Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2003).** pp. 54 – 81

Gyrogroups and left gyrogroups as transversals of a special kind

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Communicated by A. I. Kashu

Abstract. In this article we study gyrogroups and left gyrogroups as transversals in some suitable groups to its subgroups. These objects were introduced into consideration in a connection with an investigation of analogies between symmetries in the classical mechanics and in the relativistic one. The author introduce some new notions into consideration (for example, a weak gyrotransversal) and give a full description of left gyrogroups (and gyrogroups) in terms of transversal identities. Also he generalize a construction of a diagonal transversal and obtain a set of new examples of left gyrogroups.

1. Introduction

At the first time the concepts of a gyrogroup and a gyrocommutative gyrogroup were introduced into consideration in [14] in a connection with an investigation of analogies between symmetries in the classical mechanics and in the relativistic one. In [14, 15] the elementary properties of gyrogroups were established and it was shown that they are left special loops.

In [4] the concept of the gyrogroup was generalized and it was introdused a notion of a left gyrogroup; also in [4, 3, 5, 6] these objects were considered as transversals (gyrotransversals) in some groups to their subgroups.

In the present work the research of above-mentioned concepts is proceeded. In §1 the necessary definitions are introduced, among which the

2000 Mathematics Subject Classification: *20N05, 20N15.*
Key words and phrases: *loop, group, transversal, automorphism, gyrogroup.*

concepts of a weak gyrotransversal and a middle Bol loop deserves the special attention. The elementary properties of these objects are proved. In §2 it is shown that the left gyrogroups is exactly weak gyrotransversals in some groups; and some its properties are established. In §3 and §4 the transversals in groups are investigated, for which the transversal operations are gyrogroups and commutative gyrogroups, respectively. In §5, proceeding from a definition of semidirect product of a left loop and a suitable group (see [12, 11]), it is shown that weak gyrotransversals (gyrotransversals) are obtained by the natural way in the semidirect products of the left gyrogroups (gyrogroups) and the suitable groups. At last, in §6 the generalization of a construction of the diagonal transversals (see [4]) is given.

2. Necessary definitions, notations and preliminary statements

Definition 1. *[2] A system $\langle E, \cdot\rangle$ is called a* ***left (right) quasigroup*** *if the equation $a \cdot x = b$ ($y \cdot a = b$) has an unique solution in the set E for anyone given $a, b \in E$. The system $\langle E, \cdot\rangle$ is called a* ***quasigroup*** *if it is both left and right quasigroup simultaneously. A left (right) quasigroup $\langle E, \cdot\rangle$ is called a* ***left (right) loop*** *if there exists the element $e \in E$ such that $e \cdot x = x$ ($x \cdot e = x$, respectively). This element $e \in E$ is called a* ***left (right) unit****. The system $\langle E, \cdot\rangle$ is called a* ***loop*** *if it is both left and right loop simultaneously (in this case $e \cdot x = x \cdot e = x \ \ \forall x \in E$, and this element $e \in E$ is called a* ***unit*** *of the loop $\langle E, \cdot\rangle$).*

Definition 2. *[1, 10] Let $\langle G, \cdot, e\rangle$ be a group and $\langle H, \cdot, e\rangle$ be its own subgroup. A complete set $T = \{t_i\}_{i\in E}$ of representatives of the left (right) cosets of the group G to its subgroup H (exactly one representative from each coset) is called a* ***non-reduced left (right) transversal*** *in G to H. If the non-reduced left (right) transversal T satisfy the condition $e = t_1 \in T$, then the set $T = \{t_i\}_{i\in E}$ is called a* ***left (right) transversal*** *in G to H. The left transversal T in G to H is called a* ***two-sided transversal*** *in G to H, if it is a right transversal in G to H simultaneously.*

For every left transversal $T = \{t_i\}_{i\in E}$ in G to H it is possible to define correctly a following operation on the set E (a **transversal operation**):

$$x \overset{(T)}{\cdot} y = z \qquad \overset{def}{\Longleftrightarrow} \qquad t_x t_y = t_z h, \quad h \in H. \tag{1}$$

Lemma 1. *The following statements are true:*

1. *If T is a non-reduced left (right) transversal in G to H, then the system $\langle E, \overset{(T)}{\cdot} \rangle$ is a left (right) quasigroup with the right (left) unit* 1;

2. *If T is a left (right) transversal in G to H, then the system $\langle E, \overset{(T)}{\cdot} \rangle$ is a left (right) loop with the unit* 1.

Proof. The proof is similar to the proof of Lemma 1 from [10]. □

Definition 3. *A left transversal T in G to H is called a* ***loop transversal*** *if the transversal operation $\langle E, \overset{(T)}{\cdot}, 1 \rangle$ is a loop.*

Lemma 2. *For every left transversal in G to H the following statements are equivalent:*

1. *T is a loop transversal in G to H;*

2. *$\forall \pi \in G$ the set $T^{\pi} = \pi T \pi^{-1}$ is a left transversal in G to H;*

3. *$\forall \pi \in G$ the set T is a left transversal in G to $\pi H \pi^{-1}$.*

Proof. The proof it can see in [1]. □

For every left transversal T in G to H we shall denote:

$$l_{x,y} \overset{def}{=} t_{x \cdot y}^{-1} t_x t_y \in H.$$

Definition 4. *[13] A* ***left multiplication group*** *of a left quasigroup $\langle E, \cdot \rangle$:*

$$LM\left(\langle E, \cdot \rangle \right) \overset{def}{=} \langle L_a | L_a\left(x\right) = a \cdot x, \;\; a \in E \rangle.$$

A ***left inner mappings group*** *of a left loop $\langle E, \cdot, 1 \rangle$:*

$$LI\left(\langle E, \cdot, 1 \rangle \right) \overset{def}{=} \{ \alpha \in LM\left(\langle E, \cdot, 1 \rangle \right) | \alpha\left(1\right) = 1 \}.$$

It is known (see [13]) that

$$LI\left(\langle E, \cdot, 1 \rangle \right) = \langle l_{x,y} | \, x, y \in E \rangle.$$

Definition 5. *A left transversal $T = \{t_i\}_{i \in E}$ in G to H is called a*

1. ***weak gyrotransversal****, if the following conditions hold:*

   (a) T is a two-sided transversal in G to H;

(b) $LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq N_G(T)$, *i.e.* $\forall h \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ *it is true that* $hTh^{-1} \subseteq T$;

2. ***gyrotransversal*** *[4, 3], if the following conditions hold:*

   (a) $\forall t_i \in T$ *it is true that* $t_i^{-1} \in T$;

   (b) $H \subseteq N_G(T)$, *i.e.* $\forall h \in H$ *it is true that* $hTh^{-1} \subseteq T$.

Definition 6. *[4]A system* $\langle E, \cdot\rangle$ *is called a* ***left gyrogroup****, if the following conditions hold:*

1. *In the set* E *there exists an element* 1 *such that*

$$1 \cdot x = x \quad \forall x \in E.$$

2. $\forall x \in E$ *there exists an element* ${}^{-1}x \in E$ *such that*

$${}^{-1}x \cdot x = 1.$$

3. $\forall a, b, z \in E$ *the following identity holds:*

$$a \cdot (b \cdot z) = (a \cdot b) \cdot \alpha_{a,b}(z),$$

where $\alpha_{a.b} \in Aut(\langle E, \cdot\rangle)$ *is called a* ***gyroautomorphism***.

Remark 1. A left gyrogroup $\langle E, \cdot, 1\rangle$ is a left loop, i.e. the equation $a \cdot x = b$ has the unique solution in E for every fixed $a, b \in E$. Really, let

$$a \cdot x = b.$$

Then for left opposite ${}^{-1}a$ to $a \in E$ we have:

$$\begin{aligned} {}^{-1}a \cdot b &= {}^{-1}a \cdot (a \cdot x) = \left({}^{-1}a \cdot a\right) \cdot \alpha_{{}^{-1}a,a}(x) = \\ &= 1 \cdot \alpha_{{}^{-1}a,a}(x) = \alpha_{{}^{-1}a,a}(x), \end{aligned}$$

i.e. $x = \alpha^{-1}_{{}^{-1}a,a}\left({}^{-1}a \cdot b\right)$.

