(6)

The quiver in (5) is isomorphic to the quiver with adjacency matrix N_2 and the quiver in (6) is isomorphic to one with adjacency matrix N_4.

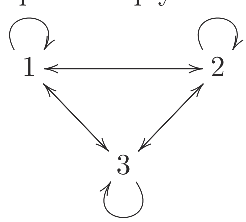
If Q has exactly 5 arrows, which are not loops, then there is a vertex, say 2, such that there is an arrow from 2 to all other vertices and there are arrows from all other vertices to 2. Without loss of generality we may assume that the last, 5-th arrow, goes from 3 to 1, whence, the quiver is as follows



(7)

This quiver is isomorphic to one with adjacency matrix N_3.

The last case is the complete simply laced quiver



(8)

Its adjacency matrix is N_1. □

Corollary 1. *There are* 5 *pairwise non-isomorphic strongly connected simply laced quivers with 3 vertices, which have no loops.*

Proof. There is a natural one to one correspondence between quivers with no loops and ones which have a loop at each the vertex. Now, the corollary follows from Remark 1. □

We shall say that two vertices i and j of a quiver Q are *similar* if the renumeration $i \to j$, $j \to i$ of the vertices of Q gives an isomorphic quiver.

Lemma 2. *There are exactly 10 pairwise non-isomorphic strongly connected simply laced quivers with 3 vertices, which have exactly* 1 *loop.*

Proof. Let Q be a quiver such as in the statement of Lemma. If we add two new loops, it will become isomorphic to one of (4),..., (8), mentioned in the proof of Remark 1.

If $\widetilde{Q}$ is of the form (4), then there is a unique possibility for Q (which we denote by Q_1), because all vertices of Q are pairwise similar.

If $\widetilde{Q}$ is of the form (5), then the three vertices are pairwise non-similar and there are three possibilities for Q (denote them Q_2, Q_3 and Q_4).

If $\widetilde{Q}$ is of the form (6), then vertices 1 and 3 are similar and 2 is not similar to them. Whence, there are two possibilities Q_5 and Q_6 for Q.

If $\widetilde{Q}$ is of the form (7), then all vertices of the quiver are pairwise non-similar and there are three possibilities Q_7, Q_8 and Q_9 for Q.

If $\widetilde{Q}$ is of the form (8), then all vertices of Q are pairwise similar, whence, there is a unique possibility, which we denote by Q_{10}. □

Corollary 2. *There are exactly 10 pairwise non-isomorphic strongly connected simply laced quivers with 3 vertices, which have exactly* 2 *loops.*

Corollary 3. *There are exactly 30 pairwise non-isomorphic strongly connected simply laced quivers with 3 vertices.*

3. Calculation of the admissible quivers with 3 vertices

From Proposition 1 it follows that for any admissible quiver there is a reduced exponent matrix, whose first row is zero. It immediately follows from the definition of the exponent matrix, that if one of its rows is zero, then all other entries are non-negative. The additive semigroup of the non-negative 3×3-exponent matrices was studied at [2]. This semigroup is finitely generated and the unique set of its generators is as follows.

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$A_4 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix};$$

$$A_7 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad A_8 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad A_9 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$A_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad A_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}; \quad A_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Whence, the semigroup of the non-negative 3×3 exponent matrices, whose first line is zero, is generated by $A_1, A_5, A_6, A_9, A_{12}$.

Notice, that the matrices A_1, A_5 and A_6 are not reduced, but A_9 and A_{12} are. It follows that the semigroup of reduced exponent matrices with first zero line is not finitely generated. For example, for any $x_1 > 0$ the exponent matrix $x_1 A_1 + A_5$ is reduced, but it can not be represented as a sum of other non-negative reduced exponent matrices.

Nevertheless, notice, that if at least two of non-negative numbers from $\{x_1, x_5, x_6, x_9, x_{12}\}$ are greater then 0, then

$$\mathcal{E} = x_1 A_1 + x_5 A_5 + x_6 A_6 + x_9 A_9 + x_{12} A_{12} \tag{9}$$

is a reduced exponent matrix.

The matrix $\mathcal{E}$, defined by (9), can be written as

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ x_1 + x_5 + x_9 + x_{12} & 0 & x_5 + x_9 \\ x_1 + x_6 + x_9 + x_{12} & x_6 + x_{12} & 0 \end{pmatrix}.$$

We are going to find all possible quivers $Q(\mathcal{E})$ depending on the values of x_i. We will find the quivers up to their equivalence classes, because if some quiver is admissible, the all those, which are equivalent to it, are also admissible.

