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An algebraic version of the Strong Black Box

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Abstract. Various versions of the prediction principle called the "Black Box" are known. One of the strongest versions can be found in [EM]. There it is formulated and proven in a model theoretic way. In order to apply it to specific algebraic problems it thus has to be transformed into the desired algebraic setting. This requires intimate knowledge on model theory which often prevents algebraists to use this powerful tool. Hence we here want to present algebraic versions of this "Strong Black Box" in order to demonstrate that the proofs are straightforward and that it is easy enough to change the setting without causing major changes in the relevant proofs. This shall be done by considering three different applications where the obtained results are actually known.

Introduction

The aim of this paper is to investigate and apply a well-known prediction principle due to Saharon Shelah. It can be found in [EM, Chapter XIII] where it is formulated and proven in a model theoretic way.

However, many known applications are in an algebraic context and thus it seems natural to transfer everything into the corresponding setting. An algebraic version of this principle, which we call "Strong Black Box", has the advantage of immediate application. Moreover, the here

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presented proofs are given in an algebraic setting and use only basic knowledge of set theory. Therefore it is relatively easy to adjust a given Strong Black Box to a different situation. To emphasize this we here present three versions and its applications. This shall be done in three independent sections: In the first section we demonstrate how to realize endomorphism rings. In Section 2 we construct E-rings, respectively $E(R)$-algebras, and in Section 3 we show the existence of a cotorsion-free module G with the additional property that, for any submodule H of G, if G/H is cotorsion-free then $H = G$ or $|H| < |G|$; such a module G is called *ultra-cotorsion-free*. Note, the reader who is only interested in one of the applications can go straight to the favorite section; it will be clear if something from before is needed.

We need to mention that the obtained results in all these applications are actually known. The classical proofs, however, are much more complicated due to the fact that they are based on the "General Black Box". We want to use the name "General Black Box" for the Black Box principle as, for example, given in [S] or [CG] to distinguish between this principle and the one presented here. Both principles hold in ordinary set theory, ZFC, but are inspired by the diamond principle due to R. Jensen which holds in the constructible universe (V=L). While the Strong Black Box provides predictions on particular successor cardinals the General Black Box includes more general cardinals, even some singular ones. On the other hand, the Strong Black Box is less complicated and its applications more straightforward. This advantage is due to the fact that the prediction principle is sharper, i.e. the disjointness condition is stronger, and hence less algebra is needed.

The reader who is familiar with constructions using the General Black Box will appreciate the achieved simplifications.

For unexplained terminology we refer the reader to [EM].

§1. Realizing endomorphism rings

Throughout this first section let R be a commutative ring with 1 and let $\mathbb{S}$ be a countable multiplicatively closed subset of R containing no units except 1 such that R is $\mathbb{S}$-reduced and $\mathbb{S}$-torsion-free. Recall, an R-module M is said to be $\mathbb{S}$-*reduced* if it satisfies $\bigcap_{s\in\mathbb{S}} sM = 0$; M is $\mathbb{S}$-*torsion-free* if $sm = 0$ $(s \in \mathbb{S}, m \in M)$ implies $m = 0$. Note that, in general, we shall skip the prefix "$\mathbb{S}$-" and use the notions torsion-free, reduced, pure etc. instead of $\mathbb{S}$-torsion-free, $\mathbb{S}$-reduced, $\mathbb{S}$-pure where M is said to be ($\mathbb{S}$-) *pure* in N (notation: $M \subseteq_* N$) if $sM = M \cap sN$ holds for all $s \in \mathbb{S}$. Hence, in this case, the $\mathbb{S}$-adic topology on M is induced by the $\mathbb{S}$-adic topology on N.

To describe the $\mathbb{S}$-adic topology with a descending chain of submodules we fix an enumeration $\mathbb{S} = \{s_n | n < \omega\}$ of $\mathbb{S}$ with $s_0 = 1$ and define a divisor chain $(q_n)_{n<\omega}$ by $q_n = s_0 \cdot \ldots \cdot s_n$. Then the $\mathbb{S}$-adic topology of any R-module M has $\{q_n M | n < \omega\}$ as a basis of neighbourhoods of zero.

Let $\widehat{R}$ denote the completion of R in the $\mathbb{S}$-adic topology. We shall assume that R satisfies $\mathrm{Hom}_R(\widehat{R}, R) = 0$, that is to say R is $\mathbb{S}$-cotorsion-free. In general, an R-module M is said to be ($\mathbb{S}$-)*cotorsion-free* if M is ($\mathbb{S}$-)torsion-free, ($\mathbb{S}$-)reduced and satisfies $\mathrm{Hom}_R(\widehat{R}, M) = 0$. Note, that the torsion-freeness of M actually follows from $\mathrm{Hom}_R(\widehat{R}, M) = 0$ and in some cases, e.g. for domains R such that the quotient field $Q(R)$ is countably generated over R, $\mathrm{Hom}_R(\widehat{R}, M) = 0$ also implies that M is reduced.

In this section we show that, given infinite cardinals κ, μ, λ satisfying $\kappa \geq |R|, \mu^\kappa = \mu, \lambda = \mu^+$ and a cotorsion-free R-algebra A with $F \subseteq_* A \subseteq_* \widehat{F}$ for some free R-module F and $|A| \leq \lambda$, there exists a cotorsion-free R-module G of cardinality λ such that $\mathrm{End}_R G = A$. This shall be done using a suitable version of the Strong Black Box, which is introduced and proven in the first subsection.

For a proof of this result using the General Black Box we refer to [CG].

1.1. The Black Box Theorem

In this subsection we shall formulate and prove the Strong Black Box in full detail. To do so let $R, \mathbb{S}, F, A$ as well as κ, μ, λ be as above.

