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RESEARCH ARTICLE

Binary coronas of balleans

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Abstract. A ballean $\mathbb{B}$ is a set X endowed with some family of subsets of X which are called the balls. We postulate the properties of the family of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. Using slow oscillating functions from X to $\{0,1\}$, we define a zero-dimensional compact space which is called a binary corona of $\mathbb{B}$. We define a class of binary normal ballean and, for every ballean from this class, give an intrinsic characterization of its binary corona. The class of binary normal balleans contains all balleans of graph. We show that a ballean of graph is a projective limit of some sequence of Čech-Stone compactifications of discrete spaces. The obtained results witness that a binary corona of balleans can be interpreted as a "generalized space of ends" of ballean.

§1. Introduction

A *ball structure* is a triple $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x,\alpha)$ is a subset of X which is called a *ball* of radius α around x. It is supposed that $x \in B(x,\alpha)$ for all $x \in X$, $\alpha \in P$. A set X is called a *support* of $\mathbb{B}$, P is called a *set of radiuses.* Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, put

$$B^{\star}(x,\alpha) = \{y \in X : x \in B(y,\alpha)\}, \quad B(A,\alpha) = \bigcup_{a\in A} B(a,\alpha).$$

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A ball structure $\mathbb{B} = (X, P, B)$ is called *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that

$$B^{\star}(x, \alpha') \subseteq B(x, \alpha), \quad B(x, \beta') \subseteq B^{\star}(x, \beta)$$

for every $x \in X$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that

$$B(x, \alpha) \subseteq B^{\star}(x, \alpha'), \quad B^{\star}(x, \beta) \subseteq B(x, \beta')$$

for every $x \in X$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \bigcap B(x, \beta)$$

for every $x \in X$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma)$$

for every $x \in X$.

Let $\mathbb{B} = (X, P, B)$ be lower symmetric , lower multiplicative ball structure. Then the family

$$\{ \bigcup_{x \in X} (B(x, \alpha) \times B(x, \alpha)) : \alpha \in P \}$$

is a fundamental system of entourages for some (uniquely determined) uniform topological space [1] . On the other hand, if X is a uniform topological space with the uniformity $\mathcal{U} \subseteq X \times X$, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, \mathrm{U}) = \{y \in X : (x, y) \in \mathrm{U}\}$ for every $\mathrm{U} \in \mathcal{U}$. Thus, the lower symmetric, lower multiplicative ball structures could be identified with the uniform topological spaces.

A ball structure $\mathbb{B}$ is called a *ballean* if $\mathbb{B}$ is upper symmetric and upper multiplicative.

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \longrightarrow X_2$ is called a *$\prec$-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. A bijection $f : X_1 \longrightarrow X_2$ is called an *isomorphism* if f and f^{-1} are *$\prec$-mappings.*

Let $\mathbb{B}_1, \mathbb{B}_2$ be balleans with common support X. We say that $\mathbb{B}_1 \subseteq \mathbb{B}_2$ if the identity mapping $f : X \longrightarrow X$ is a $\prec$-*mapping* of $\mathbb{B}_1$ to $\mathbb{B}_2$. If $\mathbb{B}_1 \subseteq \mathbb{B}_2$ and $\mathbb{B}_2 \subseteq \mathbb{B}_1$, we write $\mathbb{B}_1 = \mathbb{B}_2$.

Given a metric space (X, d), denote by $\mathbb{B}(X, d)$ the ballean $(X, \mathbb{R}^+, B_d)$, where $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$. A ballean $\mathbb{B}$ is called *metrizable* if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, d)$ for an appropriate metric space (X, d). A characterization of metrizable ballean is given in [7]. For approximation of an arbitrary balleans via metrizable ballean see [8].

A metric space is called *perfect* if every ball $B_d(x, r)$ is compact. Actually, the balleans of perfect metric spaces arose in geometrical group theory [4, Chapter IV] and are intensively investigating (under the name coarse space) in asymptotic topology [3].

For every perfect metric space X, there exists a compact space $\overline{X}$ such that X is dense in $\overline{X}$ and every continuous slow oscillating function $h : X \longrightarrow [0, 1]$ can be extended to $\overline{X}$. The space $\overline{X}$ is called a *Higson's compactification* of X and the reminder $\overline{X} \setminus X$ is called a *Higson's corona* of X. For connections between asymptotic properties of X and topological properties of its Higson's corona see the survey [3]. Note only that Higson's corona of X is very complicated even for $X = \mathbb{R}^n$.

In this paper, for every ballean $\mathbb{B} = (X, P, B)$ and every compact metric space K, we define a K-corona of $\mathbb{B}$, using ultrafilters and slow oscillating functions from X to K. In the case $K = [0, 1]$ this corona is a direct generalization of Higson's corona of perfect metric space. In the case $K = \{0, 1\}$ this corona is called a binary corona of ballean $\mathbb{B}$. A binary corona could be much more coarser than a Higson's corona, but it admits an explicit description for some class of balleans and reflects some essential features of balleans . Our main results concern the binary coronas of graphs and show that a binary corona of an arbitrary ballean can be considered as its "space of ends".

§2. Slow oscillating mappings, coronas, quasi-isomorphisms

Let $\mathbb{B} = (X, P, B)$ be a ballean, $(Y, \mathcal{U})$ be a uniform topological space. A mapping $h : X \longrightarrow Y$ is called *slow oscillating* if, for every entourage $U \in \mathcal{U}$ and every $\alpha \in P$, there exists a bounded subset V of X such that

$$(h(B(x, \alpha)), h(B(x, \alpha))) \subseteq U$$

for every $x \in X \setminus V$. A subset V is called *bounded* if there exist $x_0 \in X$, $\beta \in P$ such that $V \subseteq B(x_0, \beta)$.