Notice, that the matrices A_9 and A_{12} are type (2) equivalent. By Proposition 2, without loss of generality we may assume, that $x_9 \leqslant x_{12}$.

This means, that $x_{12} = x_9 + x_{13}$ for some non-negative x_{13}, whence the formulas for the entries of $\mathcal{E}^{(2)} = (\gamma_{ij})$ will be as follows:

$$\begin{aligned}
\gamma_{11} &= \min\{2,\, x_1 + x_5 + 2x_9 + x_{13},\, x_1 + x_6 + 2x_9 + x_{13}\};\\
\gamma_{12} &= \min\{1,\, x_6 + x_9 + x_{13}\};\\
\gamma_{13} &= \min\{1,\, x_5 + x_9\};\\
\gamma_{21} &= \alpha_{21} + \min\{1,\, x_6 + x_9\};\\
\gamma_{22} &= \min\{2,\, x_1 + x_5 + 2x_9 + x_{13},\, x_5 + x_6 + 2x_9 + x_{13}\};\\
\gamma_{23} &= \alpha_{23} + \min\{x_1 + x_9 + x_{13},\, 1\};\\
\gamma_{31} &= \alpha_{31} + \min\{1,\, x_5 + x_9 + x_{13}\};\\
\gamma_{32} &= \alpha_{32} + \min\{1,\, x_1 + x_9\};\\
\gamma_{33} &= \min\{x_1 + x_6 + 2x_9 + x_{13},\, x_5 + x_6 + 2x_9 + x_{13},\, 2\};
\end{aligned}$$

For each of the variables, which appear in the formulas for the entries of $\mathcal{E}^{(2)}$, consider cases, depending on whether it is zero, or is grater then zero. After we have made assumption about some variable, we will go on with assumptions about others. Also, if necessary, we will consider for a variable, which is earlier assumed to be positive, cases of its being equal to 1 or grater then 1.

We shall consider the following cases.

Case 1: $x_1 = 0$.

Case 1.1: $x_1 = 0$ and $x_5 = 0$.

Case 1.1.1: $x_1 = 0$, $x_5 = 0$ and $x_6 = 0$.

Case 1.1.1.1: $x_1 = 0$, $x_5 = 0$, $x_6 = 0$, and $x_9 = 0$. This leads to $x_{13} > 0$, otherwise A is the zero matrix.

Case 1.1.1.1.1: $x_1 = 0$, $x_5 = 0$, $x_6 = 0$, and $x_9 = 0$, $x_{13} = 1$. These assumptions lead to the quiver with adjacency matrix $[Q] = K_2$.

Case 1.1.1.1.2: $x_1 = 0$, $x_5 = 0$, $x_6 = 0$, and $x_9 = 0$, $x_{13} > 1$. In this case we have, that $[Q] = N_5$.

Similarly the rest of the cases are as follows.

Case 1.1.1.2: $x_1 = 0$, $x_5 = 0$, $x_6 = 0$, and $x_9 > 0$. $[Q] = N_1$.

Case 1.1.2: $x_1 = 0$, $x_5 = 0$ and $x_6 > 0$.

Case 1.1.2.1: $x_1 = 0$, $x_5 = 0$, $x_6 > 0$ and $x_9 = 0$. Notice, that in this case $x_{13} > 0$, otherwise, matrix A is not reduced.

Case 1.1.2.1.1: $x_1 = x_5x_9 = 0$, $x_6 > 0$ and $x_{13} = 1$. $[Q] = T_1$.

Case 1.1.2.1.2: $x_1 = x_5 = x_9 = 0$, $x_6 > 0$ and $x_{13} > 1$. $[Q] = N_2$.

Case 1.1.2.2: $x_1 = 0$, $x_5 = 0$, $x_6 > 0$ and $x_9 > 0$. $[Q] = N_1$.

Case 1.2: $x_1 = 0$ and $x_5 > 0$.

