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RESEARCH ARTICLE

Root vectors of the composition algebra of the Kronecker algebra

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Abstract. According to the canonical isomorphism between the positive part $\mathbf{U}_q^+(\mathbf{g})$ of the Drinfeld–Jimbo quantum group $\mathbf{U}_q(\mathbf{g})$ and the generic composition algebra $\mathcal{C}(\Delta)$ of Λ, where the Kac–Moody Lie algebra $\mathbf{g}$ and the finite dimensional hereditary algebra Λ have the same diagram, in specially, we get a realization of quantum root vectors of the generic composition algebra of the Kronecker algebra by using the Ringel–Hall approach. The commutation relations among all root vectors are given and an integral PBW–basis of this algebra is also obtained.

1. Introduction

According to Lusztig [12], a Cartan datum is a pair $\Delta = (I, (\ , \ ))$ consisting of a finite set I and a symmetric bilinear form on the free abelian group $\mathbb{Z}[I]$. It is assumed that (a) $(i,i) \in \{2,4,6,\dots\}$ for any $i \in I$; (b) $2\frac{(i,j)}{(i,i)} \in \{0,-1,-2,\dots\}$ for any $i \neq j$ in I. Denote $a_{ij} = 2\frac{(i,j)}{(i,i)}$, then $C = (a_{ij})_{i,j\in I}$ is a symmetrizable Cartan matrix. Let $\mathbf{g}$ be symmetrizable Kac–Moody Lie algebra of type $\Delta = (I, (\ , \ ))$ (see [11]). We denote by Φ^+ the set of all positive roots of $\mathbf{g}$ with respect to a set of simple roots α_i for all $i \in I$.

According to a result of Ringel [19], for any Cartan datum Δ and any finite field k, there exists a finite dimensional hereditary k–algebra Λ such that the isomorphism classes of simple Λ–modules are in bijective with

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the index set I and moreover, together with the symmetric Euler form $(\ ,\ )$ of Λ defined on the Grothendieck group $\mathbf{G}_0(\Lambda)$ give a realization of Δ. By definition, $\mathbf{G}_0(\Lambda)$ is the abelian group of all finite dimensional Λ–modules modulo exact sequences and can be identified with $\mathbb{Z}[I]$ in a natural way. For any Λ–module M, the corresponding element in $\mathbf{G}_0(\Lambda)$ is given by the dimension vector $\underline{\dim}\, M = \sum_{i\in I}[M : S_i]\underline{\dim}\, S_i$, where $[M : S_i]$ is the Jordan–Hölder multiplicity of S_i in M. It is also known from [7, 8] that there is a surjective map from the isomorphism classes of the indecomposable Λ–modules to Φ^+, by mapping the isomorphism class of an indecomposable Λ–module M onto $\sum_{i\in I}[M : S_i]\alpha_i$. This surjection induces a bijection between the isomorphism classes of indecomposable Λ–modules of discrete dimension types and the positive real roots. Moreover, there exists a family of non–isomorphic indecomposable Λ–modules corresponding to the positive imaginary roots of Δ if Δ is not of finite type.

Let $\mathbb{Q}(v)$ be the field of rational function in the variable v. The quantized enveloping algebra $\mathbf{U}_q(\mathbf{g})$, $q = v^2$, is defined as the $\mathbb{Q}(v)$–algebra generated by elements E_i, F_i, K_i and K_{-i}, $i \in I$, with the well–known defining relations.

According to Lusztig [12], there exists an action of the braid group corresponding to Δ on $\mathbf{U}_q(\mathbf{g})$. Applying the standard generators T_i, $i \in I$, of the braid group to the generators of $\mathbf{U}_q(\mathbf{g})$ in an admissible order, we obtain a family of linearly independent elements in $\mathbf{U}_q^+(\mathbf{g})$. Since those elements degenerate into a basis of $\oplus_{\alpha\in\Phi^+_{\mathrm{real}}}\mathbf{g}_\alpha$ by the specialization $q \to 1$, we call these elements the real root vectors of $\mathbf{U}_q^+(\mathbf{g})$. If Δ is of finite type they provide a complete set of root vectors.

Based on Lusztig's work [12], Green [10] proved that the positive part $\mathbf{U}_q^+(\mathbf{g})$ of $\mathbf{U}_q(\mathbf{g})$ is isomorphic to the generic composition algebra $\mathcal{C}(\Delta)$ of Λ (see Section 2 for definition) if $\mathbf{g}$ and Λ have the same Cartan datum.

Ringel [20] gave an explanation for the root vectors obtained by Lusztig's braid group action in terms of the Ringel–Hall algebra $\mathcal{H}(\Lambda)$. He showed that for preprojective and preinjective indecomposable Λ–modules V_λ, the elements u_λ in $\mathcal{H}(\Lambda)$ coincide with the corresponding real root vectors in $\mathbf{U}_q^+(\mathbf{g})$, up to the scalar $v^{-\dim_k(V_\lambda)+\dim_k \mathrm{End}_\Lambda V_\lambda}$. In [5], we have obtained an algorithm to express those elements in the composition algebra $\mathcal{C}(\Lambda)$ as linear combination of simple elements.

For affine Kac–Moody Lie algebra $\mathbf{g}$, there exist imaginary roots. Several authors have introduced imaginary root vectors for $\mathbf{U}_q(\mathbf{g})$ (see [3, 4, 6, 9]). Those imaginary root vectors cannot be obtained by Lusztig's operations.

Since the Auslander–Reiten quiver is a convenient tool to visualize the module category of a finite dimensional algebra, following Ringel [20], we may ask what kind of information about those imaginary root vectors in $\mathbf{U}_q^+(\mathbf{g})$ can be read off from the Auslander–Reiten quiver of Λ, i.e., how to interpret those imaginary root vectors in the generic composition algebra $\mathcal{C}(\Delta)$.

In this paper, we answer this question for the special case of the smallest tame hereditary algebra–the Kronecker algebra, i.e., we provide a realization of all imaginary root vectors in the generic composition algebra of this algebra. Moreover, an integral PBW–basis of this algebra is obtained. The importance of the Kronecker algebra lies in the existence of a full exact embedding from the category of regular modules of the Kronecker algebra to the category of regular modules of any tame hereditary algebra with underlining quiver $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. The corresponding quantum group is $\mathbf{U}_q(\widehat{sl_2})$. Our result is based on the representation theory of finite dimensional algebras.

The paper is organized as follows. In section 2 we give the definition of the Ringel–Hall algebra of the Kronecker algebra and recall basic facts related to this algebra. We define and interpret the imaginary root vectors of the composition algebra of the Kronecker algebra in section 3. The main result of this section is the Theorem 3.7. By using a simple combinatorial method we show that for all regular modules with fixed dimension type $n\delta = n\underline{\dim}\, S_1 + n\underline{\dim}\, S_2$, the elements

$$r_{n\delta} = \sum_{V \text{ regular, } \underline{\dim}\, V = n\delta} u$$

in $\mathcal{H}(\Lambda)$ coincide with the corresponding imaginary root vectors introduced by Beck, Chari and Pressley [3], Gavarini [9], up to the scalar v^{-2n}. In combination with Ringel's result for preprojective and preinjective indecomposable modules, we get a complete set of root vectors in $\mathcal{C}(\Delta)$. Then, in section 4, we describe the commutation relations among all root vectors based on the Auslander–Reiten quiver of the Kronecker algebra and show that all coefficients involved belong to $\mathbb{Z}[v, v^{-1}]$ and can be calculated explicitly. In the final section we show the existence of an integral PBW–basis in $\mathcal{C}(\Delta)$.

