Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 1. **(2004).** pp. 121 – 131

Green's relations on the deformed transformation semigroups

G. Y. Tsyaputa

Communicated by V. M. Usenko

Abstract. Green's relations on the deformed finite inverse symmetric semigroup $\mathcal{IS}_n$ and the deformed finite symmetric semigroup $\mathcal{T}_n$ are described.

1. Introduction

Let X and Y be nonempty sets, S a set of maps from X to Y. Let $\alpha : Y \to X$ be a fixed map.We define the multiplication of the elements from S by $\phi \circ \psi = \phi\alpha\psi$ (the compositions of the maps is from left to right). The action defined above is associative. E.S.Ljapin ([3], p. 393) formulated the problem of investigation of the properties of this semigroup depending on the restrictions to set S and map α.

Magill [4] studied this problem in the case of topological spaces and continuous maps.Under the assumption that α be onto he described the automorphisms of such semigroups and determined their isomorphism criterion.

Later Sullivan [5] proved, if $|Y| \leq |X|$ then Ljapin's semigroup is embedding into transformation semigroup on the set $X \cup \{a\}, a \notin X$.

An important case is when $X = Y$, T_X is a transformation semigroup on the set X, $\alpha \in T_X$. Symons [6] stated the isomorphism criterion for such semigroups and investigated the properties of their automorphisms.

The latter problem may be generalized to arbitrary semigroup S : for a fixed $a \in S$ the action $*_a$ is defined by $x *_a y = xay$. The obtained

Key words and phrases: *Green's relations, symmetric semigroup, inverse symmetric semigroup, deformed multiplication.*

semigroup is denoted by $(S, *_a)$ and action $*_a$ is called the multiplication deformed by element a (or just the deformed multiplication).

In [7] pairwise nonisomorphic semigroups received from finite symmetric semigroup $\mathcal{T}_n$ and finite inverse symmetric semigroup $\mathcal{IS}_n$ are classified. In particular there holds

Theorem 1.1. *Semigroups* $(\mathcal{IS}_n, *_a)$ *and* $(\mathcal{IS}_n, *_b)$ *are isomorphic if and only if* $\mathrm{rank}(a) = \mathrm{rank}(b)$.

In this article Green's relations on $(\mathcal{IS}_n, *_a)$ and $(\mathcal{T}_n, *_a)$ are described.

Recall that $\mathcal{L}-$relation is defined by $a\mathcal{L}b \iff S^1a = S^1b$. Similarly $\mathcal{R}-$relation is defined by $a\mathcal{R}b \iff aS^1 = bS^1$, and $a\mathcal{J}b \iff S^1aS^1 = S^1bS^1$. The following Green's relations are derivative: $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.

Since on finite semigroups $\mathcal{D} = \mathcal{J}$ [2, prop. 2.3], $\mathcal{J}-$ relation is not considered further below. Denote $L_a (R_a, H_a, D_a)$ the class of the corresponding relation containing a.

Remark 1.1. It is obvious, the elements a, b belong to the same $\mathcal{L}-$class (resp. $\mathcal{R}-$class) if and only if there exist such u, v from S, that $a = ub$ and $b = va$ (resp. $a = bu$ and $b = av$).

Semigroup $(S, \circ)$ with operation $a \circ b = b \cdot a$ for any a, b from S is called dual to semigroup $(S, \cdot)$. If semigroups $(S, \cdot)$ and $(S, \circ)$ are isomorphic than $(S, \cdot)$ is called self-dual.

We follow terminology and notation as in [1].

2. Green's relation on $(\mathcal{IS}_n, *_a)$.

Let $x \in \mathcal{IS}_n$. Denote x^{-1} an inverse element to x. If x is represented as a partial permutation

$$x = \begin{pmatrix} i_1 & \dots & i_k & i_{k+1} & \dots & i_n \\ j_1 & \dots & j_k & \varnothing & \dots & \varnothing \end{pmatrix}$$

then $x^{-1} = \begin{pmatrix} j_1 & \dots & j_k & j_{k+1} & \dots & j_n \\ i_1 & \dots & i_k & \varnothing & \dots & \varnothing \end{pmatrix}$. For a partial transformation $x \in \mathcal{IS}_n$ denote by $\mathrm{dom}(x)$ the domain of x, and $\mathrm{ran}(x)$ the range of x. The value $|\mathrm{ran}(x)|$ is called the rank of x and is denoted by $\mathrm{rank}(x)$.

