Algebra and Discrete Mathematics
Volume **20** (2015). Number 1, pp. 40–54
RESEARCH ARTICLE

New families of Jacobsthal and Jacobsthal-Lucas numbers

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Communicated by V. Mazorchuk

Abstract. In this paper we present new families of sequences that generalize the Jacobsthal and the Jacobsthal-Lucas numbers and establish some identities. We also give a generating function for a particular case of the sequences presented.

Introduction

Several sequences of positive integers were and still are object of study for many researchers. Examples of these sequences are the well known Fibonacci sequence and the Lucas sequence, both related with the golden mean, with so many applications in diverse fields such as mathematics, engineering, biology, physics, architecture, stock market investing, among others (see [9] and [17]). About these and other sequences like Pell sequence, Pell-Lucas sequence, Modified Pell sequence, Jacobsthal sequence and the Jacobsthal-Lucas sequence, among others, there is a vast literature where several properties are studied and well known identities are derived, see for example, [13, 18–20].

In 1965, Horadam studied some properties of sequences of the type, $w_n(a, b; p, q)$, where a, b are nonnegative integers and p, q are arbitrary

2010 MSC: 11B37, 11B83, 05A15.

Key words and phrases: Jacobsthal numbers, Jacobsthal-Lucas numbers, Binet formula, Generating matrix, Generating function.

integers, see [11] and [12]. Such sequences are defined by the recurrence relations of second order

$$w_n = pw_{n-1} - qw_{n-2}, (n \geqslant 2)$$

with initial conditions $w_0 = a$, $w_1 = b$. For example, the Fibonacci and the Lucas sequences can be considered as special cases of sequences of this type, $w_n(1,1;1,-1)$ and $w_n(2,1;1,-1)$, respectively. Also, the Jacobsthal and the Jacobsthal-Lucas sequences can be considered as $w_n(0,1;1,-2)$ and $w_n(2,1;1,-2)$, respectively. Recall that the second-order recurrence relations and the initial conditions for the Jacobsthal numbers, $J_n,\ n \geqslant 0$, and for the Jacobsthal-Lucas numbers, $j_n,\ n \geqslant 0$, respectively, are given by

$$J_{n+2} = J_{n+1} + 2J_n,\ J_0 = 0,\ J_1 = 1$$

and

$$j_{n+2} = j_{n+1} + 2j_n,\ j_0 = 2,\ j_1 = 1.$$

The Binet formulae for these sequences are

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n,$$

where 2 and -1 are the roots of the characteristic equation associated with the above recurrence relations.

More recently, some of these sequences were generalized for any positive real number k. The studies of k-Fibonacci sequence, k-Lucas sequence, k-Pell sequence, k-Pell-Lucas sequence, Modified k-Pell sequence, k-Jacobhstal and k-Jacobhstal-Lucas sequence, can be found in [1,3–7,14].

The aim of this work is to study some properties of two new sequences that generalize the Jacobhstal and the Jacobsthal-Lucas numbers. In this work we will follow closely the work of El-Mikkawy and Sogabe (see [10]) where the authors give a new family that generalizes the Fibonacci numbers, different from the one defined in [1], and establish relations with the ordinary Fibonacci numbers.

So, in this Section we start giving the new definition of generalized Jacobsthal and Jacobsthal-Lucas numbers, and we exhibit some elements of them. We also present relations of these sequences with ordinary Jacobsthal and Jacobsthal-Lucas. In Section 1 we deduce some properties of these new families, as well as in Section 2, but using different methods. In Section 3, we study a particular case, that is two sequences of the new defined families for $k = 2$. For these sequences we present some recurrence relations and generating functions.

Following our ideas, we give a new definition of generalized Jacobsthal and Jacobsthal-Lucas numbers.

Definition 1. Let n be a nonnegative integer and k be a natural number. By the division algorithm there exist unique numbers m and r such that $n = mk + r$ $(0 \leqslant r < k)$. Using these parameters we define the new generalized Jacobsthal and generalized Jacobsthal-Lucas numbers, $J_n^{(k)}$ and $j_n^{(k)}$ respectively by

$$J_n^{(k)} = \frac{1}{(r_1 - r_2)^k} \left(r_1^{m+1} - r_2^{m+1}\right)^r \left(r_1^m - r_2^m\right)^{k-r} \tag{1}$$

and

$$j_n^{(k)} = \left(r_1^{m+1} + r_2^{m+1}\right)^r \left(r_1^m + r_2^m\right)^{k-r}, \tag{2}$$

where $r_1 = 2$, $r_2 = -1$, respectively.

