Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 2. **(2004).** pp. 17 – 22

Differentially trivial and rigid right semi-artinian rings

O. D. Artemovych

Communicated by M. Ya. Komarnytskyj

ABSTRACT. We obtain a characterization of right semi-artinian rings which have only trivial derivations and prove that a rigid (i.e. has only the trivial ring endomorphisms) right semi-artinian ring R is a field or isomorphic to some $\mathbb{Z}_{p^n}$.

0. As usually, an additive mapping $D : R \to R$ is called a derivation of R if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. A ring R having no non-zero derivations will be called differentially trivial. The class of differentially trivial rings is contained in the class of ideally differential rings (i.e. rings R in which every two-sided ideal is closed with respect to all derivations of R). Ideally differential rings first appear in [1] (see also [2]). Characterization theorems for differentially trivial rings R with the additive group R^+ of finite Prüfer rank and differentially trivial left Noetherian rings was obtained by the author in [3], [4] and [5].

In this paper we study differentially trivial right semi-artinian rings. Recall that a ring R with an identity is called right semi-artinian if every non-trivial right R-module has a non-zero right socle. The main result of the present paper states as follows:

Theorem. *For a right semi-artinian ring R the following conditions are equivalent:*

(i) R is a differentially trivial ring;

2000 Mathematics Subject Classification: *16W20, 13N15.*
Key words and phrases: *Semi-artinian ring, differentially trivial ring.*

(ii) R contains a set $\{e_\alpha | \alpha \in S\}$ of local idempotents e_α such that $e_\alpha R$ is either a differentially trivial field or isomorphic to some $\mathbb{Z}_{p^n}$, $R = e_\alpha R \oplus M_\alpha$ is a ring direct sum, where M_α is an ideal of R and $\bigcap_{\alpha \in S} M_\alpha = \{0\}$.

Recall that a ring R which has only the trivial ring endomorphisms is called rigid. Our theorem we can apply to prove the following

Proposition. *A right semi-artinian ring R is rigid if and only if R is either a rigid field or isomorphic to some $\mathbb{Z}_{p^n}$.*

All rings considered here are associative and with an identity element. Throughout the paper p is a prime and $\mathbb{Z}_{p^n}$ the ring of integers modulo a prime power p^n. For convenience of the reader we recall some notation. For any ring R, we denote by $\mathcal{J}(R)$ the Jacobson radical, by $R^{(p^k)}$ the subring of R generated by its identity element and the set $\{ x^{p^k} \mid x \in R \}$ $(k \in \mathbb{N})$, by $\mathrm{char}(R)$ the characteristic, by $\mathrm{Ann}(I) = \{a \in R \mid ar = ra = 0 \text{ for every } r \in I\}$ the annihilator of an ideal I in R, by $\mathrm{soc}(R)$ the right socle and $\Omega_k(R) = \{x \in R \mid p^k x = 0\}$.

We will also use some other terminology from [6] and [7].

1. Since in any differentially trivial ring R, in particular, all inner derivations are trivial, it is commutative. Moreover by a result of Bass (see e.g. [6, Proposition 22.10A]) the Jacobson radical $\mathcal{J}(R)$ of a right semi-artinian ring R is right T-nilpotent.

Lemma 1 (see [3]). *A commutative domain R is differentially trivial if and only if at least one of the following two cases takes place:*

(1) $\mathrm{char}(R) = 0$ *and the field of quotients $Q(R)$ of R is algebraic over its prime subfield;*

(2) $\mathrm{char}(R) = p > 0$ *and $R = \{a^p | a \in R\}$.*

A ring R is said to be a right Bass ring if every non-trivial right R-module has a maximal submodule (see [6] and [8]).