Theorem 2.1. *Semigroup* $(\mathcal{IS}_n, *_a)$ *is self-dual.*

Proof. Denote $(\mathcal{IS}_n, \circ_a)$ the semigroup which is dual to $(\mathcal{IS}_n, *_a)$. We show that $(\mathcal{IS}_n, *_a)$ and $(\mathcal{IS}_n, \circ_a)$ are isomorphic to the semigroup $(\mathcal{IS}_n, *_{a^{-1}})$.

Semigroups $(\mathcal{IS}_n, *_a)$ and $(\mathcal{IS}_n, *_{a^{-1}})$ are isomorphic by the theorem 1.1. On the other hand, one can easily check the map

$$f : (\mathcal{IS}_n, *_{a^{-1}}) \Rightarrow (\mathcal{IS}_n, \circ_a),\ x \mapsto x^{-1}$$

is an isomorphism. □

Proposition 2.1. *For any x, y from $(\mathcal{IS}_n, *_a)$ such element u from $(\mathcal{IS}_n, *_a)$ that $x = y *_a u$, exists if and only if the following two conditions hold:*

$$\operatorname{dom}(x) \subseteq \operatorname{dom}(y); \qquad (1) \qquad\qquad \operatorname{ran}(y) \subseteq \operatorname{dom}(a). \qquad (2)$$

Proof. Let $x = y *_a u$ for u from $(\mathcal{IS}_n, *_a)$. Then it is clear that $\operatorname{rank}(x) \leq \operatorname{rank}(a)$. If $i \in \operatorname{dom}(x)$ then $x(i) = y *_a u(i) = u(a(y(i)))$, so $i \in \operatorname{dom}(y)$. If $y(i) \notin \operatorname{dom}(a)$ then i does not belong to the domain of x, so $\operatorname{ran}(y) \subseteq \operatorname{dom}(a)$.

Conversely, let conditions (1)-(2) hold. Consider a partial permutation, u, defined only on the elements from $\operatorname{ran}(a)$, moreover, $u(i) = x(y^{-1}(a^{-1}(i))) = a^{-1}y^{-1}x(i),\ i \in \operatorname{ran}(a)$. It is obvious that $y *_a u = yau = x$. □

Theorem 2.2. *Let $x \in (\mathcal{IS}_n, *_a)$.*

1) *If* $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$ *then*

$$R_x = \{y \mid \operatorname{dom}(y) = \operatorname{dom}(x),\ \operatorname{ran}(y) \subseteq \operatorname{dom}(a)\};$$

otherwise $R_x = \{x\}$.

2) *If* $\operatorname{dom}(x) \subseteq \operatorname{ran}(a)$ *then*

$$L_x = \{y \mid \operatorname{ran}(y) = \operatorname{ran}(x),\ \operatorname{dom}(y) \subseteq \operatorname{ran}(a)\};$$

otherwise $L_x = \{x\}$.

3) *If* $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$ *and* $\operatorname{dom}(x) \subseteq \operatorname{ran}(a)$ *then*

$$H_x = \{y \mid \operatorname{dom}(y) = \operatorname{dom}(x),\ \operatorname{ran}(y) = \operatorname{ran}(x)\};$$

otherwise $H_x = \{x\}$.

