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RESEARCH ARTICLE

On faithful actions of groups and semigroups by orientation-preserving plane isometries

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Dedicated to R. I. Grigorchuk on the occasion of his 50th birthday

Abstract. Feitful representations of two generated free groups and free semigroups by orientation-preserving plane isometries constructed.

Let $\mathcal{G}_+$ denote the group of orientation-preserving isometries of Euclidean plane. $\mathcal{G}_+$ is a locally compact Lie group, it consists of rotations and translations. Let G be a countable group or semigroup. An *action* of the (semi)group G on the plane by orientation-preserving isometries is a homomorphism $d : G \to \mathcal{G}_+$. Let x be a point in the plane. The *orbit* of x under the action d is the sequence $O_d(x) = \{d(g)x\}_{g\in G}$ indexed by elements of G. Suppose G is finitely generated and $g_1, \dots, g_k$ is some fixed set of its generators. Then the action d is uniquely determined by isometries $A_1 = d(g_1), \dots, A_k = d(g_k)$, and we denote it by $G[A_1, \dots, A_k]$. In general, the action $G[A_1, \dots, A_k]$ may not exist for some k-tuples $(A_1, \dots, A_k)$ of isometries. It does exist in the case G is the free semigroup FSG_k or the free group FG_k with k generators.

The action d is called *faithful* if it is a monomorphism. Suppose $d(g_1)x = d(g_2)x$ for some $g_1, g_2 \in G$ and a point x. If $g_1 \neq g_2$ and the action d is faithful, then $d(g_1)d(g_2)^{-1}$ is a nontrivial rotation and x is its fixed point. Thus d is faithful implies there exists a countable subset S_d of the plane such that for any $x \notin S_d$ all points of the orbit $O_d(x)$ are distinct.

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Theorem 1. *For a generic pair $(A, B) \in \mathcal{G}_+^2$ (both in the sense of measure and of category), the action $FSG_2[A, B]$ is faithful.*

Theorem 2. *Suppose A is a nonzero translation and B is a rotation by an angle φ. Then the action $FSG_2[A, B]$ is faithful if and only if $\cos\varphi$ is a transcendent number.*

The action $FG_2[A, B]$ can never be faithful for the following reason. For any group G, let G' denote the *commutant* of G, that is, the group generated by commutators $XYX^{-1}Y^{-1}$, where $X, Y \in G$. By G'' we denote the commutant of G'. It is easy to see that the group $\mathcal{G}_+'$ consists of translations, hence the group $\mathcal{G}_+''$ is trivial. Therefore every action of the group FG_2 of the form $FG_2[A, B]$ descends to an action of the group $G_2 = FG_2/FG_2''$ (the free 2-step-solvable group with two generators).

Theorem 3. *For a generic pair $(A, B) \in \mathcal{G}_+^2$ (both in the sense of measure and of category), the action $G_2[A, B]$ is faithful.*

We proceed to the proofs of Theorems 1, 2, and 3.

A finite sequence $x_0, x_1, \ldots, x_k$ of points of the lattice $\mathbb{Z}^2$ is called a *path* if $x_0 = (0, 0)$ and $|x_j - x_{j-1}| = 1$ for $j = 1, \ldots, k$. Ordered pairs (x_{j-1}, x_j), $1 \le j \le k$, are called *links* of the path. The set of all paths is denoted by P. A path $x_0, x_1, \ldots, x_k$ is *closed* if its endpoint x_k coincides with x_0. The set of all closed paths is denoted by P'.

Let x_1 and x_2 be neighboring points of the lattice $\mathbb{Z}^2$ and $\gamma \in P$. Denote by $n_\gamma(x_1, x_2)$ the number of times when the pair (x_1, x_2) occurs as a link of the path γ. Let P'' be the set of paths $\gamma \in P$ such that $n_\gamma(x_1, x_2) = n_\gamma(x_2, x_1)$ for any $x_1, x_2 \in \mathbb{Z}^2$, $|x_2 - x_1| = 1$. Clearly, $P'' \subset P'$.