We formulate the parameters of the Black Box with respect to a free R-module B and its completion $\widehat{B}$. Let $F = \bigoplus_{\varepsilon<\rho} Ra_\varepsilon$ $(\subseteq_* A, \rho \leq \lambda)$ and put $B := \bigoplus_{\alpha<\lambda} e_\alpha F \subseteq_* \bigoplus_{\alpha<\lambda} e_\alpha A =: B'$. Then, writing $e_{\alpha,\varepsilon}$ for $e_\alpha a_\varepsilon$ and using that R is commutative, we have $B = \bigoplus_{(\alpha,\varepsilon)\in\lambda\times\rho} Re_{\alpha,\varepsilon}$ and $\widehat{B} = \widehat{B'}$. For later use we put the lexicographic ordering on $\lambda \times \rho$; since λ, ρ are ordinals $\lambda \times \rho$ is well ordered.

For any $g = (g_{\alpha,\varepsilon} e_{\alpha,\varepsilon})_{(\alpha,\varepsilon)\in\lambda\times\rho} \in \widehat{B} \subseteq \prod_{(\alpha,\varepsilon)\in\lambda\times\rho} \widehat{R} e_{\alpha,\varepsilon}$ we define the *support* of g by

$$[g] = \{(\alpha, \varepsilon) \in \lambda \times \rho | g_{\alpha,\varepsilon} \neq 0\}$$

and the support of $M \subseteq \widehat{B}$ by $[M] = \bigcup_{g\in M} [g]$; note $|[g]| \leq \aleph_0$ for all $g \in \widehat{B}$. Moreover, we define the λ-*support* of g by

$$[g]_\lambda = \{\alpha \in \lambda | \exists \varepsilon \in \rho : (\alpha, \varepsilon) \in [g]\} \subseteq \lambda$$

and the A-*support* of g by

$$[g]_A = \{\varepsilon \in \rho | \exists \alpha \in \lambda : (\alpha, \varepsilon) \in [g]\} \subseteq \rho \subseteq \lambda.$$

Note, $e_{\alpha,\varepsilon} = e_\alpha a_\varepsilon$ where $a_\varepsilon \in A$, which explains the use of the notion "A-support".

Next we define a *norm* (λ-norm) on λ, respectively on $\widehat{B}$, by

$$\|\{\alpha\}\| = \alpha + 1$$

for an ordinal α in λ,

$$\|M\| = \sup_{\alpha \in M} \|\{\alpha\}\|$$

for a subset M of λ and

$$\|g\| = \| [g]_\lambda \|$$

for $g \in \widehat{B}$, i.e.

$$\|g\| = \min\{\beta \in \lambda | \, [g]_\lambda \subseteq \beta\}.$$

Note, $[g]_\lambda \subseteq \beta$ holds iff $g \in \widehat{B'_\beta}$ for $B'_\beta = \bigoplus_{\alpha<\beta} e_\alpha A$. We also define an *$A$-norm* of g by $\|g\|_A = \| [g]_A \|$. For a subset M of $\widehat{B}$ the above definitions extend naturally, e.g. $[M]_\lambda = \bigcup_{g \in M} [g]_\lambda = \{\alpha \in \lambda | \exists \varepsilon \in \rho : (\alpha, \varepsilon) \in [M]\}$.

The reader who is only interested in the case $A = R$, i.e. in realizing a ring R as endomorphism ring of an R-module, can ignore the "ρ-", respectively the "A-component", and just work with $B = \bigoplus_{\alpha<\lambda} Re_\alpha$ (cf. §2).

To formulate the Black Box we finally need to define canonical homomorphisms which shall play a crucial role in the proof of the Black Box. For this we, once and for all, fix bijections $h_\alpha : \mu \to \alpha$ for all α with $\mu \le \alpha < \lambda$ where we put $h_\mu = \mathrm{id}_\mu$ (this is possible since $\lambda = \mu^+$ and so $|\alpha| = |\mu| = \mu$ for all such αs). For technical reasons we also put $h_\alpha = h_\mu$ for $\alpha < \mu$. Note, $\mathrm{Im} h_\alpha = \mu \cup \alpha$ for all $\alpha < \lambda$.

Definition 1.1.1. *Let the bijections h_α ($\alpha < \lambda$) be as above and put $h_{\alpha,\varepsilon} = h_\alpha \times h_\varepsilon : \mu \times \mu \to \mathrm{Im} h_\alpha \times \mathrm{Im} h_\varepsilon$ for all $(\alpha, \varepsilon) \in \lambda \times \rho$.*

We define P to be a canonical summand *of B if $P = \bigoplus_{(\alpha,\varepsilon)\in I} Re_{\alpha,\varepsilon}$ for some $I \subseteq \lambda \times \rho$ with $|I| \le \kappa$ such that:*

- *if $(\alpha, \varepsilon) \in I$ then $(I \cap (\mu \times \mu))\, h_{\alpha,\varepsilon} = I \cap \mathrm{Im} h_{\alpha,\varepsilon}$;*
- *if $(\alpha, \varepsilon) \in I$ then $(\varepsilon, \varepsilon) \in I$; and*
- *if $(\alpha, \varepsilon) \in I, \alpha \in \rho$ then $(\varepsilon, \alpha) \in I$.*

Accordingly, $\varphi : P \to \widehat{B}$ is said to be a canonical homomorphism *if P is a canonical summand of B and $\mathrm{Im}\varphi \subseteq \widehat{P}$; we put $[\varphi] = [P]$, $[\varphi]_\lambda = [P]_\lambda$ and $\|\varphi\| = \|P\|$.*

Note that, by the above definition, a canonical summand P satisfies $\|P\|_A \leq \|P\|$.

Also note, that we are mainly interested in canonical homomorphisms whose norm is a limit ordinal of cofinality ω; hence we introduce the notation $\lambda^o := \{\alpha < \lambda | \mathrm{cf}(\alpha) = \omega\}$.

