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Some applications of Hasse principle for pseudoglobal fields

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Abstract. Some corollaries of the Hasse principle for Brauer group of a pseudoglobal field are obtained. In particular we prove Hasse-Minkowski theorem on quadratic forms over pseudoglobal field and the Hasse principle for quadratic forms of rank 2 or 3 over the field of fractions of an excellent two-dimensional henselian local domain with pseudofinite residue field. It is considered also the Galois group of maximal p-extensions of a pseudoglobal field.

Let K be an algebraic function field K in one variable with pseudofinite [1] constant field k. We call such a field *pseudoglobal.* For pseudoglobal fields there is an analogue of global class field theory [2,3], in particular, for such a field k we have the following exact sequence

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v\in V^K} \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \qquad (1)$$

where V^K is the set of all valuations of K (trivial on the constant field k), BrK (resp. BrK_v) is the Brauer group of K (resp. of the completion K_v of K at $v \in V^K$).

Note that I.Efrat [7] considers a more general situation where K is an algebraic function field in one variable over a perfect pseudo-algebraically closed constant field k and proves in that situation the exactness of the sequence

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v\in V^K} \mathrm{Br}(K_v^h) \longrightarrow G_k^\vee \longrightarrow 1, \qquad (2)$$

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where $G_k^\vee \simeq \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Q}/\mathbb{Z}), G_k$ being the absolute Galois group of k, and K_v^h is a fixed henselization of K at $v \in V^K$.

The exact sequence (1) shows, in particular, that for a pseudoglobal field K the map

$$\mathrm{Res} : \mathrm{Br}(K) \longrightarrow \prod_{v \in V^K} \mathrm{Br}(K_v) \tag{3}$$

is injective, i.e. the Hasse principle for Brauer group holds over K.

Our first application of the Hasse principle for Brauer group of a pseudoglobal field will be the analogue of the classical Hasse-Minkowski theorem which asserts that a quadratic form defined over a global field K is isotropic if and only if it is isotropic over all the completions of K. This fact can be quickly proved by using the following proposition.

Proposition 1. *Let K be a pseudoglobal field. Then:*

(i) An element $a \in K$ is a norm from a cyclic extension L/K if and only if it is a norm everywhere locally.

(ii) Let S be a finite set of valuations of a global field K. Let m be a positive integer, $(p, \mathrm{char}(K)) = 1$, and $a \in K^*$. If $a \in K_v^{*m}$ for all $v \notin S$, then $a \in K^{*m}$.

Proof. (i) For a cyclic extension L/K we get from (3) that there is an injective map $K^*/N_{L/K}L^* \to \prod_{v \in V^K} K_v^*/N_{L_w/K_v}L_w^*$, where for all $v \in V^K$ w is a fixed extension of the valuation v to L, and L_w is the corresponding completion.

(ii) We follow the argument used in [5, pp. 82–83, 275–276]. Let L/K be an abelian extension, and $G = \mathrm{Gal}(L/K)$. First we show that if $L_w = K_v$ for almost all $v \in V^K$ then $L = K$. Suppose that $K \neq L$. Let σ be a fixed generator of the absolute Galois group of the pseudofinite constant field k. Let $v \in V^K$, and let $k(v)$ and $k(w)$ be the residue field of K_v and L_w respectively. Since almost all valuations of K are unramified in L, we may assume v to be unramified in L. Denote by σ_w the restriction of $\sigma^{[k(v):k]}$ to the field $k(w)$. Then σ_w is a generator of the cyclic group $\mathrm{Gal}(k(w)/k(v)) \simeq \mathrm{Gal}(L_w/K_v) \subset G$, note that σ_w does not depend on the choice of extension $w|v$: if σ is fixed, then $\sigma_w \in G$ is uniquely determined by v, so we denote it by σ_v.

Let C_K (resp. C_L) be the idele class group of K (resp. L). By using the isomorphism $C_K/N_{L/K}C_L \simeq G$ (cf. [3]) we see that for any finite set of valuations $S \subset V^K$ the group G is generated by the elements $\sigma_v, v \notin S$. If there were exist only a finite set of valuations of K which does not split completely in L, then by adding them to S we would obtain that all σ_v are trivial for $v \notin S$. This contradicts to the fact that $\sigma_v, v \notin S$ generate the group G. Thus $L = K$.