For $k = 1, 2, 3$ the first seven elements of these new sequences are:

$$\{J_n^{(1)}\}_{n=0}^{5} = \{0, 1, 1, 3, 5, 11, 21\} \quad \{j_n^{(1)}\}_{n=0}^{5} = \{2, 1, 5, 7, 17, 31, 65\}$$

$$\{J_n^{(2)}\}_{n=0}^{5} = \{0, 0, 1, 1, 1, 3, 9\} \quad \{j_n^{(2)}\}_{n=0}^{5} = \{4, 2, 1, 5, 25, 35, 49\}$$

$$\{J_n^{(3)}\}_{n=0}^{5} = \{0, 0, 0, 1, 1, 1, 1\} \quad \{j_n^{(3)}\}_{n=0}^{5} = \{8, 4, 2, 1, 5, 25, 125\}.$$

We also present more elements of some of these new sequences in the tables 1 and 2. We have found some interesting regularities. In the case of the generalized Jacobsthal sequences $\{J_n^{(k)}\}_n$ it is easy to prove that:

Proposition 1. *Let $J_i^{(k)}$ be the i^{th} term of the new family of Jacobsthal numbers. Then we have:*

a) $J_i^{(k)} = 0, \quad i \in \{0, \ldots, k-1\}$;

b) $J_i^{(k)} = 1, \quad i \in \{k, \ldots, k-1\}$;

c) $J_i^{(k)} = 3^{i-2k}, \quad i \in \{2k, \ldots, 3k\}$.

To the generalized Jacobsthal-Lucas sequences $\left\{j_n^{(k)}\right\}_n$ is easy to prove that:

Table 1. $J_n^{(k)}$, for $k = 1, 2, \ldots, 9$ and $n = 0, 1, \ldots, 27$.

n\k	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	3	1	1	0	0	0	0	0	0
4	5	1	1	1	0	0	0	0	0
5	11	3	1	1	1	0	0	0	0
6	21	9	1	1	1	1	0	0	0
7	43	15	3	1	1	1	1	0	0
8	85	25	9	1	1	1	1	1	0
9	171	55	27	3	1	1	1	1	1
10	341	121	45	9	1	1	1	1	1
11	683	231	75	27	3	1	1	1	1
12	1365	441	125	81	9	1	1	1	1
13	2731	903	275	135	27	3	1	1	1
14	5461	1849	605	225	81	9	1	1	1
15	10923	3655	1331	375	243	27	3	1	1
16	21845	7225	2541	625	405	81	9	1	1
17	43691	14535	4851	1375	675	243	27	3	1
18	87381	29241	9261	3025	1125	729	81	9	1
19	174763	58311	18963	6655	1875	1215	243	27	3
20	349525	116281	38829	14641	3125	2025	729	81	9
21	699051	232903	79507	27951	6875	3375	2187	243	27
22	1398101	466489	157165	53361	15125	5625	3645	729	81
23	2796203	932295	310675	101871	33275	9375	6075	2187	243
24	5592405	1863225	614125	194481	73205	15625	10125	6561	729
25	11184811	3727815	1235475	398223	161051	34375	16875	10935	2187
26	22369621	7458361	2485485	815409	307461	75625	28125	18225	6561
27	44739243	14913991	5000211	1669647	586971	166375	46875	30375	19683

Table 2. $j_n^{(k)}$, for $k = 1, 2, \ldots, 9$ and $n = 0, 1, \ldots, 27$.

