Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2003).** pp. 95 – 101
© Journal "Algebra and Discrete Mathematics"

On the separability of the restriction functor

Th. Theohari-Apostolidi and H. Vavatsoulas

Communicated by D. Simson

June 26, 2003

Abstract. Let G be a group, $\Lambda = \bigoplus_{\sigma\in G} \Lambda_\sigma$ a strongly graded ring by G, H a subgroup of G and $\Lambda_H = \bigoplus_{\sigma\in H} \Lambda_\sigma$. We give a necessary and sufficient condition for the ring Λ/Λ_H to be separable, generalizing the corresponding result for the ring extension Λ/Λ_1. As a consequence of this result we give a condition for Λ to be a hereditary order in case Λ is a strongly graded by finite group R-order in a separable K-algebra, for R a Dedekind domain with quotient field K.

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1. Introduction

C. Năstăsescu, M. Van den Berg and F. Van Oystaeyen in their paper [5] examined, between others, the separability of the restriction functor associated to a ring homomorphism. Certainly, if $\phi : R \to S$ is a ring homomorphism, they associated to ϕ the restriction functor $\phi_* : modS \to modR$ associating to a left S-module M the left R-module defined on the set M by the ring homomorphism ϕ. They proved that the functor ϕ_* is separable if and only if the ring extension S/R is separable (Proposition 2.3). In the sequel they applied the separability of ϕ_* in case ϕ is the ring embedding $\Lambda_1 \to \Lambda$, where Λ is a strongly graded ring by a group G. As a result of this they proved that ϕ_* is separable if and only if the trace function is surjective and the group G is finite. Also, M.D. Rafael gave another version of this result ([7], Theorem 3.1).

2000 Mathematics Subject Classification: *16W50, 16G30, 16H05.*
Key words and phrases: *separable algebras, strongly graded algebras, restriction functor, induction functor.*

In this paper extending the above result for the functor $\phi_* : mod\Lambda \to mod\Lambda_H$ associating to a Λ-module M its restriction as a Λ_H-module, where H is a subgroup of G, $\Lambda = \bigoplus_{\sigma\in G}\Lambda_\sigma$ and $\Lambda_H = \bigoplus_{\sigma\in H}\Lambda_\sigma$ we prove that the ring extension Λ/Λ_H is separable if and only if the set G/H is finite and there exists an element r in the center $Z(\Lambda_H)$ of Λ_H such that $\sum_{t\in T} r^t = 1$, for a left transversal T of H in G.

Moreover, let R be a Dedekind domain with quotient field K and A a separable K-algebra. If Λ is a strongly graded ring over a finite group G, Λ is an R-order in A and H is a subgroup of G, we prove that Λ is a hereditary R-order if Λ_H is hereditary and the trace function is surjective.

2. On the separability of the restriction functor

Let Λ be a strongly graded ring by a group G, that is $\Lambda = \bigoplus_{g\in G}\Lambda_g$, where each Λ_g is a Λ_1-module and the multiplication is given by $\Lambda_\sigma\Lambda_\tau = \Lambda_{\sigma\tau}$, for all $\sigma, \tau \in G$. We refer to [6] for details on graded rings. Let H be a subgroup of G. We denote $\Lambda_H = \bigoplus_{h\in H}\Lambda_h$, then Λ_H is a strongly graded ring by H. Since $\Lambda_g\Lambda_{g^{-1}} = \Lambda_1$, for all $g \in G$, we may fix a decomposition

$$\sum_i a_g^{(i)} b_{g^{-1}}^{(i)} = 1 \tag{2.1}$$

for $a_g^{(i)} \in \Lambda_g$, $b_{g^{-1}}^{(i)} \in \Lambda_{g^{-1}}$ and i is running over a finite index set depending on g. For all $\lambda \in \Lambda$ and $\sigma \in G$ and T a left transversal of H in G, let

$$\lambda^\sigma = \sum_i a_\sigma^{(i)} \lambda b_{\sigma^{-1}}^{(i)}$$

for $a_\sigma^{(i)}$, $b_{\sigma^{-1}}^{(i)}$ as in the relation (2.1).

The ring extension S/R is separable if and only if the map $\Psi : S\bigotimes_R S \to S$, $s\otimes s' \to ss'$ splits as an (S-S)-bimodule homomorphism. All rings have unity. We refer [2], [6] and [7] for details on separable extensions.