Now let us assign a path $\gamma(g) \in P$ to an arbitrary element $g \in FG_2$. Let a and b be generators of FG_2. Introduce vectors $e_a = (1, 0)$, $e_{a^{-1}} = (-1, 0)$, $e_b = (0, 1)$, $e_{b^{-1}} = (0, -1)$. Every element $g \in FG_2$ can be represented in the form $c_k c_{k-1} \ldots c_1$, where $c_j \in \{a, b, a^{-1}, b^{-1}\}$, $j = 1, 2, \ldots, k$. Choose $\gamma(g)$ to be the path $x_0, x_1, \ldots, x_k$ such that $x_0 = (0, 0)$ and $x_j - x_{j-1} = e_{c_j}$, $1 \le j \le k$. Obviously, each path $\gamma \in P$ is assigned to a unique element of the group FG_2. However the path $\gamma(g)$ is not determined in a unique way by g. Still, some crucial features of $\gamma(g)$ depend only on an element $g \in FG_2$. These are the endpoint of $\gamma(g)$ and differences $n_{\gamma(g)}(x_1, x_2) - n_{\gamma(g)}(x_2, x_1)$ for all $x_1, x_2 \in \mathbb{Z}^2$, $|x_1 - x_2| = 1$. Given $g \in FG_2$, the set of paths assigned to g contains a unique path of the shortest length. The number of links in this shortest path is called the *length* of g.

Lemma 1. *Suppose $g \in FG_2$. Then $g \in FG_2'$ if and only if $\gamma(g) \in P'$, and $g \in FG_2''$ if and only if $\gamma(g) \in P''$.*

Proof. Let $g, h \in FG_2$. Suppose $x_0, x_1, \dots, x_k$ is the path $\gamma(g)$ and $y_0, \dots, y_m$ is the path $\gamma(h)$. Then the sequence $x_0, x_1, \dots, x_k, x_k+y_1, \dots, x_k+y_m$ is the path $\gamma(hg)$ and $x_0 = x_k - x_k, x_{k-1} - x_k, \dots, x_0 - x_k = -x_k$ is the path $\gamma(g^{-1})$. Let $N_1 : FG_2 \to \mathbb{Z}^2$ be the map taking each $g \in FG_2$ to the endpoint of the path $\gamma(g)$. It is easy to observe that N_1 is a homomorphism. Let H_1 denote the kernel of N_1. Then $g \in H_1$ if and only if $\gamma(g) \in P'$. Clearly, H_1 is a normal subgroup of FG_2 and $FG_2' \subset H_1$. Take any element $g \in H_1$ of positive length. The element g is uniquely represented as $c_k c_{k-1} \dots c_1$, where $c_j \in \{a, b, a^{-1}, b^{-1}\}$, $1 \le j \le k$, and k is the length of g. Since $g \in H_1$, we have $c_m = c_1^{-1}$ for some m, $1 < m \le k$. By construction, $m > 2$. Set $h = c_{m-1} c_{m-2} \dots c_2$. Then the element $g_1 = g c_1^{-1} h^{-1} c_1 h = c_k \dots c_{m+1} c_{m-1} \dots c_2$ is of length at most $k-2$. Moreover, $g_1 \in H_1$ since $c_1^{-1} h^{-1} c_1 h \in FG_2'$. The inductive argument yields that $H_1 = FG_2'$.