n\k	1	2	3	4	5	6	7	8	9
0	2	4	8	16	32	64	128	256	512
1	1	2	4	8	16	32	64	128	256
2	5	1	2	4	8	16	32	64	128
3	7	5	1	2	4	8	16	32	64
4	17	25	5	1	2	4	8	16	32
5	31	35	25	5	1	2	4	8	16
6	65	49	125	25	5	1	2	4	8
7	127	119	175	125	25	5	1	2	4
8	257	289	245	625	125	25	5	1	2
9	511	527	343	875	625	125	25	5	1
10	1025	961	833	1225	3125	625	125	25	5
11	2047	2015	2023	1715	4375	3125	625	125	25
12	4097	4225	4913	2401	6125	15625	3125	625	125
13	8191	8255	8959	5831	8575	21875	15625	3125	625
14	16385	16129	16337	14161	12005	30625	78125	15625	3125
15	32767	32639	29791	34391	16807	42875	109375	78125	15625
16	65537	66049	62465	83521	40817	60025	153125	39062	78125
17	131071	131327	130975	152303	99127	84035	214375	546875	390625
18	262145	261121	274625	277729	240737	117649	300125	765625	1953125
19	524287	523775	536575	506447	584647	285719	420175	1071875	2734375
20	1048577	1050625	1048385	923521	1419857	693889	588245	1500625	3828125
21	2097151	2098175	2048383	1936415	2589151	1685159	823543	2100875	5359375
22	4194305	4190209	4145153	4060225	4721393	4092529	2000033	2941225	7503125
23	8388607	8386559	8388223	8513375	8609599	9938999	4857223	4117715	10504375
24	16777217	16785409	16974593	17850625	15699857	24137569	11796113	5764801	14706125
25	33554431	33558527	33751039	34877375	28629151	44015567	28647703	14000231	20588575
26	67108865	67092481	67108097	68145025	60028865	80263681	69572993	34000561	28824005
27	134217727	134209535	133432831	133144895	125866975	146363183	168962983	82572791	40353607

Proposition 2. *Let $j_i^{(k)}$ be the i^{th} term of the new family of Jacobsthal-Lucas numbers. Then we have:*

a) $j_i^{(k)} = 2^{k-i}, \quad i \in \{0, \ldots, k-1\}$;

b) $j_i^{(k)} = 5^{i-k}, \quad i \in \{k, \ldots, 2k\}$.

The generalized Jacobsthal and Jacobsthal-Lucas numbers have the following relations with the ordinary Jacobsthal and Jacobsthal-Lucas numbers.

Lemma 1. *Given n a nonnegative integer and k a natural number*

$$J_{mk+r}^{(k)} = (J_m)^{k-r}(J_{m+1})^r$$

and

$$j_{mk+r}^{(k)} = (j_m)^{k-r}(j_{m+1})^r,$$

where m and r are nonnegative integers such that $n = mk+r$ $(0 \leqslant r < k)$.

Proof. We have

$$\begin{aligned}(J_m)^{k-r}(J_{m+1})^r &= \left(\frac{2^m - (-1)^m}{3}\right)^{k-r}\left(\frac{2^{m+1} - (-1)^{m+1}}{3}\right)^r \\ &= \frac{1}{3^k}\left(2^m - (-1)^m\right)^{k-r}\left(2^{m+1} - (-1)^{m+1}\right)^r \\ &= \frac{1}{(r_1 - r_2)^k}\left(r_1^m - r_2^m\right)^{k-r}\left(r_1^{m+1} - r_2^{m+1}\right)^r \\ &= J_{mk+r}^{(k)}.\end{aligned}$$

In a similar way we can show the second equality. □

Note that the use of the Lemma 1 allows us to conclude immediately that $J_n^{(1)}$ and $j_n^{(1)}$ are the Jacobsthal and the Jacobsthal-Lucas numbers, respectively.

1. Properties

Next we present some properties of these new families of integers.

Theorem 1. *Let k and m be fixed numbers where m is a nonnegative integer and k a natural number. The generalized Jacobsthal numbers and the ordinary Jacobsthal numbers satisfy:*

a) $\sum_{i=0}^{k-1} \binom{k-1}{i}(-1)^{-i} J_{mk+i}^{(k)} = (-2)^{k-1} J_m J_{(m-1)(k-1)}^{(k-1)}$;

b) $\sum_{i=0}^{k-1} \binom{k-1}{i} 2^{k-i-1} J_{mk+i}^{(k)} = J_m J_{(m+2)(k-1)}^{(k-1)}$;

c) $\sum_{i=0}^{k-1} J_{mk+i}^{(k)} = \frac{J_m}{2J_{m-1}} \left(J_{(m+1)k}^{(k)} - J_{mk}^{(k)} \right)$.