Let $\mathfrak{C}$ denote the set of all canonical homomorphisms. By assumption, $\mu^\kappa = \mu$ and thus $2^\kappa \leq \mu$ and $\lambda^\kappa = \lambda$ (see [J, Ch. I, (6.18)]), which implies $|\mathfrak{C}| \leq \lambda$ since $|\{I \subseteq \lambda \times \rho| \, |I| \leq \kappa\}| = \lambda^\kappa$ and, for a fixed canonical summand P, $\left|\mathrm{Hom}_R(P, \widehat{P})\right| \leq 2^\kappa$. Note, $|\mathfrak{C}| = \lambda$ then follows from the Strong Black Box Theorem.

We are now ready to formulate the main theorem of this subsection, i.e. the desired version of the Strong Black Box:

Strong Black Box Theorem 1.1.2. *Let κ, μ, λ be as before and let $E \subseteq \lambda^o$ be a stationary subset of λ.*

Then there exists a family $\mathfrak{C}^*$ of canonical homomorphisms with the following properties:

(1) *If $\varphi \in \mathfrak{C}^*$ then $\|\varphi\| \in E$.*

(2) *If φ, φ' are two different elements of $\mathfrak{C}^*$ of the same norm α then $\| \, [\varphi]_\lambda \cap [\varphi']_\lambda \, \| < \alpha$.*

(3) PREDICTION: *For any R-homomorphism $\psi : B \to \widehat{B}$ and for any subset I of $\lambda \times \rho$ with $|I| \leq \kappa$ the set*

$$\{\alpha \in E | \exists \varphi \in \mathfrak{C}^* : \|\varphi\| = \alpha, \varphi \subseteq \psi, I \subseteq [\varphi]\}$$

is stationary.

To prove the above theorem we need further definitions and other results. We begin with defining an equivalence relation on $\mathfrak{C}$ as follows:

Definition 1.1.3. *Canonical homomorphisms φ, φ' are said to be* equivalent*, or* of the same type *(notation: $\varphi \equiv \varphi'$), if*

$$[\varphi] \cap (\mu \times \mu) = [\varphi'] \cap (\mu \times \mu)$$

and there exists an order-isomorphism $f : [\varphi] \to [\varphi']$ such that

$$(x\bar{f})\varphi' = (x\varphi)\bar{f}$$

for all $x \in \mathrm{dom}\varphi$ where

$$\bar{f} : \widehat{\mathrm{dom}\varphi} \to \widehat{\mathrm{dom}\varphi'}$$

is the unique extension of the R-homomorphism defined by

$$e_{\alpha,\varepsilon}\bar{f} = e_{(\alpha,\varepsilon)f} \;\; ((\alpha, \varepsilon) \in [\varphi]) \, .$$

Note, $f : [\varphi] \to [\varphi']$ is unique since $[\varphi], [\varphi']$ are well ordered. Thus, if $\varphi \equiv \varphi'$ and $[\varphi] = [\varphi']$ then $f = \mathrm{id}$ and so $\varphi = \varphi'$.

Obviously, any type in $(\mathfrak{C}, \equiv)$ can be represented by a subset V of $\mu \times \mu$ of cardinality at most κ, an order-type of a set of cardinality κ and a homomorphism from a free R-module P of rank κ into its completion $\widehat{P}$. Therefore there are at most μ different types (equivalence classes) since $|\{V \subseteq \mu \times \mu \,|\, |V| \leq \kappa\}| = \mu^{\kappa} = \mu$, there are at most $2^{\kappa} \leq \mu$ non-isomorphic well-orderings on a set of size κ and, for a fixed P, we have $\left|\mathrm{Hom}_R(P, \widehat{P})\right| = 2^{\kappa}$.

Certain infinite sequences of canonical homomorphisms play an important role:

Definition 1.1.4. *Let $\varphi_0 \subset \varphi_1 \subset \ldots \subset \varphi_n \subset \ldots$ $(n < \omega)$ be an increasing sequence of canonical homomorphisms.*

Then $(\varphi_n)_{n<\omega}$ is said to be admissible *if*

$$[\varphi_0] \cap (\mu \times \mu) = [\varphi_n] \cap (\mu \times \mu) \text{ and } \|\varphi_n\| < \|\varphi_{n+1}\|$$

for all $n < \omega$.

Also, we say that $(\varphi_n)_{n<\omega}$ is admissible for a sequence $(\beta_n)_{n<\omega}$ of ordinals *in λ (or $(\varphi_n, \beta_n)_{n<\omega}$* is admissible*), if $(\varphi_n)_{n<\omega}$ is admissible satisfying*

$$\|\varphi_n\| \leq \beta_n < \|\varphi_{n+1}\| \text{ and } [\varphi_n] = [\varphi_{n+1}] \cap (\beta_n \times \beta_n)$$

for all $n < \omega$.

Moreover, two admissible sequences $(\varphi_n)_{n<\omega}$, $(\varphi'_n)_{n<\omega}$ are said to be equivalent, *or* of the same type, *if $\varphi_n \equiv \varphi'_n$ for all $n < \omega$.*

Note, if $(\varphi_n)_{n<\omega}$ is admissible then $\varphi = \bigcup_{n<\omega} \varphi_n$ is an element of $\mathfrak{C}$ with $\|\varphi\| = \sup_{n<\omega} \|\varphi_n\| \in \lambda^o$.

Let $\mathfrak{T}$ denote the set of all possible types of admissible sequences of canonical homomorphisms. It follows immediately from the above definition that any type in $\mathfrak{T}$ can be identified with a sequence $(\tau_n)_{n<\omega}$ for some types τ_n in $(\mathfrak{C}, \equiv)$ where the corresponding subsets of $\mu \times \mu$ are all the same. Hence we clearly deduce $|\mathfrak{T}| \leq \mu^{\aleph_0} = \mu$.