Case 1.2.1: $x_1 = 0$, $x_5 > 0$, and $x_6 = 0$.
Case 1.2.1.1: $x_1 = 0$, $x_5 > 0$, $x_6 = 0$, and $x_9 = 0$. In this case, $x_{13} > 0$, because otherwise A will not be reduced.
Case 1.2.1.1.1: $x_1 = x_6 = x_9 = 0$, $x_5 > 0$ and $x_{13} = 1$. $[Q] = (123) \circ T_1$.
Case 1.2.1.1.2: $x_1 = x_6 = x_9 = 0$, $x_5 > 0$ and $x_{13} > 1$. $[Q] = (123) \circ N_2$.
Case 1.2.1.2: $x_1 = 0$, $x_5 > 0$, $x_6 = 0$, and $x_9 > 0$. $[Q] = N_1$.
Case 1.2.2: $x_1 = 0$, $x_5 > 0$, and $x_6 > 0$.
Case 1.2.2.1: $x_1 = 0$, $x_5 > 0$, $x_6 > 0$ and $x_9 = 0$.
Case 1.2.2.1.1: $x_1 = 0$, $x_5 = 1$, $x_6 > 0$, $x_9 = 0$.
Case 1.2.2.1.1.1: $x_1 = 0$, $x_5 = 1$, $x_6 > 0$, $x_9 = 0$, and $x_{13} = 0$.
Case 1.2.2.1.1.1.1: $x_1 = x_9 = x_{13} = 0$ and $x_5 = x_6 = 1$.$[Q] = (12) \circ K_1$.
Case 1.2.2.1.1.1.2: $x_1 = x_9 = x_{13} = 0$, $x_5 = 1$ and $x_6 > 1$. $[Q] = T_2$.
Case 1.2.2.1.1.2: $x_1 = x_9 = 0$, $x_5 = 1$, $x_6 > 0$ and $x_{13} > 0$. $[Q] = N_3$.
Case 1.2.2.1.2: $x_1 = 0$, $x_5 > 1$, $x_6 > 0$ and $x_9 = 0$.
Case 1.2.2.1.2.1: $x_1 = 0$, $x_5 > 1$, $x_6 = 1$ and $x_9 = 0$. $[Q] = T_3$.
Case 1.2.2.1.2.2: $x_1 = 0$, $x_5 > 1$, $x_6 > 1$ and $x_9 = 0$. $[Q] = N_3$.
Case 1.2.2.2: $x_1 = 0$, $x_5 > 0$, $x_6 > 0$ and $x_9 > 0$. $[Q] = N_1$.
Case 2: $x_1 > 0$.
Case 2.1: $x_1 > 0$ and $x_5 = 0$.
Case 2.1.1: $x_1 > 0$, $x_5 = 0$, and $x_6 = 0$.
Case 2.1.1.1: $x_1 > 0$, $x_5 = 0$, $x_6 = 0$, and $x_9 = 0$. Notice, that in this case, $x_{13} > 0$, because otherwise A will not be reduced.
Case 2.1.1.1.1: $x_1 > 0$, $x_5 = x_6 = x_9 = 0$ and $x_{13} = 1$. $[Q] = (123) \circ T_3$.
Case 2.1.1.1.2: $x_1 > 0$, $x_5 = 0$, $x_6 = 0$, $x_9 = 0$ and $x_{13} > 1$.
$[Q] = (123) \circ N_3$.
Case 2.1.1.2: $x_1 > 0$, $x_5 = 0$, $x_6 = 0$, and $x_9 > 0$. $[Q] = N_1$.
Case 2.1.2: $x_1 > 0$, $x_5 = 0$, and $x_6 > 0$.
Case 2.1.2.1: $x_1 > 0$, $x_5 = 0$, $x_6 > 0$ and $x_9 = 0$.
Case 2.1.2.1.1: $x_1 = 1$, $x_5 = 0$, $x_6 > 0$ and $x_9 = 0$.
Case 2.1.2.1.1.1: $x_1 = 1$, $x_5 = 0$, $x_6 > 0$, $x_9 = 0$ and $x_{13} = 0$.
Case 2.1.2.1.1.1.1: $x_1 = x_6 = 1$ and $x_5 = x_9 = x_{13} = 0$. $[Q] = K_1$.
Case 2.1.2.1.1.1.2: $x_1 = 1$, $x_5 = x_9 = x_{13} = 0$ and $x_6 > 1$. $[Q] = (12) \circ T_2$.
Case 2.1.2.1.1.2: $x_1 = 1$, $x_5 = x_9 = 0$, $x_6 > 0$ and $x_{13} > 0$.
$[Q] = (132) \circ N_3$.
Case 2.1.2.1.2: $x_1 > 1$, $x_5 = 0$, $x_6 > 0$ and $x_9 = 0$.
Case 2.1.2.1.2.1: $x_1 > 1$, $x_5 = 0$, $x_6 > 0$, $x_9 = 0$ and $x_{13} = 0$.
Case 2.1.2.1.2.1.1: $x_1 > 1$, $x_5 = x_9 = x_{13} = 0$ and $x_6 = 1$.
$[Q] = (321) \circ T_2$.
Case 2.1.2.1.2.1.2: $x_1 > 1$, $x_5 = x_9 = x_{13} = 0$ and $x_6 > 1$. $[Q] = N_4$.