Proof. a) By Lemma 1 we have that $\sum_{i=0}^{k-1} \binom{k-1}{i}(-1)^{-i} J_{mk+i}^{(k)}$ is successively equal to

$$\begin{aligned}
&\sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{-i}(-1)^{k-1}(-1)^{1-k}(J_m)^{k-i}(J_{m+1})^i \\
&\quad = (-1)^{1-k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i}(J_m)^{k-1-i} J_m (J_{m+1})^i \\
&\quad = (-1)^{1-k} J_m \sum_{i=0}^{k-1} \binom{k-1}{i} (-J_m)^{k-1-i}(J_{m+1})^i,
\end{aligned}$$

that by the binomial theorem is equal to

$$(-1)^{1-k} J_m \left(J_{m+1} - J_m\right)^{k-1}.$$

Since, by the definition of the Jacobsthal sequence,

$$(-1)^{1-k} J_m \left(J_{m+1} - J_m\right)^{k-1} = (-1)^{1-k} J_m \left(2J_{m-1}\right)^{k-1}$$

and using Lemma 1 (considering $m-1$ instead of m, $k-1$ instead of k and $r=0$) we obtain

$$(-1)^{1-k} 2^{k-1} J_m J_{(m-1)(k-1)}^{(k-1)},$$

and the result follows.

b) By Lemma 1 we have

$$\begin{aligned}
\sum_{i=0}^{k-1} \binom{k-1}{i} 2^{k-i-1} J_{mk+i}^{(k)} &= \sum_{i=0}^{k-1} \binom{k-1}{i} (J_m)^{k-i} 2^{k-i-1} (J_{m+1})^i \\
&= \sum_{i=0}^{k-1} \binom{k-1}{i} (2J_m)^{k-i-1} J_m (J_{m+1})^i \\
&= J_m \sum_{i=0}^{k-1} \binom{k-1}{i} (2J_m)^{k-i-1} (J_{m+1})^i
\end{aligned}$$

and using again the binomial theorem we have

$$J_m(J_{m+1}+2J_m)^{k-1},$$

that is equal, by the definition of the Jacobsthal numbers, to

$$J_m(J_{m+2})^{k-1}$$

and by Lemma 1 (considering $m+2$ instead of m, $k-1$ instead of k and $r=0$), we get

$$J_m J^{(k-1)}_{(m+2)(k-1)}.$$

c) By Lemma 1 we can write

$$\begin{aligned}\sum_{i=0}^{k-1} J^{(k)}_{mk+i} &= \sum_{i=0}^{k-1}(J_m)^{k-i}(J_{m+1})^i \\ &= (J_m)^k \sum_{i=0}^{k-1}\left(\frac{J_{m+1}}{J_m}\right)^i \\ &= (J_m)^k \left[\frac{\left(\frac{J_{m+1}}{J_m}\right)^k - 1}{\frac{J_{m+1}}{J_m} - 1}\right] \\ &= (J_m)^k \left[\frac{(J_{m+1})^k-(J_m)^k}{(J_m)^k} \times \frac{J_m}{J_{m+1}-J_m}\right] \\ &= \frac{J_m}{J_{m+1}-J_m}\left[(J_{m+1})^k-(J_m)^k\right] \\ &= \frac{J_m}{2J_{m-1}}\left[(J_{m+1})^k-(J_m)^k\right]\end{aligned}$$

and, taking into account Lemma 1 (with $r=0$), the result follows. □

The following result for Jacobsthal-Lucas numbers can be deduced analogously:

Theorem 2. *Let k and m be fixed numbers where m is a nonnegative integer and k a natural number. The generalized Jacobsthal-Lucas numbers and the ordinary Jacobsthal-Lucas numbers satisfy:*

a) $\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{-i} j^{(k)}_{mk+i} = (-2)^{k-1} j_m j^{(k-1)}_{(m-1)(k-1)}$;

b) $\sum_{i=0}^{k-1}\binom{k-1}{i}2^{k-i-1} j^{(k)}_{mk+i} = j_m j^{(k-1)}_{(m+2)(k-1)}$;

c) $\sum_{i=0}^{k-1} j^{(k)}_{mk+i} = \frac{j_m}{2j_{m-1}}\left(j^{(k)}_{(m+1)k} - j^{(k)}_{mk}\right)$.

2. Generating matrices

In [15] the authors use a matrix method for generating the Jacobsthal numbers by defining the Jacobsthal A-matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

and they proved that

$$A^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} = A^{(n-1)} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

for any natural number n.

