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RESEARCH ARTICLE

Finite group with given c-permutable subgroups

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ABSTRACT. Following [1] we say that subgroups H and T of a group G are *c-permutable* in G if there exists an element $x \in G$ such that $HT^x = T^xH$. We prove that a finite soluble group G is supersoluble if and only if every maximal subgroup of every Sylow subgroup of G is c-permutable with all Hall subgroups of G.

Introduction

All considered in this paper groups are finite. It is interest to use some information on Sylow subgroups of a group G to determine the structure of the group. For instance, the knowledge of the maximal subgroups of Sylow subgroups often yields wealth of information about the group itself. In [2] Srinivasan proved that G is supersoluble if every maximal subgroup of every Sylow subgroup of G is normal in G. In [3] Wang introduced the concept of c-normality which is a weaker condition than the normality in deriving the same result. In [4,5] the supersolubility of groups in which all maximal subgroups of all Sylow subgroups are complemented was proved. In the paper [6] (see also [7]) Guo, Shum and Skiba proved that a group G is supersoluble if and only if every non-normal in G maximal subgroup of every Sylow subgroup of G has a supersoluble supplement in G. The analogous results for p-nilpetent and p-closed groups were obtained by Kosenok in [8]. In this paper we give in this direction two new characterizations of supersoluble groups appealing to the following concept of c-permutability which was introduced in [1].

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Let H and T be subgroups of a group G. Then, H and T are said to be *conditionally permutable* (or in brevity, c-*permutable*) in G if $HT^x = T^xH$, for some $x \in G$.

The condition of c-permutability is generally weaker than the condition of permutability, for example, one can see that a Sylow 2-subgroup of the simmetric group S_3 is not permutable, but it is c-permutable with all subgroups of S_3.

1. Preliminaries

All considered later on groups are soluble. In this section we give some known results about soluble and supersoluble groups which will be needed in proving our main results later on.

The following lemma is well known.

Lemma 1. *Let H be a Hall π-subgroup of a group G and $K \trianglelefteq G$. Then the following statements hold:*

(1) HK/K is a Hall π-subgroup of G/K;

(2) $H \cap K$ is a Hall π-subgroup of K;

(3) If $G = AB$ for some subgroups A, B of G, then there exist Hall π-subgroup G_π, A_π and B_π in G, A and B respectively such that

$$G_\pi = A_\pi B_\pi.$$

Lemma 2. *Let H be a subnormal subgroup of a group G. Then:*

(1) If $H \leq K \leq G$, then H is subnormal in K;

(2) If H is a π-group, then $H \leq O_\pi(G)$ (see [9; Corollary 7.7.2).

Now we cite some properties of the supersoluble groups from the literature.

Lemma 3. *Let G be a group. Then the following statements hold:*

(i) If G is supersoluble, then $G' \subseteq F(G)$ and G is p-closed for the largest prime divisor p of $|G|$ (see [10; VI, 9.11]);

(ii) If $L \trianglelefteq G$ and $L/\Phi(G) \cap L$ is supersoluble, then L is supersoluble (see [9; Corollary 4.2.4]);

(iii) G is supersoluble if and only if $|G : M|$ is a prime for every maximal subgroup M of G (see [10; VI, 9.3]).

(vi) *G is supersoluble if it has two normal subgroups A and B with supersoluble quotients such that $A \cap B = 1$.*

We use the symbol $F(G)$ to denote the Fitting subgroup of a group G (that is the product of all normal nilpotent subgroups of G).

Lemma 4 ([11]; A,(10.6)). *The Fitting subgroup $F(G)$ of a group G has the following properties:*

(1) *$F(G) \leq C_G(H/K)$ for every chief factor H/K of G;*

(2) *$C_G(F(G)) \subseteq F(G)$.*

Lemma 5. *If $L \trianglelefteq G$ and $L/\Phi(G) \cap L$ is nilpetent, then L is nilpotent (see [9; Corollary 4.2.1]);*

Lemma 6 ([1], Lemma 2.1). *Let G be a group. Suppose that $K \trianglelefteq G$ and $H \leq G$. Then:*

(i) *If $K \leq T \leq G$ and H is c-permutable with T in G, then KH/K is c-permutable with T/K in G/K;*