Fix a ballean $\mathbb{B} = (X, P, B)$, endow X with discrete topology and consider the Stone-Čech compactification βX of X. We take the points of βX to be the ultrafilters on X with the points of X identifying with the principal ultrafilters. The topology of βX can be defined by stating that the sets of the form $\{p \in \beta X : A \in p\}$, where A is a subset of X, are a base for the open sets. We note that the sets of this form are clopen and that, for any $p \in \beta X$ and any $A \subseteq X$, $A \in p$ if and only if $p \in \overline{A}$, where $\overline{A}$ is the closure of A in βX.

We say that an ultrafilter $p \in \beta X$ is *unbounded* if every member $A \in p$ is unbounded. Denote by $X^{\#}$ the set of all unbounded ultrafilters. Clearly, $X^{\#}$ is closed in βX.

Let K be a compact Hausdorff space, $h : X \longrightarrow K$. Then there exists unique continuous extension $h^{\beta} : \beta X \longrightarrow K$.

Let (K, d) be compact metric space, $r, q \in X^{\#}$. We say that $r \sim_K q$ if $h^{\beta}(p) = h^{\beta}(q)$ for every slow oscillating mapping $h : X \longrightarrow (K, d)$. Clearly, $\sim_K$ is a closed (in $X^{\#} \times X^{\#}$) equivalence on $X^{\#}$. A factor-space $X^{\#}/\sim_K$ is called a *K-corona* of $\mathbb{B}$ and is denoted by $\gamma(\mathbb{B}, K)$.

In the case $K = [0, 1]$ we say that $\gamma(\mathbb{B}, [0, 1])$ is a *Higson's corona* of $\mathbb{B}$. In the case $K = \{0, 1\}$ we say that $\gamma(\mathbb{B}, \{0, 1\})$ is a *binary corona* of $\mathbb{B}$.

For $r, q \in X^{\#}$, we say that $r \parallel q$ if there exists $\alpha \in P$ such that $B(R, \alpha) \in q$ for every $R \in r$. By [8, Lemma 4.1], $\parallel$ is an equivalence on $X^{\#}$. Denote by $\sim$ the minimal (by inclusion) closed (in $X^{\#} \times X^{\#}$) equivalence on $X^{\#}$ such that $\parallel \subseteq \sim$. A compact Hausdorff space $X^{\#}/\sim$ is called a *corona* of $\mathbb{B}$, it is denoted by $\gamma(\mathbb{B})$.

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be balleans, $f : X_1 \longrightarrow X_2$ be *a* $\prec$-mapping. By analogy with topology, we say that f is a *perfect* mapping if $f^{-1}(V)$ is bounded for every bounded subset $V \subseteq X_2$.

A pair of perfect mappings $f_1 : X_1 \longrightarrow X_2$, $f_2 : X_2 \longrightarrow X_1$ is called a *quasi-isomorphism* between $\mathbb{B}_1$ and $\mathbb{B}_2$ if there exist $\alpha \in P_1$, $\beta \in P_2$ such that

(i) $B_2(f_1(X_1, \beta) = X_2$, $B_1(f_2(X_2, \beta) = X_1$;

(ii) $f_2 f_1(x) \in B_1(x, \alpha)$, $f_1 f_2(y) \in B_2(y, \beta)$ for all $x \in X_1$, $y \in X_2$.

This notion of quasi-isomorphism between balleans is a generalization of the notion of quasi-isometry between metric spaces [4, Chapter IV].

We omit routine verification of the following relations between the above notions.

- Let $\mathbb{B} = (X, P, B)$ be a ballean, K be a compact metric space, $r, q \in X^{\#}$. Then $r \parallel q$ implies $r \sim_K q$. It follows that $\gamma(\mathbb{B}, K)$ is a continuous image of $\gamma(\mathbb{B})$.

- Let $\mathbb{B} = (X, P, B)$ be a ballean, K be a compact metric space, M be a closed subspace of K, $r, q \in X^{\#}$. Then $r \sim_K q$ implies $r \sim_M q$. It follows that $\gamma(\mathbb{B}, M)$ is a continuous image of $\gamma(\mathbb{B}, K)$. In particular, binary corona is a continuous image of Higson's corona.

- Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be balleans, $f : X_1 \longrightarrow X_2$ be a perfect mapping, $r, q \in X_1^{\#}$. Then $f^{\beta}(X_1^{\#}) \subseteq X_2^{\#}$, $r \parallel q$ implies $f^{\beta}(r) \parallel f^{\beta}(q)$, $r \sim q$ implies $f^{\beta}(r) \sim f^{\beta}(q)$. Moreover, if there exists $\beta \in P_2$ such that $B_2(f(X_1), \beta) = X_2$, then $\gamma(\mathbb{B}_2)$ is a continuous image of $\gamma(\mathbb{B}_1)$.

- Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be balleans, $f : X_1 \longrightarrow X_2$ be a perfect mapping. Let $(Y, \mathcal{U})$ be a uniform topological space, $h : X_2 \longrightarrow Y$ be a slow oscillating mapping. Then $hf : X_1 \longrightarrow Y$ is a slow oscillating mapping.

- Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be balleans, $f : X_1 \longrightarrow X_2$ be a perfect mapping, K a compact metric space, $r, q \in X_1^{\#}$. Then $r \sim_K q$ implies $f^{\beta}(r) \sim_K f^{\beta}(q)$. Moreover, if there exists $\beta \in P_2$ such that $B_2(f(X_1), \beta) = X_2$, then $\gamma(\mathbb{B}_2, K)$ is a continuous image of $\gamma(\mathbb{B}_1, K)$.

- Let $\mathbb{B}_1$, $\mathbb{B}_2$ be quasi-isomorphic balleans, K be a compact metric space. Then $\gamma(\mathbb{B}_1)$ is homeomorphic to $\gamma(\mathbb{B}_2)$, $\gamma(\mathbb{B}_1, K)$ is homeomorphic to $\gamma(\mathbb{B}_2, K)$.