Let us end this introduction with a summary of related works. The PBW–basis of $\mathbf{U}_q^+(\widehat{sl_2})$ are constructed by Damiani in [6]. The imaginary root vectors involved there are slight modification of $\tilde{E}_{n\delta}$, $n \in \mathbb{N} \setminus \{0\}$, defined in section 3. Zhang [24] constructed a PBW–basis of the untwisted version of the composition algebra of the Kronecker algebra. Based on the isomorphism between $\mathbf{U}_q^+(\widehat{sl_2})$ and the generic composition algebra of the Kronecker algebra (with respect to the twisted multiplication), we can interpret the root vectors explicitly in terms of Λ–modules and then obtain the integral PBW–basis. Our results and Zhang's results [24] both are obtained via the Ringel–Hall algebra approach. Therefore, some of them are equivalent but are based on different points of view and derived by different methods. More recently, Baumann and Kassel [2] described the Ringel–Hall algebra of the category of coherent sheaves on the projective line and recovered Kapranov's isomorphism between a certain subalgebra of this Ringel–Hall algebra and a certain "positive part" of $\mathbf{U}_q(\widehat{sl_2})$. In combination with the isomorphism between the generic composition algebra $\mathcal{C}(\Delta)$ and $\mathbf{U}_q^+(\widehat{sl_2})$, the real root vectors $E_{n\delta+\alpha_1}$ and the imaginary root vectors $E_{n\delta}$ defined in Section 3 are related to the locally free coherent sheaf and torsion sheaf, respectively. Some similar formulae are obtained in this paper too.

After this paper was finished I became aware of Csaba Szántó's preprint "Hall polynomials and the Hall algebra of the Kronecker algebra" where some similar results are obtained.

2. Ringel–Hall algebra of the Kronecker algebra

Most of the material on representation theory of finite dimensional algebras used in this paper can be found in Ringel's book [15].

From now on, let Λ be the Kronecker algebra over a finite field k with the underlying quiver $\overset{1}{\bullet} \Longleftarrow \overset{2}{\bullet}$ and let S_1 and S_2 be the simple Λ–modules. Let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional Λ–modules, $I = \{1, 2\} \subset \mathcal{P}$ the set of isomorphism classes of simple Λ–modules. We choose a representative $V_\alpha \in \alpha$ for any $\alpha \in \mathcal{P}$. Given Λ–modules M and N, let

$$\langle M, N\rangle = \dim_k \mathrm{Hom}_\Lambda(M, N) - \dim_k \mathrm{Ext}^1_\Lambda(M, N).$$

Since Λ is hereditary, $\langle M, N\rangle$ depends only on the dimension vectors $\underline{\dim}\, M$ and $\underline{\dim}\, N$. The Euler form on $\mathbb{Z}[I]$ $(= \mathbf{G}_0(\Lambda))$ is defined by $\langle \alpha, \beta\rangle = \langle V_\alpha, V_\beta\rangle$, where $\alpha, \beta \in \mathcal{P}$. The symmetric Euler form $(-,-)$ is given by $(\alpha, \beta) = \langle \alpha, \beta\rangle + \langle \beta, \alpha\rangle$ on $\mathbb{Z}[I]$. The index set I and the symmetric Euler form give a realization of a Cartan datum Δ whose symmetrizable Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Let $\mathbf{R}$ be a (commutative) integral domain containing $\mathbb{Q}(v)$, where $v^2 = q$, $q = |k|$ and $\mathbb{Q}(v)$ is the field of rational function of v. The Ringel–Hall algebra $\mathcal{H}(\Lambda)$ is by definition the free $\mathbf{R}$–module on a set of symbols u_α $(\alpha \in \mathcal{P})$, with an $\mathbf{R}$-bilinear (twisted) multiplication defined by setting

$$u_\alpha u_\beta = v^{\langle \alpha, \beta\rangle} \sum_{\lambda \in \mathcal{P}} g^\lambda_{\alpha\beta} u_\lambda, \text{ for all } \alpha, \beta \in \mathcal{P},$$

where $g^{\lambda}_{\alpha\beta}$ is the number of submodules X of V_λ such that V_λ/X and X lie in the isomorphism classes α and β, respectively. It is easy to verify that $\mathcal{H}(\Lambda)$ is an associative $\mathbb{N}[I]$–graded $\mathbf{R}$–algebra with the identity element u_0.

In Ringel–Hall algebra and quantum group, we use the notations

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1},$$
$$[n]! = \prod_{r=1}^{s} [r], \quad \text{and} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!},$$

here, n, m are non–negative integers, and $m < n$.

Ringel [16, 18] has proved that the elements u_i, $i \in I$, satisfy the quantum Serre relations

$$\sum_{t=0}^{3} (-1)^t \begin{bmatrix} 3 \\ t \end{bmatrix} u_i^t u_j u_i^{3-t} = 0$$

for any $i \neq j$ in I.

We denote by $\mathcal{C}(\Lambda)$ the $\mathbf{R}$–subalgebra of $\mathcal{H}(\Lambda)$ which is generated by $u_i, i \in I$: it is called the composition algebra of Λ.

Let $\bar{k}$ be the algebraic closure of k. For any $n \in \mathbb{N}$, let $F(n)$ be a subfield of $\bar{k}$ such that $[F(n) : k] = n$. If we define $\Lambda(n) = \Lambda \otimes_k F(n)$, then $\Lambda(n)$ is a finite dimensional hereditary $F(n)$–algebra corresponding to the same Cartan datum as that of Λ. We also have the Ringel–Hall algebra $\mathcal{H}_n = \mathcal{H}_n(\Lambda(n))$ of $\Lambda(n)$. Define $\Pi = \prod_{n>0} \mathcal{H}_n$. Let $v = (v_n)_n \in \Pi$ where $v_n = \sqrt{|F(n)|}$. Obviously v lies in the center of Π and is transcendental over the rational field $\mathbb{Q}$. Let $u_i = (u_i(n))_n \in \Pi$ satisfy that $u_i(n)$ is the element of $\mathcal{H}(\Lambda(n))$ corresponding to $V_i(n)$, where $V_i(n)$ is the simple $\Lambda(n)$–module which lies in the class i. The generic composition algebra $\mathcal{C}(\Delta)$ of the Cartan datum Δ is defined to be the subring of Π generated by $\mathbb{Q}$, v, v^{-1} and u_i $(i \in I)$. Let $\mathbf{U}_q^+(\widehat{sl_2})$ be the positive part of the Drinfeld–Jimbo quantum group corresponding to the Cartan datum Δ. A fundamental theorem of Green and Ringel concludes that the mapping $\eta : \mathbf{U}_q^+(\widehat{sl_2}) \to \mathcal{C}(\Delta)$ with $\eta(E_i) = u_i$ $(i \in I)$ is a bijection of associative algebras.

In the following, our results are stated only for the composition algebra $\mathcal{C}(\Lambda)$. Without any changes, the same conclusions hold for the corresponding generic composition algebra $\mathcal{C}(\Delta)$.

For simplifying our notations, in this paper, we will use α_1 and α_2 to represent the isomorphism classes of simple modules S_1 and S_2 respectively. Moreover, we still use α_1 and α_2 as the dimension vectors $\underline{\dim}\, S_1$ and $\underline{\dim}\, S_2$ in $\mathbb{Z}[I]$. Put $\delta = \alpha_1 + \alpha_2 \in \mathbb{Z}[I]$.

The Auslander–Reiten quiver of Λ consists of one preprojective component, one preinjective component and a family of homogeneous tubes of regular modules parameterized by the set of all monic irreducible polynomials over k. The indecomposable preprojective and preinjective modules have the dimension vectors $n\delta+\alpha_1$ and $n\delta+\alpha_2$ with $n \in \mathbb{N}$, respectively. Moreover, each indecomposable preprojective and preinjective module can be uniquely determined by its dimension vectors. The indecomposable regular modules have the dimension vectors $n\delta$ with $n \in \mathbb{N}\setminus\{0\}$. For convenience, we put $V_1 = V_{\alpha_1}$ and $V_2 = V_{\alpha_2}$. So V_{α_1} is simple projective module and V_{α_2} is simple injective module.