Let L denote the set of ordered pairs (x_1, x_2) such that $x_1, x_2 \in \mathbb{Z}^2$ and $|x_1 - x_2| = 1$. For any path $\gamma \in P$ the collection of numbers $n_\gamma(x_1, x_2) - n_\gamma(x_2, x_1)$, $(x_1, x_2) \in L$, can be considered as an element of the group $\mathbb{Z}^L$. Since differences $n_{\gamma(g)}(x_1, x_2) - n_{\gamma(g)}(x_2, x_1)$ depend only on $g \in FG_2$, we have a well-defined map $N_2 : FG_2 \to \mathbb{Z}^L$. The restriction of the map N_2 to the subgroup $H_1 = FG_2'$ is a homomorphism. By H_2 denote the kernel of this restriction. Clearly, $g \in H_2$ if and only if $\gamma(g) \in P''$. It is easy to observe that H_2 is a normal subgroup of FG_2 and $FG_2'' \subset H_2$. We claim that $H_2 = FG_2''$, i.e., any element $g \in H_2$ belongs to FG_2''. The claim is proved by induction on the length k of the element g. In the case $k = 0$, there is nothing to prove. Now let $k > 0$ and suppose the claim is true for all elements of length less than k. There is a unique representation $g = c_k c_{k-1} \dots c_1$ such that $c_j \in \{a, b, a^{-1}, b^{-1}\}$, $1 \le j \le k$. Denote by γ the path $x_0, x_1, \dots, x_k$ such that $x_0 = (0,0)$ and $x_j - x_{j-1} = e_{c_j}$, $1 \le j \le k$. Then $\gamma \in P''$ since $g \in H_2$. In particular, there exists an index $l > 0$ such that the points $x_0, x_1, \dots, x_{l-1}$ are distinct while $x_l = x_m$ for some $m < l$. Set $g_1 = c_{m-1} \dots c_1 c_k \dots c_m$. Then $g_1 = c_{m-1} \dots c_1 g (c_{m-1} \dots c_1)^{-1} \in H_2$ and the length of g_1 is at most k. The path $\gamma(g_1)$ can be chosen as $y_0, y_1, \dots, y_k$, where $y_i = x_{i+m} - x_m$ for $0 \le i \le k - m$ and $y_i = x_{i-k+m} - x_m$ for $i > k - m$. Since $\gamma(g_1) \in P''$, there exists $n > 0$ such that $y_{n-1} = y_1$ and $y_n = y_0$. By construction, the points $y_0, y_1, \dots, y_{l-m-1}$ are distinct and $y_{l-m} = y_0$, hence $n > l - m$. The sequences $y_0, y_1, \dots, y_{l-m}$, and $y_{l-m}, \dots, y_n$, and $y_n, \dots, y_k$ are closed paths. They are assigned to some elements $h_1, h_2, h_3 \in FG_2'$, respectively. Clearly, $g_1 = h_3 h_2 h_1$. Since $y_{n-1} = y_1$, the element $g_2 = h_3 h_1 h_2$ is of length at most $k-2$. Moreover, $g_2 = g_1 h_1^{-1} h_2^{-1} h_1 h_2 \in H_2$ as $h_1^{-1} h_2^{-1} h_1 h_2 \in FG_2''$. By the inductive assumption, $g_2 \in FG_2''$. Then $g_1 \in FG_2''$. Since g and g_1 are conjugated, we have $g \in FG_2''$. The claim is proved. $\square$

Let P_1'' denote the set of paths $\gamma \in P$ such that $n_\gamma(x, x+e_a) = n_\gamma(x+e_a, x)$ for every $x \in \mathbb{Z}^2$ and P_2'' denote the set of paths $\gamma \in P$ such that $n_\gamma(x, x+e_b) = n_\gamma(x+e_b, x)$ for every $x \in \mathbb{Z}^2$.

Lemma 2. $P_1'' \cap P' = P_2'' \cap P' = P''$.