A functor $F : \mathcal{C} \to \mathcal{D}$ between two arbitrary categories is called separable if for all objects $M, N \in \mathcal{C}$ there are maps

$$\phi_{M,N} \in Hom_{\mathcal{D}}(FM, FN) \to Hom_{\mathcal{C}}(M, N)$$

satisfying the following compatibility conditions: 1) If $\alpha \in Hom_{\mathcal{C}}(M, N)$, then $\phi_{M,N}(F(a)) = a$. 2) If there are $M', N' \in \mathcal{C}$ and $\alpha \in Hom_{\mathcal{C}}(M, N)$, $\beta \in Hom_{\mathcal{C}}(M', N')$, $f \in Hom_{\mathcal{D}}(FM, FM')$, $g \in Hom_{\mathcal{D}}(FN, FN')$ such

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that the diagram

$$\begin{array}{ccc} FM & \xrightarrow{F(\alpha)} & FN \\ f\big\downarrow & & \big\downarrow g \\ FM' & \xrightarrow{F(\beta)} & FN' \end{array}$$

is commutative, then the diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \phi(f)\big\downarrow & & \big\downarrow \phi(g) \\ M' & \xrightarrow{\beta} & N' \end{array}$$

is commutative.

We refer [4] and [5] on separable functors.

For brevity we write $\bigotimes$ instead of $\bigotimes_{\Lambda_H}$. We consider the embedding $\phi : \Lambda_H \to \Lambda$, which splits as a Λ_H-bimodule map. Hence the induction functor $\phi^* : mod\Lambda_H \to mod\Lambda$, associating to a left Λ_H-module M the Λ-module $\Lambda \bigotimes M$ is separable by ([5], Proposition 1.3,2). Now we consider the restriction functor $\phi_\star : mod\Lambda \to mod\Lambda_H$, associating to a left Λ-module M the Λ_H-module M. The next theorem essentially gives a necessary and sufficient condition for the functor $\phi_\star$ to be separable and extends ([5], Proposition 1.3).

In the following we use the above notation.

Theorem 2.1. *The ring extension Λ/Λ_H is separable if and only if the set G/H is finite and there exists an element r in the center $Z(\Lambda_H)$ of Λ_H such that $\sum_{t\in T} r^t = 1$, for a left transversal T of H in G.*

Proof. Let T be a left transversal of H in G, we get the direct sum

$$\Lambda \otimes \Lambda = \bigoplus_{k,t\in T} \Lambda_{kH} \otimes \Lambda_{Ht^{-1}}$$

Let us suppose that the ring Λ/Λ_H is separable and $\Psi : \Lambda \bigotimes \Lambda \to \Lambda$ is the multiplication map defined by

$$\Psi\left(\sum_{k,t\in T}\sum_{(i)} \lambda_{kH}^{(i)} \otimes \mu_{Ht^{-1}}^{(i)}\right) = \sum_{k,t\in T}\sum_{i} \lambda_{kH}^{(i)}\mu_{Ht^{-1}}^{(i)}$$

where $\lambda_{kH}^{(i)} \in \Lambda_{kH}$ and $\mu_{Ht^{-1}}^{(i)} \in \Lambda_{Ht^{-1}}$ and i runs over a finite index depending of k and t. Hence Ψ is a $(\Lambda - \Lambda)$-bimodule epimorphism. Then there exists a $(\Lambda - \Lambda)$-bimodule homomorphism $\Phi' : \Lambda \to \Lambda \bigotimes \Lambda$

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splitting Ψ. Let $\Phi'(1) = s'$ for some $s' \in \Lambda \bigotimes \Lambda$, where 1 is the unity of Λ, and let

$$s' = \sum_{k,t \in T} \sum_i s_{kH}^{(i)} \otimes \tau_{Ht^{-1}}^{(i)}$$

where $s_{kH}^{(i)} \in \Lambda_{kH}$ and $\tau_{Ht^{-1}}^{(i)} \in \Lambda_{Ht^{-1}}$ and i runs over a finite index set. For brevity we denote $x_{k,t} = \sum_i s_{kH}^{(i)} \otimes \tau_{Ht^{-1}}^{(i)}$ and $x_t = x_{t,t}$, then $\Phi'(1) = s' = \sum_{k,t \in T} x_{kt}$. We remark that, for $\lambda_g \in \Lambda_g$ and $g \in G$, $\lambda_g s' = \lambda_g \Phi'(1) = \Phi'(\lambda_g) = \Phi'(1)\lambda_g = s'\lambda_g$. Hence

$$\lambda_g s' = s' \lambda_g \tag{2.2}$$

for $g \in G$ and $\lambda_g \in \Lambda_g$.