§3. Binary normal spaces: intrinsic description of binary coronas

Let $\mathbb{B} = (X, P, B)$ be a ballean, K be a compact metric space.

- Which pairs of ultrafilters are identified via the equivalence $\sim_K$?

- Which subsets of $\gamma(\mathbb{B}, K)$ form topology of corona?

To answer these questions for binary corona we modify an intrinsic description of Higson's corona of normal ballean from [9].

The subsets Y, Z of X are called *asymptotically disjoint* in $\mathbb{B}$ if, for every $\alpha \in P$, there exists a bounded subset $U \subseteq X$ such that

$$B(Y \backslash U_{\alpha}, \alpha) \bigcap B(Z \backslash U_{\alpha}, \alpha) = \emptyset.$$

We say that Y, Z are *asymptotically separated* if, for every $\alpha \in P$, there exists a bounded subset $U_{\alpha} \subseteq X$ such that

$$B(Y \backslash U_{\alpha}, \alpha) \bigcap B(Z \backslash U_{\beta}, \beta) = \emptyset$$

for all $\alpha, \beta \in P$.

A ballean $\mathbb{B}$ is called *normal* if any two asymptotically disjoint subsets of X are asymptotically separated. By [9, Theorem 2.2], a ballean $\mathbb{B}$ is normal if and only if, for every subset $Y \subseteq X$ and every slow oscillating function $h : Y \longrightarrow [0,1]$, there exists a slow oscillating extension $g : X \longrightarrow [0,1]$ of h.

Let $\mathbb{B}$ be a normal ballean, $K = [0,1]$, $r, q \in X^{\#}$. By [9, Lemma 4.2], the following statements are equivalent

(i) $r \sim_K q$;

(ii) $r \sim q$;

(iii) for any $R \in r$, $Q \in q$, there exists $\alpha \in P$ such that $B(R,\alpha) \bigcap B(Q,\alpha)$ is unbounded.

Hence, $(i) \Longleftrightarrow (iii)$ gives answer to the first question and $(i) \Longleftrightarrow (ii)$ states that $\gamma(\mathbb{B}) = \gamma(\mathbb{B}, [0,1])$ for every normal ballean $\mathbb{B}$.

To answer the second question (in the case $K = [0,1]$) we use the following definition from [8]. Let $\mathbb{B} = (X, P, B)$ be ballean, $Y \subseteq X$ and let $\{U_\alpha : \alpha \in P\}$ be a family of bounded subsets of X. A set

$$\hat{Y} = \bigcup_{\alpha \in P} B(Y \setminus U_\alpha, \alpha)$$

is called a *pyramid* with the core Y determined by the family $\{U_\alpha : \alpha \in P\}$.

A ballean $\mathbb{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. Now suppose that $\mathbb{B}$ is connected and normal, $r \in X^{\#}$, $[r] = \{q \in X^{\#} : r \sim q\}$. For every $R \in r$, denote by $\Im(R)$ the set of all pyramids with the core R and put $\Im(r) = \bigcup_{R \in r} \Im(R)$. Then the family of subsets $\{[q] : q \in \overline{F}\}$, where F runs over $\Im$, is a fundamental system of neighborhoods of the element $[r] \in \gamma(\mathbb{B})$.

Now we adopt these constructions to the case of binary coronas. Let $\mathbb{B} = (X, P, B)$ be a ballean. Given any subsets Y, V of X and $\alpha \in P$, we define a subset $\Lambda(Y, V, \alpha)$ by the rule:

$x \in \Lambda(Y, V, \alpha)$ *if and only if there exist the elements* $x_1, x_2, ..., x_n \in X \setminus V$ *such that* $x_1 \in Y$, $x_n = x$ *and* $x_{i+1} \in B(x_i, \alpha)$ *for every* $i \in \{1, 2, ..., n-1\}$.

Let $Y \subseteq X$, $\{V_\alpha : \alpha \in P\}$ be a family of bounded subsets of X. A subset

$$\bigcup_{\alpha \in P} \Lambda(Y, V_\alpha, \alpha)$$

is called a *path pyramid* with the core Y determined by the family $\{V_\alpha : \alpha \in P\}$.

Let $Y \subseteq X$, $Z \subseteq X$. We say that Y is *asymptotically path disjoint* from Z if there exists a family $\{V_\alpha : \alpha \in P\}$ of bounded subsets of X such that $\Lambda \bigcap Z = \emptyset$, where Λ is a path pyramid with the core Y determined by the family $\{V_\alpha : \alpha \in P\}$. In this case, Z is also asymptotically path disjoint from Y, so we can say that Y, Z are *asymptotically path disjoint.* We say that Y, Z are *asymptotically path separated* if there exist disjoint path pyramids with the cores Y, Z.

Lemma 1. *Let $\mathbb{B} = (X, P, B)$ be a ballean $h : X \longrightarrow \{1, 0\}$ be a slow oscillating function. If $Y_0, Y_1 \subseteq X$ and $h|_{Y_0} \equiv 0$, $h|_{Y_1} \equiv 1$, then Y_0, Y_1 are asymptotically path separated.*

Proof. We may suppose that $X = Y_0 \bigcup Y_1$. For every $\alpha \in P$, choose a bounded subset V_α such that *diam* $B(h(x), \alpha) = 0$ for every $x \in X \setminus V_\alpha$. Then

$$\Lambda(Y_0, V_\alpha, \alpha) \subseteq Y_0,\ \Lambda(Y_1, V_\alpha, \alpha) \subseteq Y_1.$$

Denote by Λ_0 and Λ_1 the pyramids with the cores Y_0, Y_1 determined by the family $\{V_\alpha : \alpha \in P\}$. Then $\Lambda_0 \subseteq Y_0$, $\Lambda_1 \subseteq Y_1$ and $\Lambda_0 \bigcap \Lambda_1 = \emptyset$. $\square$

A ballean $\mathbb{B} = (X, P, B)$ is called *binary normal* if for any disjoint, asymptotically path disjoint subsets Y_0, Y_1 of X, there exists a slow oscillating function $h : X \longrightarrow 0, 1$ such that $h|_{Y_0} \equiv 0$, $h|_{Y_1} \equiv 1$. By Lemma 1, in this case Y_0, Y_1 are asymptotically path separated.