If $(\varphi_n)_{n<\omega}$, respectively $(\varphi_n, \beta_n)_{n<\omega}$, is admissible of type τ, then we also use the notion *τ-admissible*. Moreover, if $\tau = (\tau_n)_{n<\omega} \in \mathfrak{T}$ and $(\varphi_n)_{n<k}$ $(k < \omega)$ is a finite increasing sequence of canonical homomorphisms satisfying $\varphi_n \in \tau_n$ and $\|\varphi_n\| < \|\varphi_{n+1}\|$ for all $n < k$, then we shall also speak of $(\varphi_n)_{n<k}$ to be of type τ, keeping in mind that such a finite sequence could belong to different types in $\mathfrak{T}$.

We are now ready to show the following result which will play a crucial role in proving the Black Box Theorem 1.1.2. Note, that the kind of formula we use to formulate this result goes under the generic name "Svenonius sentences" or "Svenonius game" (cf. [H, p.112]). It should be noted for the reader who is familiar with game theory that the proof below uses the Gale-Stewart-Theorem, namely that in a closed (or open) game some player has a winning strategy.

Proposition 1.1.5. *Let $\psi : B \to \widehat{B}$ be an R-homomorphism, $I \subseteq \lambda \times \rho$ a set of cardinality at most κ and $\mathfrak{K} = \mathfrak{K}_{\psi,I} = \{\varphi \in \mathfrak{C} | \varphi \subseteq \psi, I \subseteq [\varphi]\}$.*

Then there exists a type $\tau \in \mathfrak{T}$ such that

$$\exists \varphi_0 \in \mathfrak{K}\ \forall \beta_0 \geq \|\varphi_0\| \ldots \exists \varphi_n \in \mathfrak{K}\ \forall \beta_n \geq \|\varphi_n\| \ldots$$

with $(\varphi_n, \beta_n)_{n<\omega}$ being τ-admissible.

Proof. Suppose, for contradiction, that the conclusion fails. Then, since the above formula is of "finite character"we have, for any type $\tau \in \mathfrak{T}$,

$$\forall \varphi_0 \in \mathfrak{K}\ \exists \beta_0(\tau, \varphi_0) \geq \|\varphi_0\| \ldots \forall \varphi_n \in \mathfrak{K}\ \exists \beta_n(\tau, \varphi_0, \ldots, \varphi_n) \geq \|\varphi_n\| \ldots$$

with $(\varphi_n, \beta_n)_{n<\omega}$ not being τ-admissible.

In the following we fix ordinals $\beta_n(\tau, \varphi_0, \ldots, \varphi_n)$ as above ($\tau \in \mathfrak{T}, \varphi_i \in \mathfrak{K}, i \leq n < \omega$). Moreover let

$$T_\alpha = \alpha \times (\alpha \cap \rho) \text{ and } B_\alpha = \bigoplus_{(\beta,\varepsilon)\in T_\alpha} Re_{\beta,\varepsilon}\ (\alpha < \lambda).$$

We define C to be the set of all ordinals $\alpha < \lambda$ such that $B_\alpha \psi \subseteq \widehat{B_\alpha}$ and $\beta_n(\tau, \varphi_0, \ldots, \varphi_n) \leq \alpha$ for each type $\tau \in \mathfrak{T}$ and for any finite sequence $(\varphi_0, \ldots, \varphi_n)$ with $\varphi_i \in \mathfrak{K}$ and $\|\varphi_i\| \leq \alpha$ (that is, iff $[\varphi_i] \subseteq \alpha \times \alpha$) for all $i \leq n$. Then C is unbounded since: Given an arbitrary $\alpha_0 < \lambda$ we inductively define ordinals $\alpha_k < \lambda$ ($k < \kappa^+ \leq \mu$) by $\alpha_k = \sup\{\alpha_l | l < k\}$ for k a limit ordinal and

$$\alpha_{k+1} = \alpha_k + 1 \vee \|B_{\alpha_k}\psi\| \vee \sup \mathfrak{B}_k$$

where

$$\mathfrak{B}_k = \{\beta_n(\tau, \varphi_0, \ldots, \varphi_n) | \tau \in \mathfrak{T}, \varphi_i \in \mathfrak{K}, \|\varphi_i\| \leq \alpha_k\}$$

is a set of cardinality at most μ since $|T| \leq \mu, |\{\varphi \in \mathfrak{K} | \|\varphi\| \leq \alpha_k\}| \leq |\alpha_k|^\kappa \leq \mu^\kappa = \mu$ provided that $\alpha_k < \lambda$. Then, using that $|[\varphi]| \leq \kappa$ for all $\varphi \in \mathfrak{K}$, it is easy to see that $\alpha = \sup\{\alpha_k | k < \kappa^+\}$ is an element of C.

