Algebra and Discrete Mathematics
Number 3. **(2003).** pp. 82 – 88

RESEARCH ARTICLE

$\mathcal{H}-$ and $\mathcal{R}-$cross-sections of the full finite semigroup T_n

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Communicated by B. V. Novikov

Abstract. All $\mathcal{H}-$ and $\mathcal{R}-$ cross-sections of the full finite semigroup T_n of all transformations of the set $N = \{1, 2, \ldots, n\}$ are described.

1. Introduction

Let ρ be an equivalence relation on a semigroup S. The subsemigroup $T \subset S$ is called a *cross-section* with respect to ρ if T contains exactly 1 element from every equivalence class. Clearly, the most interesting are the cross-sections with respect to the equivalence relations connected with the semigroup structure on S. The first candidates for such relations are congruences and the Green relations.

The *Green relations* $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on semigroup S are defined as binary relations in the following way: $a\mathcal{L}b$ if and only if $S^1a = S^1b$; $a\mathcal{R}b$ if and only if $aS^1 = bS^1$; $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$ for any $a, b \in S$ and $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.

Cross-sections with respect to the $\mathcal{H} - (\mathcal{L}-, \mathcal{R}-, \mathcal{D}-, \mathcal{J}-)$ Green relations are called $\mathcal{H} - (\mathcal{L}-, \mathcal{R}-, \mathcal{D}-, \mathcal{J}-)$ *cross-sections* in the sequel.

In the present paper all $\mathcal{H}-$ and $\mathcal{R}-$ cross-sections of the full finite semigroup T_n of all transformations of the set $N = \{1, 2, \ldots, n\}$ are described.

The study of cross-sections with respect to Green relations for the specific semigroups was initiated a few years ago. The most studied ones are cross-sections of the full inverse symmetric semigroup IS_n. For this

2000 Mathematics Subject Classification: *20M20, 20M10.*
Key words and phrases: *full finite semigroup, Green relations, cross-sections.*

semigroup the first example of an $\mathcal{H}-$cross-section has been constructed in [R]. Later, a complete description of all $\mathcal{H}-$cross-sections for IS_n, $n \neq 3$, was obtained in [CR]. After that in [GM] all $\mathcal{L}-$ and $\mathcal{R}-$cross-sections of IS_n and their disposition with respect to the $\mathcal{H}-$cross-sections of this semigroup were described.

For $a \in T_n$ we denote by $im(a)$ and ρ_a the image of the element a and the equivalence relation on the set N given by the rule $i\rho_a j$ iff $a(i) = a(j)$ respectively. We will multiply the elements in T_n from the left to the right, that is, $(ab)(x) = b(a(x))$ for all $x \in N$. The number $\mathrm{rk}(a) = |im(a)|$ is called the *rank* of a. The identity map $id_N : N \to N$ is the unit element of T_n and will be denoted by e.

For an element $a \in T_n$ one can use the usual tableaux presentation

$$a = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix},$$

where $a(i) = k_i, i = 1, 2, \ldots, n$.

It is well-known (see for example [CP]) that the Green relations on T_n can be described as follows:

$a\mathcal{R}b$ if and only if $\rho_a = \rho_b$;

$a\mathcal{L}b$ if and only if $im(a) = im(b)$;

$a\mathcal{H}b$ if and only if $\rho_a = \rho_b$ and $im(a) = im(b)$;

$a\mathcal{D}b$ if and only if $\mathrm{rk}(a) = \mathrm{rk}(b)$.

In particular, Green's $\mathcal{D}-$classes are $D_k = \{\ a \in T_n \mid \mathrm{rk}(a) = k\},\ 1 \leq k \leq n$.

2. Description of $\mathcal{H}-$ cross-sections

From the structure of Green relation $\mathcal{H}$ on the semigroup T_n it follows that each $\mathcal{H}-$class of this semigroup is uniquely determined by a disjoint decomposition $N = A_1\dot{\cup}\ldots\dot{\cup}A_k$ of the set N into k non-empty blocks and a set $P \subseteq N$ with $|P| = k$. Denote by $H_P^{A_1,\ldots,A_k}$ the $\mathcal{H}-$class determined by these data.