Definition 7. *[14, 4, 3, 15] A left gyrogroup* $\langle E, \cdot, 1\rangle$ *is called a* ***gyrogroup****, if* $\forall a, b \in E$ *the following condition holds:*

$$\alpha_{a,b} \equiv \alpha_{a \cdot b, b}.$$

Definition 8. *[14, 4, 3, 15] A gyrogroup* $\langle E, \cdot, 1\rangle$ *is called a* ***gyrocommutative gyrogroup****, if* $\forall a, b \in E$ *the following condition holds:*

$$a \cdot b = \alpha_{a,b}(b \cdot a).$$

Below we shall consider a group G as its permutation representation $\hat{G}$ by the left cosets to its subgroup H. If $T = \{t_x\}_{x\in E}$ is a left transversal in G to H, we define [10]:

$$\hat{g}(x) = y \qquad \overset{def}{\Longleftrightarrow} \qquad gt_xH = t_yH. \tag{2}$$

It is known [8] that if

$$Core_G(H) = \underset{g\in G}{\cap} gHg^{-1} = \{e\},$$

then $\hat{G} \cong G$; below we shall propose that $Core_G(H) = \{e\}$.

Lemma 3. *Let $T = \{t_x\}_{x\in E}$ is a non-reduced left transversal in G to H and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is the transversal operation. Then the following formulas are true:*

1. $\forall h \in H$: $\hat{h}(1) = 1$;
2. $\forall x, y \in E$:

$$\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \qquad \hat{t}_x(1) = x, \qquad \hat{t}_x^{-1}(1) = x \overset{(T)}{\setminus} 1,$$
$$\hat{t}_x^{-1}(y) = x \overset{(T)}{\setminus} y, \quad \hat{t}_x^{-1}(x) = 1.$$

3. *If $T = \{t_x\}_{x\in E}$ is a left transversal in G to H, then also the following identity is fulfilled:*

$$\hat{t}_1(x) = x.$$

Proof. The proof is similar to the proof of Lemma 4 from [10]. □

Lemma 4. *Let $T = \{t_x\}_{x\in E}$ be a non-reduced left transversal in G to H and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be its transversal operation. Then the following statements are equivalent:*

1. *T is a non-reduced two-sided transversal in G to H;*
2. *The equation $x \overset{(T)}{\cdot} a = 1$ has an unique solution in E for every $a \in E$;*
3. *A set $T^{-1} = \{t_x^{-1}\}_{x\in E}$ is a non-reduced two-sided transversal in G to H.*

Proof. 1) $\longleftrightarrow$ 2). The proof is similar to the proof of Lemma 7 from [10].

1) $\longleftrightarrow$ 3). Since $\forall x \in E$ it is true that

$$(Ht_x)^{-1} = t_x^{-1}H, \qquad (t_xH)^{-1} = Ht_x^{-1},$$

so if T is a non-reduced left (right) transversal in G to H then T^{-1} is a non-reduced right (left) transversal in G to H, and vice versa. □

Definition 9. *[2] A left (right) quasigroup $\langle E, \cdot\rangle$ is called a **special quasigroup at the left (at the right)**, if $\forall x, y \in E$*

$$l_{x,y} = L_{x\cdot y}^{-1}L_xL_y \in Aut\,(\,\langle E, \cdot\rangle)$$

($r_{x,y} = R_{x\cdot y}^{-1}R_yR_x \in Aut(\,\langle E, \cdot\rangle)$, respectively).

Definition 10. *[2] A left loop $\langle E, \cdot\rangle$ is called a **left Bol loop**, if the following identity (**left Bol identity**) is fulfilled $\forall x, y, z \in E$:*

$$x\,(y\,(xz)) = (x\,(yx))\,z.$$

Lemma 5. *A left Bol loop $\langle E, \cdot, 1\rangle$ satisfies the following properties:*

1. *the left inverse property, i.e. $\forall x, y \in E$: ${}^{-1}x \cdot (x \cdot z) = z$, where ${}^{-1}x \cdot x = 1$;*
2. *${}^{-1}x = x^{-1}$, i.e. the left and the right inverse elements to an element $x \in E$ coincide;*
3. *the left alternation, i.e. $\forall x, y \in E$: $x \cdot (x \cdot y) = (x \cdot x) \cdot y$;*
4. *the solution of the equation $a \cdot x = b$ is $x = a^{-1} \cdot b$, and the solution the equation $y \cdot a = b$ is $y = a^{-1} \cdot \big((a \cdot b) \cdot a^{-1}\big)$, i.e. left Bol loop $\langle E, \cdot, 1\rangle$ is a loop.*

Proof. The proof it can see in [2], chapter 6. □

Definition 11. *An operation $\langle E, \cdot\rangle$ is called a **middle Bol loop** [2], if the following identity holds:*

$$x \cdot ((yz)\,\backslash x) = (x/z) \cdot (y\backslash x)\,,$$

where "\" and "/" are left and right divisions in $\langle E, \cdot\rangle$, respectively.

Lemma 6. *Let $\langle E, \cdot\rangle$ is a middle Bol loop. Then the following statements are true:*

1. *$\langle E, \cdot\rangle$ is a loop with some unit 1;*

2. The left inverse element ${}^{-1}x$ *and the right inverse element* x^{-1} *to an element* $x \in E$ *coincide:* ${}^{-1}x = x^{-1}$*;*

3. If $\langle E, \cdot, 1\rangle$ *is a left Bol loop and "/" is the right inverse operation to the operation "* $\cdot$ *", then the operation*

$$x \circ y = x/y^{-1}$$

is a middle Bol loop $\langle E, \circ, 1\rangle$*, and everyone middle Bol loop can be obtained in a similar way from some left Bol loop.*

Proof. The proof it can see in [7]. □

3. Left gyrogroups as weak gyrotransversals

Lemma 7. *Let* T *be a left transversal in* G *to* H*. Then the following statements are equivalent:*

1. T *is a weak gyrotransversal in* G *to* H*;*

2. The transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *is a left gyrogroup.*

Proof. 1) $\longrightarrow$ 2). Let T be a left transversal in G to H and T be a weak gyrotransversal in G to H, i.e. the two following conditions hold:

1. T is a two-sided transversal in G to H;

2. $\forall h \in LI\ (\langle E, \overset{(T)}{\cdot}, 1\rangle)$ it is true that $hTh^{-1} \subseteq T$.

Let us show that the conditions 1) - 3) from Definition 6 are fulfilled for the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$.

The condition 1) is fulfilled automatically for anyone left transversal in G to H (see Lemma 1).

The condition 2) follows from the Condition 1 and Lemma 4.

Valid Condition 2 $\forall x \in E$ and $\forall h \in LI\ (\langle E, \overset{(T)}{\cdot}, 1\rangle)$ it is true that

$$ht_x h^{-1} = t_{\psi(x)}. \tag{3}$$

In virtue Lemma 3 we have:

$$\psi(x) = \hat{t}_{\psi(x)}(1) = \hat{h}\hat{t}_x\hat{h}^{-1}(1) = \hat{h}\hat{t}_x(1) = \hat{h}(x),$$

i.e. the equality (3) may be rewritten as:

$$ht_x h^{-1} = t_{\hat{h}(x)}. \tag{4}$$

Let us show that $\forall h \in LI\,(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ a mapping

$$\alpha_h : x \to \hat{h}\,(x)$$

is an automorphism of the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$. We have $\forall x, y \in E$:

$$t_x t_y = t_{x\cdot y} l_{x,y},$$

where $l_{x,y} = t_{x\cdot y}^{-1} t_x t_y \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$. Then in virtue of (4) $\forall h \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ it is true that

$$\begin{array}{rcl} h t_x h^{-1} h t_y h^{-1} & = & h t_{x\cdot y} h^{-1} h l_{x,y} h^{-1}, \\ t_{\hat{h}(x)} t_{\hat{h}(y)} & = & t_{\hat{h}(x\cdot y)} h', \end{array}$$

where $h' \in LI\,(\langle E, \overset{(T)}{\cdot}, 1\rangle)$, and

$$\hat{h}\,(x) \overset{(T)}{\cdot} \hat{h}\,(y) = \hat{h}(x \overset{(T)}{\cdot} y),$$

i.e. α_h is an automorphism of the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$.