Let $a \in K_v^{*m}$ for all $v \notin S$. As in the classical case (cf. [5], p.82-83) it is enough to consider the case where m is a power of a prime number and the m-th roots of unity are in K. In that case the extension $L = K(\sqrt[m]{a})$ is a Kummer extension, and we have $L_w = K_v$ for all $v \notin S$ where w is an extension of v to L. Then the above argument shows that $L = K$, i.e. $a \in K^{*m}$. □

Theorem 2. *A nondegenerate quadratic form q over a pseudoglobal field $K, \mathrm{char} K \neq 2$, is isotropic if and only if it is isotropic over all the completions K_v of K.*

Proof. Assume that the quadratic form q is isotropic over all the completions K_v of K. We shall argue by induction on $n = \mathrm{rank} q$ as in ([9], Appendix 3, and [10]). First, we consider the cases $n = 1, 2, 3, 4$. When $n = 1$, there is nothing to prove. When $n = 2$, we may suppose that $q = X^2 - aY^2$, and use Proposition 1 (ii) for $m = 2$. If $n = 3$, after multiplying q by nonzero element from K, we may assume that $q = X^2 - aY^2 - bZ^2$. The latter form represents zero in K if and only if b is a norm from the field $K(\sqrt{a})$, so for $n = 3$ Theorem 1 follows from Proposition 1 (i). Finally, let $n = 4$. In this case we may suppose that

$$q = X^2 - bY^2 - cZ^2 + acT^2. \tag{4}$$

Form (4) represents 0 if and only if c as an element of $K(\sqrt{ab})$ is a norm from $K(\sqrt{a}, \sqrt{b})$ ([10], 193-194). Thus Theorem 1 is established for $1 \leq n \leq 4$.

Now let $n \geq 5$. Write the form q as follows

$$q(X_1, \ldots, X_n) = a_1 X_1^2 + a_2 X_2^2 - r(X_3, \ldots, X_n). \tag{5}$$

The form r has rank $n - 2 \geq 3$. Similarly to the classical case of quadratic forms over global fields, the form r represents 0 for almost all $v \in V^K$. It suffices to show this for quadratic forms of rank 3. Let $r = b_1 Y_1^2 + b_2 Y_2^2 + b_3 Y_3^2$; let $S = \{v \in V^K \mid \exists i \in \{1, 2, 3\}\ v(b_i) \neq 0\}$. S is a finite set, and for all $v \notin S$ we can reduce r modulo v to obtain a quadratic form $\overline{r} = \overline{b}_1 Y_1^2 + \overline{b}_2 Y_2^2 + \overline{b}_3 Y_3^2$ of rank 3 over a pseudofinite field k which represents 0 over k (such statement is true over any finite field, thus it is true over a pseudofinite field k, because the pseudofinite fields are infinite models of finite fields). Henceforth, for all $v \notin S$ Hensel's lemma implies that the form r represents 0 in K_v for all $v \notin S$.

Since the subgroup K_v^{*2} is open in K_v^* with respect to v-adic topology, and r represents every element in the coset $c \cdot k^{*2}$ if it represents $c_v \in K_v^*$, then it follows that r represents the elements in a nonempty open subset of K_v^*.

Consider any $v \in S$. Since the form (5) represents 0 in K_v, there exists $c_v \in K_v^*$ such that both forms r and $a_1X_1^2 + a_2X_2^2$ represent it. So, there exist $x_1(v), \dots, x_n(v) \in K_v^*$ such that

$$a_1x_1(v)^2 + a_2x_2(v)^2 = r(x_3(v), \dots, x_n(v)) = c_v.$$

According to weak approximation theorem, we can find elements $x_1, x_2 \in K^*$ which are close enough to $x_1(v), x_2(v)$ for all $v \in S$, so that $c = a_1x_1^2 + a_2x_2^2$ is close enough to c_v to be represented by the form r.