Theorem 1. *a) T_1 contains the single $\mathcal{H}-$cross-section $H = T_1$.*

b) T_2 contains the single $\mathcal{H}-$cross-section

$$H = \{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\}.$$

c) For $n > 2$, the semigroup T_n does not contain $\mathcal{H}-$cross-sections.

Proof. a) Obvious.

b) From the structure of Green relation $\mathcal{H}$ it follows that each $\mathcal{H}-$cross-section H of this semigroup has to contain all elements of the rank 1 and one element of the rank 2. Moreover, the latter one must be idempotent. One can verify immediately that the only subsemigroup that fulfills these conditions is

$$H = \{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\}.$$

c) Consider three $\mathcal{H}-$classes
$H' = H^{\{1\},\{2,\ldots,n\}}_{\{1,3\}} = \{a', b'\}$, where

$$a' = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 3 & 3 & \cdots & 3 \end{pmatrix}, \quad b' = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

$H'' = H^{\{1,2\},\{3,\ldots,n\}}_{\{2,3\}} = \{a'', b''\}$, where

$$a'' = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & \cdots & 3 \end{pmatrix}, \quad b'' = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 3 & 2 & \cdots & 2 \end{pmatrix},$$

$H''' = H^{\{1,3\},\{2,4,\ldots,n\}}_{\{1,2\}} = \{a''', b'''\}$, where

$$a''' = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2 & 1 & 2 & \cdots & 2 \end{pmatrix}, \quad b''' = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 2 & 1 & \cdots & 1 \end{pmatrix}.$$

Let us assume that there exists an $\mathcal{H}-$cross-section H. Then from $|H \cap H'| = 1$ and $b'b' = a'$ one gets that $a' \in H$. Analogously, we can prove that $a'', a''' \in H$. Since H is a subsemigroup, the elements $c = a'a''a'''$ and c^2 also belong to H. But

$$c = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 1 & \cdots & 1 \end{pmatrix}, \quad c^2 = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & 2 \end{pmatrix},$$

therefore $c\mathcal{H}c^2$. On the other hand $c \neq c^2$. This contradicts our assumption that H is $\mathcal{H}-$cross-section and accomplishes the proof of the theorem. □

Remark. Finiteness of the set N was not used in the proof. Therefore this theorem holds true for arbitrary full infinite semigroup $\mathcal{T}_X$.

3. Description of $\mathcal{R}-$ cross-sections

Since for $a, b \in T_n$ the condition $a\mathcal{R}b$ is equivalent to the condition $\rho_a = \rho_b$, the equalities $a = b$ and $\rho_a = \rho_b$ are equivalent for elements a, b from arbitrary $\mathcal{R}-$cross-section T of T_n. We will frequently use this fact in the paper.

From the structure of Green relation $\mathcal{R}$ on the semigroup T_n it follows that each $\mathcal{R}-$class of this semigroup is uniquely determined by a disjoint decomposition $N = A_1\dot{\cup}\dots\dot{\cup}A_k$ of the set N into k non-empty blocks. Denote by $R(A_1, \dots, A_k)$ the $\mathcal{R}-$class determined by this decomposition.

Lemma 1. *Let T be an $\mathcal{R}-$cross-section of T_n and $a, b \in D_k \cap T$ for some k, $1 \le k \le n$. Then $im(a) = im(b)$.*

Proof. Let us assume the contrary, then there exist a number k and elements $a, b \in D_k \cap T$ such that $im(a) \ne im(b)$. Denote $C = im(a) \cap im(b)$, $A = im(a)\backslash C$, $B = im(b)\backslash C$ and $p = |A|$. Since $|A| = |B| = p$, there exists the disjoint decomposition of the set $A \cup B$ into the pairs $(a_1, b_1), (a_2, b_2), \dots, (a_p, b_p)$ such that $a_i \in A$, $b_i \in B$ for every i, $1 \le i \le p$. Since T is an $\mathcal{R}-$cross-section, the set $T \cap R(\{a_1, b_1\}, \dots, \{a_p, b_p\}, \{c_1\}, \dots, \{c_{k-p-1}\}, \{c_{k-p} \cup (N\backslash(A \cup B \cup C))\})$ contains precisely one element. Let c denote this element. Now on one hand the equality $im(ac) = im(c)$ and the implications $\rho_{ac} = \rho_a \Rightarrow ac = a \Rightarrow im(ac) = im(a) \Rightarrow im(a) = im(c)$ hold true and on the other hand, analogously, we can show that $im(b) = im(c)$. But the latter equality is impossible because $im(a) \ne im(b)$. This contradiction completes the proof of the lemma. □