At last $\forall a, b, z \in E$ we have in virtue of (4):

$$\begin{array}{rcl} t_a t_b t_z & = & t_{a\cdot b} l_{a,b} t_z = t_{a\cdot b} l_{a,b} t_z l_{a,b}^{-1} l_{a,b} = \\ & = & t_{a\cdot b} t_{\hat{l}_{a,b}(z)} l_{a,b} = t_{(a\cdot b)\cdot \hat{l}_{a,b}(z)} l_{a\cdot b, \hat{l}_{a,b}(z)} l_{a,b}, \end{array}$$

and, on the other hand

$$t_a t_b t_z = t_a t_{b\cdot z} l_{b,z} = t_{a\cdot(b\cdot z)} l_{a, b\cdot z} l_{b,z}.$$

So we obtain

$$t_{a\cdot(b\cdot z)} l_{a,b\cdot z} l_{b,z} = t_{(a\cdot b)\cdot \hat{l}_{a,b}(z)} l_{a\cdot b, \hat{l}_{a,b}(z)} l_{a,b}.$$

According to the definition of a left transversal it means that

$$a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} z) = (a \overset{(T)}{\cdot} b) \overset{(T)}{\cdot} \hat{l}_{a,b}\,(z)\,,$$

where $\alpha_{l_{a,b}} = \hat{l}_{a,b}$ is an automorphism of the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ (as it was shown above). The condition 3) of Definition 6 is fulfilled.

2) $\longrightarrow$ 1). Let $T = \{t_x\}_{x\in E}$ be a left transversal in G to H and the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be a left gyrogroup. Then the conditions 1)-3) from

Definition 6 are fulfilled for the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Let us show that T is a weak gyrotransversal.

By virtue of condition 2) and Lemma 4 the transversal T is a two-sided transversal, i.e. the item 1a) of Definition 5 holds.

By virtue of condition 3) it is true that $\forall a, b, z \in E$:

$$\hat{t}_a \hat{t}_b (z) = \hat{t}_{a \cdot b} (\alpha_{a,b} (z)),$$

i.e. $l_{a,b} = \alpha_{a,b}$ is an automorphism of the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Then

$$LI(\langle E, \overset{(T)}{\cdot}, 1\rangle) \subseteq Aut(\langle E, \overset{(T)}{\cdot}, 1\rangle) \tag{5}$$

Now let $h \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and we shall consider the expression $\left(ht_k h^{-1}\right)$ $\forall x \in E$. Since $\forall x \in E \quad ht_x h^{-1} \in G$ then

$$ht_x h^{-1} = t_u h_1 \tag{6}$$

for some $u \in E$ and $h_1 \in H$. Valid Lemma 3 we have:

$$u = \hat{t}_u (1) = \hat{t}_u \hat{h}_1 (1) = \hat{h} \hat{t}_x \hat{h}^{-1} (1) = \hat{h} \hat{t}_x (1) = \hat{h} (x).$$

So (6) may be rewritten as

$$ht_x h^{-1} = t_{\hat{h}(x)} h_1. \tag{7}$$

Further we have $\forall x, y \in E$:

$$\begin{aligned} t_x t_y &= t_{x \cdot y} l_{x,y}, \\ ht_x h^{-1} ht_y h^{-1} &= ht_{x \cdot y} h^{-1} h l_{x,y} h^{-1}. \end{aligned}$$

In virtue of (7) we obtain:

$$t_{\hat{h}(x)} h_1 t_{\hat{h}(y)} h_2 = t_{\hat{h}(x \cdot y)} h_3 \cdot h l_{x,y} h^{-1}, \qquad h_1, h_2, h_3 \in H.$$

Again in virtue Lemma 3 we have

$$\hat{t}_{\hat{h}(x)} \hat{h}_1 \hat{t}_{\hat{h}(y)} \hat{h}_2 (1) = \hat{t}_{\hat{h}(x \cdot y)} \hat{h}_3 \hat{h} \, \hat{l}_{x \cdot y} \hat{h}^{-1} (1),$$

$$\hat{h} (x) \overset{(T)}{\cdot} \hat{h}_1 \left(\hat{h} (y) \right) = \hat{h}(x \overset{(T)}{\cdot} y). \tag{8}$$

Since $h \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ then in virtue of (5) we have

$$\hat{h}(x \overset{(T)}{\cdot} y) = \hat{h} (x) \overset{(T)}{\cdot} \hat{h} (y).$$

Substituting the last equality in (8), we obtain

$$\hat{h}(x) \overset{(T)}{\cdot} \hat{h}_1\left(\hat{h}(y)\right) = \hat{h}(x) \overset{(T)}{\cdot} \hat{h}(y) \quad \forall x, y \in E.$$

Since T is a left transversal in G to H then system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left quasigroup; therefore we receive

$$\hat{h}_1\left(\hat{h}(y)\right) = \hat{h}(y) \quad \forall y \in E.$$

Since the mapping $\hat{h}(y)$ is a permutation on the set E then we have:

$$\hat{h}_1(z) = z \quad \forall z \in E,$$

i.e. $\hat{h}_1 = id$ and so $h_1 = e$. Then according to (7), we receive that $\forall h \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$ and $\forall x \in E$ it is true that

$$ht_x h^{-1} \in T,$$

i.e. the item 1b) of Definition 5 is fulfilled. Then the transversal T is a weak gyrotransversal. □

Corollary 11. *If a left transversal T in G to H is a gyrotransversal then the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left gyrogroup.*

Lemma 8. *Let T be a weak gyrotransversal in G to H (i.e. the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left gyrogroup). Then the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ satisfies the following properties:*

1. *$a \overset{(T)}{\cdot} b = a \overset{(T)}{\cdot} c \quad \Leftrightarrow \quad b = c$ (left cansellation);*
2. *The element $1 \in E$ is the unique unit for the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$;*
3. *$\forall a \in E$ there exist an unique left inverse element ${}^{-1}a$ $({}^{-1}a \overset{(T)}{\cdot} a = 1)$ and an unique right inverse element ${}^{-1}a$ $(a \overset{(T)}{\cdot} a^{-1} = 1)$.*
4. *If $\alpha_{a,b}$ is a gyroautomorphism then*

$$\begin{aligned} \alpha_{a,b}(z) &= (a \overset{(T)}{\cdot} b)\backslash(a \overset{(T)}{\cdot} (b \overset{(T)}{\cdot} z)), \\ \alpha_{a,b}^{-1}(z) &= b\backslash(a\backslash((a \overset{(T)}{\cdot} b) \overset{(T)}{\cdot} z)), \end{aligned}$$

and, as a corollary, $\alpha_{0,a} = \alpha_{a,0} = id$.

5. *It is true that* $\forall x, y \in E$:

$$\begin{array}{rcl} x \overset{(T)}{\cdot} (x^{-1} \overset{(T)}{\cdot} y) & = & \varphi_x(y), \\ {}^{-1}x \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} y) & = & \psi_x(y), \end{array}$$

where φ_x *and* ψ_x *are some automorphisms of operation* $\langle E, \overset{(T)}{\cdot}, 1 \rangle$.

Proof. 1) and 2) follow from Lemma 1.

3) follows from Lemma 1 and Lemma 4 (item b)).

4). According to the definition of a left gyrogroup

$$a \cdot (b \cdot z) = (a \cdot b) \cdot \alpha_{a,b}(z).$$

On the other hand, since T is a left transversal then $t_a t_b = t_{a \cdot b} l_{a,b}$, and so $\forall z \in E$

$$a \cdot (b \cdot z) = \hat{t}_a (b \cdot z) = \hat{t}_a \hat{t}_b (z) = \hat{t}_{a \cdot b} \hat{l}_{a,b}(z) = (a \cdot b) \cdot \hat{l}_{a,b}(z).$$

So $\alpha_{a,b} = \hat{h}_{a,b}$. Then, according Lemma 3

$$\begin{array}{rcl} \alpha_{a,b}(z) & = & \hat{l}_{a,b}(z) = \hat{t}_{a \cdot b}^{-1} \hat{t}_a \hat{t}_b (z) = (a \cdot b) \backslash (a \cdot (b \cdot z)), \\ \alpha_{a,b}^{-1}(z) & = & \hat{l}_{a,b}^{-1}(z) = \hat{t}_b^{-1} \hat{t}_a^{-1} \hat{t}_{a \cdot b}(z) = b \backslash (a \backslash (a \cdot b) \cdot z). \end{array}$$

5). From the item 4) it follows that

$$\begin{array}{rcl} x \cdot (x^{-1} \cdot y) & = & (x \cdot x^{-1}) \cdot \alpha_{x,x^{-1}}(y) = \varphi_x(y), \\ {}^{-1}x \cdot (x \cdot y) & = & ({}^{-1}x \cdot x) \cdot \alpha_{{}^{-1}x,x}(y) = \psi_x(y). \end{array}$$

□

Remark 2. In a left gyrogroup the left inverse property may not be fulfilled. The example it can see in [9], page 317-318.