Lemma 2. *If R is a right Bass ring with the non-zero Jacobson radical $\mathcal{J}(R)$, then $\mathcal{J}(R)^2 \neq \mathcal{J}(R)$.*

Proof. $\mathcal{J}(R)$, considered as a right R-module, contains some maximal submodule M and, as a consequence, $\mathcal{J}(R)/M$ is a simple R-module. Then there exists an element $j \in \mathcal{J}(R) \setminus M$ such that $(j+M)R = \mathcal{J}(R)/M$. It is obvious that $(j+M)\mathcal{J}(R)$ is a submodule of $(j+M)R$. Assume that $(j+M)\mathcal{J}(R) = (j+M)R$. Then there is some element $a \in \mathcal{J}(R)$ of the nilpotency index k $(k \geq 2)$ such that $(j+M)a = j+M$, and so

$$\overline{0} = (j+M)a^k = (j+M)a^{k-1} = \ldots = (j+M)a.$$

From this it follows that $j + M = \overline{0}$, a contradiction. Hence $(j + M)\mathcal{J}(R) = \{\overline{0}\}$ and therefore $\mathcal{J}(R)^2 \leq M$. This yields that $\mathcal{J}(R)^2 \neq \mathcal{J}(R)$, as desired. □

Lemma 3. *If R is a right semi-artinian local ring with the non-zero Jacobson radical $\mathcal{J}(R)$, then*

(1) $\mathrm{Ann}(\mathcal{J}(R)) \neq \{0\}$*;*

(2) $\mathrm{Ann}(\mathcal{J}(R)^2) \neq \mathrm{Ann}(\mathcal{J}(R))$*.*

Proof. (1) A right R-module $\mathcal{J}(R)$ contains a non-zero socle $\mathrm{soc}(\mathcal{J}(R))$. By Proposition 18.39 of [6] $\mathrm{soc}(\mathcal{J}(R)) = \mathrm{ann}_{\mathcal{J}(R)}\mathcal{J}(R)$, where

$$\mathrm{ann}_{\mathcal{J}(R)}\mathcal{J}(R) = \{r \in \mathcal{J}(R) \mid \mathcal{J}(R)r = \{0\}\}.$$

Since $\mathrm{Ann}(\mathcal{J}(R)) \geq \mathrm{ann}_{\mathcal{J}(R)}\mathcal{J}(R)$, we conclude that $\mathrm{Ann}(\mathcal{J}(R))$ is non-zero.

(2) If by contradiction we assume that $\mathrm{Ann}(\mathcal{J}(R)) = \mathrm{Ann}(\mathcal{J}(R)^2)$, then by the same reason as in the part (1) the annihilator $\mathrm{Ann}(\mathcal{J}(R/\mathrm{Ann}(\mathcal{J}(R))))$ contains a non-zero element $a + \mathrm{Ann}(\mathcal{J}(R))$, whence $a\mathcal{J}(R) \leq \mathrm{Ann}(\mathcal{J}(R))$ and $a\mathcal{J}(R)^2 = \{0\}$. But then in view of our assumption $a \in \mathrm{Ann}(\mathcal{J}(R))$ which leads to a contradiction. The lemma is proved.

□

2. Proof of Theorem. ($\Leftarrow$) Suppose that a ring R contains a set $\{e_\alpha | \alpha \in S\}$ of local idempotents e_α which satisfies the hypothesis of theorem. If $d : R \to R$ is some derivation, then $d(R) \subseteq M_\alpha$ for each $\alpha \in S$ and consequently $d(R) \subseteq \bigcap_{\alpha \in S} M_\alpha$. Hence d is trivial.

($\Rightarrow$) Let R be a differentially trivial right semi-artinian ring. Then it is commutative and by Theorem 3.1 of [12] the quotient ring $R/\mathcal{J}(R)$ is (Von Neumann) regular. Therefore $R/\mathcal{J}(R) = \mathrm{soc}(R/\mathcal{J}(R))$. By Proposition 22.10A of [6] $\overline{R} = R/\mathcal{J}(R)$ contains some minimal ideal $\overline{I}$ and $\overline{I} = \overline{e}\overline{R}$ for an idempotent $\overline{e}$. This idempotent can be lifted to some idempotent e of R and so $I = eR$ is a differentially trivial local ring with the identity element e and $\mathcal{J}(I) = e\mathcal{J}(R)$ is a T-nilpotent ideal.