Lemma 2. *Let T be an $\mathcal{R}-$cross-section of T_n and $a \in D_k \cap T, b \in D_{k+1} \cap T$ for some k, $1 \le k \le n-1$. Then $im(a) \subset im(b)$.*

Proof. Let

$$A = im(a) = \{a_1, \dots, a_k\}$$

and

$$c \in T \cap R(\{a_1\}, \dots, \{a_k\}, \{N\backslash A\}).$$

Then $\rho_{ac} = \rho_a$. This implies $ac = a$ and $im(ac) = im(a)$, but $im(ac) \subset im(c)$ implies that $im(a) \subset im(c)$. Since $c \in D_{k+1} \cap T$, we have that $im(b) = im(c)$ by Lemma 1. The latter equality completes the proof of the inclusion $im(a) \subset im(b)$. □

Thus from the lemmas, we can see that every $\mathcal{R}-$cross-section T of T_n defines linear order on the set N in the following way: an element $i \in N$ is less than $j \in N$ iff there exists k, $1 \le k \le n$ such that the set $im(D_k \cap T)$ contains i, but does not contain j.

Let φ denote the map that assigns to each $\mathcal{R}-$cross-section T of the semigroup T_n the linear order on the set N determined as above.

Let linear order $<$ on the set N be fixed. For every decomposition of the set $N = A_1 \dot\cup A_2 \dot\cup \dots \dot\cup A_k$ into disjoint union of non-empty blocks define the induced linear order on blocks by the rule $A_i \prec A_j$ iff $min(A_i) < min(A_j)$, where $min(A_i)$ denotes the least element of the set $(A_i, <)$.

Define

$$\begin{aligned} x_1 &= min(N), \\ x_2 &= min(N \backslash \{x_1\}), \\ &\vdots \\ x_n &= min(N \backslash \{x_1, \dots, x_{n-1}\}). \end{aligned}$$

Now construct the set $S_<$ in the following way: an element $a \in R(A_1, \dots, A_k)$, where $A_1 \prec A_2 \prec \dots \prec A_k$ belongs to the set $S_<$ if and only if the equality $a(A_i) = x_i$ holds true for every i, $1 \le i \le k$. Then it is obvious, that $S_<$ contains exactly one element from every $\mathcal{R}-$class. Moreover, the following proposition holds true.

Lemma 3. *For every linear order $<$ on the set N the set $S_<$ is closed under multiplication.*

Proof. Let $a, b \in S_<$ be arbitrary elements. Then there exist two $\mathcal{R}-$classes $R(A_1, \dots, A_k)$, $R(B_1, \dots, B_m)$ such that $a \in R(A_1, \dots, A_k)$, $b \in R(B_1, \dots, B_m)$. Without loss of generality we can assume that $A_1 \prec A_2 \prec \dots \prec A_k$ and $B_1 \prec B_2 \prec \dots \prec B_m$. By p denote the least number such that $im(a) \cap B_p = \emptyset$. Clearly, $p > 1$ and $im(ab) = \{x_1, \dots, x_{p-1}\}$ in this case. Now for every element $x \in im(ab)$ denote by C_x the set $(ab)^{-1}(x)$. The sets A_x, B_x are defined similarly. To complete the proof it is now sufficient to show that $C_x \prec C_y$ for every pair of elements $x < y$ from the set $im(ab)$. This follows immediately from the equality $min(C_x) = min(A_{min(B_x)})$ and the following sequence of implications $x < y \Rightarrow B_x \prec B_y \Rightarrow min(B_x) < min(B_y) \Rightarrow A_{min(B_x)} \prec A_{min(B_y)} \Rightarrow min(A_{min(B_x)}) < min(A_{min(B_y)}) \Rightarrow min(C_x) < min(C_y) \Rightarrow C_x \prec C_y$. □

Corollary 1. *For every linear order $<$ on the set N the set $S_<$ is an $\mathcal{R}-$cross-section in T_n.*

