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Categories of lattices, and their global structure in terms of almost split sequences

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Abstract. A major part of Iyama's characterization of Auslander-Reiten quivers of representation-finite orders Λ consists of an induction via rejective subcategories of Λ-lattices, which amounts to a resolution of Λ as an isolated singularity. Despite of its useful applications (proof of Solomon's second conjecture and the finiteness of representation dimension of any artinian algebra), rejective induction cannot be generalized to higher dimensional Cohen-Macaulay orders Λ. Our previous characterization of finite Auslander-Reiten quivers of Λ in terms of additive functions [22] was proved by means of L-functors, but we still had to rely on rejective induction. In the present article, this dependence will be eliminated.

Introduction

Let R be a complete regular local ring of dimension d. An R-algebra Λ is said to be a *Cohen-Macaulay order* if ${}_R\Lambda$ is finitely generated and free. A Λ-module M is said to be *Cohen-Macaulay* if ${}_RM$ is finitely generated and free. The category of Cohen-Macaulay modules over Λ will be denoted by Λ-**CM**. For example, if $d = 0$, then R is a field, and Λ-**CM** is the category of finite dimensional modules over the artinian algebra Λ. For $d = 1$, Λ is an order over a complete discrete valuation ring R, and Λ-**CM** is the category of Λ-lattices.

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By a theorem of Auslander [2], the category Λ-**CM** (for Λ Cohen-Macaulay) has almost split sequences if and only if Λ is either *non-singular* or an *isolated singularity*, i. e. if $\operatorname{gld}\Lambda_{\mathfrak{p}} = d$ holds for all non-maximal prime ideals $\mathfrak{p}$ of R. For Λ *representation-finite* (i. e. the number of isomorphism classes of indecomposables in Λ-**CM** is finite), it is known [2] that Λ-**CM** has almost split sequences.

Given an isolated singularity Λ, it is natural to ask what are the possible Auslander-Reiten quivers $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$ of Λ-**CM**. In the representation-finite case, this question has been answered for $d = 0$ by Igusa and Todorov [7, 8, 9] and Brenner [4], for $d = 1$ by Iyama [10, 11, 12], and for $d = 2$ by Reiten and Van den Bergh [20]. There is an essential difference between $d \leqslant 2$ and $d > 2$. Roughly speaking, the projective and injective objects play a predominant rôle for $d > 2$. To make this precise, recall that a sequence $\tau A \overset{v}{\rightarrowtail} \vartheta A \overset{u}{\longrightarrow} A$ of morphisms u, v of a Krull-Schmidt category $\mathcal{A}$ is said to be *right almost split* if u is a right almost split morphism, and $v = \ker u$ is a left almost split morphism. Now $\mathcal{A}$ is said to be a *strict τ-category* [10] if $\mathcal{A}$ has right and left almost split sequences for each object A. For $d < 2$, the projective objects P of Λ-**CM** can be characterized by $\tau P = 0$, but in case $d = 2$, the projective objects of Λ-**mod** no longer coincide with the projectives of Λ-**CM** since in that case, Λ-**CM** has no projectives at all. This means that for *all* objects A, the right almost split sequence $\tau A \rightarrowtail \vartheta A \twoheadrightarrow A$ is left almost split, and vice versa. Thus Λ-**CM** is a strict τ-category if and only if $d \leqslant 2$.

Among the dimensions $d \leqslant 2$, the characterization of finite translation quivers of the form $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$ has been most difficult in case $d = 1$. To achieve this, a rather intensive study of τ-categories was necessary [10, 11, 12]. Moreover, the theory of overorders had to be translated into the language of *rejective subcategories* of Λ-**CM** (which were invented for that purpose). In this way, the structure of $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$ was determined by induction via a decreasing chain of rejective subcategories, a non-commutative analogue to a resolution of singularities. Amazingly, the same induction led Iyama to the proof of two important conjectures in the representation theory of algebras and orders, respectively (see [13]).

Another tool for the determination of $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$ was an improved theory of ladders, initiated by Igusa and Todorov in 1984. Originally, starting with a suitable morphism, a step of a ladder is given by a commutative square

$$\begin{array}{ccc} A_{i+1} & \rightarrowtail & A_i \\ \downarrow & & \downarrow \\ B_{i+1} & \rightarrowtail & B_i \end{array}$$

such that the mapping cone sequence $A_{i+1} \to A_i \oplus B_{i+1} \to B_i$ is almost split. After a series of modifications, this ultimately led to the concept of ladder functor [22]. For a Krull-Schmidt category $\mathcal{A}$, let $\mathsf{M}(\mathcal{A})$ denote the homotopy category of two-termed complexes (see §3). A pair of *L-functors* is an adjoint pair $L^+ \dashv L^-$ of additive functors $L^\pm\colon \mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A})$ together with natural transformations $\lambda^+\colon L^+ \to 1$ and $\lambda^-\colon 1 \to L^-$, with additional properties (see §2). More generally, L-functors can be defined for any additive category $\mathcal{M}$ instead of $\mathsf{M}(\mathcal{A})$. If L-functors exist for $\mathcal{M}$, they are unique, and $\mathcal{M}$ carries a structure similar to that of a triangulated category. Therefore, we call such a category $\mathcal{M}$ *triadic* (§2).

By definition, a left L-functor $L^+\colon \mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A})$ applies to any morphism $a\colon A_1 \to A_0$ in $\mathcal{A}$. Then $\lambda_a\colon L^+a \to a$ gives rise to a pullback-pushout square in $\mathcal{A}$, which can be regarded as a step of a (generalized) ladder. In particular, if $a \in \operatorname{Ob}\mathsf{M}(\mathcal{A})$ is of the form $0 \to A$ with A indecomposable and non-projective, then λ_a gives a commutative square

$$\begin{array}{ccc} \tau A & \rightarrowtail & 0 \\ \downarrow & & \downarrow \\ \vartheta A & \twoheadrightarrow & A \end{array}$$

corresponding to the almost split sequence $\tau A \rightarrowtail \vartheta A \twoheadrightarrow A$. Thus L-functors yield almost split sequences in a functorial way, and they also apply to morphisms instead of objects A.

Using L-functors, a simplification of the statement and proof of Iyama's characterization of $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$ for representation-finite Λ with $d = 1$ became possible [22]. In particular, the complicated part of his criterion could be replaced by the existence of an additive function $l > 0$ on the vertices of $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$.

In order to extend the characterization of $\mathbb{A}(\Lambda\text{-}\mathbf{CM})$ to dimensions $d > 2$, L-functors might be useful. For arbitrary dimension d, the triadic category $\mathsf{M}(\Lambda\text{-}\mathbf{CM})$ has L-functors if and only if $\Lambda\text{-}\mathbf{CM}$ has almost split sequences [28]. However, a big obstacle came from the induction via rejective subcategories for $d = 1$. This allows no generalization to higher dimensions since a resolution of higher-dimensional non-commutative singularities Λ would not be feasable. Therefore, toward a criterion for $d > 2$, a fundamental step would be to eliminate that inductive rejection in $d = 1$. This will be done in the present paper. As a side-effect, Theorem 1 of [22] which was fundamental for the introduction of additive functions, also has been dropped now. Approximately, this latter reduction eliminates half of the use of L-functors in our treatment of $d = 1$.

As already begun in [22], we investigate $\Lambda\text{-}\mathbf{CM}$ for $d = 1$ in a more general setting. We define a Λ-*lattice* as a finitely presented Λ-module with no simple submodules. In this way, the classical situation is reduced to what is really needed. In particular, no base ring of Λ has to be specified. In §1, the structure of lattice categories Λ-**CM** (for $d = 1$) will be characterized within this general context in category-theoretic terms. L-functors will be introduced and applied to lattice categories in §§2 - 3. For a Krull-Schmidt lattice category $\mathcal{A}$, we show (Theorem 4) that $\mathsf{M}(\mathcal{A})$ has L-functors if and only if $\mathcal{A}$ is a strict τ-category.

From §4 on, we investigate strict τ-categories $\mathcal{A}$. Heuristically, this means that we study the *local* structure of $\mathcal{A}$, given by its Auslander-Reiten meshes. We will assume that $\mathcal{A}$ is *L-finite*, i. e. for any object $a \in \mathsf{M}(\mathcal{A})$, the powers $L^{\pm n}a$ stabilize for $n \gg 0$. This condition holds, e. g., if $\mathcal{A}$ is equivalent to Λ-**CM**, or its universal cover (see [31, 10]), in case Λ is representation-finite. To characterize $\mathcal{A}$ in terms of the Auslander-Reiten quiver, we have to reconstruct its *global* structure from the local mesh structure. This job will be done by the L-functors. For an object A of $\mathcal{A}$, the repeated application of L^+ to $0 \to A$ yields a cokernel $P \twoheadrightarrow A$ with P projective, which shows that $\mathcal{A}$ has enough projectives in a strict sense. Similarly, the kernel of a cokernel $c \in \mathcal{A}$ is obtained by applying L^{+n} to c, for $n \gg 0$ (Proposition 7). To show that every morphism in $\mathcal{A}$ has a kernel, more assumptions are necessary.