Given an arbitrary ballean $\mathbb{B} = (X, P, B)$, we say that the subsets Y_0, Y_1 of X are *asymptotically path connected* if Y_0, Y_1 are not asymptotically path disjoint.

Lemma 2. *Let $\mathbb{B} = (X, P, B)$ be a binary normal ballean, $D = \{0, 1\}$, $r, q \in X^{\#}$. Then the following statements are equivalent*

(i) $r \sim_D q$;

(ii) any two subsets $R \in r$, $Q \in q$ are asymptotically path connected;

(iii) $\Lambda \in q$ for every subset $R \in r$ and every path pyramid Λ with the core R.

Proof. $(i) \Longrightarrow (ii)$ follows from definition of binary normal space.

$(ii) \Longrightarrow (iii)$. Suppose the contrary. Since q is an ultrafilter, then there exist $Q \in q$ and $R \in r$ such that $\Lambda \bigcap Q = \emptyset$ for some path pyramid Λ with the core R. It follows that R, Q are asymptotically path disjoint, a contradiction.

$(iii) \Longrightarrow (i)$ follows from Lemma 1. $\square$

Lemma 3. *Let $\mathbb{B} = (X, P, B)$ be a connected binary normal ballean, $D = \{0, 1\}$, $r, q \in X^{\#}$, $x_0 \in X$. Then the following statements are equivalent*

(i) $r \sim_D q$;

(ii) for any subsets $R \in r$, $Q \in q$ there exists $\alpha \in P$ such that $Q \bigcap \Lambda(R, B(x_0, \beta), \alpha)$ is unbounded for every $\beta \in P$.

Proof. $(i) \Longrightarrow (ii)$ Suppose the contrary and choose $R \in r$, $Q \in q$ such that, for every $\alpha \in P$, there exists $\beta(\alpha) \in P$ such that the subset

$$H(\alpha) = Q \bigcap \Lambda(R, B(x_0, \beta(\alpha)), \alpha)$$

is bounded. Since $\mathbb{B}$ is connected, there exists $\gamma(\alpha) \in P$ such that $\gamma(\alpha) \geq \beta(\alpha)$ and $H(\alpha) \subseteq B(x_0, \gamma(\alpha))$. Denote by Λ a pyramid with the core R determined by the family $\{B(x_0, \gamma(\alpha)) : \alpha \in P\}$. Then $\Lambda \bigcap Q = \emptyset$, contradicting the equivalence $(i) \Longrightarrow (iii)$ of Lemma 2.

$(ii) \Longrightarrow (i)$ follows from the implication $(iii) \Longrightarrow (i)$ of Lemma 2.

□

Lemmas 2, 3 give us an explicit description of elements of the binary corona of a binary normal ballean. Now we describe the topology of binary corona.

Let $\mathbb{B} = (X, P, B)$ be a ballean, $D = \{0, 1\}$. For any $R \subseteq X$, $r \in X^\#$, put

$$[r]_D = \{q \in X^\# : q \sim_D r\}, \quad [R]_D = \{[q] : q \in X^\#,\ R \in q\}.$$

Denote by H the set of all slow oscillating functions $h : X \longrightarrow D$. For every $h \in H$, denote by D_h a copy of D and consider a mapping

$$f : \gamma(\mathbb{B}, D) \longrightarrow \prod_{h \in H} D_h,$$

defined by the rule $f([r]_D) = (h^\beta(r))_{h \in H}$. It is easy to check that f is a homeomorphic embedding of $\gamma(\mathbb{B}, D)$ into $\prod_{h \in H} D_h$ endowed with the product topology, so $\gamma(\mathbb{B}, D)$ is a compact zero-dimensional space.

Fix an arbitrary $h \in H$, put $Y_0 = h^{-1}(0)$, $Y_1 = h^{-1}(1)$ and assume that $h^\beta(r) = 0$. By Lemma 1, there exists a pyramid Λ with the core Y_0 such that

$$f([\Lambda]) \subseteq \{y \in \prod_{h' \in H} D_{h'} : pr_h y = 0\}.$$

On the other hand, fix an arbitrary path pyramid Λ with the core $R \in r$. By Lemma 2, $q \notin [r]_D$ for every $q \in (X \setminus \Lambda)^\#$. For every $q \in (X \setminus \Lambda)^\#$, choose $Q_q \in q$, a path pyramid Λ_q with the core $R_q \in r$ and a slow oscillating function $h_q : X \longrightarrow D$ such that $h_q \mid \Lambda_q \equiv 0$, $h_q \mid \Lambda_q \equiv 0$, $h_q \mid Q_q \equiv 1$. Consider a cover $\{Q_q^\# : q \in (X \setminus A)^\#\}$ and choose its finite subcover $Q_{q_1}^\#$, ..., $Q_{q_n}^\#$. Then

$$\{y \in f(\gamma(\mathbb{B}, D)) : pr_{h_1} y = 0,\ \ldots,\ pr_{h_n} y = 0\} \subseteq f([\Lambda]).$$

Thus, we have shown that the family of subsets of the form $[\Lambda]$, where Λ runs over all pyramids with the core $R \in r$, is a fundamental system of neighborhoods of the element $[r] \in \gamma(\mathbb{B}, D)$.