Proof. By Lemma 3 this set is closed under multiplication. Moreover, the multiplication is associative, because $S_< \subset T_n$. Hence $S_<$ is a subsemigroup of T_n. But from the construction of this set it also follows that

$S_<$ contains exactly 1 element from every $\mathcal{R}-$class and the statement is proven. □

Corollary 2. *The map φ is surjective.*

Proof. Follows from $S_< \in \varphi^{-1}(<)$ for every linear order $<$ on the set N. □

Lemma 4. *Let T be an $\mathcal{R}-$cross-section of T_n and $<$ denotes the linear order $\varphi(T)$. Denote by $\prec$ the induced linear order on blocks. The elements $x_i, 1 \le i \le n$ are determined as above. Then for element $a \in T \cap R(A_1, \dots, A_k)$, where $A_1 \prec A_2 \prec \cdots \prec A_k$ the equality $a(A_i) = x_i$ for every i, $1 \le i \le k$ holds true.*

Proof. Let us assume the contrary. Then there exist an element $a \in T$ and a number j such that $a(A_j) \ne x_j$. Without loss of generality we can assume that j is the minimal number with this property. Then $a(A_j) > x_j$. Consider the number p such that $min(A_j) = x_p$. Then for every $x \in A_{j+1} \cup \cdots \cup A_k$ the inequality $x > x_p$ holds true. From $A_i \prec A_j$ for every $i < j$ it follows that there exist elements $y_i \in A_i$ such that $y_i < x_p$ for every $i < j$. Now consider element $b \in T \cap D_p$. It is obvious, that $rk(ba) = j$. But $im(ba) \ne \{x_1, \dots, x_j\}$, because for every $x \in b^{-1}(x_p)$ we have that $(ba)(x) = a(b(x)) = a(x_p) > x_j$. Therefore $ba \notin T$. This contradiction completes the proof of the lemma. □

Corollary 3. *The map φ is injective.*

Proof. Let T_1, T_2 be $\mathcal{R}-$cross-sections of T_n such that $\varphi(T_1) = \varphi(T_2)$. Then by Lemma 4 sets $T_1 \cap R$, $T_2 \cap R$ coincide for every $\mathcal{R}-$class R. This implies $T_1 = T_2$ and completes the proof of the lemma. □

Theorem 2. *The map φ is a bijection between the set of all $\mathcal{R}-$cross-sections of T_n and the set of all linear orders on the set N.*

Proof. Follows from Corollaries 2 and 3. □

Theorem 3. *The semigroup T_n contains exactly $n!$ different $\mathcal{R}-$cross-sections. Every two $\mathcal{R}-$cross-sections are isomorphic.*

Proof. By Theorem 2 the number of different $\mathcal{R}-$cross-sections of the semigroup T_n equals the number of all linear orders on the set N, but the last number equals $n!$. With the linear order $x_1 < x_2 < \cdots < x_n$ we associate the permutation $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$. Let T_1, T_2 be two $\mathcal{R}-$cross-sections of T_n. Denote by π_1 and π_2 the permutations associated with the linear orders $\varphi(T_1), \varphi(T_2)$ respectively. Then the equality $\pi_1 T_1 \pi_1^{-1} = \pi_2 T_2 \pi_2^{-1}$ holds true. This means that arbitrary two $\mathcal{R}-$cross-section are conjugated and hence isomorphic. □

Acknowledgements

I would like to thank Prof. O. G. Ganyushkin for fruitful discussions.

References

[R] Renner L. E. *Analogue of Bruhat decomposition for algebraic monoids. II. The length function and the trichotomy.* J. Algebra 175 (1995), no. 2, 697-714.

[CR] Cowan D. F., Reily R. *Partial cross-sections of symmetric inverse semigroups. Internat* J. Algebra Comput. 5 (1995), no. 3, 259-287.

[GM] Ganyushkin O., Mazorchuk V. *$\mathcal{L}-$ and $\mathcal{R}-$cross-section in IS_n.* Com. in Algebra 31(2003), no. 9, 4507-4523.

[CP] Clifford A. H., Preston G. B. *The algebraic theory of semigroups*, American Mathematical Society, Providence, Rhode Island, 1964.

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Received by the editors: 01.07.2003
and final form in 24.10.2003.