4. Gyrogroups as loop transversals of a special kind

Lemma 9. *Let* $T = \{t_x\}_{x \in E}$ *be a weak gyrotransversal in* G *to* H*. Then the following statements are equivalent:*

1. *The transversal operation* $\langle E, \overset{(T)}{\cdot}, 1 \rangle$ *is a gyrogroup;*
2. $\forall x \in E$: $t_x T t_x \subseteq T$;
3. *The transversal operation* $\langle E, \overset{(T)}{\cdot}, 1 \rangle$ *is a left Bol loop.*

Proof. 1) $\longrightarrow$ 2). Let $T = \{t_x\}_{x\in E}$ be a weak gyrotransversal in G to H and transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrogroup. Then $\forall x, y \in E$:

$$\alpha_{x,y} = \alpha_{x\cdot y,y}.$$

In virtue of Definition 6 and Lemma 3 we have:

$$\begin{aligned} t_{x\cdot y}^{-1} t_x t_y &= t_{(x\cdot y)\cdot y}^{-1} t_{x\cdot y} t_y, \\ t_{x\cdot y}^{-1} t_x t_{x\cdot y}^{-1} &= t_{(x\cdot y)\cdot y}^{-1}, \end{aligned}$$

$$t_{x\cdot y} t_x^{-1} t_{x\cdot y} = t_{(x\cdot y)\cdot y} \tag{9}$$

If $y = x^{-1}$ then from (9) we obtain that

$$t_x^{-1} = e \cdot t_x^{-1} \cdot e = t_{x^{-1}}, \tag{10}$$

i.e. (9) may be rewritten as

$$t_{x\cdot y} t_{x^{-1}} t_{x\cdot y} = t_{(x\cdot y)\cdot y}. \tag{11}$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left loop, then the following replacement is correct:

$$\begin{cases} x^{-1} = u \\ x \cdot y = v \end{cases} \longrightarrow y = x\backslash v = (1/u)\backslash v = \left(u^{-1}\right)\backslash v.$$

Then (11) may be rewritten as: $\forall u, v \in E$

$$t_v t_u t_v = t_{v\cdot(({}^{-1}u)\backslash v)},$$

i.e.

$$t_v T t_v \subseteq T \qquad \forall v \in E.$$

2) $\longrightarrow$ 3). Let T be a left transversal in G to H and $\forall x \in E$

$$t_x T t_x \subseteq T,$$

Then $\forall x, y \in E$ there exists a permutation $\alpha_x(y)$ such that

$$t_x t_y t_x = t_{\alpha_x(y)}. \tag{12}$$

In virtue Lemma 3 we have

$$\alpha_x(y) = \hat{t}_{\alpha_x(y)}(1) = \hat{t}_x \hat{t}_y \hat{t}_x(1) = \hat{t}_x \hat{t}_y(x) = x \cdot (y \cdot x).$$

Then (12) may be rewritten as

$$t_x t_y t_x = t_{x\cdot(y\cdot x)}.$$

Again applying Lemma 3, we obtain:

$$\begin{aligned} x\cdot(y\cdot(x\cdot z)) &= x\cdot\left(y\cdot\hat{t}_x(z)\right) = x\cdot\hat{t}_y\hat{t}_x(z) = \\ &= \hat{t}_x\hat{t}_y\hat{t}_x(z) = \hat{t}_{x\cdot(y\cdot x)}(z) = (x\cdot(y\cdot x))\cdot z, \end{aligned}$$

i.e. the left Bol identity is fulfilled for the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Then $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left Bol loop.

3) $\longrightarrow$ 1). Let T be a weak gyrotransversal and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be a left Bol loop. Then $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left gyrogroup and the left Bol identity holds:

$$x\cdot(y\cdot(x\cdot z)) = (x\cdot(y\cdot x))\cdot z.$$

Then in virtue Lemma 3 we obtain $\forall z\in E$:

$$\begin{aligned} \hat{t}_x\hat{t}_y\hat{t}_x(z) &= x\cdot\hat{t}_y\hat{t}_x(z) = x\cdot\left(y\cdot\hat{t}_x(z)\right) = \\ &= x\cdot(y\cdot(x\cdot z)) = (x\cdot(y\cdot x))\cdot z = \hat{t}_{x\cdot(y\cdot x)}(z) \end{aligned}$$

i.e.

$$t_x t_y t_x = t_{x\cdot(y\cdot x)}. \tag{13}$$

Besides for a left Bol loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ it is true, that for every element $x\in E$ the left inverse element ${}^{-1}x$ coincides with the right inverse element x^{-1}: ${}^{-1}x = x^{-1}$. Also we know that a left Bol loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left IP-loop, i.e.

$${}^{-1}x\cdot(x\cdot y) = x\cdot\left(\left({}^{-1}x\right)\cdot y\right) = y. \tag{14}$$

Let us do a replacement:

$$\begin{cases} y^{-1} = u \\ y\cdot x = v \end{cases} \longrightarrow \quad x = y^{-1}\cdot(y\cdot x) = u\cdot v.$$

Then (13) may be rewritten as: $\forall u,v\in E$

$$t_{u\cdot v}t_{u^{-1}}t_{u\cdot v} = t_{(u\cdot v)\cdot v}. \tag{15}$$

Valid (14) we obtain $\forall x,y\in E$:

$$\begin{aligned} y &= x\cdot\left(x^{-1}\cdot y\right), \\ x\backslash y &= x^{-1}\cdot y, \\ \hat{t}_x^{-1}(y) &= \hat{t}_{x^{-1}}(y), \\ t_x^{-1} &= t_{x^{-1}}. \end{aligned}$$

By virtue of the last equality we obtain from (15): $\forall u, v \in E$

$$\begin{aligned} t_{u\cdot v}t_u^{-1}t_{u\cdot v} &= t_{(u\cdot v)\cdot v}, \\ t_{u\cdot v}^{-1}t_u t_{u\cdot v}^{-1} &= t_{(u\cdot v)\cdot v}^{-1}, \\ t_{u\cdot v}^{-1}t_u &= t_{(u\cdot v)\cdot v}^{-1}t_{u\cdot v}, \\ t_{u\cdot v}^{-1}t_u t_v &= t_{(u\cdot v)\cdot v}^{-1}t_{u\cdot v}t_v, \\ \alpha_{u,v} &= \alpha_{u\cdot v,v}, \end{aligned}$$

i.e. operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrogroup. ☐

Lemma 10. *Let $T = \{t_x\}_{x\in E}$ be a left transversal in G to H and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ be the transversal operation. Then the following statements are equivalent:*

1. *The system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrogroup;*
2. *T is a two-sided transversal in G to H and two following conditions hold:*
   (a) $\forall x \in E$: $t_x T t_x \subseteq T$;
   (b) $\forall h \in LI(\langle E, \overset{(T)}{\cdot}, 1\rangle)$: $hTh^{-1} \subseteq T$.

Proof. The proof is an evident corollary of Lemmas 7 and 9. ☐

Remark 3. Since a left Bol loop is a loop then the transversal T from Lemmas 9 and 10 is a loop transversal.

Lemma 11. *[15] In every gyrogroup $\langle E, \cdot, 1\rangle$ the following properties are fulfilled:*

1. *$\forall a \in E$ there exists the unique element $a^{-1} \in E$ such that $a\cdot a^{-1} = a^{-1}\cdot a = 1$;*
2. *$\alpha_{a,a^{-1}} = \alpha_{a^{-1},a} = \alpha_{a,a} = id$;*
3. *$\forall a, b \in E$:*

$$\begin{aligned} \alpha_{a,b}(z) &= (a\cdot b)^{-1}\cdot(a\cdot(b\cdot z)), \\ \alpha_{a,b}^{-1}(z) &= b^{-1}\cdot\left(a^{-1}\cdot((a\cdot b)\cdot z)\right). \end{aligned}$$

4. *$\alpha_{a,b}\left(b^{-1}\cdot a^{-1}\right) = (a\cdot b)^{-1}$;*

5. $\alpha_{a,b}^{-1} = \alpha_{a^{-1},a\cdot b} = \alpha_{b,a\cdot b} = \alpha_{b,a}$;

6. $\alpha_{a,b} = \alpha_{a\cdot b,a^{-1}} = \alpha_{a,b\cdot a}$;

7. $(a\cdot b)\cdot c = a\cdot(b\cdot\alpha_{b,a}(c))$;

8. *The solution of the equation* $x\cdot a = b$ *is:* $x = b\cdot(\alpha_{b,a}(a))^{-1} = b\cdot\alpha_{a,b}^{-1}\left(a^{-1}\right)$.