We call a monomorphism $m \in \mathcal{A}$ *simple* if it allows no factorization $m = ab$ into non-invertible monomorphisms a, b. Analoguously, *simple* epimorphisms are defined. For an L-finite strict τ-category $\mathcal{A}$, a simple monomorphism is either epic or a kernel. If it is epic, it need not be a simple epimorphism. We show that equality of the classes of simple monomorphisms and simple epimorphisms among the monic and epic morphisms establishes a duality between projectives and injectives in $\mathcal{A}$. This condition holds, e. g., when the Auslander-Reiten quiver of $\mathcal{A}$ admits an additive function $l > 0$ (Proposition 9). For a lattice category, such an l is given by the rational rank. If, in addition, $\mathcal{A}$ satisfies $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A} = 0$, we prove (Theorem 5) that $\mathcal{A}$ has all the relevant global properties of Λ-**CM** in case $d \leqslant 1$. In particular, Theorem 5 implies that $\mathcal{A}$ is noetherian in a strong sense (Corollary 1). As a further consequence, we get the above mentioned characterization of finite translation quivers arising as Auslander-Reiten quivers of Λ-**CM** for a Cohen-Macaulay order Λ over a complete discrete valuation ring (Corollary 3).

1. Categories of lattices

Let Λ be any ring (associative with 1). By Λ-**mod** we denote the category of finitely presented left Λ-modules. A module $E \in \Lambda$-**mod** is said to be a Λ-*lattice* [23] if E has no simple submodules. For example, if Λ is an order over a Dedekind domain R (see [19]), then Λ-lattices are just what they ought to be. Therefore, we denote the full subcategory of Λ-lattices in Λ-**mod** (for any ring Λ) by Λ-**lat**. An additive category $\mathcal{A}$ equivalent to a category Λ-**lat** with Λ left noetherian will be called a *left lattice category*. We call $\mathcal{A}$ a *right lattice category* if $\mathcal{A}^{\mathrm{op}}$ is a left lattice category. If $\mathcal{A}$ is a left and right lattice category, we simply speak of a *lattice category* $\mathcal{A}$. In the particular case where Λ is an R-order, the duality functor $E \mapsto \mathrm{Hom}_\Lambda(E, R)$ provides an equivalence

$$(\Lambda\text{-}\mathbf{lat})^{\mathrm{op}} \xrightarrow{\sim} \Lambda^{\mathrm{op}}\text{-}\mathbf{lat}, \tag{1}$$

which shows that Λ-**lat** is in fact a lattice category. In this section, we will give an intrinsic characterization of lattice categories.

A morphism in an additive category $\mathcal{A}$ is said to be *regular* if it is monic and epic. We say that $\mathcal{A}$ *has a quotient category*, denoted by $\mathrm{Q}(\mathcal{A})$, if the regular morphisms admit a calculus of left and right fractions [5]. Thus if $\mathrm{Q}(\mathcal{A})$ exists, it has the same objects as $\mathcal{A}$, and the morphisms of $\mathrm{Q}(\mathcal{A})$ are formal fractions $fr^{-1} = s^{-1}g$ with r, s regular and $sf = gr$. Moreover, there is a faithful embedding

$$\mathcal{A} \hookrightarrow \mathrm{Q}(\mathcal{A}) \tag{2}$$

which respects kernels and cokernels of morphisms.

Recall that a *short exact sequence* $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ in $\mathcal{A}$ is defined by the property $a = \ker b$ and $b = \operatorname{cok} a$. By $\rightarrowtail$ (resp. $\twoheadrightarrow$) we indicate that a morphism is a (co-) kernel. An object P of $\mathcal{A}$ is said to be *projective* if for each cokernel $c\colon B \twoheadrightarrow C$, every morphism $P \to C$ factors through c. An object C of $\mathcal{A}$ will be called a *covering object* if every $E \in \operatorname{Ob}\mathcal{A}$ admits a cokernel $C^n \twoheadrightarrow E$ for some $n \in \mathbb{N}$. The dual notions of *injective* or *cocovering* objects are defined analoguously. The full subcategories of projective (injective) objects will be denoted by $\mathbf{Proj}(\mathcal{A})$ (resp. $\mathbf{Inj}(\mathcal{A})$). An additive category $\mathcal{A}$ is said to be *preabelian* if every morphism of $\mathcal{A}$ has a kernel and a cokernel.

Proposition 1. *For a preabelian category $\mathcal{A}$, the following are equivalent.*

(a) *If a composition fg is a cokernel, then f is a cokernel.*

(b) *For given $A \overset{c}{\twoheadrightarrow} C \overset{f}{\longleftarrow} B$, the morphism $(c\ f)\colon A \oplus B \to C$ is a cokernel.*

(c) *Every composition fg of cokernels f, g is a cokernel.*

(d) *If $f\colon A \overset{c}{\twoheadrightarrow} E \overset{d}{\to} B$ is a morphism with $c = \operatorname{cok}(\ker f)$, then d is monic.*

Proof. (a) ⇒ (b): Consider the composition $c\colon A \xrightarrow{\binom{1}{0}} A \oplus B \xrightarrow{(c\,f)} C$.

(b) ⇒ (c): Let $f\colon B \twoheadrightarrow C$ be the cokernel of $d\colon D \to B$, and $g\colon A \twoheadrightarrow B$. Then property (b) implies that there is a pushout

$$\begin{CD} E @>{e}>> A @>{fg}>> C \\ @VVV \text{PO} @VV{g}V @| \\ D @>{d}>> B @>{f}>> C. \end{CD}$$

Hence $fg = \operatorname{cok} e$.

(c) ⇒ (d): Suppose that $dg = 0$. Then d factors through $e := \operatorname{cok} g$, and there exists a morphism h with $ec = \operatorname{cok} h$. Hence $fh = 0$. Therefore, h factors through the kernel of f, and thus $ch = 0$. Consequently, $c = \operatorname{cok} h$, whence e is invertible. So we get $g = 0$, which shows that d is monic.

(d) ⇒ (a): Consider a factorization $f = dc$ with $c = \operatorname{cok}(\ker f)$ and d monic. Assume that $fg = \operatorname{cok} h$. Since d is monic, this implies that $cgh = 0$. So cg factors through fg. Therefore, d is split epic, hence invertible. □

A preabelian category $\mathcal{A}$ which satisfies the equivalent properties of Proposition 1 is called *left semi-abelian* [21]. By [21], Proposition 1, a preabelian category $\mathcal{A}$ is left semi-abelian if and only if for any pullback

$$\begin{CD} A @>{a}>> B \\ @V{b}VV @VV{c}V \\ C @>{d}>> D \end{CD} \tag{3}$$

in $\mathcal{A}$ where d is a cokernel, the morphism a is epic. In [21] we called the preabelian categories $\mathcal{A}$ where a is even a cokernel for all such pullbacks (3) *left almost abelian*. Several authors use the term "quasi-abelian" instead of "almost abelian" [30, 29, 3]. Following this trend, we replace "left almost abelian" by "left quasi-abelian" in what follows. If $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ (resp. $\mathcal{A}^{\mathrm{op}}$) is left quasi-abelian, we call $\mathcal{A}$ (*right*) *quasi-abelian*. Similarly, we define (*right*) *semi-abelian*. We are grateful to Y. Kopylov for pointing out to us that the term "quasi-abelian" dates back to R. Succi Cruciani's paper [30] of 1973. There is also a Russian tradition [16, 17, 18, 15] that calls quasi-abelian categories "(Raĭkov-)semi-abelian".

Proposition 2. *Let $\mathcal{A}$ be a preabelian category, such that every object A admits a cokernel $P \twoheadrightarrow A$ with P projective. Then $\mathcal{A}$ is left quasi-abelian.*

Proof. Consider a pullback (3) where d is a cokernel. By assumption, there is a cokernel p: $P \twoheadrightarrow B$ with P projective. Hence cp factors through d, and the pullback property implies that p factors through a. Thus a is a cokernel by Proposition 1. □

By [21], Proposition 6, a semi-abelian category $\mathcal{A}$ has a quotient category if and only if for each pullback (3) with d epic, the morphism a is also epic. Semi-abelian categories with this property (i. e. that epimorphisms are stable under pullback) are called *integral* [21]. For an integral category $\mathcal{A}$, the quotient category $\mathrm{Q}(\mathcal{A})$ is abelian. Examples of quasi-abelian and integral categories are abundant (see [21], §2). We will see below that lattice categories are integral and quasi-abelian.