Thus, for any $n \geqslant 0$, $s \geqslant 0$ and $n + s \geqslant 2$, we have

$$\begin{bmatrix} J_{n+s} & 2J_{n+s-1} \\ J_{n+s-1} & 2J_{n+s-2} \end{bmatrix} = A^{(n+s-2)} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

If we compute the determinant of both sides of the previous equality we obtain

$$2J_{n+s}J_{n+s-2} - 2\left(J_{n+s-1}\right)^2 = -2\left|A^{(n+s-2)}\right|$$

which is equivalent to

$$\left(J_{n+s-1}\right)^2 - J_{n+s}J_{n+s-2} = (-2)^{n+s-2}.$$

Since, by Lemma 1 (where $m = n + s - 1$, $k = 2$ and $r = 0$)

$$J^{(2)}_{2(n+s-1)} = \left(J_{n+s-1}\right)^2,$$

we have proved the following result:

Theorem 3. *If $n, s \geqslant 0$ and $n + s \geqslant 2$, then*

$$J^{(2)}_{2(n+s-1)} - J_{n+s}J_{n+s-2} = (-2)^{n+s-2}.$$

Also, by considering the generating Jacobsthal-Lucas B-matrix given in [16] and in [8]

$$B = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$$

and proceeding in a similar way as we did for Jacobsthal numbers, we obtain for $n, s \geqslant 0$ and $n + s \geqslant 2$,

$$\begin{bmatrix} j_{n+s} & 2j_{n+s-1} \\ j_{n+s-1} & 2j_{n+s-2} \end{bmatrix} = B^{(n+s-2)} \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}.$$

Computing the determinant of both sides of this equality we get

$$2j_{n+s}j_{n+s-2} - 2\left(j_{n+s-1}\right)^2 = \left(3^2 2\right)^{(n+s-2)} \times \left(3^2 2\right)$$

which is equivalent to

$$j_{n+s}j_{n+s-2} - \left(j_{n+s-1}\right)^2 = 3^{2(n+s-1)} 2^{(n+s-2)}.$$

Using Lemma 1 again (with $m = n + s - 1$, $k = 2$ and $r = 0$) we obtain the following result:

Theorem 4. *If $n, s \geqslant 0$ and $n + s \geqslant 2$, then*

$$j_{n+s}j_{n+s-2} - j^{(2)}_{2(n+s-1)} = 3^{2(n+s-1)} 2^{(n+s-2)}.$$

3. A particular case

In this section we study the particular case of the sequences $\{J_n^{(2)}\}_n$ and $\{j_n^{(2)}\}_n$ defined by (1) and (2), respectively, with $k = 2$.

3.1. Recurrence relations

First we present a recurrence relation for these sequences.

Theorem 5. *The sequences $\{J_n^{(2)}\}_n$ and $\{j_n^{(2)}\}_n$ satisfy, respectively, the following recurrence relations:*

$$J_n^{(2)} = J_{n-1}^{(2)} + 2J_{n-3}^{(2)} + 4J_{n-4}^{(2)}, \ n = 4, 5, \ldots$$

and

$$j_n^{(2)} = j_{n-1}^{(2)} + 2j_{n-3}^{(2)} + 4j_{n-4}^{(2)}, \ n = 4, 5, \ldots$$

Proof. First, we consider n even, that is $n = 2m$, for any natural number m. In this case, using Lemma 1, we have

$$\begin{aligned} J^{(2)}_{2m} &= (J_m)^2 = J_m J_m \\ &= J_m\left(J_{m-1} + 2J_{m-2}\right) \\ &= J_m J_{m-1} + 2J_m J_{m-2} \\ &= J_m J_{m-1} + 2\left(J_{m-1} + 2J_{m-2}\right) J_{m-2} \\ &= J^{(2)}_{2m-1} + 2J_{m-1}J_{m-2} + 4(J_{m-2})^2 \\ &= J^{(2)}_{2m-1} + 2J^{(2)}_{2m-3} + 4(J_{m-2})^2 \\ &= J^{(2)}_{2m-1} + 2J^{(2)}_{2m-3} + 4J^{(2)}_{2m-4} \end{aligned}$$

as required. Now, for n odd, that is $n = 2m + 1$, for any natural number m and, using again Lemma 1, we obtain:

$$\begin{aligned} J^{(2)}_{2m+1} &= J_m J_{m+1} \\ &= J_m\left(J_m + 2J_{m-1}\right) \\ &= (J_m)^2 + 2J_{m-1}J_m \\ &= J^{(2)}_{2m} + 2J_{m-1}(J_{m-1} + 2J_{m-2}) \\ &= J^{(2)}_{2m} + 2(J_{m-1})^2 + 4J_{m-2}J_{m-1} \\ &= J^{(2)}_{2m} + 2J^{(2)}_{2m-2} + 4J_{m-2}J_{m-1} \\ &= J^{(2)}_{2m} + 2J^{(2)}_{2m-2} + 4J^{(2)}_{2m-3}. \end{aligned}$$