1) Let $\mathrm{char}(I) = \mathrm{char}(I/\mathcal{J}(I))$. Assume that $\mathcal{J}(I)$ is non-zero. By Lemma 2 $\mathcal{J}(I)^2 \neq \mathcal{J}(I)$. Writting B for $I/\mathcal{J}(I)^2$ we see that $\mathcal{J}(B) \neq \{\overline{0}\}$. From the proof of Theorem 27 of [9, Chapter VIII, §12]), Hensel's Lemma (see e.g. [10, Chapter 10, Exercises 9 and 10]) and Corollary 2 of [9, Chapter VIII, §7] it holds that B contains a subfield D such that $B = \mathcal{J}(B) + D$ is a group direct sum of the additive groups $\mathcal{J}(B)^+$ and D^+. As a consequence, for any element $\overline{b} = b + \mathcal{J}(I)^2$ of B there are unique elements $\overline{i} = i + \mathcal{J}(I)^2 \in \mathcal{J}(B)$ and $\overline{d} = d + \mathcal{J}(I)^2 \in D$ such that

$\overline{b} = \overline{i} + \overline{d}$. Then the map $\delta : B \to B$ given by $\delta(\overline{b}) = \overline{i}$ $(\overline{b} \in B)$ gives a non-zero derivation δ of B. This implies that the rule $\theta(b) = wi$ $(b \in I)$, with a fixed element w of $\mathrm{Ann}(\mathcal{J}(I)^2) \setminus \mathrm{Ann}(\mathcal{J}(I))$ (see Lemma 3), determines a non-zero derivation θ of I, a contradiction. Hence $\mathcal{J}(I) = \{0\}$.

2) Now let I be a ring of characteristic p^k $(k \geq 2)$ and $\Omega_s = \Omega_s(I)$ $(1 \leq s \leq k)$. Suppose that $\overline{I} = I/\Omega_{k-1}(I)$ has a non-zero derivation d. Then for any element $i \in I$ there exists an element $i_1 \in I$ such that $d(i + \Omega_{k-1}) = i_1 + \Omega_{k-1}$. Since $pi_1 \neq 0$ for some i_1, the rule $\mu(i) = pi_1$ $(i \in I)$ determines a non-zero derivation μ of I which gives a contradiction. This means that $\overline{I}$ is a differentially trivial ring. However $\mathrm{char}(\overline{I}/\mathcal{J}(\overline{I})) = \mathrm{char}(\overline{I})$, $\mathcal{J}(\overline{I})$ is right T-nilpotent and so $\overline{I}$ is a field. Since I is a local ring, we conclude that $\mathcal{J}(I) = \Omega_{k-1}$.

Assume that $\delta : I/\mathrm{Ann}(\Omega_1) \to I/\mathrm{Ann}(\Omega_1)$ is a non-zero derivation. Then for every $i \in I$ there is an element $a_i \in I$ such that $\delta(i + \mathrm{Ann}(\Omega_1)) = a_i + \mathrm{Ann}(\Omega_1)$, where $a_{i_0} w_0 \neq 0$ for some $i_0 \in I$ and $w_0 \in \Omega_1$. Then the rule $\mu(i) = w_0 a_i$ $(i \in I)$ gives a non-zero derivation μ of I, a contradiction. Hence $I/\mathrm{Ann}(\Omega_1)$ is differentially trivial.

We see that $pI \leq \mathrm{Ann}(\Omega_1)$ and from the part 1) $\mathcal{J}(I) \leq \mathrm{Ann}(\Omega_1)$, i.e. $\Omega_1 \mathcal{J}(I) = \{0\}$. Moreover $p\Omega_2 \mathcal{J}(I) \leq \Omega_1 \mathcal{J}(I) = \{0\}$ yields that $\Omega_2 \mathcal{J}(I) \leq \Omega_1$. Now by induction on t we can prove that $\Omega_{t+1} \mathcal{J}(I) \leq \Omega_t$ for all t $(t = 1, 2, , \ldots, k-1)$. As a consequence,

$$\mathcal{J}(I)^k = I\mathcal{J}(I)^k = \Omega_k \mathcal{J}(I)^k = (\Omega_k \mathcal{J}(I))\mathcal{J}(I)^{k-1} \leq \\ \leq \Omega_{k-1} \mathcal{J}(I)^{k-1} \leq \cdots \leq \Omega_1 \mathcal{J}(I) = \{0\}.$$

Thus $\mathcal{J}(I)^k = \{0\}$.