An additive category $\mathcal{A}$ is said to be *noetherian* if for each object of $\mathcal{A}$, the subobjects satisfy the ascending chain condition. We call $\mathcal{A}$ *bi-noetherian* if $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are noetherian. Assume that every kernel or cokernel can be completed to a short exact sequence, and that condition (a) of Proposition 1 together with its dual is satisfied. (Then (c) of Proposition 1 follows.) This holds, for example, when $\mathcal{A}$ is semi-abelian. Define the *rational length* $\rho(A) \in \mathbb{N} \cup \{\infty\}$ of an object $A \in \mathrm{Ob}\,\mathcal{A}$ as the supremum of all $n \in \mathbb{N}$ for which there exists a chain

$$0 = A_0 \overset{a_1}{\rightarrowtail} A_1 \overset{a_2}{\rightarrowtail} \cdots \overset{a_n}{\rightarrowtail} A_n = A \tag{4}$$

of non-invertible kernels $a_1, \ldots, a_n \in \mathcal{A}$. According to our assumption, this definition is self-dual, i. e. the sequence of kernels (4) can be replaced by a sequence of cokernels $A \twoheadrightarrow \cdots \twoheadrightarrow 0$. If $\mathcal{A}$ is integral, then [25], Proposition 2, implies that the rational length of an object $A \in \mathrm{Ob}\,\mathcal{A}$ is equal to $\rho(A)$ in the abelian category $\mathrm{Q}(\mathcal{A})$. We call $A \in \mathrm{Ob}\,\mathcal{A}$ *irreducible* if $\rho(A) = 1$. A chain (4) with $\mathrm{Cok}\, a_i$ irreducible will be called a *rational composition series* of A. Thus if $\mathcal{A}$ is integral, every rational composition series (4) is of length $n = \rho(A)$. For a regular morphism $r \in \mathcal{A}$, we define the *length* $\rho(r) \in \mathbb{N} \cup \{\infty\}$ as the supremum of all $n \in \mathbb{N}$ for which r can be written as a composition $r = r_1 \cdots r_n$ into non-invertible regular morphisms r_i. We say that r *has a composition series* if a factorization $r = r_1 \cdots r_n$ with $\rho(r_i) = 1$ exists. If $\mathcal{A}$ is integral and quasi-abelian, then every composition series of r is of length $\rho(r)$ by [24], Proposition 2.

Proposition 3. *An integral quasi-abelian category $\mathcal{A}$ is bi-noetherian if and only if its objects have finite rational length, and every regular morphism has finite length.*

Proof. Assume first that $\mathcal{A}$ is bi-noetherian. For any non-zero object A_0, there exists a maximal subobject a_1: $A_1 \rightarrowtail A_0$ with a_1 non-invertible.

Thus a_1 is a kernel with $\rho(\operatorname{Cok} a_1) = 1$. By induction, we get a sequence $A_0 \overset{a_1}{\leftarrowtail} A_1 \overset{a_2}{\leftarrowtail} A_2 \overset{a_3}{\leftarrowtail} \cdots$ of kernels with $\rho(\operatorname{Cok} a_i) = 1$ for all i. So there are commutative diagrams

$$\begin{array}{ccccc} A_i & \rightarrowtail & A_0 & \twoheadrightarrow & C_i \\ {\scriptstyle a_i}\downarrow & & \| & & \downarrow \\ A_{i-1} & \rightarrowtail & A_0 & \twoheadrightarrow & C_{i-1} \end{array}$$

for all i. Since $\mathcal{A}^{\mathrm{op}}$ is noetherian, we infer that $A_n = 0$ for some $n \geqslant 1$. Hence $\rho(A_0) = n < \infty$. A similar argument shows that regular morphisms of $\mathcal{A}$ have finite length. Conversely, assume that $\rho(A) < \infty$ and $\rho(r) < \infty$ for all $A \in \operatorname{Ob} \mathcal{A}$ and all regular $r \in \mathcal{A}$. Consider a strictly increasing sequence $A_0 < A_1 < A_2 < \cdots$ of subobjects of $A \in \operatorname{Ob} \mathcal{A}$. Since $\rho(A) < \infty$, almost all monomorphisms $A_i \to A_{i+1}$ must be regular. Therefore, the sequence cannot be infinite. □

For a quasi-abelian category $\mathcal{A}$, we define the *initial category* [21] as the full subcategory $\mathcal{A}_\circ$ of objects D of $\mathcal{A}$ such that every monomorphism $D' \to D$ is a kernel. The full subcategory $\mathcal{A}^\circ$ of $\mathcal{A}$ with $(\mathcal{A}^\circ)^{\mathrm{op}} = (\mathcal{A}^{\mathrm{op}})_\circ$ is called the *terminal category* of $\mathcal{A}$. By [21], Proposition 8, the categories $\mathcal{A}_\circ$ and $\mathcal{A}^\circ$ are abelian. Now we are ready to prove

Theorem 1. *An additive category $\mathcal{A}$ is a lattice category if and only if the following are satisfied.*

(a) *$\mathcal{A}$ is preabelian with a projective covering object P and an injective cocovering object I.*

(b) *$\mathcal{A}$ has a quotient category.*

(c) *$\mathcal{A}$ is bi-noetherian.*

(d) $\mathcal{A}_\circ = \mathcal{A}^\circ = 0$.

Proof. Assume that $\mathcal{A} = \Lambda$-**lat** with Λ left noetherian. Then $\mathcal{A}$ is preabelian and noetherian, and there is a hereditary torsion theory $(\mathcal{T}, \Lambda\text{-}\mathbf{lat})$ in Λ-**mod**, where $\mathcal{T}$ is the class of length-finite Λ-modules. By [21], Theorem 2, this implies that $\mathcal{A}$ is integral, whence (b) holds. Moreover, there is a short exact sequence $\Lambda_0 \rightarrowtail \Lambda \twoheadrightarrow P$ in Λ-**mod** with $\Lambda_0 \in \mathcal{T}$ and $P \in \Lambda$-**lat**. Thus P is a projective covering object. For any non-zero Λ-lattice E, there is a maximal Λ-submodule F. Therefore, if $E \in \mathcal{A}_\circ$, then i: $F \hookrightarrow E$ is a kernel in Λ-**lat**. But the cokernel of i in Λ-**lat** is zero, a contradiction. Thus $\mathcal{A}_\circ = 0$. By symmetry, this proves that every lattice category satisfies (a)-(d).

Conversely, let (a)-(d) be satisfied. By Proposition 2, this implies that $\mathcal{A}$ is quasi-abelian. By [24], Proposition 11, we have $Q_l(\mathcal{A}) \approx \Lambda$-**mod**, where $\Lambda := \mathrm{End}_{\mathcal{A}}(P)^{\mathrm{op}}$, and $Q_l(\mathcal{A})$ denotes the left abelian cover [21] of $\mathcal{A}$. From [21], Theorem 2, we infer that there is a torsion theory $(\mathcal{R}(\mathcal{A}), \mathcal{F})$ in Λ-**mod** with $\mathcal{F} \approx \mathcal{A}$, such that a finitely presented Λ-module M belongs to $\mathcal{R}(\mathcal{A})$ if and only if there exists a regular morphism $r \in \mathcal{F}$ with $M = \mathrm{Cok}\, r$ in Λ-**mod**. By (d), every simple Λ-module belongs to $\mathcal{R}(\mathcal{A})$. Moreover, $\mathcal{A}$ is integral by (b). Therefore, the regular morphisms are essentially monic and essentially epic. Hence (c) implies that $\mathcal{R}(\mathcal{A})$ is the full subcategory of length-finite modules in Λ-**mod**. Thus $\mathcal{A} \approx \mathcal{F} = \Lambda$-**lat** with Λ left noetherian by (c). □

As a consequence of Theorem 1 and Proposition 2, we get

Corollary. *Every lattice category is integral and quasi-abelian.*

Remark. *If (d) in Theorem 1 is replaced by $\mathcal{A}_\circ = \mathcal{A}^\circ = \mathcal{A}$, then the conditions (a)-(d) characterize a category $\mathcal{A}$ which is equivalent to* Λ-**mod** $\approx (\Gamma$-**mod**$)^{op}$ *with Λ, Γ left artinian. This can be regarded as the 0-dimensional analogue of a lattice category.*

2. L-functors

In this section we review the basic theory of L-functors, as far as needed for our present purpose. Functors between additive categories are always assumed to be additive. Let $\mathcal{M}$ be an additive category. For a full subcategory $\mathcal{C}$ of $\mathcal{M}$, a morphism φ: $a \to b$ of $\mathcal{M}$ is said to be $\mathcal{C}$-*epic* ($\mathcal{C}$-*monic*) if every morphism $c \to b$ (resp. $a \to c$) with $c \in \mathcal{C}$ factors through φ. By $[\mathcal{C}]$ we denote the ideal of $\mathcal{M}$ generated by the identity morphisms 1_c, $c \in \mathrm{Ob}\, \mathcal{C}$. If $\mathcal{M}/[\mathcal{C}]$ has a quotient category, we say that $\mathcal{M}_{\mathcal{C}} := Q(\mathcal{M}/[\mathcal{C}])$ *exists*. For a class Σ of morphisms, let $\mathrm{Pr}\, \Sigma$ (resp. $\mathrm{In}\, \Sigma$) denote the largest full subcategory $\mathcal{C}$ of $\mathcal{M}$ such that every $\varphi \in \Sigma$ is $\mathcal{C}$-epic ($\mathcal{C}$-monic).