Proof. Obviously, $P'' = P_1'' \cap P_2''$. For every path $\gamma \in P'$ and every $x \in \mathbb{Z}^2$ we have the equality $n_\gamma(x, x+e_a) + n_\gamma(x, x-e_a) + n_\gamma(x, x+e_b) + n_\gamma(x, x-e_b) = n_\gamma(x+e_a, x) + n_\gamma(x-e_a, x) + n_\gamma(x+e_b, x) + n_\gamma(x-e_b, x)$. If, moreover, $\gamma \in P_1''$, then $n_\gamma(x, x+e_a) = n_\gamma(x+e_a, x)$ and $n_\gamma(x, x-e_a) = n_\gamma(x-e_a, x)$, hence $n_\gamma(x, x+e_b) = n_\gamma(x+e_b, x)$ if and only if $n_\gamma(x-e_b, x) = n_\gamma(x, x-e_b)$. By the inductive argument we obtain that the equalities $n_\gamma(x, x+e_b) = n_\gamma(x+e_b, x)$ and $n_\gamma(x+ke_b, x+(k+1)e_b) = n_\gamma(x+(k+1)e_b, x+ke_b)$ are equivalent for any $\gamma \in P_1'' \cap P'$, any $x \in \mathbb{Z}^2$, and any integer k. Since $n_\gamma(x+ke_b, x+(k+1)e_b) = n_\gamma(x+(k+1)e_b, x+ke_b) = 0$ for large k, the equality $n_\gamma(x, x+e_b) = n_\gamma(x+e_b, x)$ holds. Thus, $P_1'' \cap P' \subset P_2''$. The relation $P_2'' \cap P' \subset P_1''$ is established in the same way. The lemma is proved. $\square$

Let $A, B \in \mathcal{G}_+$ be noncommuting (counterclockwise) rotations by angles φ and ψ, respectively. We assume that the angles φ and ψ are not multiples of 2π.

Lemma 3. *Suppose the action $G_2[A,B]$ is not faithful. Then there exists a nonzero polynomial Q in two variables with integer coefficients such that $Q(e^{i\varphi}, e^{i\psi}) = 0$.*

Proof. Let x_0 be the fixed point of the rotation B. Let R_α denote the rotation by an angle α around the point x_0. Let $T(y)$ denote the translation by a vector $y \in \mathbb{R}^2$. We have $B = R_\psi$ and $A = R_\varphi T(z)$, where z is a nonzero vector. Set $d = FG_2[A,B]$. Given an element $g \in FG_2$, let (m,k) be the endpoint of the path $\gamma(g)$. It is easy to observe that $d(g) = R_{m\varphi+k\psi}T(y)$ for some $y \in \mathbb{R}^2$. Then

$$\begin{array}{lclcl} d(ag) & = & Ad(g) & = & R_{(m+1)\varphi+k\psi}T(y+R_{-m\varphi-k\psi}z), \\ d(a^{-1}g) & = & A^{-1}d(g) & = & R_{(m-1)\varphi+k\psi}T(y-R_{-(m-1)\varphi-k\psi}z), \\ d(bg) & = & Bd(g) & = & R_{m\varphi+(k+1)\psi}T(y), \\ d(b^{-1}g) & = & B^{-1}d(g) & = & R_{m\varphi+(k-1)\psi}T(y). \end{array}$$

These relations along with the inductive argument allow us to calculate the isometry $d(g)$ for every $g \in FG_2$. We obtain $d(g) = R_{m_1\varphi+k_1\psi}T(y)$, where (m_1, k_1) is the endpoint of the path $\gamma(g)$ and

$$y = \sum_{(m,k)\in\mathbb{Z}^2} \Big(n_{\gamma(g)}((m,k),(m+1,k)) - n_{\gamma(g)}((m+1,k),(m,k))\Big) R_{-m\varphi-k\psi} z.$$

Suppose the isometry $d(g)$ is the identity. Then $m_1\varphi + k_1\psi$ is a multiple of 2π and $y = 0$. The first condition is equivalent to the equality $e^{i(m_1\varphi+k_1\psi)} = 1$. Since z is a nonzero vector, the condition $y = 0$ is equivalent to the equality

$$\sum_{(m,k)\in\mathbb{Z}^2} \Big(n_{\gamma(g)}((m,k),(m+1,k)) - n_{\gamma(g)}((m+1,k),(m,k))\Big) e^{-i(m\varphi+k\psi)} = 0.$$

If $\gamma(g) \notin P' \cap P''_1$, then the two equalities imply there exists a nonzero polynomial Q in two variables with integer coefficients such that $Q(e^{i\varphi}, e^{i\psi}) = 0$. On the other hand, if $\gamma(g) \in P' \cap P''_1$, then $g \in FG''_2$ due to Lemmas 1 and 2.