(ii) *If $K \leq H$, $T \leq G$ and H/K is c-permutable with KT/K in G/K, then H is c-permutable with T in G.*

Lemma 7. *([12], Lemma 1.5.6). Let H/K be an abelian chief factor of a group G. Let M be a maximal subgroup of G such that $K \subseteq M$ and $MH = G$. Then*

$$G/M_G \simeq [H/K](G/C_G(H/K)).$$

2. Main results

Theorem 1. *A soluble group G is supersoluble if and only if it has a non-identity normal subgroup N with supersoluble quotient such that every maximal subgroup of every Sylow subgroup of N is c-permutable with all Hall subgroups of G.*

Proof. First suppose that G has a non-identity normal subgroup N with supersoluble quotient such that every maximal subgroup of every Sylow subgroup of N is c-permutable with all Hall subgroups of G. We shall show that G is supersoluble. Assume that it is false and let G be a counterexample of minimal order. Then the following statements hold.

(a) G/K is supersoluble for every non-identity normal in G subgroup K.

By hypothesis, it is true if $K = N$. Let $K \neq N$. We shall show that the hypothesis of the theorem is true for G/K. First of all we note that KN/N is such a non-identity normal in G/K subgroup that the quotient

$$(G/K)(KN/K) \simeq G/KN \simeq (G/N)/(KN/N)$$

is supersoluble.

Let P/K be a Sylow p-subgroup of KN/K and P_1/K be a maximal in P/K subgroup. Let E/K be a Hall π-subgroup of KN/K. We have to prove that P_1/K is c-permutable with E/K. Let R be a Sylow p-subgroup of KN such that $RK/K = P/K$. By Lemma 1 there exist Sylow p-subgroups N_p and K_p in N and K respectively such that $R = N_pK_p$. Hence $P/K = N_pK/K$. We will show that $P_1 \cap N_p$ is a maximal subgroup of N_p. First of all we note that $P_1 \cap N_p \neq N_p$. Indeed, if $P_1 \cap N_p = N_p$, then $N_p \subseteq P_1$ and so

$$P/K = N_pK/K = P_1/K,$$

this contradicts the choice of the subgroup P_1/K. Next assume that G has a subgroup T such that $P_1 \cap N_p \subset T \subset N_p$. Then

$$P_1 = K(P_1 \cap N_p) \subseteq TK \subseteq KN_p = P.$$

But P_1 is a maximal subgroup of P and so we have either $P_1 = TK$ or $TK = KN_p$. If $P_1 = TK$, we have $T \subseteq P_1 \cap N_p \subseteq T$, that is impossible. Hence $TK = N_pK$ and therefore

$$N_p = N_p \cap TK = T(N_p \cap K) \subseteq T(P_1 \cap N_p) = T,$$

a contradiction. Hence we have to conclude that $P_1 \cap N_p$ is a maximal subgroup of N_p. By hypothesis it follows that $P_1 \cap N_p$ is c-permutable with all Hall subgroups of G. By Lemma 1, G has a Hall π-subgroup G_π such that $E/K = G_\pi K/K$. Let x be an element of G such that $G_\pi^x(P_1 \cap N_p) = (P_1 \cap N_p)G_\pi^x$. Then

$$G_\pi^x(P_1 \cap N_p)K/K = G_\pi^x P_1/K = (G_\pi^x K/K)(P_1/K) =$$

$$= (E/K)^{xK}(P_1/K) = (P_1/K)(E/K)^{xK} = (P_1 \cap N_p)G_\pi^x K/K.$$

(b) If H is a minimal normal subgroup of G, then for some prime p we have

$$H = O_p(G) = F(G) = C_G(H) \nsubseteq \Phi(G)$$

and $|H| \neq p$.

By Lemma 3 and Statement (a), H is the unique minimal normal subgroup of G and $H \nsubseteq \Phi(G)$. Let M be a maximal subgroup of G such that $H \nsubseteq M$. Since G is soluble, H is an elementary abelian p-group for some prime p. Hence $G = [H]M$. Now let $C = C_G(H)$. Then

$$C = C \cap HM = H(C \cap M).$$

But, evidently, $C \cap M \trianglelefteq G$ and so $C \cap M = 1$. It follows that $H = C$. Besides, since by Lemma 4 we have $F(G) \subseteq C$, then $H = F(G) = O_p(G)$. It is also clear that $|H| \neq p$.