Let $B = I/\mathcal{J}(I)^2$. Since $B/\mathcal{J}(B) \cong I/\mathcal{J}(I)$ and $I/\mathcal{J}(I)$ is a differentially trivial ring, $B/\mathcal{J}(B) = (B/\mathcal{J}(B))^{(p^k)}$ for all $k \in \mathbb{N}$ by Lemma 1. Hence $B = \mathcal{J}(B) + D$, where D is the subring of B generated by an identity element and the set $\{x^{p^2} | x \in B\}$. Writting D_1 for an image of D in B/pB we see that D and D_1 are differentially trivial. Since $D_1 \cong D/(D \cap pB)$ and $\mathrm{char}(D_1) = p$, we deduce that $\mathcal{J}(D) = D \cap pB = pD$. This gives that D is an Artinan ring and $D \cong \mathbb{Z}_{p^2}$ by Lemma 2.8 of [3]. Hence $I = A + \mathcal{J}(I)$, where $A \cong \mathbb{Z}_{p^n}$.

In view of Corollary 27.9 from [11] $\mathcal{J}(I)^+ = A_1 + T$ is a group direct sum of a cyclic subgroup A_1 isomorphic pA and some subgroup T. Certainly T is an ideal of I. Since $I^+ = A + T$ is a group direct sum, each element i of I can be uniquelly written in the form $i = a + g$ with $a \in A$ and $g \in T$. Let ρ be a fixed element of $\mathrm{Ann}(T^2) \setminus \mathrm{Ann}(T)$. It is not hard to check that the rule $\chi(i) = \rho g$ $(i \in I)$ determines a non-zero derivation χ in R, a contradiction. Hence $I \cong \mathbb{Z}_{p^n}$ for some $n \in \mathbb{N}$.

3) Combining previous remarks, we see that R contains a set $\{e_\alpha | \alpha \in S\}$ of local idempotents e_α such that $e_\alpha R$ is a differentially trivial field or isomorphic to some $\mathbb{Z}_{p^n}$, $R = e_\alpha R \oplus M_\alpha$ is a ring direct sum, where M_α is an ideal of R $(\alpha \in S)$. If $\alpha \neq \beta$, then $M_\alpha + M_\beta = R$ and so $\bigcap_{\alpha \in S} M_\alpha = \{0\}$.

From Theorem 3.1 of [12] and our theorem it follows

Corollary 4. *R is a differentially trivial right semi-artinian ring if and only if $R/\mathcal{J}(R) = soc(R/\mathcal{J}(R))$ and R contains a set $\{e_\alpha | \alpha \in S\}$ of local idempotents e_α such that $e_\alpha R$ is a differentially trivial field or isomorphic to some $\mathbb{Z}_{p^n}$, $R = e_\alpha R \oplus M_\alpha$ is a ring direct sum, where M_α is an ideal of R and $\bigcap_{\alpha \in S} M_\alpha = \{0\}$.*

3. Proof of Proposition. $(\Leftarrow)$ is obvious.

$(\Rightarrow)$ Let R be a rigid right semi-artinian ring. Suppose that R is not a field. Theorem 2.2 of [13] implies that R contains only trivial idempotents and consequently every minimal ideal of R is nilpotent. Hence $\mathcal{J}(R)$ is non-zero. If D is any non-zero derivation of R and $tD(R) = \{0\}$ for every non-zero nilpotent element t of nilpotency index $< n-1$ and $wD(R) \neq \{0\}$ for some non-zero nilpotent element w of nilpotency index n, then the rule $\sigma(r) = r + wD(r)$ $(r \in R)$ determines a non-trivial endomorphism σ of R, a contradiction. Therefore without loss of generality we may assume that

$$\mathcal{J}(R)D(R) = \{0\} \tag{1}$$

for any derivation D of R.