Thus the form $cY^2 - r$ represents 0 in K_v for $v \in S$. Since r represents 0 in K_v for $v \notin S$, it represents all elements in K_v for $v \notin S$. So, $cY^2 - r$ represents 0 in K_v for all $v \in V^K$. By induction, $cY^2 - r$ represents 0 in K. It follows that q represents 0 in K. □

Recall that two quadratic forms are said to be equivalent if one can be obtained from the other by an invertible change of variables.

Corollary 3. *Two nondegenerate quadratic forms q and q' over a pseudoglobal field K are equivalent over K if and only if q and q' are equivalent over all the completions $K_v, v \in V^K$.*

Proof. Use induction on $n = \operatorname{rank} q = \operatorname{rank} q'$ exactly as in the case of global field (cf. [9], p.150 or [10], p.209). □

Corollary 4. *Any nondegenerate 5-dimensional quadratic form over a pseudoglobal field K is isotropic.*

Proof. Let $q = r(X_1, \dots, X_4) - aX_5^2$. Using the local class field theory for general local field [13] it is easy to prove that a nondegenerate 4-dimensional quadratic form over a general local field F (i.e. complete discrete valued field with quasifinite residue field represents every nonzero element of F. It follows that a nondegenerate 5-dimensional quadratic form over a pseudoglobal field K represents 0 over all the completions $K_v, v \in V^K$. □

Corollary 5. *Let A be a central simple algebra of exponent a power of 2 over a pseudoglobal field K. Then over any finite extension of K the exponent of A is equal to the index of A.*

Proof. This follows from [6, Prop. 7]. □

Remark 6. Any pseudoglobal field is a C_2-field (cf. [4]), and this implies Corollaries 4 and 5. Moreover, the exponent of every central simple algebra over a pseudoglobal field is equal to its index.

Let k be a field, and let X be a curve defined over k. The Brauer group $\mathrm{Br}(X)$ of X is the kernel of homomorphism $\mathrm{Br}K \longrightarrow \oplus_{v\in V_K}\mathrm{Br}K_v$, where K is the function field of X (cf. [11], Appendix A).

Proposition 7. *Let K be a pseudoglobal field over constant field k, then the following equivalent properties hold:*

i) the reciprocity law holds for K/k;

ii) for any finite cyclic extension L/K the sequence

$$\mathrm{Br}(L/K) \longrightarrow \oplus_{v\in V^K}\mathrm{Br}(L_w/K_v) \to [L:K]^{-1}\mathbb{Z}/\mathbb{Z} \longrightarrow 0$$

is exact;

iii) for any finite cyclic extension L/K, $H^1(\mathrm{Gal}(L/K), \mathrm{Br}(Y)) = 0$, where $\mathrm{Br}(Y)$ is the Brauer group of a smooth projective curve Y with function field L;

iv) for any finite cyclic extension L/K the map

$$K^*/N_{L/K}L^* \longrightarrow \bigoplus_{v\in V^K} K_v^*/N_{L_w/K_v}L_w^*$$

is injective;

v) $H^1(G(k), \mathrm{Jac}_C(k_s)) = 0$, where $G(k)$ is the absolute Galois group of k, and $\mathrm{Jac}_C(k_s)$ is the Jacobian of any complete smooth curve C over k;

vi) $\mathrm{Br}(C) = 0$ for any complete smooth curve C over k.

Proof. For a pseudoglobal field K/k property i) was proved in [3] as well as the equivalence of i) and iv), property iv) was also stated in Proposition 1 (i). The equivalence of i), ii), and iii) was proved in Proposition A.12 [11, p.167], and the equivalence of iv),v),vi) in Proposition A.13 [11, p.168]. □

Condition vi) of Proposition 7 has important applications to the quadratic forms and to the period-index problem of algebras on curves over discretely valued fields. Namely, using the results from [12] we have.