Finally, we can guarantee that at least one of the following conditions holds: (i) there exists a nonzero polynomial Q in two variables with integer coefficients such that $Q(e^{i\varphi}, e^{i\psi}) = 0$; (ii) the isometry $FG_2[A,B](g)$ is the identity if and only if $g \in FG''_2$. The condition (ii) means the action $G_2[A,B]$ is faithful. □

Lemma 4. *There exist F_σ-sets $S_1, S_2 \in \mathbb{R}^2$ such that:*

(i) the section $\{\beta \mid (\alpha,\beta) \in S_1\}$ is at most countable for any $\alpha \in \mathbb{R}$,

(ii) the section $\{\alpha \mid (\alpha,\beta) \in S_2\}$ is at most countable for any $\beta \in \mathbb{R}$,

(iii) the action $G_2[A,B]$ is faithful whenever $(\varphi,\psi) \notin S_1 \cup S_2$.

Proof. Let Q be a nonzero polynomial in two variables with integer coefficients. Clearly, the set $Z(Q) = \{(z_1,z_2) \in \mathbb{C}^2 \mid Q(z_1,z_2) = 0\}$ is closed. The expression $Q(z_1,z_2)$ is uniquely represented in the form

$$p_0(z_2)z_1^m + p_1(z_2)z_1^{m-1} + \cdots + p_{m-1}(z_2)z_1 + p_m(z_2),$$

where $p_0, p_1, \ldots, p_m$ ($m \geq 0$) are polynomials in one variable with integer coefficients and, moreover, p_0 is a nonzero polynomial. Set $P(Q) = \{(z_1,z_2) \in \mathbb{C}^2 \mid p_0(z_2) = 0\}$, $Z_1(Q) = Z(Q) \cap P(Q)$, and $Z_2(Q) = Z(Q) \setminus Z_1(Q)$. Since p_0 is a nonzero polynomial, the set $P(Q)$ is the union of a finite number of parallel planes in $\mathbb{C}^2$. Then the set $Z_1(Q)$ is closed and the section $\{z_2 \mid (z_1,z_2) \in Z_1(Q)\}$ is at most finite for any $z_1 \in \mathbb{C}$. Given $\epsilon > 0$, let $P_\epsilon(Q)$ denote ϵ-neighborhood of the set $P(Q)$. Obviously, the set $Z_2(Q) \setminus P_\epsilon(Q)$ is closed for any $\epsilon > 0$, therefore $Z_2(Q)$ is an F_σ-set. Take any $z_2 \in \mathbb{C}$. If $p_0(z_2) \neq 0$, then the section

$\{z_1 \mid (z_1, z_2) \in Z_2(Q)\} = \{z_1 \mid (z_1, z_2) \in Z(Q)\}$ contains at most m elements. If $p_0(z_2) = 0$, then the section $\{z_1 \mid (z_1, z_2) \in Z_2(Q)\}$ is empty.

Set $Z_1 = \bigcup_Q Z_1(Q)$ and $Z_2 = \bigcup_Q Z_2(Q)$, where both unions are over all nonzero polynomials in two variables with integer coefficients. Since there are only countably many such polynomials, it follows from the above that Z_1 and Z_2 are F_σ-sets. Moreover, for any $z_1 \in \mathbb{C}$ the section $\{z_2 \mid (z_1, z_2) \in Z_1\}$ is at most countable, and for any $z_2 \in \mathbb{C}$ the section $\{z_1 \mid (z_1, z_2) \in Z_2\}$ is at most countable.