Assume that $\mathcal{M}_{\mathcal{C}}$ exists. For a morphism $\alpha \in \mathcal{M}$, we denote the (co-)kernel of α in $\mathcal{M}_{\mathcal{C}}$ by $\ker_{\mathcal{C}}\alpha$ (resp. $\mathrm{cok}_{\mathcal{C}}\alpha$) and call this a *local (co-)kernel*. As a counterpart, we call $\alpha \in \mathcal{M}$ a *global kernel* of $\beta \in \mathcal{M}_{\mathcal{C}}$ if $\beta\alpha = 0$ holds in $\mathcal{M}_{\mathcal{C}}$, and for each $\alpha' \in \mathcal{M}$ with $\beta\alpha' = 0$ in $\mathcal{M}_{\mathcal{C}}$ there exists a unique $\gamma \in \mathcal{M}$ with $\alpha\gamma = \alpha'$. By this universal property, the global kernel and its dual, the *global cokernel*, are unique up to isomorphism. We write $\alpha = \ker^{\mathcal{C}}\beta$ (resp. $\mathrm{cok}^{\mathcal{C}}\beta$) for the global (co-)kernel of β.

We call an object s of $\mathcal{M}$ *left* (*right*) *semisimple* if every monomorphism $a \to s$ (epimorphism $s \to a$) splits. The full subcategory of left (right) semisimple objects is denoted by $S_l(\mathcal{M})$ (resp. $S_r(\mathcal{M})$), and the objects of $\mathrm{S}(\mathcal{M}) := \mathrm{S}_l(\mathcal{M}) \cap \mathrm{S}_r(\mathcal{M})$ will be called *semisimple*. Note that for a module category $\mathcal{M}$, the semisimple objects coincide with the semisimple modules ([1], Theorem 9.6).

Definition 1. Let $\mathcal{M}$ be an additive category. By Σ we denote the class of regular morphisms which are $\mathrm{S}_l(\mathcal{M})$-epic and $\mathrm{S}_r(\mathcal{M})$-monic. The morphisms in Σ will be called *(absolutely) exact*. We call $\mathcal{M}$ *(absolutely) triadic* if the following are satisfied for $\mathcal{P} := \mathrm{Pr}\,\Sigma$ and $\mathcal{I} := \mathrm{In}\,\Sigma$.

(T1) $\mathcal{M}_\mathcal{P}$ and $\mathcal{M}_\mathcal{I}$ exist and are abelian, $\mathcal{M}_\mathcal{P}$ has enough projectives, and $\mathcal{M}_\mathcal{I}$ has enough injectives.

(T2) Every morphism in $\mathcal{M}_\mathcal{P}$ (resp. $\mathcal{M}_\mathcal{I}$) has a global (co-)kernel.

(T3) Every global kernel is a global cokernel, and vice versa.

Remark. *By [28], Corollary of Theorem 1, the global kernels in $\mathcal{M}$ coincide with the exact morphisms. In [28], we define a triadic category with respect to arbitrary full subcategories $\mathcal{P}, \mathcal{I}$ of $\mathcal{M}$. Then it can be shown that $Pr\Sigma \subset \mathcal{P}$ and $In\Sigma \subset \mathcal{I}$. Thus in the absolute case of Definition 1, $\mathcal{P}$ and $\mathcal{I}$ are as small as possible. In the wider sense of [28], every additive category $\mathcal{M}$ is triadic with respect to the pair $\mathcal{P} = \mathcal{I} = \mathcal{M}$. The reason why we introduced triadic categories for arbitrary $\mathcal{P}$ and $\mathcal{I}$ comes from the observation that they naturally arise in the study of orders over a two-dimensional regular ring.*

By [28], Theorem 1, we have

Theorem 2. *Let $\mathcal{M}$ be a triadic category. There is an equivalence T: $\mathcal{M}_\mathcal{P} \xrightarrow{\sim} \mathcal{M}_\mathcal{I}$ such that every exact morphism β can be completed to a triad [28], i. e. a sequence*

$$Td \xrightarrow{\alpha} b \xrightarrow{\beta} c \xrightarrow{\gamma} d \tag{5}$$

with $\alpha = \ker_\mathcal{I}\beta$, $\gamma = \mathrm{cok}_\mathcal{P}\beta$, and $\beta = \mathrm{cok}^\mathcal{I}\alpha = \ker^\mathcal{P}\gamma$, such that each commutative diagram $\psi\beta = \beta'\chi$ with exact β' induces a morphism of triads

$$\begin{array}{ccccccc} Td & \xrightarrow{\alpha} & b & \xrightarrow{\beta} & c & \xrightarrow{\gamma} & d \\ \downarrow{\scriptstyle T\omega} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \omega} \\ Td' & \xrightarrow{\alpha'} & b' & \xrightarrow{\beta'} & c' & \xrightarrow{\gamma'} & d'. \end{array} \tag{6}$$

Remark. *Using (T2), Theorem 2 implies that every epimorphism $\gamma \in \mathcal{M}_\mathcal{P}$, and every monomorphism $\alpha \in \mathcal{M}_\mathcal{I}$ can be extended to a triad (5). Moreover, each of the commutative squares in (6) extends to a morphism of triads.*

Recall that a *pointed functor* [14] of $\mathcal{M}$ is defined as a functor L^-: $\mathcal{M} \to \mathcal{M}$ together with a natural transformation λ^-: $1 \to L^-$. Dually, we define an *augmented functor* of $\mathcal{M}$ as an endofunctor L^+ with a natural transformation λ^+: $L^+ \to 1$. For an augmented or pointed functor $L^\pm$, let $\operatorname{Pr} L^\pm$ (resp. $\operatorname{In} L^\pm$) denote the largest full subcategory $\mathcal{C}$ of $\mathcal{M}$ such that $\lambda_a^\pm$ is $\mathcal{C}$-epic (resp. $\mathcal{C}$-monic) for every $a \in \operatorname{Ob} \mathcal{M}$. For an adjoint pair of endofunctors $L^+ \dashv L^-$ with adjunction Φ: $\operatorname{Hom}_{\mathcal{M}}(L^+a, b) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{M}}(a, L^-b)$, an augmentation λ^+: $L^+ \to 1$ of L^+ makes L^- into a pointed functor via $\lambda_a^- = \Phi(\lambda_a^+)$. In other words, the right adjoint of an augmented functor is pointed, and the left adjoint of a pointed functor is augmented. If $\mathcal{M}$ is triadic, we define a *left triadic* functor of $\mathcal{M}$ as an augmented functor L^+: $\mathcal{M} \to \mathcal{M}$ such that λ_a^+ is exact for all $a \in \operatorname{Ob} \mathcal{M}$. Thus if L^+ is left triadic, every object a of $\mathcal{M}$ gives rise to a triad

$$TSa \xrightarrow{\sigma_a} L^+a \xrightarrow{\lambda_a^+} a \xrightarrow{\pi_a} Sa \tag{7}$$

with a functor S: $\mathcal{M} \to \mathcal{M}_{\mathcal{P}}$. Dually, a pointed functor L^- of $\mathcal{M}$ with λ_a^- exact for all $a \in \operatorname{Ob} \mathcal{M}$ will be called *right triadic*.

Definition 2. Let $\mathcal{M}$ be a triadic category. We define a *left L-functor* of $\mathcal{M}$ as a left triadic functor L^+: $\mathcal{M} \to \mathcal{M}$ such that the inclusions $\operatorname{Pr} L^+ \subset \operatorname{Pr} \Sigma$, $\operatorname{In} L^+ \subset \operatorname{In} \Sigma$ hold, and Sa is semisimple for every $a \in \operatorname{Ob} \mathcal{M}$. Dually, a pointed functor L^- of $\mathcal{M}$ will be called a *right L-functor* if it induces a left L-functor $\mathcal{M}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$. We say that an additive category *has L-functors* if it is triadic and admits a left L-functor L^+ and a right L-functor L^-.

By [28], Proposition 15, we have

Theorem 3. *If an additive category $\mathcal{M}$ has L-functors, then L^+ is left adjoint to L^-. The right adjoint of a left L-functor is a right L-functor, and vice versa. A left or right L-functor of a triadic category is unique, up to isomorphism.*

3. L-functors for lattice categories

Now we will show how triadic categories arise in the context of lattice categories. Let $\mathcal{A}$ be a *Krull-Schmidt category*, i. e. an additive category such that every object of $\mathcal{A}$ is a finite direct sum of objects with local endomorphism rings. The ideal $\operatorname{Rad} \mathcal{A}$ of $\mathcal{A}$ generated by the non-invertible morphisms between indecomposable objects is called the *radical* of $\mathcal{A}$. Let $\operatorname{Mor}(\mathcal{A})$ be the category of two-termed complexes $0 \to A_1 \xrightarrow{a} A_0 \to 0$

in $\mathcal{A}$. So the objects of $\mathrm{Mor}(\mathcal{A})$ can be regarded as morphisms $a \in \mathcal{A}$, and the morphisms in $\mathrm{Mor}(\mathcal{A})$ are tantamount to commutative squares in $\mathcal{A}$. If we identify $A \in \mathrm{Ob}\,\mathcal{A}$ with the identity morphism $1_A \in \mathrm{Mor}(\mathcal{A})$, then $\mathcal{A}$ becomes a full subcategory of $\mathrm{Mor}(\mathcal{A})$, such that the ideal $[\mathcal{A}]$ of $\mathrm{Mor}(\mathcal{A})$ consists of the morphisms $\varphi\colon\ a \to b$ in $\mathrm{Mor}(\mathcal{A})$ which are homotopic to zero. Since every morphism $f \in \mathcal{A}$ has a decomposition $f = e \oplus r$ into an isomorphism e and some $r \in \mathrm{Rad}\,\mathcal{A}$, the factor category $\mathrm{Mor}(\mathcal{A})/[\mathcal{A}]$ is equivalent to its full subcategory $\mathsf{M}(\mathcal{A})$ of objects $A_1 \stackrel{a}{\to} A_0$ with $a \in \mathrm{Rad}\,\mathcal{A}$. For any $A \in \mathrm{Ob}\,\mathcal{A}$, there are two corresponding objects $A^+\colon\ 0 \to A$ and $A^-\colon\ A \to 0$ of $\mathsf{M}(\mathcal{A})$. So we get two equivalences $(\ \ )^+\colon\ \mathcal{A} \xrightarrow{\sim} \mathcal{A}^+$ and $(\ \ )^-\colon\ \mathcal{A} \xrightarrow{\sim} \mathcal{A}^-$ between $\mathcal{A}$ and full subcategories of $\mathsf{M}(\mathcal{A})$.