(c) H is not a Sylow subgroup of N.

Suppose that H is a Sylow p-subgroup of N. Let M be a maximal subgroup of H, Q be a Sylow q-subgroup of G where $q \neq p$. By hypothesis for some $x \in G$ we have $D = MQ^x = Q^xM$. Since M is subnormal in G, then by Lemma 2, M is subnormal in D too. But M is a Sylow subgroup of D, and so by Lemma 2, $M \trianglelefteq D$. Hence $q \nmid |G : N_G(M)|$. Thus $|G : |G : N_G(M)| = p^\alpha$ for some $\alpha \in \{0\} \cup \mathbb{N}$. But in view of the minimality of H and by Statement (b), $N_G(M) \neq G$. Hence $p \mid |G : N_G(M)|$ and so $p \mid n$ where n is the number of all maximal in H subgroups. This contradicts Statement 8.5 from [10; III]. So we have (c).

(d) $N = G$.

Indeed, suppose that $N \neq G$. Let N_q be a Sylow subgroup of N, P_1 be a maximal subgroup of N_q. And let T be a Hall π-subgroup of N.

Let us choose in G a Hall π-subgroup G_π such that $T \subseteq G_\pi$. Then $T = N \cap G_\pi$. By hypothesis G has an element x such that $D = P_1G_\pi^x = G_\pi^xP_1$. It follows that

$$N \cap D = P_1(N \cap G_\pi^x) = (N \cap G_\pi^x)P_1.$$

But by Lemma 1, $N \cap G_\pi^x$ is a Hall π-subgroups of N. Thus every maximal subgroup of every Sylow subgroup of N is c-permutable with all Hall subgroup of N. Thus the hypothesis of this theorem is true for N. But $|N| < |G|$ and so by the choice of G we have to conclude that N is supersoluble. Let q be the largest prime divisor of $|N|$. Then by Lemma 3, a Sylow q-subgroup N_q of N is normal in N. Since N_q char $N \trianglelefteq G$ it follows that $N_q \trianglelefteq G$. Then $N_q \subseteq F(G)$ and hence by (b) we have $N_q \subseteq H$. Thus $N_q = H$, this contradicts Statement (c). This contradiction completes the proof of Statements (d).

(e) Conclusion contradiction.

Let P be a Sylow p-subgroup of G. Then $H \subseteq P$ and by Statement (c), $H \neq P$. If $H \leq \Phi(P)$, then by Statement 3.2 from [10; III], $H \subseteq \Phi(G)$, contrary to Statement (b). Hence one can choose in P a maximal subgroup P_1 such that $H \nsubseteq P_1$. Let D be a Hall p'-subgroup of G. Then by hypothesis there exists an element x such that $T = P_1D^x = D^xP_1$. It is clear that $|G : T| = |P : P_1| = p$. Besides, evidently, $H \nsubseteq T$ and so $1 \neq H \cap T \neq H$. But $H \cap T \trianglelefteq G$, this contradicts the minimality of H. This contradiction completes the proof of supersolubility G.

Finally suppose that G is a supersoluble group. Let N be a minimal normal subgroup of G. Then $|N| = p$ for some prime p. Thus a maximal subgroup of N is normal in G. □

Theorem 2. *A soluble group G is supersoluble if and only if it has a non-identity normal subgroup N with supersoluble quotient such that every non-complemented in G maximal subgroup of every Sylow subgroup of $F(N)$ is c-permutable with all subgroups of G.*

Proof. We need only prove the "if" part (see the proof of Theorem 1). Assume that this is false and let G be a counterexample of minimal order.