Define a map $E : \mathbb{R}^2 \to \mathbb{C}^2$ by the relation $E(\alpha, \beta) = (e^{i\alpha}, e^{i\beta})$ for any $\alpha, \beta \in \mathbb{R}^2$. Set $S_1 = E^{-1}(Z_1)$ and $S_2 = E^{-1}(Z_2)$. The map E is continuous and the preimage $E^{-1}(z)$ of any point $z \in \mathbb{C}^2$ is at most countable. It follows that S_1 and S_2 are F_σ-sets satisfying conditions (i) and (ii).

Recall that A and B are noncommuting rotations by the angles φ and ψ, respectively. Suppose $(\psi, \varphi) \notin S_1 \cup S_2$. Then $Q(e^{i\varphi}, e^{i\psi}) \neq 0$ for each nonzero polynomial Q in two variables with integer coefficients. By Lemma 3, the action $G_2[A, B]$ is faithful. Thus condition (iii) holds. □

Proof of Theorem 3

Let x_0 be a point in Euclidean plane. For any $\alpha \in \mathbb{R}$ and any $y \in \mathbb{R}^2$, let R_α denote the (counterclockwise) rotation by the angle α around the point x_0 and $T(y)$ denote the translation by the vector y. Define a map $D : \mathbb{R} \times \mathbb{R}^2 \to \mathcal{G}_+$ by the relation $(\alpha, y) \mapsto R_\alpha T(y)$. The map D descends to a map $D_0 : \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^2 \to \mathcal{G}_+$, which is a diffeomorphism. Let $S_1, S_2 \subset \mathbb{R}^2$ be F_σ-sets satisfying conditions (i), (ii), and (iii) of Lemma 4. We can assume without loss of generality that S_1 and S_2 are invariant under translations from $(2\pi\mathbb{Z})^2$. Set $S_0 = \mathbb{R}^2 \setminus (S_1 \cup S_2)$. It follows from the conditions (i) and (ii) that S_0 is a G_δ-subset of $\mathbb{R}^2$ which is dense and of full measure. Finally, let $\mathcal{S}$ denote the set of pairs $(A, B) \in \mathcal{G}_+^2$ such that $A = R_\varphi T(y)$ and $B = R_\psi T(z)$, where $(\varphi, \psi) \in S_0$, φ and ψ are not multiples of 2π, and $y \neq z$. Since D_0 is a diffeomorphism, it follows that $\mathcal{S}$ is a dense G_δ-subset of full measure of $\mathcal{G}_+^2$. This means that a pair $(A, B) \in \mathcal{S}$ is generic both in the sense of measure and of category. By construction, A and B are nontrivial rotations that do not commute. By Lemma 4, the action $G_2[A, B]$ is faithful. □

Lemma 5. *Generators of the group FG_2/FG_2'' generate a free subsemigroup.*

Proof. Let a and b be generators of the group FG_2. By H denote the semigroup generated by a and b. Suppose $g_1, g_2 \in H$. We have to prove that $g_2^{-1} g_1 \in FG_2''$ only if $g_1 = g_2$. Let $x_0, x_1, \ldots, x_k$ be the path $\gamma(g_1)$

and $y_0, y_1, \ldots, y_n$ be the path $\gamma(g_2)$. Without loss of generality it can be assumed that all links (x_{j-1}, x_j) and (y_{j-1}, y_j) are of the form $(x, x+e_a)$ or $(x, x+e_b)$. If $g_2^{-1}g_1 \in FG_2'$, then $n = k$, $x_k = y_k$, and $x_0, x_1, \ldots, x_k = y_k, \ldots, y_1, y_0$ is the path $\gamma(g_2^{-1}g_1)$. Obviously, $n_{\gamma(g_2^{-1}g_1)}(x_{j-1}, x_j) = 1$ for any $j = 1, \ldots, k$, while $n_{\gamma(g_2^{-1}g_1)}(x_j, x_{j-1}) = 1$ only if (x_{j-1}, x_j) is a link of the path $\gamma(g_2)$. It follows from Lemma 1 that $g_2^{-1}g_1 \in FG_2''$ only if $g_1 = g_2$. □