A morphism $f\colon\ A \to B$ in $\mathcal{A}$ is said to be *right almost split* if $f \in \mathrm{Rad}\,\mathcal{A}$, and every morphism $A' \to B$ in $\mathrm{Rad}\,\mathcal{A}$ factors through f. If f is right almost split in $\mathcal{A}^{\mathrm{op}}$, then f is called *left almost split*. A sequence

$$\tau A \stackrel{v_A}{\rightarrowtail} \vartheta A \xrightarrow{u_A} A \tag{8}$$

in $\mathcal{A}$ is said to be *right almost split* if u_A is right almost split, and $v_A = \ker u_A$ is left almost split. Note that a right almost split sequence (8) is uniquely determined by the object A, up to isomorphism. Similarly, a sequence

$$A \xrightarrow{u^A} \vartheta^- A \stackrel{v^A}{\twoheadrightarrow} \tau^- A \tag{9}$$

is said to be *left almost split* if it is right almost split in $\mathcal{A}^{\mathrm{op}}$. A Krull-Schmidt category $\mathcal{A}$ with left and right almost split sequences for all $A \in \mathrm{Ob}\,\mathcal{A}$ is called a *strict τ-category* [10].

To each strict τ-category $\mathcal{A}$, a (valued) translation quiver $\mathbb{A}(\mathcal{A})$ can be associated as follows. The class of vertices of $\mathbb{A}(\mathcal{A})$ is given by a representative system $\mathrm{ind}\,\mathcal{A}$ of the isomorphism classes of indecomposable objects in $\mathcal{A}$. For $A, B \in \mathrm{ind}\,\mathcal{A}$, let d_{AB} be the multiplicity of A in a direct decomposition of ϑB, and d'_{AB} the multiplicity of B in $\vartheta^- A$. Then there is an arrow $A \to B$ with valuation (d_{AB}, d'_{AB}) whenever $d_{AB} \neq 0$ (or equivalently, $d'_{AB} \neq 0$). The translation quiver $\mathbb{A}(\mathcal{A})$ is called the *Auslander-Reiten quiver* [10] of $\mathcal{A}$. Iyama [10] has shown that $\mathbb{A}(\mathcal{A})$, together with its natural modulation, determines the associated completely graded τ-category of $\mathcal{A}$ up to equivalence.

Proposition 4. *Let $\mathcal{A}$ be a lattice category with the Krull-Schmidt property. For every indecomposable projective object P, there exists a unique maximal subobject $\vartheta P < P$.*

Proof. Since $\mathcal{A}$ is noetherian, there exists a maximal subobject $E < P$. Assume that there is a different maximal subobject $F < P$. The corresponding monomorphisms $E \to P \leftarrow F$ induce a morphism $g\colon E \oplus F \to P$. Since $\mathcal{A}$ is semi-abelian, g has a decomposition $g = mq$ with m monic and q a cokernel. Thus m defines a subobject of P which contains E and F. Hence m is invertible. Since P is projective with $\mathrm{End}_{\mathcal{A}}(P)$ local, we infer that either $E \to P$ or $F \to P$ is split epic, a contradiction. □

Remark. *Let $P = P_1 \oplus \cdots \oplus P_n$ be any projective object in a Krull-Schmidt lattice category, with P_i indecomposable. Then the monomorphisms $u_{P_i}\colon \vartheta P_i \to P_i$ define a monomorphism $u_P\colon \vartheta P \to P$ such that*

$$0 \longrightarrow \vartheta P \xrightarrow{u_P} P \tag{10}$$

is a right almost split sequence. By duality, every injective object I of $\mathcal{A}$ gives rise to a left almost split sequence $I \xrightarrow{u^I} \vartheta^- I \to 0$.

Proposition 5. *Let $\mathcal{A}$ be a lattice category with the Krull-Schmidt property. An object of $\mathsf{M}(\mathcal{A})$ is left semisimple if and only if it is isomorphic to $u_P \oplus E^-$ for some projective object P, and an arbitrary object E of $\mathcal{A}$.*

Proof. Let $a\colon A_1 \to A_0$ be a left semisimple object in $\mathsf{M}(\mathcal{A})$. Then there is an exact square

$$\begin{array}{ccc} A & \twoheadrightarrow & A_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle a} \\ P & \xrightarrow{f} & A_0 \end{array}$$

with P projective. By [22], Proposition 2 and its dual, this represents a regular morphism $\varepsilon\colon p \to a$ in $\mathsf{M}(\mathcal{A})$. Hence ε is split monic, and thus invertible. Therefore, f is invertible, which shows that A_0 is projective. Now the proof can be completed as in the proof of [27], Proposition 8. □

Recall that a commutative square (3) is said to be *exact* if it is a pullback and a pushout. Proposition 5 allows us to determine the exact morphisms (see Definition 1) of $\mathsf{M}(\mathcal{A})$.

Proposition 6. *Let $\mathcal{A}$ be a lattice category with the Krull-Schmidt property. Then $\mathsf{M}(\mathcal{A})$ is triadic. A morphism $\varphi\colon b \to c$ in $\mathsf{M}(\mathcal{A})$, given by a commutative square (3), is exact if and only if (3) is an exact square.*

Proof. Assume that $b \to c$ is exact. By Proposition 5, this implies that φ is $\mathcal{A}^-$-epic. Hence (3) is exact by [22], Proposition 2. Conversely, let φ be given by an exact square (3). Then φ is $\mathcal{A}^-$-epic and $\mathcal{A}^+$-monic by [22], Proposition 2. Hence φ is exact by Proposition 5 and its dual. Now [28], Corollary 2 of Theorem 3, implies that $\mathsf{M}(\mathcal{A})$ is triadic. □

Theorem 4. *Let $\mathcal{A}$ be a lattice category with the Krull-Schmidt property. Then $\mathsf{M}(\mathcal{A})$ has L-functors if and only if $\mathcal{A}$ is a strict τ-category.*

Proof. This follows by Proposition 4 and the above remark, and Proposition 6 together with [28], Theorem 5. $\square$

4. L-finiteness

For a strict τ-category $\mathcal{A}$, the homotopy category $\mathsf{M}(\mathcal{A})$ need not be triadic. Nevertheless, by [22], §3, there exists an augmented functor L^+: $\mathsf{M}(\mathcal{A}) \to \mathsf{M}(\mathcal{A})$ with a right adjoint L^- such that $L^\pm$ become L-functors when $\mathcal{A}$ is triadic. An object a of $\mathsf{M}(\mathcal{A})$ belongs to $\operatorname{Pr} L^+$ (resp. $\operatorname{Pr} L^-$) if and only if λ_a^+ (resp. λ_a^-) is invertible. Therefore, we call $\mathcal{A}$ *left* (*right*) *L-finite* if for each $a \in \operatorname{Ob} \mathsf{M}(\mathcal{A})$, there is an integer $n \in \mathbb{N}$ such that $L^{+n}a \in \operatorname{Pr} L^+$ (resp. $L^{-n}a \in \operatorname{In} L^-$). If $\mathcal{A}$ is left and right L-finite, we just say that $\mathcal{A}$ is *L-finite.* Thus if $\mathcal{A}$ is left L-finite, every $a \in \operatorname{Ob} \mathsf{M}(\mathcal{A})$ gives rise to an exact square

$$\begin{array}{ccc} B & \xrightarrow{\quad c \quad} & A_1 \\ \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle a} \\ P & \xrightarrow{\quad p \quad} & A_0 \end{array} \tag{11}$$

with $b = L^{+n}a \in \operatorname{Pr} L^+$. Hence $\tau P = 0$.