So for every $n = 4, 5, \ldots$ the result is true. In a similar way we can prove the result for $j^{(2)}_n$. □

We also note that if we consider separately the even and the odd terms of the above defined sequences we can obtain shorter recurrence relations. In fact, for $n = 2m$, for any natural number m, by Theorem 3 (with $n = m$ and $s = 1$) we have

$$J^{(2)}_{2m} - J_{m+1}J_{m-1} = (-2)^{m-1}$$

and so

$$\begin{aligned} J^{(2)}_{2m} &= J_{m-1}J_{m+1} + (-2)^{m-1} \\ &= J_{m-1}(J_m + 2J_{m-1}) + (-2)^{m-1} \\ &= J_{m-1}J_m + 2(J_{m-1})^2 + (-2)^{m-1} \\ &= J^{(2)}_{2m-1} + 2J^{(2)}_{2m-2} + (-2)^{m-1}. \end{aligned}$$

In a similar way, if we consider $n = 2m+1$, for any natural number m, we have $J_{2m+1}^{(2)} = J_m J_{m+1}$ that is equal to

$$J_m \left(J_m + 2J_{m-1}\right) = \left(J_m\right)^2 + 2J_{m-1}J_m = J_{2m}^{(2)} + 2J_{2m-1}^{(2)}.$$

Hence, in this case, we can conclude that

$$J_{2m+1}^{(2)} = J_{2m}^{(2)} + 2J_{2m-1}^{(2)}.$$

Therefore we can conclude the following:

Proposition 3. *A shorter recurrence relation for the sequence* $\left\{J_n^{(2)}\right\}_n$ *is given by*

$$\begin{cases} J_{2m}^{(2)} = J_{2m-1}^{(2)} + 2J_{2m-2}^{(2)} + (-2)^{m-1} \\ \\ J_{2m+1}^{(2)} = J_{2m}^{(2)} + 2J_{2m-1}^{(2)} \end{cases}$$

for the even and the odd terms.

In a similar way we obtain a shorter recurrence relation to $\left\{j_n^{(2)}\right\}_n$.

Proposition 4. *A shorter recurrence relation for the sequence* $\left\{j_n^{(2)}\right\}_n$ *is given by*

$$\begin{cases} j_{2m}^{(2)} = j_{2m-1}^{(2)} + 2j_{2m-2}^{(2)} - 3^{2m}2^{m-1} \\ \\ j_{2m+1}^{(2)} = j_{2m}^{(2)} + 2j_{2m-1}^{(2)} \end{cases}$$

for the even and the odd terms.

Proof. The proof of the second identity is similar to the one in the previous proposition. To the first identity, by Theorem 4 we have:

$$j_{m+1}j_{m-1} - j_{2m}^{(2)} = 3^{2m}2^{m-1}.$$

Hence

$$\begin{aligned} j_{2m}^{(2)} &= j_{m+1}j_{m-1} - 3^{2m}2^{m-1} \\ &= j_{m-1}(j_m + 2j_{m-1}) - 3^{2m}2^{m-1} \\ &= j_{m-1}j_m + 2(j_{m-1})^2 - 3^{2m}2^{m-1} \\ &= j_{2m-1}^{(2)} + 2j_{2m-2}^{(2)} - 3^{2m}2^{m-1}. \end{aligned}$$

□

3.2. Generating Functions

Next we find generating functions for these sequences. Let us suppose that the terms of the sequences $\{J_n^{(2)}\}_n$ and $\{j_n^{(2)}\}_n$ are the coefficients of a power series centred at the origin, that is convergent in $\left]-\frac{1}{r_1}, \frac{1}{r_1}\right[$, according the Proposition 2.5 in [14] and [2], respectively, for $k = 2$.