Lemma 2.1. *For any n, $m \in \mathbb{N}$, we have*

$$\begin{aligned}
\dim_k \operatorname{Hom}_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1}) &= \max\{0, m-n+1\},\\
\dim_k \operatorname{Hom}_\Lambda(V_{n\delta+\alpha_2}, V_{m\delta+\alpha_2}) &= \max\{0, n-m+1\},\\
\dim_k \operatorname{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1}) &= \max\{0, n-1-m\},\\
\dim_k \operatorname{Ext}^1_\Lambda(V_{n\delta+\alpha_2}, V_{m\delta+\alpha_2}) &= \max\{0, m-1-n\}.
\end{aligned}$$

Let $\alpha, \beta, \lambda \in \mathcal{P}$. According to Peng [14] and Riedtmann [21], there is a homological formula to calculate the filtration number $g^\lambda_{\alpha,\beta}$:

Lemma 2.2. *For any $V_\lambda, V_\alpha, V_\beta \in \Lambda$–mod, we have*

$$g^\lambda_{\alpha,\beta} = \frac{|\operatorname{Ext}^1_\Lambda(V_\alpha, V_\beta)_{V_\lambda}||\operatorname{Aut}_\Lambda(V_\lambda)|}{|\operatorname{Aut}_\Lambda(V_\alpha)||\operatorname{Aut}_\Lambda(V_\beta)||\operatorname{Hom}_\Lambda(V_\alpha, V_\beta)|}$$

where $\operatorname{Ext}^1_\Lambda(V_\alpha, V_\beta)_{V_\lambda}$ is the set of all exact sequences in $\operatorname{Ext}^1_\Lambda(V_\alpha, V_\beta)$ with middle term V_λ.

For any $\lambda \in \mathcal{P}$, we let $a_\lambda = |\operatorname{Aut}_\Lambda(V_\lambda)|$. The following lemma is well-known (see [17]):

Lemma 2.3.

(1) Let V_λ be an indecomposable Λ–module with $\dim_k \operatorname{End}_\Lambda V_\lambda = s$ and $\dim_k \operatorname{rad}\operatorname{End}_\Lambda V_\lambda = t$. Then $a_\lambda = (q^{s-t}-1)q^t$.

(2) Let $V_\lambda \simeq \oplus_{l=1}^t s_l V_{\lambda_l}$ such that $V_{\lambda_i} \not\simeq V_{\lambda_j}$ for any $i \neq j$. Then $a_\lambda = q^s a_{s_1\lambda_1} \cdots a_{s_t\lambda_t}$, where $s = \sum_{i\neq j} s_i s_j \dim_k \operatorname{Hom}_\Lambda(V_{\lambda_i}, V_{\lambda_j})$.

(3) Let $V_\lambda = sV_\rho$ with $\operatorname{End}_\Lambda V_\rho = F$ a field. Then $a_\lambda = |GL_s(F)| = \prod_{1\leq t\leq s}(d^s - d^{t-1})$, where $d = |F| = q^{[F:k]}$.

Ringel has pointed out in [20] that the Auslander–Reiten translates τ and τ^{-1} play very important rule in Ringel–Hall algebras and quantum groups. Recall that there exist two τ orbits in the preprojective and preinjective component of Λ, i.e., $\tau^{-1}(V_{n\delta+\alpha_1}) \simeq V_{(n+2)\delta+\alpha_1}$, $\tau(V_{n\delta+\alpha_2}) \simeq V_{(n+2)\delta+\alpha_2}$ for $n \in \mathbb{N}$. For any regular module M, $\tau^{\pm}(M) \simeq M$. For any $\lambda \in \mathcal{P}$, define $\tau^{\pm}(u_\lambda)$ in $\mathcal{H}(\Lambda)$ to be the element corresponding to $\lambda' \in \mathcal{P}$ such that $\tau^{\pm}(V_\lambda) \in \lambda'$. Moreover, we define $\tau^{\pm}(u) = \sum_{\lambda\in\mathcal{P}} c_\lambda \tau^{\pm}(u_\lambda)$ for any element $u = \sum_{\lambda\in\mathcal{P}} c_\lambda u_\lambda$ in $\mathcal{H}(\Lambda)$, where $c_\lambda \in \mathbf{R}$.

Since Λ is hereditary, we have the following well–known result:

Lemma 2.4.

(1) If both V_α and V_β for $\alpha, \beta \in \mathcal{P}$ have no projective direct summands, then in $\mathcal{H}(\Lambda)$ we have $\tau(u_\alpha u_\beta) = \tau(u_\alpha)\tau(u_\beta)$.

(2) If both V_α and V_β for $\alpha, \beta \in \mathcal{P}$ have no injective direct summands, then in $\mathcal{H}(\Lambda)$ we have $\tau^{-1}(u_\alpha u_\beta) = \tau^{-1}(u_\alpha)\tau^{-1}(u_\beta)$.

3. Quantum root vectors

According to Ringel [20], we can define the following root vectors in $\mathcal{C}(\Lambda)$ which correspond to positive real roots:

$$\begin{aligned} E_1 &= E_{\alpha_1} = u_1, \qquad E_2 = E_{\alpha_2} = u_2, \\ E_{n\delta+\alpha_1} &= v^{-dim_k V_{n\delta+\alpha_1} + dim_k \operatorname{End}_\Lambda(V_{n\delta+\alpha_1})} = v^{-2n} u_{n\delta+\alpha_1} \text{ for any } n \in \mathbb{N}, \\ E_{n\delta+\alpha_2} &= v^{-dim_k V_{n\delta+\alpha_2} + dim_k \operatorname{End}_\Lambda(V_{n\delta+\alpha_2})} = v^{-2n} u_{n\delta+\alpha_2} \text{ for any } n \in \mathbb{N}. \end{aligned}$$

Note that by using the result of Ringel [20] and the result of Xiao [22], the braid group actions defined by Lusztig in [12] can be realized through BGP–reflection functors. Hence, we know that the above real root vectors were well–defined. Indeed, we have for $i \neq j \in \{1, 2\}$

$$\begin{aligned} E_{n\delta+\alpha_1} &= \underbrace{T_1 T_2 T_1 \cdots T_i}_{n \text{ times}}(E_j), \\ E_{n\delta+\alpha_2} &= \underbrace{T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_i^{-1}}_{n \text{ times}}(E_j) \end{aligned}$$

where T_1 and T_2 are the braid group operations defined in [12].

Put

$$r_{n\delta} = \sum_{V \text{ regular}, \ \dim V = n\delta} u$$

for later use.

Following Damiani [6] with a slight modification, we define the first group of imaginary root vectors:

$$\tilde{E}_{n\delta} = E_{(n-1)\delta+\alpha_2}E_1 - v^{-2}E_1E_{(n-1)\delta+\alpha_2} \qquad \text{for } n \in \mathbb{N}.$$

Later in Lemma 3.9, we will show that they are well defined .

The following two lemmas can be read off directly from the Auslander–Reiten quiver of the Kronecker algebra without using the braid group actions.

Lemma 3.1. $\tilde{E}_\delta = E_2E_1 - v^{-2}E_1E_2 = v^{-2}r_\delta$.

Lemma 3.2. *For any $n \in \mathbb{N}$, we have*

$$\begin{aligned}[\tilde{E}_\delta, E_{n\delta+\alpha_1}] &= [2]E_{(n+1)\delta+\alpha_1},\\ [E_{n\delta+\alpha_2}, \tilde{E}_\delta] &= [2]E_{(n+1)\delta+\alpha_2}.\end{aligned}$$

For future use, we need the notations of generalized permutations and partitions.

Definition 3.3. If λ_i $(i = 1, 2, \cdots, k)$, λ, and l are $k+2$ nonnegative integers such that $\lambda_1 + \lambda_2 + \cdots \lambda_k = \lambda \leq l$, define

$$\mathrm{P}(l; \lambda_1, \lambda_2, \cdots, \lambda_k) := \frac{\mathrm{P}(l, \lambda)}{\lambda_1!\lambda_2! \cdots \lambda_k!}$$

where $\mathrm{P}(l, \lambda) = \dfrac{l!}{(l-\lambda)!}$. For convenience, we write $\mathrm{P}(l; \lambda_1, \lambda_2, \cdots, \lambda_k) = 0$ when $l < 0$ or $\lambda_i < 0$ for some $1 \leq i \leq k$.