Proposition 7. *Let $\mathcal{A}$ be a left L-finite strict τ-category. Then every cokernel has a kernel, and for each $A \in \operatorname{Ob} \mathcal{A}$, there is a cokernel $P \twoheadrightarrow A$ with P projective. An object P of $\mathcal{A}$ is projective if and only if $\tau P = 0$, and a morphism of $\mathcal{A}$ is a cokernel if and only if it is* **Proj**$(\mathcal{A})$*-epic.*

Proof. If we set $A_1 = 0$ in (11), we get a short exact sequence $B \rightarrowtail P \twoheadrightarrow A_0$ with $\tau P = 0$. The proof of [22], Theorem 3, shows that P is projective. Therefore, [22], Proposition 11, implies that the projective objects P are characterized by the property $\tau P = 0$. Thus if a in (11) is a cokernel, then p factors through a, which implies that b is split epic. Since $b \in \operatorname{Rad} \mathcal{A}$, we infer that $P = 0$, whence $c = \ker a$. Finally, let b: $B \to C$ be **Proj**$(\mathcal{A})$-epic. Consider a short exact sequence $C' \overset{c}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} C$ and a cokernel q: $Q \twoheadrightarrow B$ with P, Q projective. Then $bq = pd$ for some d: $Q \to P$, and it is easily verified that $(b\ p)$: $B \oplus P \to C$ is a cokernel of $\begin{pmatrix} q & 0 \\ -d & c \end{pmatrix}$: $Q \oplus C' \to B \oplus P$. Hence $(b\ p)$ has a kernel, which gives an exact square

$$\begin{array}{ccc} E & \xrightarrow{e} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{b} & C. \end{array} \tag{12}$$

Hence p factors through b, and the pullback property implies that e is split epic. Since idempotents split in $\mathcal{A}$, we infer that e has a kernel $g\colon K \rightarrowtail E$. Thus by the pushout property of (12), it follows that $b = \operatorname{cok}(fg)$. □

Proposition 7 shows that the short exact sequences of an L-finite strict τ-category $\mathcal{A}$ make $\mathcal{A}$ into an Ext-category (see [28]), that is, an exact category with enough projectives and enough injectives such that every split epimorphism has a kernel.

Corollary. *Let $\mathcal{A}$ be an L-finite strict τ-category. Then $\mathsf{M}(\mathcal{A})$ has L-functors.*

Proof. By [22], Proposition 2, and [27], Proposition 8, a morphism in $\mathsf{M}(\mathcal{A})$ is exact if and only if it corresponds to an exact square in $\mathcal{A}$. Therefore, the corollary follows by [28], Theorem 5 and Corollary 2 of Theorem 3. □

Remark. *By [22], §6, and [10], §7, it follows that whether a strict τ-category $\mathcal{A}$ is L-finite can be read off from the Auslander-Reiten quiver $\mathbb{A}(\mathcal{A})$. For the rest of this section, we will derive further consequences of L-finiteness. Since $\mathbb{A}(\mathcal{A}) = \mathbb{A}(\mathcal{A}/\bigcap_{n=1}^{\infty} Rad^n \mathcal{A})$, we eventually assume that $\bigcap_{n=1}^{\infty} Rad^n \mathcal{A} = 0$.*

Lemma 1. *Let $\mathcal{A}$ be a Krull-Schmidt category with a commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{e} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ J & \xrightarrow{e'} & I. \end{array}$$

Assume that e is split monic, J injective, and $\operatorname{cok} b \in \operatorname{Rad}\mathcal{A}$. Then every retraction of e can be lifted to a retraction of e'.

Proof. If $fe = 1$, then there is a morphism $f'\colon I \to J$ with $f'a = bf$. Hence $(1 - f'e')b = 0$, and thus $1 - f'e' \in \operatorname{Rad}\mathcal{A}$. Therefore, $f'e'$ is invertible, and it follows easily that $(f'e')^{-1}f'$ is the desired lifting. □

Lemma 2. *Let $\mathcal{A}$ be an L-finite strict τ-category with $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A} = 0$. Then every object A of $\mathcal{A}$ admits a rational composition series.*

Proof. By duality, it is enough to show that an infinite strictly ascending sequence $A_0 < A_1 < \cdots$ of subobjects $A_i \rightarrowtail A$ cannot exist. We may assume, without loss of generality, that A is injective. Suppose first that A is indecomposable. Then all A_i are indecomposable by Lemma 1. Therefore, the inclusions $A_i \rightarrowtail A_{i+1}$ are in $\operatorname{Rad}\mathcal{A}$, whence $A_0 \rightarrowtail A$ belongs to $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A}$, a contradiction. Now let A be decomposable. Since $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A} = 0$, there exists an integer $n \in \mathbb{N}$ such that $A_n \rightarrowtail A_i$ does not belong to $\operatorname{Rad}\mathcal{A}$ for all $i \geqslant n$. Hence there exists an indecomposable direct summand B of A_n such that the composition e_i: $B \rightarrowtail A_n \rightarrowtail A_i$ is split monic for all $i \geqslant n$. So there are commutative diagrams with short exact rows

$$\begin{array}{ccccc}
B & \overset{e_n}{\rightarrowtail} & A_n & \twoheadrightarrow & C_n \\
\| & & \downarrow & & \downarrow \\
B & \overset{e_i}{\rightarrowtail} & A_i & \twoheadrightarrow & C_i \\
{\scriptstyle b}\downarrow & & \downarrow & & \downarrow{\scriptstyle c_i} \\
J & \overset{e}{\rightarrowtail} & A & \twoheadrightarrow & C,
\end{array}$$

with J injective and $\operatorname{cok} b \in \operatorname{Rad}\mathcal{A}$, where all the e_i are split monic. The lifting e of e_n is split monic by Lemma 1. Moreover, Lemma 1 implies that every retraction of e_i lifts to a retraction of e. Therefore, the c_i are kernels. So we get an infinite strictly ascending sequence $C_n < C_{n+1} < \cdots$ of subobjects $C_i \rightarrowtail C$. By induction, this leads to a contradiction. ☐

Lemma 3. *Let $\mathcal{A}$ be a left L-finite strict τ-category. If a pullback is made up of two commutative squares*

$$\begin{array}{ccccc}
A & \xrightarrow{i} & E & \xrightarrow{e} & B \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle c} \\
C & \xrightarrow{j} & F & \xrightarrow{f} & D,
\end{array} \tag{13}$$

where the left-hand square is exact, then the right-hand square is a pullback.

Proof. Let z: $Z \to E$ be a morphism with $ez = gz = 0$. Since the left-hand square is a pullback, there is an x: $Z \to A$ with $ix = z$ and $bx = 0$. Hence $\binom{b}{ei} x = 0$, and thus $x = 0$. So we get $z = 0$.

Next let p: $P \to B$ and q: $P \to F$ be morphisms with $cp = fq$. Assume first that P is projective. Since the left-hand square is a pushout, there are morphisms p': $P \to E$ and q': $P \to C$ with $q = gp' + jq'$. Hence

$c(p - ep') = fj \cdot q'$. So we get a morphism $h\colon P \to A$ with $p - ep' = eih$ and $q' = bh$. Thus $h' := p' + ih$ satisfies $p = eh'$ and $q = gh'$. Now let P be non-projective. Then there is a cokernel $r\colon Q \twoheadrightarrow P$ with Q projective, and we get a morphism $h''\colon Q \to E$ with $pr = eh''$ and $qr = gh''$. Hence eh'' and gh'' annihilate the kernel k of r. Consequently, $h''k = 0$, and thus $h'' = h'r$ for some $h'\colon P \to E$. So we get $p = eh'$ and $q = gh'$. □

Let $\mathcal{A}$ be an L-finite strict τ-category. For a monomorphism $A \to B$, let B/A denote the poset of subobjects E of B with $A \leqslant E \leqslant B$. Then every exact square (3) with monomorphisms a, d gives rise to an isomorphism of posets $B/A \cong D/C$. In fact, if $E \in B/A$ is given, then the corresponding $F \in D/C$ is obtained via (13) by taking the pushout of i and b. By Lemma 3 and its dual, this correspondence $E \mapsto F$ is bijective.

Let us call a monomorphism $A \to B$ *simple* if B/A has exactly two elements. Dually, we call an epimorphism *simple* if it is a simple monomorphism in $\mathcal{A}^{\mathrm{op}}$.

Proposition 8. *Let $\mathcal{A}$ be an L-finite strict τ-category. A morphism $a\colon A_1 \to A_0$ is a cokernel if and only if there exists no factorization $a = me$ with a simple monomorphism m.*

Proof. Assume that a is not a cokernel. By Proposition 7, there exists an exact square (11) with P projective and $b = u_P d$ for some $d\colon B \to \vartheta P$. Since u_Q is a simple monomorphism for any indecomposable direct summand Q of P, there exists a factorization $b = st$ with a simple monomorphism s. By Lemma 3, the pushout of t and c yields an exact square

$$\begin{array}{ccc} D & \longrightarrow & E \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle m} \\ P & \longrightarrow & A_0 \end{array}$$

with a simple monomorphism m such that a factors through m. Conversely, let $a = me$ be a cokernel with m monic. Then e factors through a. Thus m is split epic, hence invertible. □

Corollary. *Let $\mathcal{A}$ be an L-finite strict τ-category. A simple monomorphism $m\colon A \to B$ is either epic or a kernel.*

Proof. Suppose that $fm = 0$ with $f \neq 0$. We show that $m = \ker f$. Assume that $fg = 0$. Since $(g\ m)$ is not a cokernel, it factors through a simple monomorphism. Hence g factors through m. □

5. The dualizing property

Let $\mathcal{A}$ be an L-finite strict τ-category. For any simple monomorphism $a\colon A_1 \to A_0$ in $\mathcal{A}$, there is an exact square (11) with $b = u_P$ for some indecomposable projective P. If $a \in \operatorname{Rad}\mathcal{A}$, we may regard (11) as an exact morphism in $\mathsf{M}(\mathcal{A})$. Since $u_P \in \operatorname{Pr} L^+$, we necessarily have $u_P \cong L^{+n}a$ for $n \gg 0$. Therefore, up to isomorphism, P is uniquely determined by a.