For $\{J_n^{(2)}\}_n$ we obtain the following result:

Theorem 6. *The generating function $f^{(2)}(x)$ for $J_n^{(2)}$ is given by*

$$f^{(2)}(x) = \frac{x^2 + 2x^3}{1 - x - 2x^3 - 4x^4}.$$

Proof. To the sum of this power series,

$$f^{(2)}(x) = \sum_{n=0}^{\infty} J_n^{(2)} x^n,$$

we call generating function of the generalized Jacobsthal sequence of numbers $\{J_n^{(2)}\}_n$.

Then

$$f^{(2)}(x) - xf^{(2)}(x) - 2x^3 f^{(2)}(x) - 4x^4 f^{(2)}(x)$$

is equal to

$$\left(J_0^{(2)} + J_1^{(2)}x + J_2^{(2)}x^2 + J_3^{(2)}x^3\right) - \left(J_0^{(2)}x - J_1^{(2)}x^2 - J_2^{(2)}x^3\right) \\ - 2J_0^{(2)}x^3 + \sum_{n=4}^{\infty} \left(J_n^{(2)} - J_{n-1}^{(2)} - 2J_{n-3}^{(2)} - 4J_{n-4}^{(2)}\right) x^n.$$

Hence, taking into account the initial conditions of the sequence $\{J_n^{(2)}\}_n$, we have

$$\left(1 - x - 2x^3 - 4x^4\right) f^{(2)}(x) = \left(0 + 0x + x^2 + x^3\right) - \left(0x - 0x^2 - x^3\right) \\ - 2 \times 0x^3 + \sum_{n=4}^{\infty} \left(J_n^{(2)} - \left(J_{n-1}^{(2)} + 2J_{n-3}^{(2)} + 4J_{n-4}^{(2)}\right)\right) x^n.$$

Now, by Theorem 5, this is equivalent to

$$\left(1 - x - 2x^3 - 4x^4\right) f^{(2)}(x) = x^2 + 2x^3 + \sum_{n=4}^{\infty} \left(J_n^{(2)} - J_n^{(2)}\right)$$

and therefore

$$f^{(2)}(x) = \frac{x^2 + 2x^3}{1 - x - 2x^3 - 4x^4}. \qquad \square$$

Theorem 7. *The generating function $g^{(2)}(x)$ for $j_n^{(2)}$ is given by*

$$g^{(2)}(x) = \frac{4 - 2x + 3x^2 - 2x^3}{1 - x - 2x^3 - 4x^4}.$$

Proof. To the sum of this power series,

$$g^{(2)}(x) = \sum_{n=0}^{\infty} j_n^{(2)} x^n$$

we call generating function of the generalized Jacobsthal-Lucas sequence of numbers $\{j_n^{(2)}\}_n$.

Then, in a similar way as in the proof of the previous theorem, we obtain

$$\begin{aligned}\left(1 - x - 2x^3 - 4x^4\right) g^{(2)}(x) &= \left(j_0^{(2)} + j_1^{(2)}x + j_2^{(2)}x^2 + j_3^{(2)}x^3\right) \\ &\quad - \left(j_0^{(2)}x - j_1^{(2)}x^2 - j_2^{(2)}x^3\right) - 2j_0^{(2)}x^3 \\ &\quad + \sum_{n=4}^{\infty} \left(j_n^{(2)} - j_{n-1}^{(2)} - 2j_{n-3}^{(2)} - 4j_{n-4}^{(2)}\right) x^n.\end{aligned}$$

Taking into account the initial conditions of the sequence $\{j_n^{(2)}\}_n$, we have

$$\begin{aligned}&\left(1 - x - 2x^3 - 4x^4\right) g^{(2)}(x) = \left(4 + 2x + x^2 + 5x^3\right) \\ &- \left(4x - 2x^2 - x^3\right) - 8x^3 + \sum_{n=4}^{\infty} \left(j_n^{(2)} - \left(j_{n-1}^{(2)} + 2j_{n-3}^{(2)} + 4j_{n-4}^{(2)}\right)\right) x^n.\end{aligned}$$

Now, by Theorem 5, this is equivalent to

$$\left(1 - x - 2x^3 - 4x^4\right) g^{(2)}(x) = 4 - 2x + 3x^2 - 2x^3 + \sum_{n=4}^{\infty} \left(j_n^{(2)} - j_n^{(2)}\right) x^n$$

and therefore

$$g^{(2)}(x) = \frac{4 - 2x + 3x^2 - 2x^3}{1 - x - 2x^3 - 4x^4}. \qquad \square$$

4. Conclusion

In this paper we have presented new families of sequences, $J_n^{(k)}$ and $j_n^{(k)}$, that generalize the Jacobsthal and the Jacobsthal-Lucas sequences and we have established some identities involving them.