4) *If* $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$ *and* $\operatorname{dom}(x) \nsubseteq \operatorname{ran}(a)$ *then* $D_x = R_x$;
if $\operatorname{ran}(x) \nsubseteq \operatorname{dom}(a)$ *and* $\operatorname{dom}(x) \subseteq \operatorname{ran}(a)$ *then* $D_x = L_x$;
if $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$ *and* $\operatorname{dom}(x) \subseteq \operatorname{ran}(a)$ *then*

$$D_x = \{y \mid \operatorname{dom} y \subseteq \operatorname{ran}(a),\ \operatorname{ran}(y) \subseteq \operatorname{dom}(a)\};$$

otherwise $D_x = \{x\}$.

In particular if $\operatorname{rank}(a) \leq 1$ *then all Green's relations classes on semigroup* $(\mathcal{IS}_n, *_a)$ *are one element.*

Proof. 1) By remark 1.1 for x and y from semigroup $(\mathcal{IS}_n, *_a)$ belong to the same $\mathcal{R}-$class there should exist such u and v from $(\mathcal{IS}_n, *_a)$ that

$$x = y *_a u, \tag{3}$$

$$y = x *_a v. \tag{4}$$

By Lemma 2.1 equality (3) holds if and only if

$$\operatorname{dom}(x) \subseteq \operatorname{dom}(y); \quad (5) \qquad \operatorname{ran}(y) \subseteq \operatorname{dom}(a), \quad (6)$$

and (4) holds if and only if

$$\operatorname{dom}(y) \subseteq \operatorname{dom}(x); \quad (7) \qquad \operatorname{ran}(x) \subseteq \operatorname{dom}(a). \quad (8)$$

Condition (8) implies if $\operatorname{ran}(x) \nsubseteq \operatorname{dom}(a)$ then $R_x = \{x\}$. Let $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$. Following conditions (5) and (7), one gets for all $y \in R_x$ there holds an equality $\operatorname{dom}(x) = \operatorname{dom}(y)$. Finally condition (6) implies $\operatorname{ran}(y) \subseteq \operatorname{dom}(a)$. Besides conditions (5) – (8) are sufficient so statement 1) is proved.

Obviously, if $\operatorname{rank}(a) \leq 1$ then $|R_x| = 1$.

2) By Theorem 2.1 L_x- class description is obtained from R_x- class description by the interchanges of domains and ranges.

3) Statement about H_x- class follows from 1) and 2) statements of the theorem and $\mathcal{H}-$ relation definition.

4) $x\mathcal{D}y$ if and only if there exists such z from $(\mathcal{IS}_n, *_a)$ that $x\mathcal{L}z$ and $z\mathcal{R}y$. Consider the possible cases.

a) $|L_x| = 1$ and $|R_x| > 1$. As mentioned above, this case holds if and only if $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$ and $\operatorname{dom}(x) \nsubseteq \operatorname{ran}(a)$; then $D_x = R_x$.

b) $|L_x| > 1$ and $|R_x| = 1$. Then $\operatorname{ran}(x) \nsubseteq \operatorname{dom}(a)$ and $\operatorname{dom}(x) \subseteq \operatorname{ran}(a)$, hence $D_x = L_x$;

c) $|L_x| > 1$ and $|R_x| > 1$. In this case $\operatorname{ran}(x) \subseteq \operatorname{dom}(a)$ and $\operatorname{dom}(x) \subseteq \operatorname{ran}(a)$; so $y\mathcal{D}x$ if and only if $\operatorname{ran}(y) \subseteq \operatorname{dom}(a)$ and $\operatorname{dom}(y) \subseteq \operatorname{ran}(a)$.

d) Now if $|L_x| = 1$ and $|R_x| = 1$ then it is obvious that $D_x = \{x\}$.

□

Proposition 2.2. *Let* $p = \operatorname{rank}(a)$, $p > 1$. *Then in semigroup* $(\mathcal{IS}_n, *_a)$

the number of one element $\mathcal{R}-$classes ($\mathcal{L}-$classes) equals

$$\sum_{k=0}^{n}\sum_{m=1}^{k}\binom{n-p}{m}\binom{p}{k-m}\binom{n}{k}k!\,.$$

The number of multi-element $\mathcal{R}-$classes ($\mathcal{L}-$classes) equals $\sum\limits_{k=1}^{p}\binom{n}{k}$. The cardinality of a multi-element class is

$$[p]_k = p(p-1)\cdots(p-k+1),\ \text{where}\ 1 \leq k \leq p,$$

moreover the number of $\mathcal{R}-$classes ($\mathcal{L}-$classes) of the cardinality $[p]_k$ equals $\binom{n}{k}$.