Proof. 1). Since a gyrogroup is a left Bol loop then in virtue Lemma 5, item 2) it is true that ${}^{-1}x = x^{-1}$, so

$$x\cdot x^{-1} = x^{-1}\cdot x = 1.$$

2). In virtue Lemma 5, items 1) and 2) for a left Bol loop it is true that

$$x^{-1}\cdot(x\cdot z) = x\cdot\left(x^{-1}\cdot z\right) = z \qquad \forall x,z\in E.$$

Therefore

$$\alpha_{a,a^{-1}}(z) = \left(a\cdot a^{-1}\right)\setminus\left(a\cdot\left(a^{-1}\cdot z\right)\right) = a\cdot\left(a^{-1}\cdot z\right) = z,$$

i.e. $\alpha_{a,a^{-1}} = id$. Similarly, $\alpha_{a^{-1},a} = id$. Further we have in virtue Lemma 5, item 3)

$$\alpha_{a,a}(z) = (a\cdot a)\setminus(a\cdot(a\cdot z)) == (a\cdot a)\setminus((a\cdot a)\cdot z) = z,$$

i.e. $\alpha_{a,a} = id$.

3). By the definition 6

$$a\cdot(b\cdot z) = (a\cdot b)\cdot\alpha_{a,b}(z),$$

so in virtue Lemma 5, item 3) we have

$$\alpha_{a,b}(z) = (a\cdot b)^{-1}\cdot(a\cdot(b\cdot z)). \tag{16}$$

Making the replacement $z = \alpha_{a,b}^{-1}(u)$, we obtain from (12):

$$u = (a\cdot b)^{-1}\cdot(a\cdot(b\cdot\alpha_{a,b}^{-1}(u))), \tag{17}$$

and again using Lemma 5, item 3) we obtain:

$$\alpha_{a,b}^{-1}(u) = b^{-1}\cdot(a^{-1}\cdot((a\cdot b)\cdot u)).$$

4). Using item 3) of present Lemma and Lemma 5, item 3) we obtain:

$$\begin{aligned}\alpha_{a,b}\left(b^{-1}\cdot a^{-1}\right) &= (a\cdot b)^{-1}\cdot(a\cdot(b\cdot(b^{-1}\cdot a^{-1}))) = \\ &= (a\cdot b)^{-1}\cdot\left(a\cdot a^{-1}\right) = (a\cdot b)^{-1}.\end{aligned}$$

5). According to item 3) of present Lemma

$$\alpha_{a,b}^{-1}(z) = b^{-1} \cdot (a^{-1} \cdot ((a \cdot b) \cdot z)). \tag{18}$$

Let us do a replacement

$$\begin{cases} a^{-1} = c \\ a \cdot b = d \end{cases} \longrightarrow \begin{cases} c = a^{-1} \\ c \cdot d = a^{-1} \cdot (a \cdot b) = b \end{cases}$$

Then (18) may be rewritten as

$$\alpha_{a,b}^{-1}(z) = (c \cdot d)^{-1} \cdot (c \cdot (d \cdot z)) = \alpha_{c,d}(z) = \alpha_{a^{-1},a \cdot b}(z),$$

i.e. $\alpha_{a,b}^{-1} \equiv \alpha_{a^{-1},a \cdot b}$.

Further in virtue Lemma 9 if a transversal $T = \{t_x\}_{x \in E}$ corresponds to a gyrogroup $\langle E, \cdot, 1\rangle = \langle E, \overset{(T)}{\cdot}, 1\rangle$, then it is true that

$$t_x t_y t_x = t_{x\cdot(y\cdot x)}, \quad \forall x, y \in E.$$

Then we have

$$t_{x\cdot(y\cdot x)} = t_x t_y t_x = t_x t_{y\cdot x} \alpha_{y,x} = t_{x\cdot(y\cdot x)} \alpha_{x,y\cdot x} \alpha_{y,x}.$$

So we obtain

$$\alpha_{x,y\cdot x} \alpha_{y,x} = id,$$

i.e.

$$\alpha_{a,b}^{-1} = \alpha_{b,a\cdot b}. \tag{19}$$

At last since by the Definition 6 for every gyrogroup it is true that

$$\alpha_{a,b} = \alpha_{a\cdot b,b},$$

then from (19) we obtain:

$$\alpha_{a\cdot b,b}^{-1} = \alpha_{a,b}^{-1} = \alpha_{b,a\cdot b}. \tag{20}$$

Making the replacement

$$\begin{cases} a \cdot b = c \\ b = d \end{cases}$$

we obtain from (20) $\forall c, d \in E$:

$$\alpha_{c,d}^{-1} = \alpha_{d,c}.$$

6). We have from item 5) of present Lemma

$$\alpha_{a,b} = \alpha^{-1}_{a^{-1},a\cdot b} = \alpha_{a\cdot b,a^{-1}}. \tag{21}$$

By virtue of the last equality in the item 5) we have

$$\alpha_{b,\,a\cdot b} = \alpha_{b,a},$$

i.e.

$$\alpha_{a,b} = \alpha_{a,b\cdot a}.$$

7). By virtue of items 3) and 5) of present Lemma and Lemma 5, item 1) we have:

$$(a\cdot b)\cdot c = a\cdot (b\cdot \alpha^{-1}_{a,b}(c)\,) = a\cdot (b\cdot \alpha_{b,a}(c))\,.$$

8). According to items 1) and 3) of present Lemma and Lemma 5, item 4) the solution of the equation $x\cdot a = b$ is

$$\begin{aligned} x &= a^{-1}\cdot\left((a\cdot b)\cdot a^{-1}\right) = b\cdot\left(b^{-1}\cdot\left(a^{-1}\cdot\left((a\cdot b)\cdot a^{-1}\right)\right)\right) = \\ &= b\cdot\alpha^{-1}_{a,b}\left(a^{-1}\right) = b\cdot\alpha_{b,a}\left(a^{-1}\right). \end{aligned} \tag{22}$$

But since

$$1 = \alpha_{b,a}\left(a^{-1}\cdot a\right) = \alpha_{b,a}\left(a^{-1}\right)\cdot\alpha_{b,a}(a)\,, \tag{23}$$

then we obtain from (22):

$$x = b\cdot\alpha_{b,a}\left(a^{-1}\right) = b\cdot\left(\alpha_{b,a}(a)\right)^{-1}.$$

□

It is very interesting to investigate operations, which are inverse ones to a gyrogroup operation $\langle E,\cdot,1\rangle$. The left inverse operation coincides with operation $\langle E,\cdot,1\rangle$ (because of the left Bol loop $\langle E,\cdot,1\rangle$ is a *LIP*-loop). Let us study the right inverse operation.