§4. Binary coronas of graph balleans

Let $Gr(V, E)$ be a connected graph with the set of vertices V and the set of edges E. Given any $x, y \in V$, $n \in \omega$ denote by $d(x, y)$ a length of the shortest path between x, y and put $B(x, n) = \{v \in V : d(x, v) \leq n\}$. A ballean of the metric space (V, d) is denoted by $\mathbb{B}(Gr)$. A ballean $\mathbb{B}$ is called a *graph ballean* if $\mathbb{B}$ is isomorphic to $\mathbb{B}(Gr)$ for an appropriate graph Gr. By [6, Theorem 1], $\mathbb{B} = (X, P, B)$ is a graph ballean if and only if $\mathbb{B}$ is metrizable and there exist $\alpha \in P$ and $f : P \longrightarrow \omega$ such that, for any $x, y \in X$, $\beta \in P$ with $y \in B(x, \beta)$, there exist the elements $x_1, x_2, ..., x_n$ from X, $n \leq f(\beta)$ such that $x_1 = x$, $x_n = y$ and $x_{i+1} \in B(x_i, \alpha)$ for every $i \in \{1, ..., n-1\}$.

Lemma 4. *Let $Gr(V, E)$ be a connected graph, $x_0 \in V$, $h : V \longrightarrow \{0, 1\}$. Then the following statements are equivalent*

(i) h is slow oscillating;

(ii) there exists $m \in \omega$ such that diam $h(B(x, 1)) = 0$ for every $x \in V \setminus B(x_0, m)$.

Proof. $(i) \Longrightarrow (ii)$ is trivial.

$(ii) \Longrightarrow (i)$. If $k \in \omega$ then *diam* $h(B(x, k)) = 0$ for every $x \in V \setminus B(x_0, m + k)$. □

Lemma 5. *Every graph ballean $\mathbb{B}(Gr)$ is binary normal.*

Proof. Let Y, Z be disjoint, asymptotically path disjoint subsets of V, $x_0 \in V$. Choose $m \in \omega$ such that

$$Z \bigcap \Lambda(Y, B(x_0, m), 1) = \emptyset$$

and put

$$h(x) = \begin{cases} 0, & x \in Y \bigcup \Lambda(Y, B(x_0, m), 1); \\ 1, & \text{otherwise}. \end{cases}$$

If $x \in V \setminus B(x_0, m + 1)$, then *diam* $h(B(x, 1)) = 0$. By Lemma 4, h is slow oscillating. □

Now, assume that a graph $Gr(V, E)$ is unbounded, i.e. $V \neq B(x, n)$ for all $x \in V$, $n \in \omega$. For every $A \subseteq V$, denote by $Gr[A]$ a graph with the set of vertices A and the set of edges $E \bigcap (A \times A)$. Fix $x_0 \in V$ and, for every $n \in \omega$, denote by $\mathcal{P}_n$ the set of all connected components of the graph $Gr[V \setminus B(x_0, n)]$. Every element y of $\mathcal{P}_{n+1}$ belongs to some (uniquely determined) element $\pi_n(y) \in \mathcal{P}_n$. Endow $\mathcal{P}_n$ with the discrete topology and consider the Stone-Čech extension $\pi_n^\beta : \beta\mathcal{P}_{n+1} \longrightarrow \beta\mathcal{P}_n$

of the mapping $\pi_n : \mathcal{P}_{n+1} \longrightarrow \mathcal{P}_n$. Thus, we have defined a projective sequence of compact topological spaces $< \beta\mathcal{P}_n, \pi_n^\beta >_{n\in\omega}$.

Theorem 1. *For every unbounded graph $Gr(V,E)$, the binary corona $\gamma(\mathbb{B}(Gr), \{0,1\})$ is a projective limit of the sequence $< \beta\mathcal{P}_n, \pi_n^\beta >_{n\in\omega}$.*

Proof. For every $n \in \omega$, define a mapping $f_n : V \setminus B(x_0, n) \longrightarrow \mathcal{P}_n$ assigning to every $x \in V \setminus B(x_0, n)$ the connected component $f_n(x)$ of $Gr[V \setminus B(x_0, n)]$ containing the vertex x. Since $\pi_n f_{n+1} = f_n$ and $\pi_n^\beta f_{n+1}^\beta = f_n^\beta$, there exists a mapping

$$f : V^\# \longrightarrow \lim_{\longleftarrow} < \beta\mathcal{P}_n, \pi_n^\beta >_{n\in\omega}$$

such that $pr_n f = f_n^\beta$. Since every mapping f_n^β is continuous, f is also continuous.

Show that ker $f = \sim_D$. Let $r, q \in V^\#$ and $q \notin [r]$. By Lemmas 5,3, there exist $R \in r$, $Q \in q$ and $m \in \omega$ such that

$$Q \bigcap \Lambda(R, B(x_0, m), 1) = \emptyset.$$

Then $f_n^\beta(q) \neq f_n^\beta(r)$. On the other hand, if $[r] = [q]$, by Lemma 2, $f_n^\beta(r) = f_n^\beta(q)$ for every $n \in \omega$. Thus, $f/\ker f$ is a homeomorphism between $\gamma(\mathbb{B}(Gr), D)$ and

$$\lim_{\longleftarrow} < \beta\mathcal{P}_n, \pi_n^\beta >_{n\in\omega} .$$

□

Theorem 2. *Let $< X_n >_{n\in\omega}$ be a sequence of discrete spaces, $\pi_n : X_{n+1} \longrightarrow X_n$. Then there exists a tree $Tr(V,E)$ such that $\gamma(\mathbb{B}(Tr), \{0,1\})$ is homeomorphic to*

$$\lim_{\longleftarrow} < X_n^\beta, \pi_n^\beta >_{n\in\omega} .$$

Proof. Put $V = \bigcup_{n\in\omega} X_n$, $E = \{(x, \pi_n(x)) : x \in X_{n+1}, n \in \omega\}$. Identify $\mathcal{P}_n$ with the set X_{n+1} and apply Theorem 1. □