Proof. By Theorem 2.1 semigroup $(\mathcal{IS}_n, *_a)$ is self-dual so the number and the cardinalities of $\mathcal{R}-$classes as well as of $\mathcal{L}-$classes are the same. Hence it is enough to calculate the number and the cardinalities of $\mathcal{R}-$classes.

Let $\mathrm{rank}(x) = k$.

The proof of the statement 1) of the Theorem 2.2 implies R_x- class is one element provided $\mathrm{ran}(x) \nsubseteq \mathrm{dom}(a)$. For all such x, $\mathrm{dom}(x)$ can be chosen arbitrary. Denote by m the number of such points $i \in \mathrm{dom}(x)$ that $x(i) \notin \mathrm{dom}(a)$. Then $1 \leq m \leq k$. For a fixed m set $\mathrm{ran}(x)\setminus \mathrm{dom}(a)$ can be chosen in $\binom{n-p}{m}$ ways. Similarly, set $\mathrm{ran}(x) \cap \mathrm{dom}(a)$ can be chosen in $\binom{p}{k-m}$ different ways. Hereby $\mathrm{ran}(x)$ can be defined in $\sum\limits_{m=1}^{k}\binom{n-p}{m}\binom{p}{k-m}$ ways. Then the number of one element $\mathcal{R}-$ classes equals

$$\sum_{k=0}^{n}\sum_{m=1}^{k}\binom{n-p}{m}\binom{p}{k-m}\binom{n}{k}k!$$

where $\binom{n}{k}$ is the number of ways to chose $\mathrm{dom}(x)$, $k!$ is the number of different x when $\mathrm{dom}(x)$ and $\mathrm{ran}(x)$ are defined.

By Theorem 2.2 each multi-element $\mathcal{R}-$class is defined by the domain of its representative x, moreover, $|R_x| > 1$ if and only if $\mathrm{ran}(x) \subseteq \mathrm{dom}(a)$. Thus for a fixed k the number of multi-element $\mathcal{R}-$classes is $\binom{n}{k}$ and $1 \leq k \leq p$. It is clear the total number of multi-element classes is equal to $\sum\limits_{k=1}^{p}\binom{n}{k}$.

Now count the cardinality of class R_x. By Theorem 2.2.1) $y \in R_x$ if and only if $\mathrm{dom}(y) = \mathrm{dom}(x)$ and $\mathrm{ran}(y) \subseteq \mathrm{dom}(a)$. So $\mathrm{ran}(y)$ can be chosen in $\binom{p}{k}$ ways. Then one defines the map from $\mathrm{dom}(x)$ to $\mathrm{ran}(y)$ in $k!$ ways. Hence $|R_x| = \binom{p}{k}k! = [p]_k$. □

3. Green's relation on $(\mathcal{T}_n, *_a)$.

Let $\mathcal{T}_n$ be full symmetric semigroup of all transformations of the set $\{1, 2, \ldots, n\}$. For any $x \in \mathcal{T}_n$ denote by $\operatorname{ran}(x)$ the range of transformation x. The value $|\operatorname{ran}(x)|$ is called the range of x and is denoted by $\operatorname{rank}(x)$.

Denote by ρ_x the partition of the set $\{1, 2, \ldots, n\}$ induced by transformation x, that is i and j belong to the same block of the partition ρ_x provided $x(i) = x(j)$. By $x^{-1}(i)$ denote the full pre-image of the point $i \in \operatorname{ran}(x)$.