Definition 3. Let $\mathcal{A}$ be a strict τ-category. A function $l\colon \operatorname{Ob}\mathcal{A} \to \mathbb{N}$ is said to be *additive* if for $A, B \in \operatorname{Ob}\mathcal{A}$,

$$l(A \oplus B) = l(A) + l(B) \tag{14}$$

$$l(A) = l(\vartheta A) - l(\tau A) = l(\vartheta^- A) - l(\tau^- A). \tag{15}$$

If $l(A) > 0$ for $A \neq 0$, then we write $l > 0$. We say that $\mathcal{A}$ *is dualizing* if there is a one-to-one correspondence between the isomorphism classes of indecomposable projective P with u_P epic and the isomorphism classes of indecomposable injective I with u^I monic, given by an exact square

$$\begin{array}{ccc} \vartheta P & \longrightarrow & I \\ \big\downarrow{\scriptstyle u_P} & & \big\downarrow{\scriptstyle u^I} \\ P & \longrightarrow & \vartheta^- I. \end{array} \tag{16}$$

Assume that $\mathcal{A}$ is L-finite. Then the correspondence (16) is explicitly given by $u_P = L^{+n}u^I$ and $u^I = L^{-n}u_P$ for $n \gg 0$. Therefore, the dualizing property merely depends on the Auslander-Reiten quiver $\mathbb{A}(\mathcal{A})$ (cf. [10], §7). An additive function $l\colon \operatorname{Ob}\mathcal{A} \to \mathbb{N}$ admits a natural extension to $\operatorname{Ob}\mathsf{M}(\mathcal{A})$. Namely, for an object $a\colon A_1 \to A_0$ of $\mathsf{M}(\mathcal{A})$, we define $l(a) := l(A_0) - l(A_1)$. Thus $l(\mathcal{A}^+) \geqslant 0$ and $l(\mathcal{A}^-) \leqslant 0$. Moreover, we have $l(L^+a) = l(L^-a) = l(a)$ for all $a \in \operatorname{Ob}\mathsf{M}(\mathcal{A})$.

For an indecomposable projective $P \in \operatorname{Ob}\mathcal{A}$, the monomorphism u_P is obviously simple. Assume that u_P is epic (which happens, e. g., if $\mathcal{A}$ is a lattice category), and let us try to prove that u_P is a simple epimorphism. For a factorization $u_P = ab$, there are two possibilities. If $a \in \operatorname{Rad}\mathcal{A}$, then b is invertible. Otherwise, a is split epic. So b is a regular morphism of the form $b\colon \vartheta P \to P \oplus C$. For lattices over an order, of course, this is not possible, unless $C = 0$. The reason is that the rational rank of lattices is an additive function.

Proposition 9. *For an L-finite strict τ-category $\mathcal{A}$, the following are equivalent.*

(a) *$\mathcal{A}$ is dualizing.*

(b) *A regular $r \in \mathcal{A}$ is a simple monomorphism if and only if it is a simple epimorphism.*

If $\mathcal{A}$ has an additive function $l > 0$, then $\mathcal{A}$ is dualizing.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows by the above. Assume that $\mathcal{A}$ has an additive function $l > 0$. We show that u_P is a simple epimorphism for any indecomposable projective P. By the dual of Proposition 8, there is a factorization $u_P = ab$ with a simple epimorphism b. If $a \in \operatorname{Rad} \mathcal{A}$, then a has a factorization $a = u_P c$. Hence $u_P(1 - cb) = 0$, and thus $cb = 1$, a contradiction. So we infer that a is of the form $a\colon P \oplus C \twoheadrightarrow P$. Since $l(b) = l(L^{-n}b) = l(u^I) = 0$ for $n \gg 0$ and some injective I, we get $l(C^-) = l(a) = l(u_P) - l(b) = 0$. Hence a is invertible. □

Lemma 4. *Let $\mathcal{A}$ be an L-finite dualizing strict τ-category. If a morphism $c\colon A \to C$ does not factor through a regular morphism $r\colon A \to B$ of length 1, then there is an exact square*

$$\begin{array}{ccc} A & \xrightarrow{\;c\;} & C \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r'} \\ B & \xrightarrow{\;d\;} & D \end{array} \tag{17}$$

with $\rho(r') = 1$. A regular $f \in \mathcal{A}$ is a simple epimorphism if and only if $\rho(f) = 1$.

Proof. Assume first that r is a simple epimorphism. If c does not factor though r, then the dual of Proposition 8 implies that $\binom{c}{r}$ is a kernel. Hence there exists an exact square (17) with a simple epimorphism r'. Now let $f\colon E \to B$ be a regular morphism with $\rho(f) = 1$. By Proposition 9, it remains to show that f is a simple monomorphism. By Proposition 8, there exists a factorization $f = rs$ with a simple monomorphism r. We show that s is epic. Thus let c be a morphism with $cs = 0$. If c factors through r, then $c = 0$ since rs is regular. Otherwise, by Proposition 9, the above argument yields a commutative diagram (17) with a simple monomorphism r'. Thus $df = drs = r'cs = 0$, and therefore, $d = 0$. Hence $r'c = 0$, which gives $c = 0$. This shows that r, s are regular, whence s is invertible. □

Lemma 5. *Let $\mathcal{A}$ be an L-finite dualizing strict τ-category. If B is an irreducible object and $r\colon A \to B$ a simple monomorphism, then $\rho(A) \leqslant 1$.*

Proof. If r is not regular, then r is a kernel by the Corollary of Proposition 8. Then $A = 0$. So let r be regular, and let $c\colon A \twoheadrightarrow C$ be a cokernel. Then Lemma 4 yields a commutative diagram (17) with r' regular and $\rho(r') \leqslant 1$. If $\rho(r') = 0$, then d is a cokernel, whence $C \cong D = 0$. Otherwise, we may assume that (17) is exact. Then $d = \operatorname{cok}(r \cdot \ker c)$, and thus d is invertible. Hence c is invertible. $\square$

Lemma 6. *Let $\mathcal{A}$ be an L-finite dualizing strict τ-category with $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A} = 0$. Every non-zero morphism $f\colon A \to B$ with B irreducible admits a factorization $f = rc$ with a cokernel c and a regular morphism r having a composition series.*

Proof. If f is not a cokernel, then Proposition 8 yields a factorization $f = r_1 f'$ with a simple monomorphism $r_1\colon B_1 \to B$. By Lemma 5, $\rho(B_1) = 1$. So we can apply the same argument to f', which leads to a strictly decreasing sequence $B > B_1 > B_2 \cdots$ of subobjects. As the B_i are irreducible, the inclusions $B_{i+1} \to B_i$ belong to $\operatorname{Rad} \mathcal{A}$. Therefore, we end up with a factorization $f\colon A \overset{c}{\twoheadrightarrow} B_n \overset{r}{\to} B$, where r is regular with a composition series of length n. $\square$

Now we are ready to prove our main theorem.

Theorem 5. *Let $\mathcal{A}$ be an L-finite dualizing strict τ-category with $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A} = 0$. Then $\mathcal{A}$ is a bi-noetherian integral quasi-abelian category.*

Proof. Let $A_0 < A_1 < \cdots$ be a strictly increasing infinite sequence of subobjects of A. If A is irreducible, then Lemma 6 implies that the inclusions $A_i \to A$ are regular with a composition series. Therefore, the A_i with $i \geqslant 1$ are irreducible by Lemma 5. So the inclusions $A_i \to A_{i+1}$ belong to $\operatorname{Rad} \mathcal{A}$, whence $A_0 \to A$ is in $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A}$, a contradiction. Now we proceed by induction. By Lemma 2, there exists a rational composition series $0 \to \cdots \rightarrowtail B \rightarrowtail A$. This gives a short exact sequence $B \rightarrowtail A \twoheadrightarrow C$ with C irreducible. If $A_i \leqslant B$ for all i, we are done. Otherwise, the composition $p_i\colon A_i \to A \twoheadrightarrow C$ is non-zero for some $n \in \mathbb{N}$. By Lemma 6, $p_i = r_i c_i$ with a cokernel c_i and regular r_i. So there are commutative diagrams

$$\begin{array}{ccccc} B_i & \rightarrowtail & A_i & \overset{c_i}{\twoheadrightarrow} & C_i \\ \downarrow & & \downarrow & & \downarrow \\ B_j & \rightarrowtail & A_j & \overset{c_j}{\twoheadrightarrow} & C_j \end{array} \qquad (18)$$

with short exact rows and monic vertical morphisms for $n \leqslant i \leqslant j$. By the inductive hypothesis, the ascending sequences of subobjects $B_n \leqslant B_{n+1} \leqslant \cdots \leqslant B$ and $C_n \leqslant C_{n+1} \leqslant \cdots \leqslant C$ become stationary, i. e. $B_i = B_{i+1}$ and $C_i = C_{i+1}$ for some $i \geqslant n$. Taking a cokernel $P \twoheadrightarrow A_{i+1}$ with P projective, it follows easily that the left-hand square in (18) with $j = i + 1$ is a pushout. Hence $A_i = A_{i+1}$, a contradiction. By duality, this proves that $\mathcal{A}$ is bi-noetherian.