Now we choose an increasing sequence $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ ($n < \omega$) in C with $\alpha_0 \geq \mu, \|I\|, \|I\|_A$ and put $\alpha = \sup_{n<\omega} \alpha_n$. Note, since $\alpha_n \in C$ for all $n < \omega$ we also have that $B_\alpha \psi \subseteq \widehat{B_\alpha}$. Moreover, let $\{\varepsilon_n | n < \omega\}$ be an arbitrary but fixed set of elements of $\alpha \cap \rho$. Using these α_ns and

ε_ns we inductively define subsets I_n of $\alpha \times (\alpha \cap \rho) = T_\alpha = [B_\alpha]$ $(n < \omega)$ of cardinality at most κ by:

$$I_0 = I \cup \{(\alpha_n, \varepsilon_n) | n < \omega\}$$

and

$$I_{n+1} = I_n \cup \overline{I_n} \cup \overline{I_n}^{\psi} \cup \overline{I_n}^{h}$$

where

$$\overline{I_n} = \{(\varepsilon, \varepsilon) | \exists \beta \in \lambda : (\beta, \varepsilon) \in I_n\} \cup \{(\varepsilon, \beta) | \exists \beta \in \rho : (\beta, \varepsilon) \in I_n\},$$

$$\overline{I_n}^{\psi} = \bigcup_{(\beta,\varepsilon) \in I_n} [e_{\beta,\varepsilon}\psi],$$

and

$$\overline{I_n}^{h} = \bigcup_{(\beta,\varepsilon) \in I_n} \Big((I_n \cap (\mu \times \mu))\, h_{\beta,\varepsilon} \cup (I_n \cap \mathrm{Im} h_{\beta,\varepsilon})\, h_{\beta,\varepsilon}^{-1} \Big).$$

These I_ns really satisfy the required conditions since: For $n = 0$ we have $|I_0| \leq \kappa$ and $I_0 \subseteq T_\alpha$ by $\|I\|, \|I\|_A \leq \alpha_0 < \alpha$. Next suppose $|I_n| \leq \kappa$ and $I_n \subseteq T_\alpha$. Then, clearly, $|I_{n+1}| \leq \kappa$; also $\overline{I_n} \subseteq T_\alpha$ is obvious, $\overline{I_n}^{\psi} \subseteq T_\alpha$ holds since $B_\alpha \psi \subseteq \widehat{B_\alpha}$, $\overline{I_n}^{h} \subseteq T_\alpha$ follows from the definition of the $h_{\beta,\varepsilon}$s, and so $I_{n+1} \subseteq T_\alpha$ as required.

Now we put $I^* = \bigcup_{n<\omega} I_n$ and $P = \bigoplus_{(\beta,\varepsilon)\in I^*} R e_{\beta,\varepsilon}$. Then $|I^*| \leq \kappa$, $I^* \subseteq T_\alpha = \alpha \times (\alpha \cap \rho)$ and $I \subseteq I^* = [P]$. Moreover, $(I^* \cap (\mu \times \mu))\, h_{\beta,\varepsilon} = I^* \cap \mathrm{Im} h_{\beta,\varepsilon}$ for all $(\beta, \varepsilon) \in I^*$ by the definition of the $\overline{I_n}^{h}$s and I^* also satisfies: if $(\beta, \varepsilon) \in I^*$ then $(\varepsilon, \varepsilon) \in I^*$ and if $(\beta, \varepsilon) \in I^*, \beta \in \rho$ then $(\varepsilon, \beta) \in I^*$ (see definition of the $\overline{I_n}$s). Therefore P is a canonical summand of B (see Definition 1.1.1) with $P\psi \subseteq \widehat{P}$ where the latter follows from the definition of the $\overline{I_n}^{\psi}$s. Thus $\varphi := \psi \restriction P$ is a canonical homomorphism with $I \subseteq I^* = [\varphi]$, i.e. $\varphi \in \mathfrak{K}$.

Finally, we put $\varphi_n = \varphi \restriction (P \cap B_{\alpha_n})$, that is $[\varphi_n] = [\varphi] \cap (\alpha_n \times \alpha_n)$. Using the definitions of the B_{α_n}s and of the set $C \subseteq C_\psi$ it is easy to check that $\varphi_n \in \mathfrak{K}$ $(n < \omega)$ and that then $(\varphi_n)_{n<\omega}$ is an admissible sequence with $\|\varphi_n\| \leq \alpha_n < \|\varphi_{n+1}\|$. Let $\tau \in \mathfrak{T}$ be the type of $(\varphi_n)_{n<\omega}$. By the definition of C we also have that $\beta_n(\tau, \varphi_0, \ldots, \varphi_n) \leq \alpha_n$ since $\|\varphi_n\| \leq \alpha_n$ for any $n < \omega$. Therefore, $\|\varphi_n\| \leq \beta_n \leq \alpha_n$ and hence $[\varphi_n] = [\varphi] \cap (\alpha_n \times \alpha_n) = [\varphi] \cap (\beta_n \times \beta_n) = [\varphi_{n+1}] \cap (\beta_n \times \beta_n)$, i.e. $(\varphi_n, \beta_n)_{n<\omega}$ is τ-admissible, contradicting the assumption that it is not for $\beta_n = \beta_n(\tau, \varphi_0, \ldots, \varphi_n)$. Hence the original conclusion holds and so the proof is finished. $\square$

In order to prove the Strong Black Box Theorem we also need the following known lemma. We include the proof for the convenience of the reader; it can also be found in [EM]. First recall that, for an ordinal α, a mapping $\eta_\alpha : \omega \to \alpha$ is a *ladder on* α if it is strictly increasing and $\sup \mathrm{Im}\eta_\alpha = \alpha$; an indexed family of such ladders on different αs is called a *ladder system*.

Lemma 1.1.6. *Let $E \subseteq \lambda^o$ be a stationary subset of $\lambda = \mu^+$ for some μ such that $\mu^{\aleph_0} = \mu$.*

Then there is a ladder system $\{\eta_\alpha | \alpha \in E\}$ such that, for all cubs C, the set $\{\alpha \in E | \mathrm{Im}\eta_\alpha \subseteq C\}$ is stationary.

Proof. For any $\alpha \in E$ let $\{\eta_\alpha^i | i < \mu\}$ be an enumeration of all ladders on α (if necessary with repetition); this is possible since $|{}^\omega\alpha| = |\alpha|^{\aleph_0} \leq \mu^{\aleph_0} = \mu$ for all $\alpha < \lambda = \mu^+$. Moreover, for each $i < \mu$, let $\overline{\eta}_i$ be the ladder system given by $\overline{\eta}_i := \{\eta_\alpha^i | \alpha \in E\}$.