Suppose that $\rho : R/\operatorname{Ann}(\mathcal{J}(R)) \longrightarrow R/\operatorname{Ann}(\mathcal{J}(R))$ is a non-zero derivation of
$R/\operatorname{Ann}(\mathcal{J}(R))$. Then for every element $u \in R$ there exists an element $v_u \in R$ such that

$$\rho(u + \operatorname{Ann}(\mathcal{J}(R))) = v_u + \operatorname{Ann}(\mathcal{J}(R)). \tag{2}$$

Since for some $u_0 \in R$ there exists an element $v_{u_0} \notin \operatorname{Ann}(\mathcal{J}(R))$ such that

$$m_0 v_{u_0} \neq 0 \tag{3}$$

for some element $m_0 \in \mathcal{J}(R)$, we obtain that a map $\theta : R \longrightarrow R$ given by $\theta(u) = m_0 v_u$ $(u \in R)$, with m_0 and u_1 as in (2) and (3), determines a non-zero derivation θ in R. Then in view of (1) $\theta(a)\theta(b) = 0$ for all elements $a, b \in R$ and so the rule $\beta(a) = a + \theta(a)$ $(a \in R)$, determines a non-trivial ring endomorphism β of R, a contradiction. This shows that the quotient ring $R/\operatorname{Ann}(\mathcal{J}(R))$ is differentially trivial and therefore $D(R) \subseteq \operatorname{Ann}(\mathcal{J}(R))$. Inasmuch as $\operatorname{Ann}(\mathcal{J}(R)) \leq \mathcal{J}(R)$, $(D(R))^2 = \{0\}$

for each derivation D of R. So the rule $\delta(r) = r + D(r)$ $(r \in R)$ determines a non-trivial ring endomorphism δ of R, a contradiction. Hence R is a differentially trivial ring and we can apply Theorem to complete the proof.

References

[1] M. Ya. Komarnytsky and O. D. Artemovych, On ideally differential rings, *Visnyk of Lviv University*, **21**(1983), 35-40. (in Ukrainian)

[2] O. D. Artemovych, Ideally differential and perfect rigid rings, *Dopovidi AN of Ukraine*, (1985), no. 4, 3-5. (in Ukrainian)

[3] O. D. Artemovych, Differentially trivial and rigid rings of finite rank, *Periodica Math. Hungar.*, **36**(1998), 1-16.

[4] O. D. Artemovych, Differentially trivial left Noetherian rings, *Comment. Math. Univ. Carolinae*, **40**(1999), 201-208.

[5] O. D. Artemovych, Differentially trivial Noetherian semiperfect rings, *Math. Pannonica*, **13**(2002), 207-216.

[6] C. Faith, *Algebra* II. *Ring theory*, Springer-Verlag, Berlin Heidelberg New York, 1976.

[7] J. Lambek, *Lectures on rings and modules*, Blaisdell Publ. Co., Waltham Toronto London, 1966.

[8] A. A. Tuganbaev, Rings over which each module possesses a maximal submodule, *Mat. Zametki*, **61** (1997), no. 3, 407-415 (in Russian) (English translated in *Math. Notes*, **61** (1997), no. 3, 333-339).

[9] O. Zariski and P. Samuel, *Commutative algebra.* Vol. II, D.Van Nostrand C.,1960.

[10] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publ. Co., Reading, 1969.

[11] L. Fuchs, *Infinite abelian groups.* Vol.II, Academic Press, New York London, 1970.

[12] C. Năstăsescu et N. Popescu, Anneaux semi-artinies, *Bull. Soc. Math. France*, **96** (1968), 357-368.

[13] C. J. Maxson, Rigid rings, *Proc. Edinburgh Math. Soc.*, **21** (1979), 95-101.

Contact information

O. D. Artemovych Department of Algebra and Logic, Faculty of Mechanics and Mathematics, Ivan Franko National University of Lviv, 1 University St, Lviv 79000 UKRAINE
E-Mail: artemovych@franko.lviv.ua

Received by the editors: 08.04.2004
and final form in 01.06.2004.