Let for $g \in G$, $t \in T$, $gtH = t'H$ for some $t' \in T$. Comparing the $\Lambda_{t'H} \bigotimes \Lambda_{Ht^{-1}}$ component of both sides of the relation (2.2), we get

$$\lambda_g x_t = x_{t'} \lambda_g \tag{2.3}$$

Summing both sides of the relation (2.3) over all $t \in T$, we have

$$\sum_{t \in T} \lambda_g x_t = \sum_{t' \in T} x_{t'} \lambda_g \tag{2.4}$$

Moreover, since $1 = \Psi\Phi'(1) = \Psi(s') = \Psi\left(\sum_{k,t \in T} x_{k,t}\right)$ and the kHt^{-1} -indexed coefficient of 1 in Λ is zero, we remark that

$$\sum_i s_{kH}^{(i)} \tau_{Ht^{-1}}^{(i)} = 0$$

for $k \neq t$.

If we define $s = \sum_{t \in T} x_t$, then $\Psi(s) = 1$. Moreover, from the relation (2.4) it follows that

$$\lambda_g s = s \lambda_g \tag{2.5}$$

for all $\lambda_g \in \Lambda_g$, $g \in G$.

Now we define the map $\Phi : \Lambda \to \Lambda \otimes \Lambda$ by $\Phi(\lambda) = \lambda s$. Because of the relation (2.5), Φ is a $(\Lambda - \Lambda)$-bimodule homomorphism such that $\Psi\Phi(\lambda) = \lambda$, for all $\lambda \in \Lambda$, that is Φ splits Ψ.

Let Z be the finite subset of T such that $s = \sum_{z \in Z} x_z$ with $x_z \neq 0$. We denote $r_z = \Psi(x_z)$ then

$$\sum_{z \in Z} r_z = 1 \tag{2.6}$$

We prove that $r_z \in C_{\Lambda_1}(\Lambda) = \{\lambda \in \Lambda : \lambda x = x\lambda, \text{ for all, } \lambda \in \Lambda_1\}$, for $z \in Z$.

From the relation (2.3) for $g = 1$, we get that $\lambda_1 x_t = x_t \lambda_1 \Leftrightarrow \Psi(\lambda_1 x_t) = \Psi(x_t \lambda_1) \Leftrightarrow \lambda_1 r_t = r_t \lambda_1$, for all $t \in Z$. Hence, $r_z \in C_{\Lambda_1}(\Lambda)$. Now we prove that $r_1 \in Z(\Lambda_H)$, the center of Λ_H. Indeed; $r_1 = \Psi(x_1) \in \Lambda_H$. Moreover, from the relation (2.3) for $g \in H$ and $t = 1$, we get $\lambda_g x_1 = x_1 \lambda_g \Rightarrow \Psi(\lambda_g x_1) = \Psi(x_1 \lambda_g) \Rightarrow \lambda_g r_1 = r_1 \lambda_g$. Hence $r_1 \in Z(\Lambda_H)$. Now we prove that r_1^t is zero for almost all $t \in T$. Indeed; from the relation (2.3) we get $\lambda_g \Psi(x_t) = \Psi(x_{t'})\lambda_g$ if and only if

$$\lambda_g r_t = r_{t'} \lambda_g \tag{2.7}$$

for $g \in G$, $t \in T$, $\lambda_g \in \Lambda_g$ and $gtH = t'H$. If $r_t = 0$, for some $t \in T$, then, from the relation (2.7) and for $g = t^{-1}$, we get $0 = r_1 \lambda_{t^{-1}}$, for all $\lambda_{t^{-1}} \in \Lambda_{t^{-1}}$. Hence $r_1 = 0$. Now again from the relation (2.7) for $t = 1$ and $g \in Z$, we get $0 = r_g \lambda_g$, for all $\lambda_g \in \Lambda_g$ and $g \in Z$. Hence $r_g = 0, \quad \forall g \in Z$, but this is impossible because of the relation (2.6). Therefore the set T is finite. From the relations (2.7) and (2.1) and since $r_t \in C_{\Lambda_1}(\Lambda)$ we get that for all $\lambda_g \in \Lambda_g$, and for all $g \in G$,

$$\sum_i a_g^{(i)} b_{g^{-1}}^{(i)} \lambda_g r_t = r_{t'} \lambda_g \Leftrightarrow \sum_i a_g^{(i)} r_t b_{g^{-1}}^{(i)} \lambda_g - r_{t'} \lambda_g = 0 \Leftrightarrow$$

$$\left(r_t^g - r_{t'}\right) \lambda_g = 0 \Leftrightarrow r_t^g - r_{t'} = 0, \text{ for all, } g \in G.$$

For $t = 1$ the last relation becomes $r_1^g = r_g$, for all $g \in G$. Finally, from the relation (2.6) we get $\sum_{t \in T} r^t = 1$.