Next let $f\colon\ A \to B$ be any non-zero morphism in $\mathcal{A}$. Consider a rational composition series $0 \to \cdots \rightarrowtail D \overset{d}{\rightarrowtail} B$, and let $c\colon\ B \twoheadrightarrow C$ be the cokernel of d. Then C is irreducible. We shall prove, by induction, that f has a kernel. By Lemma 6, there is a factorization $cf = rc'$ with a cokernel c' and a regular morphism r. This gives a commutative diagram

$$\begin{array}{ccccc} D' & \overset{d'}{\rightarrowtail} & A & \overset{c'}{\twoheadrightarrow} & C' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle r} \\ D & \overset{d}{\rightarrowtail} & B & \overset{c}{\twoheadrightarrow} & C \end{array}$$

with exact rows. By our inductive hypothesis and Lemma 6, there is a kernel $k\colon\ K \rightarrowtail D'$ of g. Now it is easily verified that $d'k = \ker f$. By duality, this shows that $\mathcal{A}$ is preabelian, hence quasi-abelian by Proposition 2.

Since $\mathcal{A}$ is bi-noetherian, it follows that regular morphisms have a composition series. By Lemma 4 and [21], Proposition 6, this implies that $\mathcal{A}$ is integral. □

Let us call an additive category $\mathcal{A}$ *strongly noetherian* if the category $\mathbf{mod}(\mathcal{A})$ of coherent functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is abelian and noetherian. (For equivalent descriptions of $\mathbf{mod}(\mathcal{A})$, see [21], and [28], Proposition 5.) More explicitly, this property can be expressed as follows. A non-empty class Σ of morphisms $f\colon\ A_f \to A$ in $\mathcal{A}$ is said to be an (additive) *sieve* [6] of A if for $f, g \in \Sigma$, the morphism $(f\ g)\colon\ A_f \oplus A_g \to A$ and each composite morphism fh with $h \in \mathcal{A}$ belongs to Σ. Now $\mathcal{A}$ is strongly noetherian if and only if every sieve Σ of any object of $\mathcal{A}$ is *principal*, i. e. every morphism in Σ factors through a fixed $f \in \Sigma$. We call $\mathcal{A}$ *strongly bi-noetherian* if $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ is strongly noetherian. For example, a ring R is left noetherian if and only if the category R-$\mathbf{proj}$ of finitely generated projective left R-modules is strongly noetherian. $\mathcal{A}$ will be called a *strong* lattice category if $\mathcal{A} \approx \Lambda$-$\mathbf{mod} \approx (\Gamma\text{-}\mathbf{mod})^{\mathrm{op}}$ with Λ, Γ noetherian.

Corollary 1. *Let $\mathcal{A}$ be an L-finite dualizing strict τ-category with $\bigcap_{n=1}^{\infty} \mathrm{Rad}^n \mathcal{A} = 0$. Then $\mathcal{A}$ is strongly bi-noetherian.*

Proof. Let Σ be a sieve of $A \in \operatorname{Ob}\mathcal{A}$. Since $\mathcal{A}$ is semi-abelian, every $f\colon A_f \to A$ in Σ admits a factorization $f = m_f c_f$ with a cokernel c_f and a monomorphism m_f. Since $\mathcal{A}$ is noetherian, there exists an $h \in \Sigma$ such that every $f \in \Sigma$ factors through $m_h\colon B \to A$. Therefore, replacing A by B, we may assume, without loss of generality, that there is a cokernel $f_0 \in \Sigma$. By [22], Corollary of Proposition 9, there exists an integer $n \in \mathbb{N}$ such that every morphism $f\colon A_f \to A$ in $\operatorname{Rad}^n\mathcal{A}$ belongs to $[\mathbf{Proj}(\mathcal{A})]$, and thus factors through f_0. Now we construct a finite sequence $f_0, f_1, \dots, f_m$ in Σ such that every $f \in \Sigma$ factors through $(f_0, f_1, \dots, f_m)\colon A_{f_0} \oplus \cdots \oplus A_{f_m} \to A$. Define $R^i := \operatorname{Rad}^i\mathcal{A} \smallsetminus \operatorname{Rad}^{i+1}\mathcal{A}$ and $u_i\colon \vartheta^i A \to \cdots \xrightarrow{u_{\vartheta A}} \vartheta A \xrightarrow{u_A} A$. Let i be the greatest integer $< n$ with $\Sigma \cap R^i \neq \varnothing$. Then there is a morphism $f_1 = u_i d_1 \in \Sigma$ with $d_1\colon D_1 \rightarrowtail \vartheta^i A$ split monic and D_1 indecomposable. So we have $\vartheta^i A = D_1 \oplus C$. Denote the injection $\binom{0}{1}\colon C \rightarrowtail D_1 \oplus C$ by d_1'. If there exists a morphism in $\Sigma \cap R^i$ which factors through $u_i d_1'$, then there is an $f_2 = u_i d_1' d_2 \in \Sigma$ with a split monomorphism $d_2\colon D_2 \rightarrowtail C$ and D_2 indecomposable. After finitely many steps, we get a sequence $f_0, f_1, \dots, f_j$ such that every $f \in \Sigma \cap R^i$ factors through $(f_0, f_1, \dots, f_j)$. Therefore, modulo $(f_0, f_1, \dots, f_j)$, we can replace i by a smaller integer. By induction, this proves the corollary. $\square$

Corollary 2. *Let $\mathcal{A}$ be a Krull-Schmidt category with finitely many isomorphism classes of indecomposable objects. The following are equivalent.*

(a) *$\mathcal{A}$ is a strong lattice category.*

(b) *$\mathcal{A}$ is an L-finite strict τ-category with $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n\mathcal{A} = 0$, having an additive function $l > 0$.*

Proof. (a) $\Rightarrow$ (b): By [26], $\mathcal{A}$ is a strict τ-category. The remaining assertions follow by [26], and the implication (b) $\Rightarrow$ (c) of [22], Theorem 4 (see also [22], Proposition 10).

(b) $\Rightarrow$ (a): Proposition 9, Theorem 5, and Proposition 7 imply that $\mathcal{A}$ satisfies (a)-(c) of Theorem 1. Suppose that there is a non-zero object A in $\mathcal{A}_\circ$. Since A is noetherian, there exists a simple monomorphism $m\colon B \to A$. Since $\mathcal{A}$ is L-finite, $l(m) = l(u_P) = 0$ for some projective object P. On the other hand, $A \in \operatorname{Ob}\mathcal{A}_\circ$ implies that m is a kernel, which gives a contradiction. Hence $\mathcal{A}$ is a lattice category. By Corollary 1, $\mathbf{Proj}(\mathcal{A})$ and $\mathbf{Inj}(\mathcal{A})$ are strongly bi-noetherian. Thus $\mathcal{A}$ is a strong lattice category. $\square$

By the remark of §4, L-finiteness of a strict τ-category is a property of its Auslander-Reiten quiver. Therefore, we may speak of an *L-finite* translation quiver Q. By [10], Theorem 7.1, this property of Q can be checked easily. A translation quiver Q with valuation (d, d') is said to be *admissible* [12] if there exists a function c: $Q \to \mathbb{N} \setminus \{0\}$ with $c_X = c_{\tau X}$ for non-projective vertices X, and

$$c_X d_{XY} = d'_{XY} c_Y$$

for all $X, Y \in Q$.

Corollary 3. *For a finite admissible translation quiver Q, the following are equivalent.*

(a) *There exists an order Λ over a complete discrete valuation ring R with $Q = \mathbb{A}(\Lambda\text{-}\mathbf{CM})$.*

(b) *Q is L-finite and admits an additive function $l > 0$.*

Proof. By [12], 4.2.1, there exists a modulation for Q, and the mesh category $\mathcal{A}$ is R-linear for some complete discrete valuation ring R. Moreover, $\bigcap_{n=1}^{\infty} \operatorname{Rad}^n \mathcal{A} = 0$. By [22], Proposition 8, the existence of an additive function $l > 0$ implies that $\mathcal{A}$ is a strict τ-category. Therefore, the equivalence $(a) \Leftrightarrow (b)$ follows by Corollary 2. □

Remark. *There are 0-dimensional analogues of Corollary 2 and Corollary 3 that also follow by Theorem 5. Here the additive function l has to be replaced by a function $l > 0$ with $l(P) = l(\vartheta P)+1$ and $l(I) = l(\vartheta^- I)+1$ for indecomposable P, I with P projective and I injective. Furthermore, the condition of L-finiteness can be dropped since $Rad\,\mathcal{A}$ is nilpotent in this case.*

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