We have the following result:

Lemma 3.4. *If λ_i $(i = 1, 2, \cdots, k)$ and l are $k+1$ nonnegative integers such that $l = \sum_{j=1}^k \lambda_j$, then we have*

$$\mathrm{P}(l; \lambda_1, \lambda_2, \cdots, \lambda_k) = \sum_{j=1}^{k} \mathrm{P}(l-1; \lambda_1, \lambda_2, \cdots, \lambda_j - 1, \cdots, \lambda_k).$$

Let $\mathbb{P}_n$ be the set of all partitions of n and let $\mathbf{p} \in \mathbb{P}_n$ be one partition. Sometimes we use the notation which indicates the number of times each integer occurs as a part:

$$\mathbf{p} = (1^{\lambda_1}2^{\lambda_2} \cdots n^{\lambda_n})$$

where exactly λ_i of the parts of $\mathbf{p}$ are equal to i with each λ_i nonnegative. For the basic theory of partitions, the reader is referred to [1, 13]. Recall that the length of the partition $\mathbf{p}$, denoted by $l(\mathbf{p})$, is the number of parts, i.e., $l(\mathbf{p}) = \sum_{i=1}^n \lambda_i$.

Lemma 3.5 (see [23]). *In $\mathcal{H}(\Lambda)$, we have*

$$\tilde{E}_{n\delta} = v^{-3n+1} \sum_{\mathbf{p}=(1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})\in\mathbb{P}_n} c(\mathbf{p}) r_{1\delta}^{\lambda_1} r_{2\delta}^{\lambda_2} \cdots r_{n\delta}^{\lambda_n}$$

where $c(\mathbf{p}) = (-1)^{l(\mathbf{p})-1} \sum_{j=1}^{n} \frac{1-v^{2j}}{1-v^2} \mathrm{P}(l(\mathbf{p})-1; \lambda_1, \lambda_2, \ldots, \lambda_j - 1, \ldots, \lambda_n)$.

Remark 3.6. In [23], Zhang has calculated the term $E_{(n-1)\delta+\alpha_2}E_1$ under the untwisted multiplication. In the above lemma, we rewrite Zhang's result under the twisted multiplication in a simple combinatorial form.

Following [3, 9], we introduce another group of imaginary root vectors $E_{n\delta}$ in Ringel-Hall algebra $\mathcal{H}(\Lambda)$ by $E_{0\delta} = 1$ and

$$E_{n\delta} = \frac{1}{[n]} \sum_{i=0}^{n-1} v^{-i} \tilde{E}_{(n-i)\delta} E_{i\delta} = \frac{1}{[n]} \sum_{i=1}^{n} v^{i-n} \tilde{E}_{i\delta} E_{(n-i)\delta}.$$

The following is the main result of this section:

Theorem 3.7. *For any $n \in \mathbb{N} \setminus \{0\}$, in $\mathcal{H}(\Lambda)$ we have $E_{n\delta} = v^{-2n} r_{n\delta}$.*

Proof. To show the equality in the theorem is true, we only need to find the coefficient of the term $r_{1\delta}^{\lambda_1} r_{2\delta}^{\lambda_2} \cdots r_{n\delta}^{\lambda_n}$ corresponding to the partition $\mathbf{p} = (1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ in each multiplication $v^{-i}\tilde{E}_{(n-i)\delta}E_{i\delta}$ for $0 \le i \le n-1$, and then combine all these coefficients.

When $i \neq 0$, the term $E_{i\delta} = v^{-2i} r_{i\delta}$ will contribute one copy of $r_{i\delta}$ for the term $r_{1\delta}^{\lambda_1} r_{2\delta}^{\lambda_2} \cdots r_{n\delta}^{\lambda_n}$. Then at first we should find the coefficient of $r_{1\delta}^{\lambda_1} r_{2\delta}^{\lambda_2} \cdots r_{i\delta}^{\lambda_i - 1} \cdots r_{n\delta}^{\lambda_n}$ in term $\tilde{E}_{(n-i)\delta}$. For convenience, in the expression of each term $\tilde{E}_{(n-i)\delta}$, we will consider the index set $\mathbb{P}_{n-i}$ as a subset of $\mathbb{P}_n$ by adding $(n-i)^0(n-i+1)^0 \cdots n^0$ to each partition of $\mathbb{P}_{n-i}$.

The coefficient of the term $r_{1\delta}^{\lambda_1} r_{2\delta}^{\lambda_2} \cdots r_{n\delta}^{\lambda_n}$ in $\frac{1}{[n]} v^{-i} \tilde{E}_{(n-i)\delta} E_{i\delta}$ is

$$\begin{aligned}
&\frac{1}{[n]} v^{-i} v^{-2i} v^{-3(n-i)+1} (-1)^{l(\mathbf{p})-2} \\
&\Big(\sum_{j=1}^{i-1} \frac{1-v^{2j}}{1-v^2} \mathrm{P}(l(\mathbf{p})-2; \lambda_1, \lambda_2, \ldots, \lambda_j - 1, \ldots, \lambda_i - 1, \ldots, \lambda_n) \\
&+ \frac{1-v^{2i}}{1-v^2} \mathrm{P}(l(\mathbf{p})-2; \lambda_1, \lambda_2, \ldots, \ldots, \lambda_i - 2, \ldots, \lambda_n) \\
&+ \sum_{j=i+1}^{n} \frac{1-v^{2j}}{1-v^2} \mathrm{P}(l(\mathbf{p})-2; \lambda_1, \lambda_2, \ldots, \lambda_i - 1, \ldots, \lambda_j - 1, \ldots, \lambda_n) \Big).
\end{aligned}$$

When $i = 0$, the coefficient is

$$\frac{1}{[n]} v^{-3n+1} (-1)^{l(\mathbf{p})-1} \sum_{j=1}^{n} \frac{1-v^{2j}}{1-v^2} \mathrm{P}(l(\mathbf{p})-1; \lambda_1, \lambda_2, \ldots, \lambda_j - 1, \ldots, \lambda_n).$$

If $l(\mathbf{p}) > 1$, for all $1 \le j \le n$, we will use the following equality

$$\begin{aligned} & \mathrm{P}(l(\mathbf{p})-1; \lambda_1, \lambda_2, \ldots, \lambda_j - 1, \ldots, \lambda_n) \\ = & \sum_{k=1}^{j-1} \mathrm{P}(l(\mathbf{p})-2; \lambda_1, \lambda_2, \ldots, \lambda_k - 1, \ldots, \lambda_j - 1, \ldots, \lambda_n) \\ & + \mathrm{P}(l(\mathbf{p})-2; \lambda_1, \lambda_2, \ldots, \lambda_j - 2, \ldots, \lambda_n) \\ & + \sum_{k=j+1}^{n} \mathrm{P}(l(\mathbf{p})-2; \lambda_1, \lambda_2, \ldots, \lambda_j - 1, \ldots, \lambda_k - 1, \ldots, \lambda_n) \end{aligned}$$

Then in $\frac{1}{[n]} \sum_{i=0}^{n-1} v^{-i} \tilde{E}_{(n-i)\delta} E_{i\delta}$, we cancel all the terms corresponding to the partitions $\mathbf{p} \in \mathbb{P}_n$ with $l(\mathbf{p}) > 1$.