We claim that there is an $i < \mu$ such that $\overline{\eta}_i$ satisfies the conclusion of the theorem. Suppose not. Then, for any $i < \mu$, there is a cub $C_i \subseteq \lambda$ such that the set $T_i := \{\alpha \in E | \mathrm{Im}\eta_\alpha^i \subseteq C_i\}$ is not stationary, i.e. there is a cub D_i with $T_i \cap D_i = \emptyset$. Replacing C_i by the cub $C_i \cap D_i$ we may assume that $T_i = \emptyset$ for any $i < \mu$, i.e. $\mathrm{Im}\eta_\alpha^i \not\subseteq C_i$ for all $\alpha \in E$ ($i < \mu$). We put $C = \bigcap_{i<\mu} C_i$. Then C is also a cub in λ (cf. [EM, II, Proposition 4.3]). We choose an ordinal $\alpha \in C \cap E$ which is a limit point of C, i.e. $\alpha = \sup_{n<\omega} \alpha_n$ for some $\alpha_n \in C \cap \alpha$ with $\alpha_n < \alpha_{n+1}$ (α, respectively the α_ns, exist since the set of all limit points of a cub is also a cub; see [EM, p.35]). Therefore the map $\eta_\alpha : \omega \to \alpha$ defined by $\eta_\alpha(n) = \alpha_n$ is a ladder on α with $\mathrm{Im}\eta_\alpha \subseteq C$. By the above enumeration $\eta_\alpha = \eta_\alpha^i$ for some $i < \mu$ contradicting $\mathrm{Im}\eta_\alpha^i \not\subseteq C_i \supseteq C$. □

Finally, we prove the main theorem of this subsection.

Proof of the Strong Black Box Theorem 1.1.2. First we decompose the given stationary set E into $|\mathfrak{T}| \leq \mu$ pairwise disjoint stationary subsets, say $E = \bigcup_{\tau \in \mathfrak{T}} E_\tau$.

For each $\tau \in \mathfrak{T}$ we choose a ladder system $\{\eta_\alpha | \alpha \in E_\tau\}$ such that the set $\{\alpha \in E_\tau | \mathrm{Im}\eta_\alpha \subseteq C\}$ is stationary for any cub C (cf. Lemma 1.1.6).

For any $\alpha \in E_\tau$, we define $\mathfrak{C}_\alpha \subseteq \mathfrak{C}$ to be the set of all canonical homomorphisms φ such that $\|\varphi\| = \alpha$ and $\varphi = \bigcup_{n<\omega} \varphi_n$ for some τ-admissible sequence $(\varphi_n)_{n<\omega}$ with $[\varphi_n] = [\varphi] \cap (\eta_\alpha(n) \times \eta_\alpha(n))$ $(n < \omega)$. Note, for $\varphi, \varphi' \in \mathfrak{C}_\alpha$ with $\mathrm{dom}\varphi = \mathrm{dom}\varphi'$ (iff $[\varphi] = [\varphi']$), we clearly deduce $\varphi_n = \varphi'_n$ for all $n < \omega$ and so $\varphi = \varphi'$ (cf. Definition 1.1.3).

Now we define $\mathfrak{C}^*$ to be the union of all these $\mathfrak{C}_\alpha$s, i.e. $\mathfrak{C}^* = \bigcup_{\alpha \in E} \mathfrak{C}_\alpha$.

First note that condition (1) obviously holds.

Next we show that condition (2) is satisfied. To do so let $\varphi, \varphi' \in \mathfrak{C}^*$ with $\|\varphi\| = \|\varphi'\| = \alpha$. Then $\varphi, \varphi' \in \mathfrak{C}_\alpha$ where $\alpha \in E_\tau$ for some $\tau \in \mathfrak{T}$ and so $\varphi = \bigcup_{n<\omega} \varphi_n,\ \varphi' = \bigcup_{n<\omega} \varphi'_n$ for some τ-admissible sequences $(\varphi_n)_{n<\omega},\ (\varphi'_n)_{n<\omega}$.

Suppose that $\|\,[\varphi]_\lambda \cap [\varphi']_\lambda\,\| = \alpha$. Then there are $\alpha_n \in [\varphi]_\lambda \cap [\varphi']_\lambda$ with $\sup_{n<\omega} \alpha_n = \alpha$; w.l.o.g. we may assume $\alpha_n \geq \mu$. Let $\varepsilon_n, \varepsilon'_n \in \rho$ such that $(\alpha_n, \varepsilon_n) \in [\varphi]$ and $(\alpha_n, \varepsilon'_n) \in [\varphi']$.

We consider two cases.

Firstly assume that $\rho < \lambda$, i.e. $\rho \leq \mu$. Then $h_{\alpha_n,\varepsilon_n} = h_{\alpha_n} \times h_\mu = h_{\alpha_n,\varepsilon'_n}$ $(n < \omega)$. Since $(\varphi_n)_{n<\omega},\ (\varphi'_n)_{n<\omega}$ are of the same type τ we know that

$$[\varphi] \cap (\mu \times \mu) = [\varphi_0] \cap (\mu \times \mu) = \left[\varphi'_0\right] \cap (\mu \times \mu) = \left[\varphi'\right] \cap (\mu \times \mu).$$