We can define the following operations on a set E (see [15]):

$$\begin{aligned} a\oplus b &\overset{def}{=} a\cdot\alpha_{a,b^{-1}}(b)\,, \\ a\odot b &\overset{def}{=} a\oplus b^{-1}. \end{aligned} \tag{24}$$

Lemma 12. *Let $\langle E,\cdot,1\rangle$ be a gyrogroup. Then the following statements are true:*

1. *$a\odot b = a/b$, $\quad a\oplus b = a/b^{-1}$, where "/" is a right division in the gyrogroup $\langle E,\cdot,1\rangle$,*

   $a\cdot b = a\,//\,b^{-1}$, where "//" is a right division in a system $\langle E,\oplus,1\rangle$;

2. $a \oplus \alpha_{a,b}(b) = a \cdot b$*;*

3. The system $\langle E, \oplus, 1\rangle$ *is a loop with the unit* 1*, and* $\forall x \in E$ *the left and right inverse elements to an element* x *in* $\langle E, \oplus, 1\rangle$ *coincide. Moreover, both of them are equal to* x^{-1} *(where* x^{-1} *is an inverse element to an element* x *in* $\langle E, \cdot, 1\rangle$*);*

4. $(a \oplus b)^{-1} = b^{-1} \oplus a^{-1}$*;*

5. $Aut(\langle E, \oplus, 1\rangle) = Aut(\langle E, \cdot, 1\rangle)$*;*

6. The system $\langle E, \oplus, 1\rangle$ *is a middle Bol loop, i.e. the following identity holds:*

$$x \oplus ((y \oplus z) \setminus\setminus x) = (x // z) \oplus (y \setminus\setminus x),$$

where "\\" and "//" are left and right division in $\langle E, \oplus, 1\rangle$*, respectively.*

Proof. 1). According to (23), (24) and Lemma 11, item 8), we obtain

$$a \odot b = a \oplus b^{-1} = a \cdot \alpha_{a,b}\left(b^{-1}\right) = a \cdot (\alpha_{a,b}(b))^{-1} = a/b.$$

Then

$$a \oplus b = a \odot b^{-1} = a/b^{-1}.$$

Further we have

$$(a \cdot b) \oplus b^{-1} = (a \cdot b)/b = a,$$

i.e.

$$a \cdot b = a//b^{-1},$$

where "//" is a right division in $\langle E, \oplus, 1\rangle$.

2). From item 1) and (23) it follows that

$$\begin{aligned} (a \cdot b) // (\alpha_{a,b}(b)) &= (a \cdot b) \cdot (\alpha_{a,b}(b))^{-1} = \\ &= (a \cdot b) \cdot \alpha_{a,b}\left(b^{-1}\right) = a \cdot \left(b \cdot b^{-1}\right) = a, \end{aligned}$$

i.e.

$$a \oplus \alpha_{a,b}(b) = a \cdot b.$$

3). In virtue of the item 1) the system $\langle E, \oplus, 1\rangle$ is an inverse operation to the loop $\langle E, \cdot, 1\rangle$, therefore it is a quasigroup. Further we have $\forall x \in E$:

$$\begin{aligned} 1 \oplus x &= 1/x^{-1} = x, \\ x \oplus 1 &= x/1^{-1} = x, \end{aligned}$$

i.e. $\langle E, \oplus, 1\rangle$ is a loop. At last,

$$\begin{aligned} x \oplus x^{-1} &= x/\left(x^{-1}\right)^{-1} = x/x = 1, \\ x^{-1} \oplus x &= x^{-1}/x^{-1} = 1. \end{aligned}$$

4). We have $\forall a, b \in E$:

$$(a \oplus b)^{-1} = \left(a/b^{-1}\right)^{-1}, \qquad b^{-1} \oplus a^{-1} = b^{-1}/a.$$

But $a = c \cdot b^{-1}$ for some $c \in E$, therefore

$$(a \oplus b)^{-1} = \left(\left(c \cdot b^{-1}\right)/b^{-1}\right)^{-1} = c^{-1} = \left(c^{-1} \cdot a\right)/a = b^{-1}/a = b^{-1} \oplus a^{-1}.$$

5). According to the item 1), we have

$$a \oplus b = a/b^{-1}, \qquad a \cdot b = a//b^{-1}.$$

Then every automorphism α of the operation $\langle E, \cdot, 1\rangle$ will be an automorphism of the inverse operation $\langle E, /\rangle$, and so α will be an automorphism of the operation $\langle E, \oplus, 1\rangle$; and vice versa.

6). It is an evident corollary of Lemma 6, item 3). □

Let us note also the folowing identities

$$(x//y)^{-1} = z^{-1}\backslash\backslash x^{-1}, \qquad (x\backslash\backslash y)^{-1} = y^{-1}//x^{-1}.$$

5. Gyrocommutative gyrogroups

Lemma 13. *Let $T = \{t_x\}_{x\in E}$ be a left transversal in G to H such that the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrogroup. Then the following statements are equivalent:*

1. *$\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrocommutative gyrogroup;*
2. *$\forall x, y \in E: \quad (x \cdot y)\cdot(x \cdot y) = x \cdot (y \cdot (y \cdot x))$, - the Bruck identity;*
3. *$\forall x, y \in E: \quad t_x t_y^2 t_x = t_{x\cdot y}^2$;*
4. *$\forall x, y \in E: \quad (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$, - automorphic inverse property.*

Proof. Let the conditions of Lemma hold; then $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a left Bol loop.

1) $\longrightarrow$ 2). Let $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrocommutative gyrogroup. Then by the definition 8 $\forall x, y \in E$:

$$x \cdot y = \alpha_{x,y}\left(y \cdot x\right).$$

Then by the definition 6 of the automorphism $\alpha_{x,y}$ and in virtue Lemma 5 we have:

$$x \cdot y = (x \cdot y)^{-1} \cdot (x \cdot (y \cdot (y \cdot x))),$$
$$(x \cdot y) \cdot (x \cdot y) = x \cdot (y \cdot (y \cdot x)).$$

2) $\longrightarrow$ 3). Let the following identity holds $\forall x, y \in E$:

$$(x \cdot y) \cdot (x \cdot y) = x \cdot (y \cdot (y \cdot x)).$$

Then in virtue Lemmas 5 and 9 we have $\forall x, y \in E$:

$$t_x t_y^2 t_x = t_x t_{y \cdot y} t_x = t_{x \cdot ((y \cdot y) \cdot x)} = t_{x \cdot (y \cdot (y \cdot x))} = t_{(x \cdot y) \cdot (x \cdot y)} = t_{x \cdot y}^2.$$

3) $\longrightarrow$ 4). Let $\forall x, y \in E$

$$t_x t_y^2 t_x = t_{x \cdot y}^2.$$

Then in virtue Lemma 5 we have:

$$t_x t_y t_y t_x = t_{x \cdot y} t_{x \cdot y},$$
$$l_{x,y} = t_{x \cdot y}^{-1} t_x t_y = t_{x \cdot y} t_x^{-1} t_y^{-1},$$
$$\hat{l}_{x,y}(1) = \hat{t}_{x \cdot y} \hat{t}_x^{-1} \hat{t}_y^{-1}(1),$$
$$1 = (x \cdot y) \cdot \left(x^{-1} \cdot y^{-1}\right),$$
$$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}.$$

4) $\longrightarrow$ 1). Let $\forall x, y \in E$:

$$(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}.$$

Then in virtue Lemma 11, item 4) we have

$$(x \cdot y)^{-1} = \alpha_{x \cdot y}\left(y^{-1} \cdot x^{-1}\right) = \alpha_{x,y}((y \cdot x)^{-1}).$$

Then we obtain

$$\begin{aligned} 1 &= \alpha_{x,y}\left(1\right) = \alpha_{x,y}((y \cdot x) \cdot (y \cdot x)^{-1}) = \\ &= \alpha_{x,y}\left(y \cdot x\right) \cdot \alpha_{x,y}((y \cdot x)^{-1}) = \alpha_{x,y}\left(y \cdot x\right) \cdot (x \cdot y)^{-1}, \end{aligned}$$

i.e.

$$\alpha_{x,y}\left(y\cdot x\right)=x\cdot y,$$

and the system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrocommutative gyrogroup. □

Lemma 14. *Let $T=\{t_x\}_{x\in E}$ is a left transversal in G to H and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a transversal operation. Then the following statements are equivalent:*

1. *$\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrocommutative gyrogroup;*
2. *T is a two-sided transversal in G to H and the following three conditions hold:*
   (a) *$\forall x\in E: \quad t_xTt_x\subseteq T$;*
   (b) *$\forall h\in LI\left(\langle E, \overset{(T)}{\cdot}, 1\rangle\right)$: $\qquad hTh^{-1}\subseteq T$;*
   (c) *$\forall x\in E: \quad t_xt_y^2t_x=t_{x\cdot y}^2$.*

Proof. The proof is an evident corollary from Lemmas 10 and 13. □

Lemma 15. *If $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a gyrocommutative gyrogroup then the operation "$\oplus$" (determined in (24)) satisfies the following properties: $\forall x,y\in E$*

1. *$x\oplus y=y\oplus x$;*
2. *$x//y=y\backslash\backslash x$.*

Proof. 1). According Lemma 12 we have

$$x\oplus y=x/y^{-1},$$

so it is necessary to prove that $\forall x,y\in E$

$$x/y^{-1}=y/x^{-1}.$$

But since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop then $x=z\cdot y^{-1}$ for some $z\in E$. In virtue Lemma 13 we obtain

$$\left(x/y^{-1}\right)\cdot x^{-1}=\left(\left(z\cdot y^{-1}\right)/y^{-1}\right)\cdot\left(z\cdot y^{-1}\right)^{-1}=z\cdot\left(z^{-1}\cdot y\right)=y,$$

as it was required.