The only exception in the calculation is the term $r_{n\delta}$ corresponding to partition $\mathbf{p} = (1^0 2^0 \cdots n^1)$ that appears in $\frac{1}{[n]} \tilde{E}_{n\delta} E_{0\delta} = \frac{1}{[n]} \tilde{E}_{n\delta}$ only. It follows that

$$E_{n\delta} = \frac{1}{[n]} v^{-3n+1} \frac{1-v^{2n}}{1-v^2} r_{n\delta} = v^{-2n} r_{n\delta}.$$

□

Corollary 3.8. *For any n, $m \in \mathbb{N}$, we have $[\tilde{E}_{n\delta}, \tilde{E}_{m\delta}] = 0 = [E_{n\delta}, E_{m\delta}]$.*

Proof. The statement follows immediately from the fact $r_{n\delta} r_{m\delta} = r_{m\delta} r_{n\delta}$. □

Another immediate consequence of the above theorem is that $E_{n\delta}$ (or $r_{n\delta}$) is in $\mathcal{C}(\Lambda)$.

Lemma 3.9. *For any $n \in \mathbb{N}$ and $0 \le i \le n$, we have*

$$\tilde{E}_{(n+1)\delta} = E_{(n-i)\delta+\alpha_2} E_{i\delta+\alpha_1} - v^{-2} E_{i\delta+\alpha_1} E_{(n-i)\delta+\alpha_2}.$$

Proof. By definition, we know that $\tilde{E}_{(n+1)\delta} = E_{n\delta+\alpha_2} E_1 - v^{-2} E_1 E_{n\delta+\alpha_2}$. If $n = 0$, the equality is nothing but the definition. Put $n \ge 1$. First of all, we will show

$$(\star) \quad E_{n\delta+\alpha_2} E_1 - v^{-2} E_1 E_{n\delta+\alpha_2} = E_{(n-1)\delta+\alpha_2} E_{\delta+\alpha_1} - v^{-2} E_{\delta+\alpha_1} E_{(n-1)\delta+\alpha_2},$$

after that we can apply the Auslander-Reiten translate τ^{-1} on both sides of the above equality. Using Lemma 3.2, we have

$$
\begin{aligned}
& E_{(n-1)\delta+\alpha_2}E_{\delta+\alpha_1} - v^{-2}E_{\delta+\alpha_1}E_{(n-1)\delta+\alpha_2} \\
= \; & E_{(n-1)\delta+\alpha_2}(\frac{1}{[2]}\tilde{E}_\delta E_1 - \frac{1}{[2]}E_1\tilde{E}_\delta) \\
& -v^{-2}(\frac{1}{[2]}\tilde{E}_\delta E_1 - \frac{1}{[2]}E_1\tilde{E}_\delta)E_{(n-1)\delta+\alpha_2} \\
= \; & \frac{1}{[2]}E_{(n-1)\delta+\alpha_2}\tilde{E}_\delta E_1 - \frac{1}{[2]}\tilde{E}_\delta E_{(n-1)\delta+\alpha_2}E_1 + \frac{1}{[2]}\tilde{E}_\delta E_{(n-1)\delta+\alpha_2}E_1 \\
& -\frac{1}{[2]}E_{(n-1)\delta+\alpha_2}E_1\tilde{E}_\delta - v^{-2}\frac{1}{[2]}\tilde{E}_\delta E_1 E_{(n-1)\delta+\alpha_2} \\
& +v^{-2}\frac{1}{[2]}E_1\tilde{E}_\delta E_{(n-1)\delta+\alpha_2} - v^{-2}\frac{1}{[2]}E_1E_{(n-1)\delta+\alpha_2}\tilde{E}_\delta \\
& +v^{-2}\frac{1}{[2]}E_1E_{(n-1)\delta+\alpha_2}\tilde{E}_\delta \\
= \; & \frac{1}{[2]}(E_{(n-1)\delta+\alpha_2}\tilde{E}_\delta - \tilde{E}_\delta E_{(n-1)\delta+\alpha_2})E_1 + \frac{1}{[2]}\tilde{E}_\delta E_{(n-1)\delta+\alpha_2}E_1 \\
& -\frac{1}{[2]}E_{(n-1)\delta+\alpha_2}E_1\tilde{E}_\delta - v^{-2}\frac{1}{[2]}\tilde{E}_\delta E_1E_{(n-1)\delta+\alpha_2} \\
& -v^{-2}\frac{1}{[2]}E_1(E_{(n-1)\delta+\alpha_2}\tilde{E}_\delta - \tilde{E}_\delta E_{(n-1)\delta+\alpha_2}) \\
& +v^{-2}\frac{1}{[2]}E_1E_{(n-1)\delta+\alpha_2}\tilde{E}_\delta \\
= \; & E_{n\delta+\alpha_2}E_1 - v^{-2}E_1E_{n\delta+\alpha_2} \\
& +\frac{1}{[2]}\tilde{E}_\delta(E_{(n-1)\delta+\alpha_2}E_1 - v^{-2}E_1E_{(n-1)\delta+\alpha_2}) \\
& -\frac{1}{[2]}(E_{(n-1)\delta+\alpha_2}E_1 - v^{-2}E_1E_{(n-1)\delta+\alpha_2})\tilde{E}_\delta
\end{aligned}
$$

$$
\begin{aligned}
= \; & E_{n\delta+\alpha_2}E_1 - v^{-2}E_1E_{n\delta+\alpha_2} + \frac{1}{[2]}\tilde{E}_\delta\tilde{E}_{(n-1)\delta} - \frac{1}{[2]}\tilde{E}_\delta\tilde{E}_{(n-1)\delta} \\
= \; & E_{n\delta+\alpha_2}E_1 - v^{-2}E_1E_{n\delta+\alpha_2} + \frac{1}{[2]}[\tilde{E}_\delta, \tilde{E}_{(n-1)\delta}].
\end{aligned}
$$

By corollary 3.8, we know that $[\tilde{E}_\delta, \tilde{E}_{(n-1)\delta}] = 0$. The equality $(\star)$ follows.

If $n > 1$ and for any $k \le \lfloor \frac{n}{2} \rfloor$, by Lemma 2.4, we can apply the Auslander–Reiten translate τ^{-1} on the left hand side of the equality $(\star)$

k times. Then,

$$\begin{array}{rl} & \tau^{-k}(\tilde{E}_{(n+1)\delta}) = \tau^{-k}(E_{n\delta+\alpha_2}E_1) - \tau^{-k}(v^{-2}E_1E_{n\delta+\alpha_2}) \\ = & \tau^{-k}(E_{n\delta+\alpha_2})\tau^{-k}(E_1) - v^{-2}\tau^{-k}(E_1)\tau^{-k}(E_{n\delta+\alpha_2}). \end{array}$$

If $n-1>1$ and $k \leq \lfloor \frac{n-1}{2} \rfloor$, we can apply τ^{-1} on the right hand side of the equality $(\star)$ k times. Then,

$$\begin{array}{rl} & \tau^{-k}(\tilde{E}_{(n+1)\delta}) = \tau^{-k}(E_{(n-1)\delta+\alpha_2}E_{\delta+\alpha_1}) - \tau^{-k}(v^{-2}E_{\delta+\alpha_1}E_{(n-1)\delta+\alpha_2}) \\ = & \tau^{-k}(E_{(n-1)\delta+\alpha_2})\tau^{-k}(E_{\delta+\alpha_1}) - v^{-2}\tau^{-k}(E_{\delta+\alpha_1})\tau^{-k}(E_{(n-1)\delta+\alpha_2}). \end{array}$$

Based on the Auslander–Reiten quiver of the Kronecker algebra, it is easy to see that the following identities hold in the Ringel–Hall algebra $\mathcal{H}(\Lambda)$:

$$\begin{array}{l} \tau^{-k}(\tilde{E}_{(n+1)\delta}) = \tilde{E}_{(n+1)\delta}, \\ \tau^{-k}(E_1) = E_{2k\delta+\alpha_1}, \tau^{-k}(E_{\delta+\alpha_1}) = E_{(1+2k)\delta+\alpha_1}, \\ \tau^{-k}(E_{n\delta+\alpha_2}) = E_{(n-2k)\delta+\alpha_2}, \text{ when } k \leq \lfloor \frac{n}{2} \rfloor, \\ \tau^{-k}(E_{(n-1)\delta+\alpha_2}) = E_{(n-1-2k)\delta+\alpha_2}, \text{ when } k \leq \lfloor \frac{n-1}{2} \rfloor. \end{array}$$