Corollary 1. *For every unbounded graph Gr, there exists a tree Tr such that $\gamma(\mathbb{B}(Gr), \{0,1\})$ and $\gamma(\mathbb{B}(Tr), \{0,1\})$ are homeomorphic.*

Corollary 2. *For every zero-dimensional compact space K of countable weight, there exists a tree Tr such that $\gamma(Tr, \{0,1\})$ is homeomorphic to K.*

Proof. To apply Theorem 1 note that K is a projective limit of some sequence of finite spaces. □

Now we consider a relation between a binary corona of graph ballean and a set of ends of graph. An injective sequence $< v_n >_{n\in\omega}$ of vertices of a connected graph $Gr(V, E)$ is called a *ray* if $d(x_n, x_{n+1}) = 1$ for every $n \in \omega$. Two rays R_1, R_2 are called *end equivalent* if there exists a ray R which meets both R_1 and R_2 infinitely often. Let $\mathcal{E}(Gr)$ denote the set of the corresponding equivalence classes, the *ends* of Gr. Suppose that $Gr(V, E)$ is a tree, $x_0 \in V$. Then $\mathcal{E}(Gr)$ can be identified with the set of all rays starting from the root x_0. Assume that $Gr(V, E)$ is locally finite tree, i.e. every ball $B(x, 1)$, $x \in V$ is finite. Then $\mathcal{P}_n$ can be identified with the set $\{x \in V : d(x_0, x) = n + 1\}$ and $\mathcal{P}^\beta_n = \mathcal{P}_n$. By Theorem 1, $\gamma(\mathbb{B}(Gr), \{0, 1\})$ can also be described as a set of all rays starting from x_0. Thus, in the case of locally finite tree the binary corona is naturally identified with the set of ends. Now we extend this correspondence to all locally finite graphs.

For every tree Tr with the root x_0, there exists a partial ordering on the set of vertices defined by the rule: $x \le y$ if and only if the path from x_0 to y goes over x. A rooted spanning tree Tr of a connected graph Gr is called *normal* if every pair of adjacent vertices of Gr is comparable in the partial ordering defined by Tr. Every locally finite connected graph has a normal rooted spanning tree (see [2, Chapter 6], [6, Section 4]).

If Tr is a spanning tree of Gr and R_1, R_2 are end equivalent rays in Tr, then clearly R_1, R_2 are also end equivalent in Gr. We therefore have a natural mapping $\mathcal{E}(Tr) \longrightarrow \mathcal{E}(Gr)$ assigning to each end of Tr the end of Gr containing it. In general this mapping needs not to be neither one-to-one nor onto; if it is both, then Tr is called *end-faithful.* Every normal rooted spanning tree of a connected locally finite graph is end faithful.

Theorem 3. *Let $Gr(V, E)$ be an infinite locally finite connected graph, $x_0 \in V$, Tr be its normal x_0-rooted spanning tree.*

Then

(i) $\gamma(\mathbb{B}(Gr), \{0, 1\})$ and $\gamma(\mathbb{B}(Tr), \{0, 1\})$ are homeomorphic;

(ii) $|\gamma(\mathbb{B}(Gr), \{0, 1\})| = |\mathcal{E}(Gr)|$.

Proof. (i) It suffices to show that a mapping $h : V \longrightarrow \{0, 1\}$ is slow oscillating with respect to $\mathbb{B}(Gr)$ if and only if h is slow oscillating with respect to $\mathbb{B}(Tr)$. For any $x \in V$, $m \in \omega$, denote by $B(x, m)$ and $B'(x, m)$ the balls of radius m around x in Gr and Tr.

Suppose that h is slow oscillating with respect to $\mathbb{B}(Gr)$ and choose $m \in \omega$ such that *diam* $h(B(x, 1)) = 0$ for every $x \in V \setminus B(x, m)$. Since Gr is locally finite, there exist $n \in \omega$ such that $B(x_0, m) \subseteq B'(x_0, m)$. Then *diam* $h(B'(x, 1)) = 0$ for every $x \in V \setminus B'(x_0, n)$ and, by Lemma 4, h is slow oscillating with respect to $\mathbb{B}(Tr)$.

Assume that h is slow oscillating with respect to $\mathbb{B}(Tr)$ and choose $m \in \omega$ such that $diam\ h(B'(x,1)) = 0$ for every $x \in V \setminus B'(x_0,m)$. Choose $n \in \omega$ such that $B(B'(x_0,m),1) \subseteq B(x_0,n)$. Take an arbitrary $x \in V \setminus B(x_0,n)$. Since $B(x,1)$ lies on the rays in Tr starting from x_0 and going over x, $diam\ h(B(x,1)) = 0$, so h is slow oscillating with respect to $\mathbb{B}(Gr)$.

(ii) Since Tr is end-faithful, $|\mathcal{E}(Gr)| = |\mathcal{E}(Tr)|$. Apply (i). □

Every countable connected graph also has a normal rooted spanning tree [2, Chapter 6], but Theorem 3 can not be extended to all countable connected graph. Let X, Y be countable disjoint subsets $x_0 \in X$, $Y = \{y_n : n \in \omega\}$. Consider a graph Gr with the set of vertices $V = X \bigcup Y$ and the set of edges

$$E = \{(x,x') : x, x' \in X, x \neq x'\} \bigcup \{(y_n, y_{n+1}) : n \in \omega\} \bigcup \{(x_0, y_0)\}.$$

By Theorem 1, $|\gamma(\mathbb{B}(Gr), \{0,1\})| = 1$, but Gr has two ends.