Proof of Theorem 1

Let a and b be generators of the free group FG_2. Let $p : FG_2 \to G_2 = FG_2/FG_2''$ be the natural projection. The elements $p(a)$ and $p(b)$ are generators of the group G_2. By Lemma 5, the semigroup generated by $p(a)$ and $p(b)$ is free. It follows easily that for any $A, B \in \mathcal{G}_+$ the action $FSG_2[A, B]$ is faithful whenever the action $G_2[A, B]$ is faithful. Thus Theorem 1 is a corollary of Theorem 3. □

Proof of Theorem 2

Let x_0 be the fixed point of the rotation B. Denote by R_α the rotation by an angle α around the point x_0. Denote by $T(y)$ the translation by a vector $y \in \mathbb{R}^2$. We have $B = R_\varphi$ and $A = T(z)$, where z is a nonzero vector. Let a and b be generators of the semigroup FSG_2. An arbitrary element $g \in FSG_2$ can be uniquely represented in the form $b^{m_k}ab^{m_{k-1}}a \ldots b^{m_1}ab^{m_0}$, where $m_0, m_1, \ldots, m_k$ are nonnegative integers. It is easy to observe that $FSG_2[A, B](g) = R_{\alpha_g}T(y_g)$, where $\alpha_g = \varphi \sum_{j=0}^{k} m_j$ and

$$y_g = R_{-m_0\varphi}z + R_{-(m_0+m_1)\varphi}z + \cdots + R_{-(m_0+m_1+\cdots+m_{k-1})\varphi}z.$$

Let $h = b^{l_s}ab^{l_{s-1}}a \ldots b^{l_1}ab^{l_0}$ be an element of FSG_2 different from g. Suppose that $FSG_2[A, B](h) = FSG_2[A, B](g)$. Then $\alpha_h - \alpha_g$ is a multiple of 2π and $y_h = y_g$. The first condition is equivalent to the equality $e^{i(l_0+l_1+\cdots+l_s)\varphi} = e^{i(m_0+m_1+\cdots+m_k)\varphi}$, while the second condition is equivalent to the equality

$$e^{-il_0\varphi} + e^{-i(l_0+l_1)\varphi} + \cdots + e^{-i(l_0+\cdots+l_{s-1})\varphi} = e^{-im_0\varphi} + e^{-i(m_0+m_1)\varphi} + \cdots + e^{-i(m_0+\cdots+m_{k-1})\varphi}.$$

Since the sequences $m_0, m_1, \ldots, m_k$ and $l_0, l_1, \ldots, l_s$ are different, the two equalities imply $e^{-i\varphi}$ is an algebraic number. Thus the action $FSG_2[A, B]$ can be not faithful only if the number $e^{-i\varphi}$ is algebraic.

Now suppose $e^{-i\varphi}$ is an algebraic number. Then there exist two different nondecreasing sequences $m_0, m_1, \ldots, m_k$ and $l_0, l_1, \ldots, l_s$ of nonnegative integers such that

$$e^{-im_0\varphi} + e^{-im_1\varphi} + \cdots + e^{-im_k\varphi} = e^{-il_0\varphi} + e^{-il_1\varphi} + \cdots + e^{-il_s\varphi}.$$

Choose a positive integer M such that $m_k \leq M$ and $l_s \leq M$. We can observe that $FSG_2[A,B](g) = FSG_2[A,B](h)$, where

$$\begin{aligned} g &= b^{M-m_k}ab^{m_k-m_{k-1}}a\ldots b^{m_1-m_0}ab^{m_0}, \\ h &= b^{M-l_s}ab^{l_s-l_{s-1}}a\ldots b^{l_1-l_0}ab^{l_0}. \end{aligned}$$

The elements g and h of the semigroup FSG_2 are different, therefore the action $FSG_2[A,B]$ is not faithful.

It remains to observe that, given a real number α, the numbers $e^{-i\alpha}$, $\sin\alpha$ and $\cos\alpha$ are either all algebraic or all transcendent. □

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