According to the above facts, it is immediate to show that

$$\tilde{E}_{(n+1)\delta} = E_{(n-i)\delta+\alpha_2}E_{i\delta+\alpha_1} - v^{-2}E_{i\delta+\alpha_1}E_{(n-i)\delta+\alpha_2}$$

for any $n \in \mathbb{N}$ and $0 \leq i \leq n$. □

4. Commutation relations

Lemma 4.1. *For any $m \in \mathbb{N}$, we have*

$$\begin{array}{rcl} E_{(m+1)\delta+\alpha_1}E_{m\delta+\alpha_1} & = & v^2E_{m\delta+\alpha_1}E_{(m+1)\delta+\alpha_1} \\ E_{m\delta+\alpha_2}E_{(m+1)\delta+\alpha_2} & = & v^2E_{(m+1)\delta+\alpha_2}E_{m\delta+\alpha_2}. \end{array}$$

Proof. The claim is an immediate consequence of Lemma 2.1, Lemma 2.2 and Lemma 2.3. □

Proposition 4.2. *For any n, $m \in \mathbb{N}$, and $n>m$, we have*

$$\begin{array}{rcl} E_{n\delta+\alpha_1}E_{m\delta+\alpha_1} & = & \displaystyle\sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} a_h^{(n-m)}E_{(m+h)\delta+\alpha_1}E_{(n-h)\delta+\alpha_1}, \\ E_{m\delta+\alpha_2}E_{n\delta+\alpha_2} & = & \displaystyle\sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} a_h^{(n-m)}E_{(n-h)\delta+\alpha_2}E_{(m+h)\delta+\alpha_2} \end{array}$$

where

$$\begin{array}{ll} a_h^{(n-m)} = v^2 & \text{if } h = 0 \\ a_h^{(n-m)} = v^{2(h-1)}(v^4-1) & \text{if } 0 < h < \frac{n-m}{2} \\ a_h^{(n-m)} = v^{2(h-1)}(v^2-1) & \text{if } 0 < h = \frac{n-m}{2} \end{array}$$

Proof. We only prove the first equality, the proof for the second equality is similar. It is well–known that for the Kronecker algebra, each indecomposable preprojective module has the dimension vector $l\delta + \alpha_1$ with $l \in \mathbb{N}$. Given any two indecomposable preprojective modules $V_{n\delta+\alpha_1}$ and $V_{m\delta+\alpha_1}$, the extension of $V_{n\delta+\alpha_1}$ and $V_{m\delta+\alpha_1}$ is a preprojective module also.

By Lemma 2.1, we have the short exact sequence

$$0 \to V_{m\delta+\alpha_1} \to V_{(m+h)\delta+\alpha_1} \oplus V_{(n-h)\delta+\alpha_1} \to V_{n\delta+\alpha_1} \to 0$$

and the split exact sequence

$$0 \to V_{(n-h)\delta+\alpha_1} \to V_{(m+h)\delta+\alpha_1} \oplus V_{(n-h)\delta+\alpha_1} \to V_{(m+h)\delta+\alpha_1} \to 0$$

where $n > m$ and $0 \leq h \leq \lfloor \frac{n-m}{2} \rfloor$.

By definition, we know

$$\begin{aligned} & u_{n\delta+\alpha_1} u_{m\delta+\alpha_1} \\ = \;& v^{\langle n\delta+\alpha_1, m\delta+\alpha_1\rangle} \sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} g_{n\delta+\alpha_1, m\delta+\alpha_1}^{((m+h)\delta+\alpha_1)\oplus((n-h)\delta+\alpha_1)} u_{(m+h)\delta+\alpha_1 \oplus (n-h)\delta+\alpha_1} \end{aligned}$$

and

$$\begin{aligned} & u_{(m+h)\delta+\alpha_1} u_{(n-h)\delta+\alpha_1} \\ = \;& v^{\langle (m+h)\delta+\alpha_1, (n-h)\delta+\alpha_1\rangle} g_{(m+h)\delta+\alpha_1, (n-h)\delta+\alpha_1}^{((m+h)\delta+\alpha_1)\oplus((n-h)\delta+\alpha_1)} u_{(m+h)\delta+\alpha_1 \oplus (n-h)\delta+\alpha_1}. \end{aligned}$$

It is easy to check that $\langle n\delta+\alpha_1, m\delta+\alpha_1\rangle = -n+m+1$ and $\langle (m+h)\delta+\alpha_1, (n-h)\delta+\alpha_1\rangle = n-m-2h+1$.

Put $V = V_{(m+h)\delta+\alpha_1} \oplus V_{(n-h)\delta+\alpha_1}$. By Lemma 2.2, we know that

$$\begin{aligned} & g_{n\delta+\alpha_1, m\delta+\alpha_1}^{((m+h)\delta+\alpha_1)\oplus((n-h)\delta+\alpha_1)} \\ = \;& \frac{|\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_V| \, |\mathrm{Aut}_\Lambda(V)|}{|\mathrm{Aut}_\Lambda(V_{n\delta+\alpha_1})| \, |\mathrm{Aut}_\Lambda(V_{m\delta+\alpha_1})| \, |\mathrm{Hom}_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})|}. \end{aligned}$$

We know that $|\mathrm{Aut}_\Lambda(V_{n\delta+\alpha_1})| = |\mathrm{Aut}_\Lambda(V_{m\delta+\alpha_1})| = q-1 = v^2-1$ and $\mathrm{Hom}_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1}) = 0$. By Lemma 2.3, we have

$$\begin{aligned} & |\mathrm{Aut}_\Lambda(V)| = q^{\dim_k \mathrm{Hom}(V_{(m+h)\delta+\alpha_1}, V_{(n-h)\delta+\alpha_1})}(q-1)(q-1) \\ = \;& q^{n-m-2h+1}(q-1)(q-1) \end{aligned}$$

Since $\mathrm{Ext}^1_\Lambda(V_{(m+h)\delta+\alpha_1}, V_{(n-h)\delta+\alpha_1}) = 0$, we have

$$g^{((m+h)\delta+\alpha_1)\oplus((n-h)\delta+\alpha_1)}_{(m+h)\delta+\alpha_1,(n-h)\delta+\alpha_1} = 1,$$

and thus,

$$u_{(m+h)\delta+\alpha_1\oplus(n-h)\delta+\alpha_1} = v^{-(n-m-2h+1)}u_{(m+h))\delta+\alpha_1}u_{(n-h)\delta+\alpha_1}.$$

Then, we have

$$\begin{aligned}
& u_{n\delta+\alpha_1}u_{m\delta+\alpha_1} \\
= \ & v^{\langle n\delta+\alpha_1, m\delta+\alpha_1\rangle}\sum_{h=0}^{\lfloor\frac{n-m}{2}\rfloor} g^{((m+h)\delta+\alpha_1)\oplus((n-h)\delta+\alpha_1)}_{n\delta+\alpha_1,m\delta+\alpha_1}u_{(m+h)\delta+\alpha_1\oplus(n-h)\delta+\alpha_1} \\
= \ & v^{-n+m+1}\sum_{h=0}^{\lfloor\frac{n-m}{2}\rfloor} |\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_{V_{(m+h)\delta+\alpha_1}\oplus V_{(n-h)\delta+\alpha_1}}| \\
& q^{n-m-2h+1}u_{(m+h)\delta+\alpha_1\oplus(n-h)\delta+\alpha_1} \\
= \ & v^{-n+m+1}\sum_{h=0}^{\lfloor\frac{n-m}{2}\rfloor} |\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_{V_{(m+h)\delta+\alpha_1}\oplus V_{(n-h)\delta+\alpha_1}}| \\
& q^{n-m-2h+1}v^{-(n-m-2h+1)}u_{(m+h))\delta+\alpha_1}u_{(n-h)\delta+\alpha_1} \\
= \ & v^{-2(h-1)}\sum_{h=0}^{\lfloor\frac{n-m}{2}\rfloor} |\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_{V_{(m+h)\delta+\alpha_1}\oplus V_{(n-h)\delta+\alpha_1}}| \\
& u_{(m+h))\delta+\alpha_1}u_{(n-h)\delta+\alpha_1}.
\end{aligned}$$

If $h = 0$, we have $|\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_V| = 1$, then $a^{(n-m)}_0 = v^2$.