§5. Binary coronas of cellular balleans

Given any ballean $\mathbb{B} = (X,P,B)$, $x, y \in X$ and $\alpha \in P$, we say that x, y are *α-path connected* if there exist the elements $x_0, x_1, ..., x_n$ from X such that $x = x_0$, $y = x_n$ and $x_{i+1} \in B(x_i, \alpha)$ for every $i \in \{0, 1, ..., n-1\}$. For any $x \in X$, $\alpha \in P$, put

$B^{\square}(x,\alpha) = \{y \in X : x, y$ are α -path connected$\}$.

A ballean $B^{\square} = (X, P, B^{\square})$ is called a *cellularization* of $\mathbb{B}$. If $\mathbb{B} = \mathbb{B}^{\square}$, we say that $\mathbb{B}$ is cellular. By [7, Theorem 3], a ballean of metric space (X,d) is cellular if and only if (X,d) is non-Archimedian.

Suppose that $\mathbb{B} = (X,P,B)$ is a cellular, binary normal ballean and show that $\mathbb{B}$ is normal. It suffices to check only that, for every subset Y of X and every pyramid $\widehat{Y} = \bigcup_{\alpha \in P} B(Y \setminus U_\alpha, \alpha)$ defined by the family of bounded subsets $\{U_\alpha : \alpha \in P\}$, there exists a path pyramid Λ with the core Y such that $\Lambda \subseteq \widehat{Y}$. We may assume that $B^{\square}(x,\alpha) = B(x,\alpha)$ for all $x \in X$, $\alpha \in P$. Put $V_\alpha = B(U_\alpha, \alpha)$ and

$$\Lambda = \bigcup_{\alpha \in P} \Lambda(Y, V_\alpha, \alpha).$$

Then $\Lambda(Y, V_\alpha, \alpha) \subseteq B(Y \setminus U_\alpha, \alpha)$, so $\Lambda \subseteq \widehat{Y}$.

For every ballean $\mathbb{B} = (X,P,B)$, there is a standard preodering $\leq$ on P defined by the rule: $\alpha \leq \beta$ if and only if $B(x,\alpha) \subseteq B(x,\beta)$ for all $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists

$\beta \in P'$ such that $\beta \geq \alpha$. A *cofinality* $cf\ \mathbb{B}$ is the minimal cardinality of cofinal subsets of P.

A connected ballean $\mathbb{B} = (X, P, B)$ is called *ordinal* if there exists a cofinal, well-ordered by $\leq$ subset of P. Clearly, every metrizable ballean is ordinal. By [8, § 3], every ordinal ballean is normal and every ordinal ballean of uncountable cofinality is cellular.

Theorem 4. *Every cellular ordinal ballean* $\mathbb{B} = (X, P, B)$ *is binary normal and binary corona of* $\mathbb{B}$ *coincides with Higson's corona of* $\mathbb{B}$.

Proof. It suffices to show that, for any disjoint, asymptotically disjoint subsets Y_0, Y_1 of X, there exists a slow oscillating function $h : X \longrightarrow \{0, 1\}$ such that $h|_{Y_0} \equiv 0$, $h|_{Y_1} \equiv 1$. We may suppose that P is well-ordered and $B^{\square}(x, \alpha) = B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. Since Y_0, Y_1 are asymptotically disjoint, there exists an increasing family $\{U_\alpha : \alpha \in P\}$ of bounded subsets such that

$$B(Y_0 \backslash U_\alpha, \alpha) \bigcap Y_1 = \emptyset$$

for every $\alpha \in P$. For every $x \in X$, put $h(x) = 0$ if and only if

$$x \in Y_0 \bigcup \bigcup_{\alpha \in P} B(Y_0 \backslash U_\alpha, \alpha).$$

If $x \in X \backslash B(U_\alpha, \alpha)$ and $h(x) = 0$, then $h\ |_{B(x,\alpha)} \equiv 0$, so h is slow oscillating. $\square$

Now we consider another class of balleans with equal binary and Higson's coronas.

A connected ballean $\mathbb{B} = (X, P, B)$ is called *pseudodiscrete* if, for every $\alpha \in P$, there exists a bounded subset U_α of X such that $B(x, \alpha) = \{x\}$ for every $x \in X \backslash U_\alpha$.

Let X be a set, φ be a filter on X such that $\bigcap \varphi = \emptyset$. For any $x \in X$, $F \in \varphi$, put

$$B_\varphi(x, F) = \begin{cases} X \backslash F, & \text{if } x \notin F; \\ \{x\}, & \text{if } x \in F; \end{cases}$$

A ballean $\mathbb{B}(X, \varphi) = (X, \varphi, B_\varphi)$ is called a *a ballean of filter* φ. Clearly, $B(X, \varphi)$ is cellular and pseudodiscrete.

Theorem 5. $\mathbb{B} = (X, P, B)$ *be a pseudodiscrete ballean. Then*

(i) there exists a filter φ *on* X *such that* $\mathbb{B} = \mathbb{B}(X, \varphi)$*;*

(ii) $\gamma(\mathbb{B}, \{0, 1\}) = X^{\#}$.

Proof. (*i*) Put $\varphi = \{F \subseteq X : X \setminus F$ is bounded in $\mathbb{B}\}$. Since $\mathbb{B}$ is connected, φ is a filter on X and $\bigcap \varphi = \emptyset$. Take an arbitrary $F \in \varphi$ and choose $\alpha \in P$ such that $X \setminus F \subseteq B(x, \alpha)$ for every $x \in X \setminus F$. Then

$$B_\varphi(x, F) \subseteq B(x, \alpha)$$

for every $x \in X$. Fix an arbitrary $\alpha \in P$ and choose a bounded in $\mathbb{B}$ subset U_α such that $B(x, \alpha) = \{x\}$ for every $x \in X \setminus U_\alpha$. Put $F = B(U_\alpha, \alpha)$. Then

$$B(x, \alpha) \subseteq B_\varphi(x, F)$$

for every $x \in X$. Thus, $\mathbb{B} = \mathbb{B}(X, \varphi)$.