Hence

$$\begin{aligned} [\varphi] \cap (\alpha_n \times \mu) = [\varphi] \cap \mathrm{Im} h_{\alpha_n,\varepsilon_n} = \\ = ([\varphi] \cap (\mu \times \mu))\, h_{\alpha_n,\varepsilon_n} = \left(\left[\varphi'\right] \cap (\mu \times \mu)\right) h_{\alpha_n,\varepsilon'_n} = \\ = \left[\varphi'\right] \cap \mathrm{Im} h_{\alpha_n,\varepsilon'_n} = \left[\varphi'\right] \cap (\alpha_n \times \mu) \end{aligned}$$

for all $n < \omega$ because $\mathrm{dom}\varphi, \mathrm{dom}\varphi'$ are canonical summands of B. Therefore,

$$[\varphi] = \bigcup_{n<\omega} ([\varphi] \cap (\alpha_n \times \mu)) = \bigcup_{n<\omega} \left(\left[\varphi'\right] \cap (\alpha_n \times \mu)\right) = \left[\varphi'\right].$$

Secondly assume $\rho = \lambda$. Then $(\alpha_n, \varepsilon_n) \in [\varphi]$ implies $(\varepsilon_n, \alpha_n) \in [\varphi]$ and so $(\alpha_n, \alpha_n) \in [\varphi]$ (see Definition 1.1.1). Similarly, we obtain $(\alpha_n, \alpha_n) \in [\varphi']$ and so, as in the first case,

$$\begin{aligned} [\varphi] = \bigcup_{n<\omega} ([\varphi] \cap (\alpha_n \times \alpha_n)) = \bigcup_{n<\omega} ([\varphi] \cap (\mu \times \mu))\, h_{\alpha_n,\alpha_n} = \\ = \bigcup_{n<\omega} \left(\left[\varphi'\right] \cap (\mu \times \mu)\right) h_{\alpha_n,\alpha_n} = \bigcup_{n<\omega} \left(\left[\varphi'\right] \cap (\alpha_n \times \alpha_n)\right) = \left[\varphi'\right]. \end{aligned}$$

In either case we deduce $\varphi = \varphi'$ and thus (2) is proven.

It remains to show (3). So let $\psi : B \to \widehat{B}$ be an R-homomorphism and let $I \subseteq \lambda \times \rho$ with $|I| \leq \kappa$. By Proposition 1.1.5 there is a type $\tau \in \mathfrak{T}$ such that:

$$\exists \varphi_0 \in \mathfrak{K}\ \forall \beta_0 \geq \|\varphi_0\| \ldots \exists \varphi_n \in \mathfrak{K}\ \forall \beta_n \geq \|\varphi_n\| \ldots$$

with $(\varphi_n, \beta_n)_{n<\omega}$ is τ-admissible where $\mathfrak{K} = \mathfrak{K}_{\psi,I} = \{\varphi \in \mathfrak{C} | \varphi \subseteq \psi, I \subseteq [\varphi]\}$.

We define a subset C of λ as follows: An ordinal α belongs to C if and only if $\alpha \geq \mu$, $\alpha \geq \|\varphi_0\|$, $B_\alpha \psi \subseteq \widehat{B_\alpha}$ (recall: $B_\alpha = \bigoplus_{(\beta,\varepsilon)\in T_\alpha} Re_{\beta,\varepsilon}$, $T_\alpha = \alpha \times (\alpha \cap \rho)$), and, if $(\varphi_0, \beta_0, \ldots, \varphi_n, \beta_n)$ is a finite part of one of the above τ-admissible sequences with $\beta_n < \alpha$, then there is also $\varphi_{n+1} \in \mathfrak{K}$ with $[\varphi_{n+1}] \subseteq \alpha \times \alpha$ and $(\varphi_0, \beta_0, \ldots, \varphi_n, \beta_n, \varphi_{n+1})$ is τ-admissible. Clearly, C is a cub and therefore the set $E'_\tau = \{\alpha \in E_\tau | \mathrm{Im}\eta_\alpha \subseteq C\}$ is stationary by Lemma 1.1.6.

In the following let $\alpha \in E'_\tau$ be fixed, i.e. $\eta_\alpha(n) \in C$ for all $n < \omega$. By the definition of C we have $\|\varphi_0\| \leq \eta_\alpha(0) < \eta_\alpha(1)$ and so there is $\varphi_1 \in \mathfrak{K}$ with $\|\varphi_1\| \leq \eta_\alpha(1)$ such that $(\varphi_0, \eta_\alpha(0), \varphi_1)$ is τ-admissible. We proceed like this along $n < \omega$, i.e. whenever we have the τ-admissible sequence $(\varphi_0, \eta_\alpha(0), \ldots, \varphi_n, \eta_\alpha(n))$ with $\|\varphi_n\| \leq \eta_\alpha(n) < \eta_\alpha(n+1)$ we can find $\varphi_{n+1} \in \mathfrak{K}$ with $\|\varphi_{n+1}\| \leq \eta_\alpha(n+1)$ such that $(\varphi_0, \eta_\alpha(0), \ldots, \varphi_n, \eta_\alpha(n), \varphi_{n+1})$ is τ-admissible. Therefore we obtain an infinite τ-admissible sequence $(\varphi_n, \eta_\alpha(n))_{n<\omega}$, i.e. $\|\varphi_n\| \leq \eta_\alpha(n) < \|\varphi_{n+1}\|$ and $[\varphi_{n+1}] \cap (\eta_\alpha(n) \times \eta_\alpha(n)) = [\varphi_n]$ (cf. Definition 1.1.4). We put $\varphi = \bigcup_{n<\omega} \varphi_n$; then

$$\|\varphi\| = \sup_{n<\omega} \|\varphi_n\| = \sup_{n<\omega} \eta_\alpha(n) = \alpha$$

and

$$[\varphi] \cap (\eta_\alpha(n) \times \eta_\alpha(n)) = \bigcup_{k\geq n} ([\varphi_k] \cap (\eta_\alpha(n) \times \eta_\alpha(n))) = [\varphi_n]\,.$$

Hence $\varphi = \bigcup_{n<\omega} \varphi_n \in \mathfrak{C}_\alpha \subseteq \mathfrak{C}^*$. Since $\alpha \in E'_\tau$ was arbitrary and E'_τ is stationary the proof is finished. □

We finish this subsection with an "enumerated" version of the Strong Black Box Theorem 1.1.2, which can then directly be applied in Subsection 1.2.