For the converse, let us suppose that there exists an element r in the center of Λ_H such that $\sum_{t \in T} r^t = 1$. We define the map

$$\Phi : \Lambda \to \Lambda \otimes \Lambda \text{ defined by } \lambda \mapsto \sum_i \sum_{t \in T} a_t^{(i)} \otimes r b_{t^{-1}}^{(i)} \lambda$$

for $a_t^{(i)} \in \Lambda_t$ and $b_{t^{-1}}^{(i)} \in \Lambda_{t^{-1}}$ defined as in the relation (2.1). It is easy to see that the definition of Φ is independent of the election of the elements $a_t^{(i)}$ and $b_{t^{-1}}^{(i)}$ and for the election of the set T and Φ is a left Λ-homomorphism which splits the map Ψ. The technique is analogous to that using in the proof of ([9], Theorem 2.2). □

Corollary 2.2. *i) The induction functor ϕ^* is separable.*

ii) The restriction functor ϕ_* is separable if and only if there exists an element r in the center $Z(\Lambda_H)$ of Λ_H such that $\sum_{t \in T} r^t = 1$

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Let R be a Dedekind domain with quotient field K and A a separable K-algebra. An R-order Λ in A is a subring of A containing R in its center which is finite generated R-module and $K\Lambda = A$. If Λ is an R-order in A, a Λ-lattice is a left Λ-module which is a finitely generated and projective R-module. Let Λ be an R-order in A which is a strongly graded ring by a finite group G and H be a subgroup of G. We refer [1] for details about R-orders in A. Using the above notation and moreover setting $Tr_{G/H}(r) = \sum_{t\in T} r^t$, for an element $r \in \Lambda$ and a left transversal T of H in G, we get the following.

Corollary 2.3. *i) If Λ_H is hereditary and there exists an element $r \in C_{\Lambda_1}(\Lambda_H)$ such that $Tr_{G/H}(r) = 1$, then Λ is hereditary.*

ii) If Λ is hereditary order, then Λ_H is hereditary.

Proof. i) Under the second condition the restriction functor $\phi_\star$ is separable by Corollary (2.2). Let us suppose that Λ_H is hereditary. If M is a left Λ-lattice, then M_H is a left Λ_H-lattice and hence projective. By ([5], Proposition 1.2,3) M is also projective and hence Λ is hereditary.

ii) By the same argument using that the induction functor $\phi^\star$ is separable. (see also [3], Proposition 2.3) $\square$

Remarks. J. Haefner and G. Janusz have proved in ([2], Proposition 2.2 statement 2) that for finite G the restriction functor $\phi_\star : mod\Lambda \to mod\Lambda_H$ is separable if H is normal subgroup of G and $[G : H]$ is a unit in Λ_1. Moreover they claimed that the converse of the statement 2 is also true. The above theorem corrects this result of J. Haefner and G. Janusz and also corrects the statement 3 of the same Proposition 2.2 refering on crossed products and the separability of the restriction functor.

References

[1] C. Curtis and I. Reiner, Methods of representation theory with applications to finite groups and orders, Vol 1, John Wiley and Sons, New York, 1981

[2] F. De Meyer and E. Ingraham, Separable Algebras over Commutative Rings, Lecture Notes in Mathematics, Vol 181, Springer-Verlag, Berlin.

[3] J. Haefner and G. Janusz, Hereditary Crossed Products, Transaction of AMS, Vol. 352, No7, 3381-3410 (2000)

[4] L. Le Bruyn, M. Van Den Bergh and F. Van Oystaeyen, Graded Orders, Birkhauser, Boston, 1988.

[5] C. Nastasescu, M. Van Den Bergh and F. Van Oystaeyen, Separable Functors Applied to Graded Rings, J. of Algebra, 123, 397-413 (1989).

[6] C. Nastasescou and F. Van Oystaeyen, Graded Ring Theory, Mathematics Library, Vol 28, North-Holland, Amsterdam, 1982.

[7] M. D. Rafael, Separable functors Revisited, Comm.Alg., 18(5), 1445-1459 (1990).

[8] I. Reiner, Maximal Orders, Academic Press, New York, 1975

Journal Algebra Discrete Math.

[9] Th. Theohari-Apostolidi and H. Vavatsoulas, On Induced Modules over Strongly Group-Graded Algebras, Beitrage zur Algebra und Geometrie, 40(1999), No2, 373-383.

Contact information

Th. Theohari-Apostolidi
Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54124 Greece
E-Mail: theohari@math.auth.gr

H. Vavatsoulas
Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54124 Greece
E-Mail: vava@math.auth.gr

Received by the editors: 12.05.2003
and final form in 23.10.2003.

Journal Algebra Discrete Math.