Let Φ be a minimal normal subgroup of G contained in N. Suppose that $\Phi \subseteq \Phi(G)$. Then (since by hypothesis G/N is supersoluble) in view of Lemma 3, $\Phi \neq N$. Consider the quitient G/Φ. Let $T/\Phi = F(N/\Phi)$. Since T/Φ is nilpotent, then by Lemma 5, T is a nilpotent normal subgroup of N . Hence $T \subseteq F(N)$. On the other hand, since $F(N)/\Phi \subseteq F(N/\Phi)$, we have $F(N) \leq T$ and so $T = F(N)$. Thus

$$F(N/\Phi) = T/\Phi = F(N)/\Phi.$$

Now let P/Φ be a Sylow p-subgroup of T/Φ, M/Φ be a maximal in P/Φ subgroup and P_p be a Sylow p-subgroup of P. Then $P_p\Phi = P$ and $L = M \cap P_p$ is a maximal subgroup of P_p (see the proof of Theorem 1). Hence by hypothesis L is either complemented in G or c-permutable with all subgroups of G. Assume we have the first case and let T be a subgroup of G such that $G = LT$ and $L \cap T = 1$. We shall show that then $T\Phi/\Phi$ is a complement to M/Φ in G/Φ. It is clear that $G/\Phi = (L\Phi/\Phi)(T\Phi/\Phi)$. Suppose that Φ is a q-group where $q \neq p$. Then, evidently, $\Phi \subseteq T$ and so

$$(L\Phi/\Phi) \cap (T/\Phi) = (L\Phi \cap T)/\Phi = \Phi(L \cap T)/\Phi = \Phi/\Phi$$

. Analogously one can show that $T\Phi/\Phi$ is a complement to M/Φ in G/Φ in the case when Φ is a p-group. Finally, if we have the second case then by Lemma 6 we see that $M/\Phi = L\Phi/\Phi$ is c-permutable with all subgroups of G/Φ. Therefore G/Φ has a non-identity normal subgroup N/Φ with supersoluble quotient

$$(G/\Phi)/(N/\Phi) \simeq G/N$$

such that every non-complemented in G/Φ maximal subgroup of every Sylow subgroup of $F(N/\Phi) = F(N)/\Phi$ is c-permutable with all subgroups of G/Φ. Thus the hypothesis is true for G/Φ. Since $\Phi \neq 1$, $|G/\Phi| < |G|$ and so by the choice of G, G/Φ is supersoluble. But then by Lemma 3, G is supersoluble too, a contradiction. Hence $\Phi \nsubseteq \Phi(G)$. Thus $\Phi(G) \cap N = 1$. Since $N \trianglelefteq G$ and $F(N)$ char N, $F(N) \trianglelefteq G$. It follows $\Phi(F(N)) \subseteq \Phi(G)$ and so $\Phi(F(N)) = 1$. Thus $F(N)$ is an abelian group.

Now let D be the least (by inclusion) normal subgroup of G with supersoluble quotient. Then $D \leq N$ and so if R is a minimal normal subgroup of G contained in D, then $R \subseteq F(N)$. Since $\Phi(G) \cap N = 1$, G has a maximal subgroup M such that $G = [R]M$ and so $|G : M| = |R|$. First assume that $|R|$ is a prime. Then by Lemma 7,

$$G/M_G \simeq [RM_G/M_G]C_G(RM_G/M_G)$$

is a metacyclic group. Hence G/M_G is a supersoluble group and so

$$R \subseteq D \subseteq M_G \subseteq M.$$

But then $G = RM = M$. This contradiction shows that $|R|$ is not a prime.

Let R_1 be a maximal subgroup of R. Then $R_1 \neq 1$. Let R be a p-group and $D = O_p(N)$. Let us choose a maximal subgroup E of G such that $G = ER$. Then $G = DE$ and so $K = E \cap D \trianglelefteq G$. Since, evidently, $E \cap R = 1$, $D = R \times K$. Let $M = R_1K$. Then since $|D : M| = |R : R_1| = p$, M is a maximal subgroup of D. Hence by hypothesis M is either c-permutable with all subgroups of G or it is complemented in G. If we have the first case, then, in particular,

$$R_1KE^x = R_1E^x = E^xM$$

for some $x \in G$. It follows that $|G : E| = |R_1| = |R|$. This contradiction shows that we have the second case. Let T be a complement to M in G. It is not difficult to show that $V = T \cap D$ is a group of prime order p. It is clear also that $V \trianglelefteq G$. Because D/K is a chief factor of G and since KV is a normal in G subgroup such that $K \subseteq KV \subseteq D$, it follows that $KV = D$ and so $p = |V| = |D/K| = |R|$. This contradiction completes the proof of this theorem. □

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