Theorem 3.1. *Let $n > 1$ and $x \in (\mathcal{T}_n, *_a)$*

1) *If $\operatorname{rank}(x) \leq \operatorname{rank}(a)$ and for every block M of the partition ρ_a, $|\operatorname{ran}(x) \cap M| \leq 1$ then*

$$R_x = \{y \mid \rho_y = \rho_x \text{ and for every block } M \text{ of the partition } \rho_a, |\operatorname{ran}(y) \cap M| \leq 1\}$$

and $|R_x| > 1$. Otherwise $R_x = \{x\}$.

2) *If $\operatorname{rank}(a) > 1$, $\operatorname{rank}(x) \leq \operatorname{rank}(a)$ and for every block B_x of the partition ρ_x, $B_x \cap \operatorname{ran}(a) \neq \varnothing$ then*

$$L_x = \{y \mid \operatorname{ran}(y) = \operatorname{ran}(x) \text{ and for every block } B_y \text{ of the partition } \rho_y, B_y \cap \operatorname{ran}(a) \neq \varnothing\}.$$

Otherwise $L_x = \{x\}$.

If $\operatorname{rank}(a) = 1$ then all $\mathcal{L}-$classes are one element.

3) *If $\operatorname{rank}(x) \leq \operatorname{rank}(a)$, for every block M of the partition ρ_a, $|\operatorname{ran}(x) \cap M| \leq 1$ and for every block B_x of the partition ρ_x, $B_x \cap \operatorname{ran}(a) \neq \varnothing$ then*

$$H_x = \{y \mid \rho_y = \rho_x, \operatorname{ran}(y) = \operatorname{ran}(x), \text{ for every block } M \text{ of the partition } \rho_a, |\operatorname{ran}(y) \cap M| \leq 1 \text{ and for every block } B_y \text{ of the partition } \rho_y, B_y \cap \operatorname{ran}(a) \neq \varnothing\}.$$

Otherwise $H_x = \{x\}$.

4) *If $\operatorname{rank}(x) \leq \operatorname{rank}(a)$, for every block M of the partition ρ_a, $|\operatorname{ran}(x) \cap M| \leq 1$ and either there exists block B_x of the partition ρ_x, such that $B_x \cap \operatorname{ran}(a) = \varnothing$ or $\operatorname{rank}(a) = 1$ then $D_x = R_x$;*

if $\operatorname{rank}(x) \leq \operatorname{rank}(a)$, *for every block* M *of the partition* ρ_a, $|\operatorname{ran}(x) \cap M| > 1$ *and for every block* B_x *of the partition* ρ_x, $B_x \cap \operatorname{ran}(a) \neq \varnothing$ *then* $D_x = L_x$;

if $\operatorname{rank}(x) \leq \operatorname{rank}(a)$, *for every block* M *of the partition* ρ_a, $|\operatorname{ran}(x) \cap M| \leq 1$ *and for every block* B_x *of the partition* ρ_x, $B_x \cap \operatorname{ran}(a) \neq \varnothing$ *then*

$$D_x = \{y \mid \text{for every block } B_y \text{ of the partition } \rho_y,\ B_y \cap \operatorname{ran}(a) \neq \varnothing, \text{ and for every block } M \text{ of the partition } \rho_a,\ |\operatorname{ran}(y) \cap M| \leq 1\};$$

in all other cases $D_x = \{x\}$.

Proof. 1) Let different elements x and y from semigroup $(\mathcal{T}_n, *_a)$ belong to the same $\mathcal{R}-$ class. By Remark 1.1 this holds if and only if there exist such u and v from $(\mathcal{T}_n, *_a)$ that:

$$x = y *_a u, \tag{9}$$

$$y = x *_a v. \tag{10}$$

Since there holds $\operatorname{rank}(xa) \leq \operatorname{rank}(a)$ for any transformation $x \in \mathcal{T}_n$ the condition (9) implies $\operatorname{rank}(x) = \operatorname{rank}(yau) \leq \operatorname{rank}(y)$. Analogously, from (10) one gets $\operatorname{rank}(y) = \operatorname{rank}(xav) \leq \operatorname{rank}(x)$. Hence for all $y \in R_x$ there holds an equality, $\operatorname{rank}(x) = \operatorname{rank}(y)$, moreover $\operatorname{rank}(x) \leq \operatorname{rank}(a)$. This means the points from $\operatorname{ran}(x)$ belong to different blocks of the partition ρ_a, that is for every block M of the partition ρ_a, $|\operatorname{ran}(x) \cap M| \leq 1$. So $R_x \subseteq P$, where P is a set from the right hand side of (9). Now assume that $\operatorname{rank}(x) \leq \operatorname{rank}(a)$ and for every block M of the partition ρ_a there holds inequality $|\operatorname{ran}(x) \cap M| \leq 1$. Consider $y \in P$ and $u \in \mathcal{T}_n$ which is defined on every point $(ya)(i)$ as $x(i)$ and is defined arbitrary on other points. Analogously, choose $v \in \mathcal{T}_n$ which is defined on every point $(xa)(i)$ as $y(i)$. Then the equalities (9) and (10) hold. Hereby $y \in R_x$, and the reverse statement is proved. In this case $|R_x| > 1$.

If at least one of the conditions on x fails the above yields $|R_x| = 1$.

2) Let $\operatorname{rank}(a) > 1$ and different elements, x and y, from $(\mathcal{T}_n, *_a)$ belong to the same $\mathcal{L}-$ class. By Remark 1.1 this holds if and only if there exist such u and v from $(\mathcal{T}_n, *_a)$ that

$$x = u *_a y \tag{11}$$

$$y = v *_a x \tag{12}$$

Hence $\operatorname{rank}(x) = \operatorname{rank}(y)$. Since $\operatorname{rank}(x) \leq \operatorname{rank}(a)$ equalities (11) and (12) immediately imply $\operatorname{ran}(x) = \operatorname{ran}(y)$. The last means $\operatorname{rank}(ax) = \operatorname{rank}(x)$

that is for every block B_x of the partition ρ_x, $B_x \cap \mathrm{ran}(a) \neq \varnothing$. Then for all $y \in L_x$ one gets that for every block B_y of the partition ρ_y, $B_y \cap \mathrm{ran}(a) \neq \varnothing$. Thus there is an inclusion $L_x \subseteq Q$ where Q is a set from the right hand side of (10). Conversely, let for x there hold $\mathrm{rank}(x) \leq \mathrm{rank}(a)$, for every block B_x of the partition ρ_x, $B_x \cap \mathrm{ran}(a) \neq \varnothing$, and let $y \in Q$. Then the statement that for every block B_y of the partition ρ_y, $B_y \cap \mathrm{ran}(a) \neq \varnothing$ implies $\mathrm{ran}(ay) = \mathrm{ran}(y)$. From $\mathrm{ran}(y) = \mathrm{ran}(x)$ one gets $\mathrm{ran}(ay) = \mathrm{ran}(x)$. So there exist transformations u and v, satisfying the following conditions: for any $i \in N$ $u(i) \in (ay)^{-1}(x(i))$ and respectively for any $i \in N$ $v(i) \in (ax)^{-1}(y(i))$.

The straightforward check shows that u and v satisfy equalities (11) and (12). If the conditions of the statement 2) of the theorem fail then $L_x = \{x\}$.

Let $\mathrm{rank}(a) = 1$. Assume that L_x- class contains $y \neq x$. Then (11) and (12) imply $\mathrm{rank}(x) = \mathrm{rank}(y) = 1$ and $\mathrm{ran}(x) = \mathrm{ran}(y)$. The last contradicts our assumption. Hence for all x $|L_x| = 1$.

3) Statement about H_x- class follows from 1) and 2) statements of the theorem and $\mathcal{H}-$ relation definition.