(*ii*) Observe that $X^\# = \overline{\varphi}$, where $\overline{\varphi} = \{r \in \beta X : \varphi \subseteq r\}$, X is endowed with discrete topology. Let $r, q \in X^\#$, $r \neq q$. we have to define a slow oscillating function $h : X \longrightarrow \{0, 1\}$ such that $h^\beta(r) \neq h^\beta(q)$. Choose $R \in r$ such that $R \notin q$ and put $h(x) = 0$ if and only if $x \in R$. Since $B_\varphi(x, F) = \{x\}$ for every $x \in X \setminus F$, h is slow oscillating in $\mathbb{B}(X, \varphi)$. □

Corollary 3. *Every pseudodiscrete ballean is binary normal and its binary corona coincides with Higson's corona.*

Corollary 4. *Let X be a discrete space and let H be a closed subspace of $\beta X \setminus X$. Then there exists a pseudodiscrete ballean $\mathbb{B}$ such that its binary corona is homeomorphic to H.*

Proof. To apply Theorem 5, it suffice to note that $H = \overline{\varphi}$ for some filter φ on X such that $\bigcap \varphi = \emptyset$. □

§6. Comments and open questions

Is every metrizable ballean $\mathbb{B}$ binary normal? This is so if $\mathbb{B}$ is either a graph ballean (Lemma 5) or a cellular metrizable ballean (Theorem 4). However, M. Zarichniy constructed a counterexample in general case.

Example. For every $n \in \omega$, put $L_n = \{(x, n^2) : x \in \mathbb{R}\}$, $L_n^+ = \{(x, n^2) : x > 0\}$, $L_n^- = \{(x, n^2) : x < 0\}$. Endow $X = \bigcup_{n \in \omega} L_n$ with Euclidian metric and consider a ballean $\mathbb{B} = \mathbb{B}(X, d)$. Fix any $n \in \omega$ and note that

$$\Lambda(L_0^+, B(0, n+1), n) \subseteq L_0^+ \bigcup L_1^+ \bigcup \dots \bigcup L_n^+,$$

so L_0^+, L_0^- are disjoint and asymptotically path disjoint. Suppose that $\mathbb{B}$ is binary normal and take a slow oscillating function $h : X \longrightarrow \{0, 1\}$ such that $h|_{L_0^+} \equiv 1$, $h|_{L_0^-} \equiv 0$. Choose a bounded subset V of X so that *diam* $h(B(x, 1)) = 0$ for every $x \in X \setminus V$. Then there exists $m \in \omega$ such that $L_m \bigcap V = 0$, so $h|_{L_m} \equiv$ *const*. Since $h|_{L_0^+} \equiv 1$, $h|_{L_0^-} \equiv 0$, we conclude that h is not slow oscillating, a contradiction.

Question 1. Characterize a class of metric spaces with binary normal balleans.

Question 2. Is every cellular normal ballean binary normal?

Let us say that a ballean $\mathbb{B} = (X, P, B)$ is *path normal* if any two disjoint, asymptotically path disjoint subsets are asymptotically path separated.

By Lemma 1, every binary normal ballean is path normal. It is easy to verify, that every metrizable ballean is path normal, so Zarichniy's example shows that a class of path normal balleans is wider that a class of binary normal balleans.

Question 3. Characterize a class of metric spaces (X, d) such that every two asymptotically disjoint subsets of X are asymptotically path disjoint.

Note that this class contains all non-Archimedian metric spaces.

Question 4. Let Gr be a connected graph. Does there exist a spanning tree Tr of Gr such that the binary coronas of Gr and Tr coincide?

In view of Theorem 1, the next question concerns a purely topological characterization of the class of binary coronas of graph balleans.

Question 5. Find necessary and sufficient conditions under which, for a compact zero-dimensional space K, there exists a projective sequence $< X_n, \pi_n >_{n\in\omega}$ of discrete spaces such that K is homeomorphic to

$$\lim_{\leftarrow} < \beta X_n, \pi_n^\beta >_{n\in\omega} .$$

We conclude the paper with another look at the binary coronas of balleans.

Let $\mathbb{B} = (X, P, B)$ be a ballean. A subset $Y \subseteq X$ is called *almost invariant* if $B(Y, \alpha) \setminus Y$ is bounded for every $\alpha \in P$. A filter φ on X is called almost invariant if every member of φ is an almost invariant subset of X. A filter φ on X which is a maximal (by inclusion) element in the family of all almost invariant filters is called an *end*. Denote by $\widehat{X}$ a family of all ends of X. We identify the elements of X with principal filters, so $X \subseteq \widehat{X}$. Given an arbitrary almost invariant subset F of X, put $\widehat{F} = \{\varphi \in \widehat{X} : F \in \varphi\}$. Then the family of all these subsets $\widehat{F}$ forms a base of compact topology on $\widehat{X}$ and $\widehat{X} \setminus X$ is homeomorphic to binary corona of $\mathbb{B}$.

This approach is going from Freudental-Hopf compactification of groups [5]. A subset A of a group G is called almost invariant if $gA \setminus A$ is finite for every $g \in G$. A set of all maximal almost invariant filters on G with the standard topology is a Freudental-Hopf compactification of a group (or a space of ends). This particular case can be easily included to the ballean's scheme. Indeed, every group G defines a ballean $\mathbb{B}(G) = (G, \Im_e, B)$, where $\Im_e$ is a family of all finite subsets of G with the identity e, $B(g, F) = gF$. Thus, a binary corona of $\mathbb{B}(G)$ is a remainder of Freudental-Hopf compactification of G.

Let G be an infinite, finitely generated group. By Freudental-Hopf theorem [4], G has 1,2 or infinitely many ends.

Question 6. Let G be an infinite, finitely generated group with infinitely many ends. Is a binary corona of $\mathbb{B}(G)$ homeomorphic to $\{0,1\}^\omega$?

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