2). According Lemma 12 we have

$$x//y = x \cdot y^{-1},$$

then we obtain

$$y\backslash\backslash x = \left(x^{-1}//y^{-1}\right)^{-1} = \left(x^{-1} \cdot y\right)^{-1} = x \cdot y^{-1} = x//y.$$

□

6. Semidirect products of gyrogroups, left gyrogroups and suitable groups

Let us remind a definition of a semidirect product of a left loop $\langle E, \cdot, 1\rangle$ and a suitable permutation group H (see [11, 12]).

Definition 12. *Let $\langle E, \cdot, 1\rangle$ be a left loop with two-sided unit 1, and H be a subgroup of the permutation group $St_1(S_E)$ such that the following conditions are fulfilled:*

1. $\forall a, b \in E$: $\quad l_{a,b} = L_{a \cdot b}^{-1} L_a L_b \in H$;

2. $\forall a \in E$ and $\forall h \in H$: $\quad \varphi(a, h) = L_{h(a)}^{-1} h L_a h^{-1} \in H$,

where $L_a(x) = a \cdot x$ is a left translation by an element $a \in E$. Then on a set $E \times H$ of pairs (u, h) it is possible to define an operation:

$$(u, h_1) * (v, h_2) \stackrel{def}{=} \left(u \cdot h_1(v), l_{a,h_1(v)} \varphi(v, h_1) h_1 h_2\right), \tag{25}$$

and an action on the set E:

$$(u, h)(x) \stackrel{def}{=} u \cdot h(x). \tag{26}$$

It is possible to show (see [11, 12]) that:

1. A system $G = \langle E \times H, *, (1, id)\rangle$ is a group (a **semidirect product** of the left loop $\langle E, \cdot, 1\rangle$ and the group H);
2. It is true that $(u, h)^{-1} = (h^{-1}(u\backslash 1), (L_u h L_{h^{-1}(u\backslash 1)})^{-1})$;
3. A set $T = \{(u, id) \,|u \in E\}$ is a left transversal in the group G to its subgroup $H^* = \{(1, h) \,|h \in H\} \cong H$, and the transversal operation $\langle E, \stackrel{(T)}{\cdot}, 1\rangle$ coincides with the operation $\langle E, \cdot, 1\rangle$.

The following special case of the above-described construction will be important for us: when the left loop $\langle E,\cdot,1\rangle$ is a **left special loop (left A_l-loop)**, that is

$$LI\left(\langle E,\cdot,1\rangle\right)\subseteq H\subseteq Aut\left(\langle E,\cdot,1\rangle\right). \qquad (27)$$

The formula (25) of the semidirect product may be rewritten as

$$(u,h_1)*(v,h_2)\overset{def}{=}\left(u\cdot h_1(v),l_{a,h_1(v)}h_1h_2\right). \qquad (28)$$

Then all above-mentioned properties are correct and the following formula holds:

$$(u,h)^{-1}=\left(h^{-1}(u\backslash 1),(L_uL_{u\backslash 1}h)^{-1}\right). \qquad (29)$$

Remark 4. The formula (27) coincides with the formula of a gyrosemidirect product of a left gyrogroup and its gyroautomorphism group (see [4, 14]).

Lemma 16. *Every left gyrogroup $\langle E,\cdot,1\rangle$ may be represented as a weak gyrotransversal in the group $\langle E\times H,*,(1,id)\rangle$ to a subgroup H, if H satisfies the conditions of the Definition 12.*

Proof. The proof obviously follows from Lemma 7 and above-mentioned properties of the semidirect product. □

Corollary 12. *Every left gyrogroup $\langle E,\cdot,1\rangle$ may be represented as a weak gyrotransversal in the group $\langle E\times H,*,(1,id)\rangle$ to a subgroup H_0, which satisfies the condition (27) (and semidirect product is defined under the formula (28)).*

Lemma 17. *Every gyrogroup $\langle E,\cdot,1\rangle$ may be represented as a gyrotransversal in the group $G=\langle E\times H,*,(1,id)\rangle$ to its gyroautomorphism group H_0 (i.e. H_0 satisfies the condition (27)).*

Proof. (See also [4]) According the Corollary 12 a set $T=\{(a,id)\,|a\in E\}$ is a weak gyrotransversal in G to H_0. Since $\langle E,\cdot,1\rangle$ is a gyrogroup then in virtue Lemma 9 $\langle E,\cdot,1\rangle$ is a left Bol loop; therefore it satisfies the left inverse property, i.e. $\forall u\in E$

$$L_uL_{u\backslash 1}=id. \qquad (30)$$

Then in virtue of (29)

$$(u,id)^{-1}=(u\backslash 1,id)=\left(u^{-1},id\right),$$

i.e. $T^{-1} = T$.

Further $\forall u \in E$ and $\forall h \in H_0$ in virtue of the formula (28)

$$(1, h) * (u, id) * \left(1, h^{-1}\right) = (1, h) * \left(u, l_{u,1} h^{-1}\right) =$$
$$= (1, h) * \left(u, h^{-1}\right) = \left(h(u), l_{1,h(u)} h h^{-1}\right) = (h(u), id),$$

i.e. $\forall u \in E$ and $\forall h \in H_0$

$$(1, h) * T * (1, h)^{-1} = T.$$

It means that T is a gyrotransversal in G to H_0. □

Remark 5. A left gyrogroup may not be represented as a gyrotransversal in the group $G = \langle E \times H, *, (1, id)\rangle$ to its gyroautomorphism group (since in a left gyrogroup, not being a gyrogroup, it is not necessarily satisfied the condition (30)).

Lemma 18. *A left gyrogroup $\langle E, \cdot, 1\rangle$ may be represented as a gyrotransversal in the group $G = \langle E \times H, *, (1, id)\rangle$ to a group H_0, which satisfies the condition (27) $\Leftrightarrow$ $\langle E, \cdot, 1\rangle$ is a LIP-loop.*

Proof. The proof is evident, because a left gyrogroup $\langle E, \cdot, 1\rangle$ is always a weak gyrotransversal in the group $G = \langle E \times H, *, (1, id)\rangle$ to the subgroup H_0 (see a Corollary 12), and the condition (30) is equivalent to a definition of LIP-loop. □

7. Generalized diagonal transversals

Definition 13. *Let K be a group, G be a semidirect product*

$$G = K \leftthreetimes Inn(K),$$

where

$$Inn(K) = \left\{\alpha_n | \alpha_n(x) = kxk^{-1}, \quad k, x \in K\right\}$$

is a group of internal automorphisms of the group K. Then a ***generalized diagonal transversal*** *D_m* ***of degree*** *m is a set*

$$D_m = \{(k, \alpha_k^m) | k \in K\}. \tag{31}$$

We shall denote

$$D_m(K) = (k, \alpha_k^m). \tag{32}$$

A **diagonal transversals**, which were investigated in [4, 5, 6, 3], are obtained in a case when $m = 1$.

Lemma 19. *The generalized diagonal transversal D_m of degree m is a gyrotransversal in G to $H = Inn(K)$.*

Proof. For every element $(k, \alpha_h) \in G$ (where $k, h \in K$) we have:

$$(k, \alpha_h) = (k, \alpha_k^m) \cdot (1, \alpha_{k^{-m}h}),$$

and this decomposition is an unique one. It means that the set D_m is a left transversal in G to H.

Further we have:

$$(D_m(k))^{-1} = (k, \alpha_k^m)^{-1} = \left(\alpha_k^{-m}\left(k^{-1}\right), \alpha_k^{-m}\right) = D_m\left(k^{-1}\right),$$

i.e. $(D_m)^{-1} \equiv D_m$.