Case 2.1.2.1.2.2: $x_1 > 1$, $x_5 = x_9 = 0$, $x_6 > 0$, and $x_{13} > 0$.
$[Q] = (132) \circ N_3$.
Case 2.1.2.2: $x_1 > 0$, $x_5 = 0$, $x_6 > 0$ and $x_9 > 0$. $[Q] = N_1$.
Case 2.2: $x_1 > 0$ and $x_5 > 0$.
Case 2.2.1: $x_1 > 0$, $x_5 > 0$ and $x_6 = 0$.
Case 2.2.1.1: $x_1 > 0$, $x_5 > 0$, $x_6 = 0$ and $x_9 = 0$.
Case 2.2.1.1.1: $x_1 = 1$, $x_5 > 0$, $x_6 = 0$ and $x_9 = 0$.
Case 2.2.1.1.1.1: $x_1 = 1$, $x_5 > 0$, $x_6 = 0$, $x_9 = 0$ and $x_{13} = 0$.
Case 2.2.1.1.1.1.1: $x_1 = x_5 = 1$ and $x_6 = x_9 = x_{13} = 0$. $[Q] = (23) \circ K_1$.
Case 2.2.1.1.1.1.2: $x_1 = 1$, $x_5 > 1$, $x_6 = x_9 = x_{13} = 0$. $[Q] = (132) \circ T_2$.
Case 2.2.1.1.1.2: $x_1 = 1$, $x_5 > 0$, $x_6 = x_9 = 0$ and $x_{13} > 0$.
$[Q] = (132) \circ N_3$.
Case 2.2.1.1.2: $x_1 > 1$, $x_5 > 0$, $x_6 = 0$ and $x_9 = 0$.
Case 2.2.1.1.2.1.1: $x_1 > 1$, $x_5 = 1$, and $x_6 = x_9 = x_{13} = 0$.
$[Q] = (13) \circ T_2$.
Case 2.2.1.1.2.1.2: $x_1 > 1$, $x_5 > 1$ and $x_6 = x_9 = x_{13} = 0$. $[Q] = (23) \circ N_4$.
Case 2.2.1.1.2.2: $x_1 > 1$, $x_5 > 0$, $x_6 = x_9 = 0$ and $x_{13} > 0$.
$[Q] = (231) \circ N_3$.
Case 2.2.1.2: $x_1 > 0$, $x_5 > 0$, $x_6 = 0$ and $x_9 > 0$. $[Q] = N_1$.
Case 2.2.2: $x_1 > 0$, $x_5 > 0$ and $x_6 > 0$. $[Q] = N_1$.

Now for each adjacency matrix $[Q]$ from Theorem 10 point out a case, in which either $[Q]$, or $\sigma \circ [Q]$ (for some permutation σ) appears.

N_1 : 1.1.1.2;	N_2 : 1.1.2.1.2;	N_3 : 1.2.2.1.1.2;
N_4 : 2.1.2.1.2.1.2;	N_5 : 1.1.1.1.2;	
K_1 : 1.2.2.1.1.1.1;	K_2 : 1.1.1.1;	
T_1 : 1.1.2.1.1;	T_2 : 1.2.2.1.1.1.2;	T_3 : 1.2.2.1.2.1.

The above list shows, that all quivers from Theorem 10 are admissible.

We also see, that the matrix N_5 is obtained only in Case 1.1.1.1.2. In this case one of coefficients (precisely, x_{12} of A_{12}) is greater than 1. This gives the following example.

Example 1. For the admissible quiver Q with adjacency matrix

$$N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

there is no $\alpha_1, \ldots, \alpha_{12}$ such that $\alpha_i \in \{0, 1\}$ for all i and Q is the quiver of $A = \sum\limits_{i=1}^{12} \alpha_i A_i$, where $\{A_1, \ldots, A_{12}\} = \mathcal{G}_3$.

This example shows, that condition $\alpha_i \in \{0, 1, 2\}$ in Theorem 8 can not be changed to $\alpha_i \in \{0, 1\}$.

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Received by the editors: 18.10.2015
and in final form 18.10.2015.