If $0 < h < \frac{n-m}{2}$, we have $|\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_V| = q^{2h} - q^{2(h-1)} = v^{4(h-1)}(v^4-1)$, then $a^{(n-m)}_h = v^{2(h-1)}(v^4-1)$.

If $h = \frac{n-m}{2}$, we have $|\mathrm{Ext}^1_\Lambda(V_{n\delta+\alpha_1}, V_{m\delta+\alpha_1})_V| = q^{2(h-1)+1} - q^{2(h-1)} = v^{4(h-1)}(v^2-1)$, then $a^{(n-m)}_h = v^{2(h-1)}(v^2-1)$.

Since $E_{t\delta+\alpha_1} = v^{-2t}u_{t\delta+\alpha_1}$ for all $t \geq 0$, the first equality follows immediately. □

Proposition 4.3. *For any $n, m \in \mathbb{N}$, we have*

$$\begin{aligned}
& E_{n\delta+\alpha_2}E_{m\delta+\alpha_1} \\
& = v^{-2}E_{m\delta+\alpha_1}E_{n\delta+\alpha_2} + \sum_{\mathbf{p}\in\mathbb{P}_{n+m+1}} c_{\mathbf{p}}E^{\lambda_1}_{1\delta}E^{\lambda_2}_{2\delta}\cdots E^{\lambda_{n+m+1}}_{(n+m+1)\delta}
\end{aligned}$$

where $\mathbf{p} = (1^{\lambda_1}2^{\lambda_2}\cdots(n+m+1)^{\lambda_{n+m+1}})$ *and* $c_{\mathbf{p}} = v^{-n-m}(-1)^{l(\mathbf{p})-1}\cdot$ $\big(\sum_{j=1}^{n+m+1}\frac{1-v^{2j}}{1-v^2}\mathrm{P}(l(\mathbf{p})-1;\lambda_1,\lambda_2,\cdots,\lambda_j-1,\cdots,\lambda_{n+m+1})\big)$.

Proof. The clam is an immediate consequence of Lemma 3.5, Theorem 3.7 and Lemma 3.9. $\square$

Proposition 4.4. *Let* $n, m \in \mathbb{N}$. *Then,*

$$\begin{aligned}
& [\tilde{E}_{n\delta}, E_{m\delta+\alpha_1}] \\
= \ & \sum_{k=1}^{n-1} v^{2(k-1)}(v^2-v^{-2})E_{(k+m)\delta+\alpha_1}\tilde{E}_{(n-k)\delta} + v^{2(n-1)}[2]E_{(m+n)\delta+\alpha_1}, \\
& [E_{m\delta+\alpha_2}, \tilde{E}_{n\delta}] \\
= \ & \sum_{k=1}^{n-1} v^{2(k-1)}(v^2-v^{-2})\tilde{E}_{(n-k)\delta}E_{(k+m)\delta+\alpha_2} + v^{2(n-1)}[2]E_{(n+m)\delta+\alpha_2}.
\end{aligned}$$

Proof. To prove the claim, we need to prove the following equalities at first:

$$\begin{aligned}
(1)\quad & [\tilde{E}_{n\delta}, E_1] \\
& = \sum_{k=1}^{n-1} v^{2(k-1)}(v^2-v^{-2})E_{k\delta+\alpha_1}\tilde{E}_{(n-k)\delta} + v^{2(n-1)}[2]E_{n\delta+\alpha_1}, \\
(2)\quad & [\tilde{E}_{n\delta}, E_{\delta+\alpha_1}] \\
& = \sum_{k=1}^{n-1} v^{2(k-1)}(v^2-v^{-2})E_{(k+1)\delta+\alpha_1}\tilde{E}_{(n-k)\delta} + v^{2(n-1)}[2]E_{(n+1)\delta+\alpha_1}, \\
(3)\quad & [E_2, \tilde{E}_{n\delta}] \\
& = \sum_{k=1}^{n-1} v^{2(k-1)}(v^2-v^{-2})\tilde{E}_{(n-k)\delta}E_{k\delta+\alpha_2} + v^{2(n-1)}[2]E_{n\delta+\alpha_2}, \\
(4)\quad & [E_{\delta+\alpha_2}, \tilde{E}_{n\delta}] \\
& = \sum_{k=1}^{n-1} v^{2(k-1)}(v^2-v^{-2})\tilde{E}_{(n-k)\delta}E_{(k+1)\delta+\alpha_2} + v^{2(n-1)}[2]E_{(n+1)\delta+\alpha_2}.
\end{aligned}$$

Then, applying τ^{-1} for (1) and (2) and τ for (3) and (4), we can complete the proof. By duality, we only prove (1) and (2). We will prove them simultaneously by induction on n.

By Lemma 3.9, we have

$$\begin{aligned}
& [\tilde{E}_{n\delta}, E_1] \\
= & (E_{(n-2)\delta+\alpha_2} E_{\delta+\alpha_1} - v^{-2} E_{\delta+\alpha_1} E_{(n-2)\delta+\alpha_2}) E_1 \\
& - E_1 (E_{(n-2)\delta+\alpha_2} E_{\delta+\alpha_1} - v^{-2} E_{\delta+\alpha_1} E_{(n-2)\delta+\alpha_2}) \\
= & E_{(n-2)\delta+\alpha_2} v^2 E_1 E_{\delta+\alpha_1} - v^{-2} E_{\delta+\alpha_1} (\tilde{E}_{(n-1)\delta} + v^{-2} E_1 E_{(n-2)\delta+\alpha_2}) \\
& -(-v^2 \tilde{E}_{(n-1)\delta} + v^2 E_{(n-2)\delta+\alpha_2} E_1) E_{\delta+\alpha_1} + v^{-2} v^{-2} E_{\delta+\alpha_1} E_1 E_{(n-2)\delta+\alpha_2} \\
= & -v^{-2} E_{\delta+\alpha_1} \tilde{E}_{(n-1)\delta} + v^2 \tilde{E}_{(n-1)\delta} E_{\delta+\alpha_1} \\
= & v^2 [\tilde{E}_{(n-1)\delta}, E_{\delta+\alpha_1}] + (v^2 - v^{-2}) E_{\delta+\alpha_1} \tilde{E}_{(n-1)\delta}
\end{aligned}$$

and

$$\begin{aligned}
& [\tilde{E}_{n\delta}, E_{\delta+\alpha_1}] \\
= & (E_{(n-3)\delta+\alpha_2} E_{2\delta+\alpha_1} - v^{-2} E_{2\delta+\alpha_1} E_{(n-3)\delta+\alpha_2}) E_{\delta+\alpha_1} \\
& - E_{\delta+\alpha_1} (E_{(n-3)\delta+\alpha_2} E_{2\delta+\alpha_1} - v^{-2} E_{2\delta+\alpha_1} E_{(n-3)\delta+\alpha_2}) \\
= & v^2 [\tilde{E}_{(n-1)\delta}, E_{2\delta+\alpha_1}] + (v^2 - v^{-2}) E_{2\delta+\alpha_1} \tilde{E}_{(n-1)\delta},
\end{aligned}$$

from which the claim follows immediately. □

Proposition 4.5. *Let $n, m \in \mathbb{N}$. Then,*

$$\begin{aligned}
& [E_{n\delta}, E_{m\delta+\alpha_1}] \\
= & \sum_{k=0}^{n-1} [n+1-k] E_{(m+n-k)\delta+\alpha_1} E_{k\delta} = \sum_{i=m+1}^{m+n} [1-m+i] E_{i\delta+\alpha_1} E_{(m+n-i)\delta}, \\
& [E_{m\delta+\alpha_2}, E_{n\delta}] \\
= & \sum_{k=0}^{n-1} [n+1-k] E_{k\delta} E_{(m+n-k)\delta+\alpha_2} = \sum_{i=m+1}^{m+n} [1-m+i] E_{(m+n-i)\delta} E_{i\delta+\alpha_2}
\end{aligned}$$