Proposition 8. *Let C be a curve defined over a general local field K with pseudofinite residue field k. Let $K(C)$ be its function field, and let $(n, \mathrm{char}k) = 1$. Let $\alpha \in \mathrm{Br}(K(C))$ be an element of order n in the Brauer group of $K(C)$. Then the index of α divides n^2.*

Proof. i) By Proposition 7 vi) $\mathrm{Br}(C) = 0$ for any smooth projective curve C defined over k. Then by [12], Theorem 3.5 the index of α divides n^2. □

On the other hand, Theorem 3.1 from [6] on quadratic forms over fields of fractions of excellent two-dimensional henselian local domains with either separably closed or finite residue field k holds also in the case of pseudofinite residue field.

Proposition 9. *Let A be an excellent two-dimensional henselian local domains with pseudofinite residue field k in which 2 is invertible. Let K be the field of fraction of A, and let q be a quadratic form of rank 2 or 3 over K. Then q is isotropic over K if and only if it is isotropic over all completions of K with respect to rank 1 discrete valuations.*

Proof. The only step in the proof of the corresponding result in [6] (Theorem 3.1) which uses the specific of the field K is the assertion that if certain element of exponent 2 in $\mathrm{Br}(K)$ is unramified, then it is trivial. Denoting the unramified Brauer group by $\mathrm{Br}_{\mathrm{nr}}(K)$ we have the natural inclusions $\mathrm{Br}_{\mathrm{nr}}(K) \subset \mathrm{Br}(X) \subset \mathrm{Br}(K)$, where X is a regular model of A with special fiber $X_0 \to \mathrm{Spec}(k)$. By Theorem 1.3 of [6] the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(X_0)$ induces an isomorphism on l-primary subgroups for any prime l different from $p = \mathrm{char} k$. Further, Proposition 7 vi) implies that $\mathrm{Br}(X_0) = 0$, so $\mathrm{Br}_{\mathrm{nr}}(K)$ is a p-primary group. □

Now let us turn to the Galois group of maximal p-extensions of a pseudoglobal field. The cohomological approach for describing the Galois groups for p-extensions of local and global fields was elaborated by Koch in [8]. It is known that any group can be described in terms of generators and relations. We recall some definitions from [7] and [8].

Let p be a prime number, G be a pro-p-group, $H^n(G, \mathbb{Z}/p\mathbb{Z}) := H^n(G)$. The number of generators of G is $\dim_{\mathbb{Z}/p\mathbb{Z}} H^1(G)$. The number of relations of G is $\dim_{\mathbb{Z}/p\mathbb{Z}} H^2(G)$. Let G_K be the absolute Galois group of the field K, and $G_K(p)$ be its p-component.We denote $H^n(K) = H^n(G_K(p), \mathbb{Z}/p\mathbb{Z})$.

In particular, a pro-p-group G is free if and only if $H^2(G) = 0$.

Recall that a field k is called pseudo-algebraically closed (PAC) if each nonempty variety over k has a k-rational point (pseudofinite field is a perfect PAC field whose absolute Galois group is isomorphic to $\widehat{\mathbb{Z}}$). I.Efrat [7] considers the Hasse principle for Brauer group of arbitrary extension of a perfect PAC field of relative trancendence degree 1 and proves the following result.

Proposition 10. ([7], COROLLARY 3.6) *Let k be a PAC field and let K be an extension of k of relative trancendence degree 1. Then the restriction homomorphism*

$$\mathrm{Res}: \ H^2(K) \longrightarrow \prod_{v \in V^K} H^2(\widehat{K}_v)$$

is injective, where $\widehat{K}_v = K(p) \cap K_v^h$, $K(p)$ *is the composite of all finite Galois extensions of* p*-power degree, and* K_v^h *is a henselization of* K *at* v.

As an immediate corollary, we have the following theorem.