Also we obtain:

$$\begin{aligned}(1, \alpha_h)\, D_m(k)\,(1, \alpha_h)^{-1} &= (1, \alpha_h)(k, \alpha_k^m)(1, \alpha_{h^{-1}}) = \\ &= (1, \alpha_h)(k, \alpha_{k^m h^{-1}}) = (\alpha_h(k), \alpha_{hk^m h^{-1}}) = \\ &= \left(\alpha_h(k), \alpha_{\alpha_h(k^m)}\right) = \left(\alpha_h(k), \alpha_{(\alpha_h(k))^m}\right) = \\ &= \left(\alpha_h(k), \alpha^m_{\alpha_h(k)}\right) = D_m(\alpha_h(k)).\end{aligned}$$

According to Definition 5, item 2) the set D_m is a gyrotransversal. ☐

Let us study the transversal operation $\langle E, \overset{(D_m)}{\cdot}, 1\rangle$. We have:

$$\begin{aligned}D_m(k_1)\, D_m(k_2) &= (k_1, \alpha_{k_1}^m)(k_2, \alpha_{k_2}^m) = \\ &= \left(k_1\alpha_{k_1}^m(k_2), \alpha^m_{k_1\alpha^m_{k_1}(k_2)}\right) \cdot \left(1, \alpha_{(k_1\alpha^m_{k_1}(k_2))^{-m} k_1^m k_2^m}\right) = \\ &= D_m(k_1\alpha_{k_1}^m(k_2)) \cdot \left(1, \alpha_{k_1^m (k_1k_2)^{-m} k_2^m}\right),\end{aligned}$$

because of

$$\begin{aligned}&\left(k_1\alpha_{k_1}^m(k_2)^{-m}\, k_1^m k_2^m\right)^{-m} k_1^m k_2^m = \left(k_1^m k_2^{-1}k_1^{-1}k_1^{-m}\right)^m k_1^m k_2^m = \\ &= \underbrace{\left(k_1^m k_2^{-1}k_1^{-1}k_1^{-m}\right)\left(k_1^m k_2^{-1}k_1^{-1}k_1^{-m}\right) \cdot \ldots \cdot \left(k_1^m k_2^{-1}k_1^{-1}k_1^{-m}\right)}_{m} \cdot k_1^m k_2^m = \\ &= k_1^m \left(k_2^{-1}k_1^{-1}\right)^m k_2^m = k_1^m (k_1k_2)^{-m} k_2^m.\end{aligned}$$

It means that

$$k_1 \overset{(D_m)}{\cdot} k_2 = k_1\alpha_{k_1}^m(k_2) \tag{33}$$

and

$$l_{k_1,k_2} = \left(1, \alpha_{k_1^m (k_1k_2)^{-m} k_2^m}\right). \tag{34}$$

Lemma 20. *A left gyrogroup* $\langle E, \overset{(D_m)}{\cdot}, 1\rangle$ *is a group* $\Leftrightarrow$ *the following identity*

$$(ab)^m = b^m a^m \qquad \forall a, b \in K/Z(K). \tag{35}$$

is fulfilled in a factor-group $K/Z(K)$.

Proof. According to the formula (34), the left gyrogroup $\langle E, \overset{(D_m)}{\cdot}, 1\rangle$ is a group if and only if when

$$\alpha_{k_1^m (k_1 k_2)^{-m} k_2^m} = id \qquad \forall k_1, k_2 \in K.$$

It is equivalent to a fact that $\forall a, b \in K/Z(K)$ it is true that

$$a^m (ab)^{-m} b^m = 1,$$
$$(ab)^{-m} = a^{-m} b^{-m},$$
$$\left(b^{-1}a^{-1}\right)^m = \left(a^{-1}\right)^m \left(b^{-1}\right)^m,$$
$$(cd)^m = d^m c^m \qquad \forall c, d \in K/Z(K).$$

□

Lemma 21. *The left gyrogroup* $\langle E, \overset{(D_m)}{\cdot}, 1\rangle$ *is a gyrogroup* $\Leftrightarrow$ *the following identity*

$$b^{2m} a^m = \left(a^{-m} b\, a^{m+1} b\right)^m \qquad \forall a, b \in K/Z(K) \tag{36}$$

is fulfilled in the factor-group $K/Z(K)$.

Proof. According to the formula (36), the left gyrogroup $\langle E, \overset{(D_m)}{\cdot}, 1\rangle$ is a gyrogroup if and only if when

$$l_{a \overset{(D_m)}{\cdot} b, b} = l_{a,b},$$
$$\alpha_{(a \overset{(D_m)}{\cdot} b)^m ((a \overset{(D_m)}{\cdot} b) b)^{-m} b^m} = \alpha_{a^m (ab)^{-m} b^m},$$
$$\alpha_{(a a^m b a^{-m})^m (a a^m b a^{-m} b)^{-m} b^m} = \alpha_{a^m (ab)^{-m} b^m}.$$

It is equivalent to a fact that in the factor-group $K/Z(K)$ it is true that

$$\left(a^{m+1} b a^{-m}\right)^m \left(a^{m+1} b a^{-m} b\right)^{-m} b^m = a^m (ab)^{-m} b^m,$$
$$\left(a^m a b a^{-m}\right) \cdot \left(a^m a b a^{-m}\right) \cdot \ldots \cdot \left(a^m a b a^{-m}\right) \cdot \left(a^{m+1} b a^{-m} b\right)^{-m} = a^m (ab)^{-m},$$
$$(ab)^m a^{-m} \left(a^{m+1} b\, a^{-m} b\right)^{-m} = (ab)^{-m},$$
$$a^{-m} \left(a^{m+1}\, b\, a^{-m} b\right)^{-m} = (ab)^{-2m},$$
$$(ab)^{2m} = \left(a^{m+1} b\, a^{-m} b\right)^m a^m.$$

Let us replace: $c = ab$, $d = a^{-1}$; then

$$c^{2m} = \left(d^{-m-1} dc\, d^m dc\right)^m d^{-m},$$
$$c^{2m} d^m = \left(d^{-m} c\, d^{m+1} c\right)^m \qquad \forall c, d \in K/Z\left(K\right).$$

□

Remark 6. If $m = 1$ then we obtain the results from [4].

References

[1] **Baer R.**: *Nets and groups. 1.* Trans. Amer. Math. Soc., **46** (1939), 110-141.

[2] **Belousov V. D.**: *Foundations of quasigroup and loop theory* (Russian), Moscow, "Nauka", 1967.

[3] **T. Foguel**: *Groups, Transversals and loops*, Comment. Math. Univ. Carolinae, **41, 2** (2000), 261-269.

[4] **T. Foguel, A.A. Ungar**: *Transversals, Loops and Gyrogroups*, unpublished letter to the author from May 20, 1998.

[5] **T. Foguel, A.A. Ungar**: *Involutory decomposition of groups into twisted subgroups and subgroups*, J. Group Theory, **3** (2000), No. 1, 27-46.

[6] **T. Foguel, A.A. Ungar**: *Gyrogroups and the decompoziotion of groups into twisted subgroups and subgroups*, Pasific J. Math., **197** (2001), 1-11.

[7] **Gwaramija A. A.**: *About one class of loops* (Russian), Uchen. Zapiski, **375** (1978), 25-34.

[8] **Kargapolov M. I., Merzlyakov Yu. I.**, *Foundations of group theory* (Russian), Moscow, "Nauka", 1982.

[9] **Kiechle H.**: *Relatives of K-loops: Theory and examples*, Comm. Math. Univ. Carolinae, **41, 2** (2000), No. 2, 201-324.

[10] **Kuznetsov E. A.**: *Transversals in groups. 1. Elementary properties*, Quasigroups and related systems, **1** (1994), 22-42.

[11] **Kuznetsov E.A.**: *Transversals in groups. 3. Semidirect product of a transversal operation and subgroup*, Quasigroups and related systems, **8** (2001), 37-44.

[12] **Sabinin L. V.**: *On the equivalence of categories of loop and homogeneous spaces.* Soviet Math. Dokl., **13** (1972), No. 4, 970-974.

[13] **Sabinin L. V., Mikheev O. I.**: *Quasigroups and differential geometry*, Chapter XII in the book "*Quasigroups and loops: Theory and Applications*", Berlin, Helderman-Verlag, 1990, 357-430.

[14] **Abraham A. Ungar**: *Thomas rotation and the parametrization of the Lorentz transformation group*, Found. Phys. Lett., **1** (1988), 57-89.

[15] **Ungar A.A.**: *Thomas Precession: Its Underlying Gyrogroup Axioms and Their Use in Hyperbolic Geometry and Relativistic Physics*, Foundations of Physics, **27** (1997), No. 6, 881-951.

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Received by the editors: 15.06.2003
and final form in 04.11.2003.