4) As it is known, $x\mathcal{D}y$ if and only if there exist $z \in (\mathcal{T}_n, *_a)$ such that $x\mathcal{L}z$ and $z\mathcal{R}y$. Consider possible cases.

a) $|L_x| = 1$ and $|R_x| > 1$. By statements 1) and 2) this means that $\mathrm{rank}(x) \leq \mathrm{rank}(a)$, for every block M of the partition ρ_a, $|\mathrm{ran}(x) \cap M| \leq 1$ and either there exists block B_x of the partition ρ_x, such that $B_x \cap \mathrm{ran}(a) = \varnothing$ or $\mathrm{rank}(a) = 1$. Then $D_x = R_x$.

b) $|L_x| > 1$ and $|R_x| = 1$. In this case $\mathrm{rank}(a) > 1$, $\mathrm{rank}(x) \leq \mathrm{rank}(a)$, for every block M of the partition ρ_a, $|\mathrm{ran}(x) \cap M| > 1$ and for every block B_x of the partition ρ_x, $B_x \cap \mathrm{ran}(a) \neq \varnothing$. Then $D_x = L_x$;

c) $|L_x| > 1$ i $|R_x| > 1$. Then $\mathrm{rank}(a) > 1$, $\mathrm{rank}(x) \leq \mathrm{rank}(a)$, for every block M of the partition ρ_a, $|\mathrm{ran}(x) \cap M| \leq 1$ and for every block B_x of the partition ρ_x, $B_x \cap \mathrm{ran}(a) \neq \varnothing$. In this case $y\mathcal{D}x$ if and only if for every block B_y of the partition ρ_y, $B_y \cap \mathrm{ran}(a) \neq \varnothing$ and for every block M of the partition ρ_a, $|\mathrm{ran}(y) \cap M| \leq 1$.

d) If $|L_x| = 1$ and $|R_x| = 1$ then obviously $D_x = \{x\}$.

□

Let $S(n, k)$ be Stirling's number of the second type, that is the number of (unordered) decompositions of an $n-$element set into k subsets.

Proposition 3.1. *Let $\mathcal{T} = (\mathcal{T}_n, *_a)$, $n > 1$ and $p = \mathrm{rank}(a)$.*

1. If $p = 1$ then all $\mathcal{L}-$classes in $\mathcal{T}$ are one element. The number of one element $\mathcal{L}-$classes is equal to n^n.

Let $p > 1$. Then in $\mathcal{T}$ the number of one element $\mathcal{L}-$ classes equals

$$n^n - \sum_{m=1}^{p} \binom{n}{m} S(p,m) \sum_{j=1}^{m} S(n-p,j) \binom{m}{j} j!;$$

the number of multi-element $\mathcal{L}-$ classes equals $\sum\limits_{m=1}^{p} \binom{n}{m}$ moreover there are $\binom{n}{m}$ multi-element $\mathcal{L}-$ classes of the cardinality

$$S(p,m) \sum_{j=1}^{m} S(n-p,j) \binom{m}{j} j!, \ 1 \leq m \leq p.$$

2. Let $\{a_1, \dots, a_p\}$ be the range of transformation $a \in \mathcal{T}$. Denote $n(a_i) = |a^{-1}(a_i)|$, $1 \leq i \leq p$. The number of one element $\mathcal{R}-$ classes is equal to

$$n^n - \sum_{m=1}^{p} S(n,m) \sum_{1 \leq i_1 < i_2 < \dots < i_m \leq p} n(a_{i_1}) \cdots n(a_{i_m}) m!;$$

the number of multi-element $\mathcal{R}-$ classes is equal to $\sum\limits_{m=1}^{p} S(n,m)$, moreover, there are $S(n,m)$ multi-element $\mathcal{R}-$ classes of the cardinality

$$\sum_{1 \leq i_1 < i_2 < \dots < i_m \leq p} n(a_{i_1}) \cdots n(a_{i_m}) m!.$$

Proof. 1. The case when $p \leq 1$ is considered in the proof of statement 2) of Theorem 3.1.