We also gave generating functions for generalized Jacobsthal and Jacobsthal-Lucas sequences $\{J_n^{(2)}\}_n$ and $\{j_n^{(2)}\}_n$.

When we were looking for more elements of these new families we have found, first, that these families were not in the Encyclopedia of Integer Sequences [21]. Furthermore, we have found some interesting regularities, stated in Propositions 1 and 2.

Acknowledgements

This work has been supported by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia - under the project PEst-OE/MAT/ UI4080/2014.

References

[1] C. Bolat, H. Kőse, *On the Properties of k-Fibonacci Numbers*. Int. J. Contemp. Math. Sci. 5 (22) (2010) 1097-1105.

[2] H. Campos, P. Catarino, A. P. Aires, P. Vasco, A. Borges, *On some Identities of k-Jacobsthal-Lucas Numbers*. Int. J. Math. Anal. (Ruse) 8 (10) (2014) 489-494.

[3] P. Catarino, *On Some Identities and Generating Functions for k-Pell Numbers*. Int. J. Math. Anal. (Ruse) 7 (38) (2013) 1877 - 1884.

[4] P. Catarino, *On some identities for k-Fibonacci sequence*. Int. J. Contemp. Math. Sci. 9 (1) (2014) 37-42.

[5] P. Catarino, P. Vasco, *On some Identities and Generating Functions for k-Pell-Lucas sequence*. Appl. Math. Sci. 7 (98) (2013) 4867-4873.

[6] P. Catarino, P. Vasco, *Modified k-Pell Sequence: Some Identities and Ordinary Generating Function*. Appl. Math. Sci. 7 (121) (2013) 6031 - 6037.

[7] P. Catarino, P. Vasco, *Some basic properties and a two-by-two matrix involving the k-Pell Numbers*. Int. J. Math. Anal. (Ruse) 7 (45) (2013) 2209-2215.

[8] A. Dasdemir, *On the Jacobsthal numbers by matrix method*. Fen Derg. 7 (1) (2012) 69-76.

[9] R. A. Dunlap, *The Golden Ratio and Fibonacci numbers*. World Scientific Press, Singapore, 1997.

[10] M. El-Mikkawy, T. Sogabe, *A new family of k-Fibonacci numbers*. Appl. Math. Comput. 215 (12) (2010) 4456-4461.

[11] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*. Fibonacci Quart. 3 (3) (1965) 161-176.

[12] A. F. Horadam, *Generating functions for powers of a certain generalized sequences of numbers*. Duke Math. J. 32 (3) (1965) 437-446.

[13] A. F. Horadam, *Pell identities*. Fibonacci Quart. 9 (3) (1971) 245-263.

[14] D. Jhala, K. Sisodiya, G. P. S. Rathore, *On Some Identities for k-Jacobsthal Numbers*. Int. J. Math. Anal. (Ruse) 7 (12) (2013) 551-556.

[15] F. Koken, D. Bozkurt, *On the Jacobsthal numbers by matrix method*. Int. J. Contemp. Math. Sci. 3 (13) (2008) 605-614.

[16] F. Koken, D. Bozkurt, *On the Jacobsthal-Lucas numbers by matrix method*. Int. J. Contemp. Math. Sci. 3 (33) (2008) 1629-1633.

[17] T. Koshy, *Fibonacci and Lucas numbers with applications*. John Wiley and Sons. New York, 2001.

[18] T. Koshy, *Fibonacci, Lucas, and Pell numbers, and Pascal's triangle*. Applied Probability Trust. (2011) 125-132.

[19] F. Lu, Z. Jiang, *The sum and product of Fibonacci numbers and Lucas numbers, Pell numbers and Pell-Lucas numbers representation by matrix method*. Wseas Transactions on Mathematics. 12 (4) (2013) 449-458.

[20] D. Marques, *The Order of Appearance of the Product of Consecutive Lucas Numbers*. Fibonacci Quart. 51 (1) (2013) 38-43.

[21] OEIS Foundation Inc., *The On-line Encyclopedia of Integer Sequences*. 2011. http://oeis.org.

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Received by the editors: 18.11.2014
and in final form 31.12.2014.