Theorem 11. *Suppose that* K *is a pseudoglobal field. Let* p *be a prime number. Let* G *be the Galois group of the maximal* p*-extension of* K*, and for* $v \in V^K$ *let* G_v *be the corresponding decomposition group. Then the restriction homomorphism defines an injective map*

$$\varphi^* : \; H^2(G) \longrightarrow \sum_{v \in V^K} H^2(G_v).$$

Proof. It suffices to note that by Lemma 3.3 [7] the image of the restriction map $H^2(G) \to \prod_{v \in V^K} H^2(G_v)$ actually lies in $\sum_{v \in V^K} H^2(G_v)$. □

Corollary 12. *Let* w *be any valuation of pseudoglobal field* K*, and let*

$$\varphi_w^* : \; H^2(G) \to \sum_{v \neq w} H^2(G_v)$$

be the map induced by φ^**, where the item* $H^2(G_w)$ *is omitted in the direct sum. Suppose that* K *contains the* p*-th roots of 1. Then the map* φ_w^* *is injective.*

Proof. It suffices to note that by the Hasse principle the map

$$H^2(G, \widehat{K}^*)_p \to \sum_{v \neq w} H^2(G_v, \widehat{K}_{\mathfrak{p}}^*)_p$$

remains injective. □

Finally, consider the maximal p-extensions of a pseudoglobal field with given ramification.

Let K be an algebraic function field in one variable over constant field k, S be any set of valuations of the field K. Let G_S be the Galois group of the maximal p-extension K_S of K, unramified outside S. The field K_S is the composite of all finite p-extensions of K with ramification only in the set S. To the map φ^* from Theorem 11 there corresponds the map $\varphi_S^* : \; H^2(G_S) \to \sum_{v \in S} H^2(G_v)$, induced by the morphisms $\varphi_v^* : \; G_v \to G \to G_S$. Denote the kernel of φ_S^* by Ш$_S$. The group Ш$_S$ can be nontrivial, but in the case of a global field it is finite. Moreover, it is a subgroup of the finite group $\mathrm{B}_S := \mathrm{Char}(V_S/K^{*p})$, where $V_S = \{\alpha \in K^* \mid (\alpha) = \mathfrak{a}^p, \; \alpha \in k_v^p \; \forall v \in S\}$, and (α) is a principal divisor corresponding to α.

It is known [8] that in the case of a global field there is a natural embedding of the group $Ш_S$ into the group B_S. Here two natural questions arise. Is there such an embedding in the case of a pseudoglobal field? Is the group $Ш_S$ finite for a pseudoglobal field?

References

[1] J.Ax, *The elementary theory of finite fields*// Ann. of Math., (1968), **88**, 2, 239-271.

[2] V.Andriychuk, *On the algebraic tori over some function fields* // Mat.Studii, (1999), **12**, 2, p.115-126.

[3] Андрійчук В.І. *Псевдоскінченні поля і закон взаємності*// Матем. студії. 2, 1993, 14-20.

[4] V.Andriychuk, *On the Brauer group and Hasse principle for pseudoglobal fields* // Вісник Львів. ун-ту, Серія мех.-мат. (2003), **61**, 2, с.3-12.

[5] E.Artin, J.Tate, *Class field theory*. Harvard, 1961.

[6] J.-L. Colliot-Théléne, M. Ojanguren and R. Parimala, *Quadratic forms over fraction fields of two-dimensional henselian rings and Brauer groups of related schemes*, Algebra, Arithmetic and Geometry, Proceedings of the Mumbai Colloquium 2000.

[7] I.Efrat, *A Hasse principle for function field over PAC fields* // Israel J. Math, (2001), **122**, 43-60.

[8] H.Koch, *Galoische Theorie der p-Erweiturungen* // Berlin,1970. (Russian translation: Х. Кох, *Теория Галуа p-расширений*. Москва, Мир, 1973.)

[9] J.Millnor, D.Husemoller, *Symmetric bilinear forms*. Springer-Verlag, Berlin Heidelberg New York, 1973 (Russian translation: Дж. Милнор, Д.Хьюзмоллер. *Симметрические билинейные формы*. Москва, Наука, 1986).

[10] J.S.Milne, *Class field theory*, Course notes, http// www.jmilne.org/math

[11] J.S.Milne, *Arithmetic Duality Theorems*, Academic Press, Inc. Boston, 1986.

[12] D.J.Saltman, *Division algebras over p-adic curves*, J. Ramanujan Math. Soc., **12**, No.1. (1997), pp. 25-47; **13**, No.2. (1998), pp. 125-129.

[13] J.-P. Serre, Corps locaux. Paris, 1962.

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