Proof. We only prove the first equality. By using analogous method we can show the second one. By the same reason as stated in Proposition 4.4, it is sufficient to prove the following:

(1) $[E_{n\delta}, E_1]$
$$= \sum_{k=0}^{n-1} [n+1-k] E_{(n-k)\delta+\alpha_1} E_{k\delta} = \sum_{i=1}^{n} [1+i] E_{i\delta+\alpha_1} E_{(n-i)\delta},$$

(2) $[E_{n\delta}, E_{\delta+\alpha_1}]$
$$= \sum_{k=0}^{n-1} [n+1-k] E_{(1+n-k)\delta+\alpha_1} E_{k\delta} = \sum_{i=2}^{1+n} [i] E_{i\delta+\alpha_1} E_{(1+n-i)\delta}$$

We will prove (1) by induction on n. The proof of (2) is similar. By Lemma 3.2, we know that $[\tilde{E}_\delta, E_1] = [2]E_{\delta+\alpha_1}$ and $[\tilde{E}_\delta, E_{\delta+\alpha_1}] = [2]E_{2\delta+\alpha_1}$. Since $\tilde{E}_\delta = E_\delta$, the claim is true for $n = 1$.

By definition, we know that $E_{n\delta} = \frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\tilde{E}_{i\delta}E_{(n-i)\delta}$.

Then, we have that

$$
\begin{aligned}
[E_{n\delta}, E_1] &= \frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}[\tilde{E}_{i\delta}E_{(n-i)\delta}, E_1] \\
&= \frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}(\tilde{E}_{i\delta}[E_{(n-i)\delta}, E_1] + [\tilde{E}_{i\delta}, E_1]E_{(n-i)\delta}).
\end{aligned}
$$

By induction and Proposition 4.4, we obtain that

$$
\begin{aligned}
&[E_{n\delta}, E_1] \\
= &\frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\Big(\sum_{k=0}^{n-i-1}[n-i+1-k]\cdot \\
&\big(\sum_{h=1}^{i-1} v^{2(h-1)}(v^2-v^{-2})E_{(h+n-i-k)\delta+\alpha_1}\tilde{E}_{(i-h)\delta}E_{k\delta}\big)\Big) \\
&+\frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\big(\sum_{k=0}^{n-i-1}[n-i+1-k]v^{2(i-1)}[2]E_{(n-k)\delta+\alpha_1}E_{k\delta}\big) \\
&+\frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\big(\sum_{k=0}^{n-i-1}[n-i+1-k]E_{(n-i-k)\delta+\alpha_1}\tilde{E}_{i\delta}E_{k\delta}\big) \\
&+\frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\big(\sum_{l=1}^{i-1} v^{2(l-1)}(v^2-v^{-2})E_{l\delta+\alpha_1}\tilde{E}_{(i-l)\delta}E_{(n-i)\delta}\big) \\
&+\frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\big(v^{2(i-1)}[2]E_{i\delta+\alpha_1}E_{(n-i)\delta}\big).
\end{aligned}
$$

After arranging the index sets of each term appeared in the above summation, we have that

$$\begin{aligned}
&[E_{n\delta}, E_1]\\
=\ & \frac{1}{[n]}\sum_{i=1}^{n-1} E_{i\delta+\alpha_1}\Big(\sum_{j=1}^{n-i}\cdot\\
&\big(\sum_{t=1}^{i-1} v^{2(t-1)}v^{(j+t-n)}(v^2-v^{-2})[i-t+1]\big)\tilde{E}_{j\delta}E_{(n-j-i)\delta}\Big)\\
&+\frac{[2]}{[n]}\sum_{i=2}^{n}\big(\sum_{t=1}^{i-1} v^{(i-t-n)}v^{2(i-t-1)}[t+1]\big)E_{i\delta+\alpha_1}E_{(n-i)\delta}\\
&+\frac{1}{[n]}\sum_{i=1}^{n-1}[i+1]E_{i\delta+\alpha_1}\big(\sum_{j=1}^{n-i} v^{(j-n)}\tilde{E}_{j\delta}E_{(n-j-i)\delta})\big)\\
&+\frac{1}{[n]}\sum_{i=1}^{n-1} E_{i\delta+\alpha_1}\big(\sum_{j=1}^{n-i} v^{(i+j-n)}v^{(2i-1)}(v^2-v^{-2})\tilde{E}_{j\delta}E_{(n-j-i)\delta})\big)\\
&+\frac{1}{[n]}\sum_{i=1}^{n} v^{i-n}\big(v^{2(i-1)}[2]E_{i\delta+\alpha_1}E_{(n-i)\delta}\big).
\end{aligned}$$

By using $E_{(n-i)\delta} = \dfrac{1}{[n-i]}\displaystyle\sum_{j=1}^{n-i} v^{j-(n-i)}\tilde{E}_{i\delta}E_{(n-i)\delta}$, we can simplify the above equality. We omit the detail of calculating the coefficient of each term appeared in the above equality. The coefficient of the term $\tilde{E}_{i\delta}E_{(n-i)\delta}$ is $[i+1]$. The proof is completed. □

5. An integral Poincaré–Birkhoff-Witt basis

Let us summarize the results of proceeding paragraphs. Firstly, we define a total order on Φ^+ adapted for the structure of the Auslander–Reiten quiver of the Kronecker algebra. The order is given by

$$\begin{aligned}
&\alpha_1 < \delta+\alpha_1 < \cdots < (n-1)\delta+\alpha_1 < n\delta+\alpha_1 < \cdots < \delta < 2\delta < \cdots\\
&< (n-1)\delta < n\delta < \cdots < n\delta+\alpha_2 < (n-1)\delta+\alpha_2 < \cdots < \delta+\alpha_2 < \alpha_2.
\end{aligned}$$

For any root vector E_α, consider the element $E_\alpha^{(t)} = \frac{E_\alpha^t}{[t]!}$. These elements are called divided powers. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and let $C_\star(\Delta)$ be the $\mathcal{A}$–subalgebra of $C(\Delta)$ generated by the elements $E_i^{(t)}$. We obtain the following result.

Theorem 5.1. *The set* $\mathcal{B}^+ = \{E_{\beta_1}^{(r_1)}E_{\beta_2}^{(r_2)}\cdots E_{\beta_n}^{(r_n)} | n \in \mathbb{N}, \beta_1 < \beta_2 < \cdots < \beta_n \in \Phi^+\}$ *is an* $\mathcal{A}$*–basis of* $C_\star(\Delta)$.

Proof. It is more or less the same as Theorem 2 in [3] or Theorem 5.1 in [24]. □

The imaginary root vectors play a very important rule for the canonical basis of quantum groups. Beck, Chari and Pressley [3] gave an algebraic characterization of the affine canonical basis by its behavior with respect to a symmetric bilinear form. They used the theory of symmetric functions [24] to modify the imaginary root vectors $E_{n\delta}$ in order to get the canonical basis. In [19], Ringel defined a symmetric bilinear form on Ringel–Hall algebra as follows: For $\alpha, \beta \in P$, let

$$(u_\alpha, u_\beta) = \begin{cases} \frac{|V_\alpha|}{a_\alpha} & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

The coefficients $\frac{|V_i|}{a_i}$ are crucial in Lusztig's description of the canonical basis of quantum groups. An open problem is how to realize the canonical basis of the composition algebra of the Kronecker algebra based on the realization of the imaginary root vectors shown in this paper.

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