Let $p > 1$. Denote $\operatorname{rank}(x) = m$. By statement 2) of the Theorem 3.1 the multi-element $\mathcal{L}-$class in semigroup $\mathcal{T}$ is uniquely defined by the range of its representative, x, moreover $\operatorname{rank}(x) \leq \operatorname{rank}(a)$, that is $1 \leq m \leq p$. So the number of multi-element $L-$classes equals $\sum\limits_{m=1}^{p} \binom{n}{m}$. Calculate the cardinality of this class if m is fixed. By Theorem 3.1 $y \in L_x$ provided $\operatorname{ran}(y) = \operatorname{ran}(x)$ and for every block B_y of the partition ρ_y, $B_y \cap \operatorname{ran} a \neq \varnothing$. Hereby in every block of the partition ρ_y there is at least one point from $\operatorname{ran}(a)$. The number of distributions of the points from $\operatorname{ran}(a)$ into the blocks of the partition ρ_y, equals $S(p,m)$. Transformation y maps other $n-p$ points of set $\{1, 2, \dots, n\}$ in $\sum\limits_{j=1}^{m} S(n-p,j)\binom{m}{j} j!$ ways, where $\binom{m}{j}$ is the number of ways to choose blocks in which $n-p$ points are distributed in $S(n-p,j)$ ways. Moreover each time in $j!$ ways the chosen blocks can be shuffled. Hence the total number of transformations in a single multi-element $\mathcal{L}-$class equals

$$S(p,m)\sum_{j=1}^{m} S(n-p,j)\binom{m}{j}j!.$$

Remaining elements from $\mathcal{T}$ form the set of one element $\mathcal{L}-$classes. Their number is equal to

$$n^n - \sum_{m=1}^{p}\binom{n}{m}S(p,m)\sum_{j=1}^{m} S(n-p,j)\binom{m}{j}j!.$$

2. By statement 1) of Theorem 3.1 each multi-element $\mathcal{R}-$class is defined by the partition ρ of the set $\{1,2,\ldots,n\}$ such that the number of partition blocks is less or equal to rank(a). For every $1 \le m \le p$ there are $S(n,m)$ decompositions of the set $\{1,2,\ldots,n\}$ into m unordered blocks.

Calculate the number of elements in $\mathcal{R}-$class defined by the partition into blocks $X_1, X_2, \ldots, X_m$. By statement 1) of Theorem 3.1 this number depends solely on the number of blocks in the partition, moreover for every $1 \le i \le p$ $a^{-1}(a_i)$ contains the range of at most one of these blocks. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ be such elements from ran(a) that $a^{-1}(a_{i_j})$ contains the range of a certain block. Then there are $m!$ different ways to map the blocks $X_1, X_2, \ldots, X_m$ to sets $a^{-1}(a_{i_1}), \ldots, a^{-1}(a_{i_m})$. Then the total number of the ways to map $X_1, X_2, \ldots, X_m$ to $a^{-1}(a_{i_1}), \ldots, a^{-1}(a_{i_m})$ is equal to $n(a_{i_1})\cdots n(a_{i_m})m!$. Hence for a fixed m the cardinality of a multi-element $\mathcal{R}-$class is equal to

$$\sum_{1\le i_1<i_2<\cdots<i_m\le p} n(a_{i_1})\cdots n(a_{i_m})m!.$$

The total number of elements in all multi-element $\mathcal{R}-$classes equals

$$\sum_{m=1}^{p} S(n,m) \sum_{1\le i_1<i_2<\cdots<i_m\le p} n(a_{i_1})\cdots n(a_{i_m})m!.$$

Now it is clear that the number of one element $\mathcal{R}-$classes is equal to

$$n^n - \sum_{m=1}^{p} S(n,m) \sum_{1\le i_1<i_2<\cdots<i_m\le p} n(a_{i_1})\cdots n(a_{i_m})m!.$$

□

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Contact information

G. Y. Tsyaputa

Kyiv National Taras Shevchenko University, Faculty of Mechanics and Mathematics, Volodymyrska, 64, Kyiv, Ukraine 01033
E-Mail: gtsyaputa@univ.kiev.ua

Received by the editors: 03.11.2003
and final form in 30.01.2004.