Corollary 1.1.7. *Let the assumptions be the same as in the Strong Black Box Theorem 1.1.2.*

Then there exists a family $(\varphi_\beta)_{\beta<\lambda}$ of canonical homomorphism such that:

(i) $\|\varphi_\beta\| \in E$ *for all* $\beta < \lambda$*;*

(ii) $\|\varphi_\gamma\| \leq \|\varphi_\beta\|$ *for all* $\gamma \leq \beta < \lambda$*;*

(iii) $\|\,[\varphi_\gamma]_\lambda \cap [\varphi_\beta]_\lambda\,\| < \|\varphi_\beta\|$ *for all* $\gamma < \beta < \lambda$*;*

(iv) PREDICTION: *For any homomorphism $\psi : B \to \widehat{B}$ and for any subset I of $\lambda \times \rho$ with $|I| \leq \kappa$ the set*

$$\{\alpha \in E | \exists \beta < \lambda : \|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi, I \subseteq [\varphi_\beta]\}$$

is stationary.

Proof. By the Strong Black Box Theorem 1.1.2 there is a class $\mathfrak{C}^*$ of canonical homomorphisms satisfying the conditions (i) and (iv) which are obviously independent of the enumeration (cf. conditions (1) and (3) in Theorem 1.1.2). Moreover, we put an arbitrary well-ordering on the sets $\mathfrak{C}_\alpha = \{\varphi \in \mathfrak{C}^* | \|\varphi\| = \alpha\}$ $(\alpha \in E)$ and define $\varphi \in \mathfrak{C}_\alpha$ to be less than $\varphi' \in \mathfrak{C}_{\alpha'}$ if $\alpha < \alpha'$. This defines a well-ordering on $\mathfrak{C}^*$ and hence there is a corresponding ordinal λ^* such that the condition (ii) is satisfied. In fact, $\lambda^* = \lambda$ since $|\mathfrak{C}_\alpha| \leq \mu$ for all $\alpha < \lambda$ and thus all initial segments of the above defined well-ordering are of cardinality less than λ.

Condition (iii) also easily follows since $\| [\varphi]_\lambda \cap [\varphi']_\lambda \| < \|\varphi'\|$ is obvious for $\|\varphi\| < \|\varphi'\|$ and it coincides with condition (2) in Theorem 1.1.2 for $\|\varphi\| = \|\varphi'\|$. □

1.2. The Realization Theorem

In this subsection we shall apply the Strong Black Box as given in Corollary 1.1.7 to prove the following theorem.

Theorem 1.2.1. *Let $R, \mathbb{S}, A$ and κ, μ, λ be as before.*

Then there exists an $\mathbb{S}$-cotorsion-free R-module G of cardinality λ such that

$$\mathrm{End}_R G = A.$$

Before we can construct the desired module we need the following lemma, which basically tells us how to obtain the module "step by step".

Step Lemma 1.2.2. *Let $P = \bigoplus_{(\alpha,\varepsilon)\in I^*} Re_{\alpha,\varepsilon}$ for some $I^* \subseteq \lambda \times \rho$ and let M be an A-module as well as an $\mathbb{S}$-cotorsion-free R-module with $P \subseteq_* M \subseteq_* \widehat{B}$. Also suppose that there is a set $I = \{(\alpha_n, \varepsilon_n) | n < \omega\} \subseteq [P] = I^*$ such that $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ and $I_\lambda \cap [g]_\lambda$ is finite for all $g \in M$ $(I_\lambda = [I]_\lambda = \{\alpha_n | n < \omega\})$.*

Moreover, let $\varphi : P \to M$ be such an R-homomorphism which is not multiplication by an element of A.

Then there exists an element y of $\widehat{P}$ such that $y\varphi \notin M' := \langle M, yA\rangle_*$ where '$*$' denotes the ($\mathbb{S}$-) purification in $\widehat{B}$ and φ is identified with its unique extension $\varphi : \widehat{P} \to \widehat{M}$.

Moreover, M' is again an A-module as well as a $\mathbb{S}$-cotorsion-free R-module with $M \subseteq_* M' \subseteq_* \widehat{B}$.

(*The element y can be chosen to be either $y = x$ or $y = x + \pi b$ for suitable $\pi \in \widehat{R}, b \in P$ and for $x = \sum_{n<\omega} q_n e_{\alpha_n,\varepsilon_n}$.*)

Proof. Let the assumptions be as above. Either $x = \sum_{n<\omega} q_n e_{\alpha_n,\varepsilon_n}$ satisfies $x\varphi \notin \langle M, xA\rangle_*$ or not. In the latter case there are $k < \omega$ and $a \in A$ such that

$$q_k x\varphi - xa \in M. \tag{+}$$

Since $M \subseteq_* \widehat{B}$ is $\mathbb{S}$-torsion-free and $\varphi \notin A$ we also have that $q_k\varphi \notin A$, and thus there is an element b of P such that $q_k b\varphi = b(q_k\varphi) \neq ba$. Hence, by the cotorsion-freeness of M, there is $\pi \in \widehat{R}$ such that

$$\pi(q_k b\varphi - ba) \notin M. \tag{++}$$

Let $z = x + \pi b$ and suppose $z\varphi \in \langle M, zA\rangle_*$. Then

$$q_l z\varphi - za' \in M$$

for some $l \geq k, a' \in A$. Therefore, using (+), we obtain that

$$\begin{aligned}(q_l z\varphi - za') - \frac{q_l}{q_k}(q_k x\varphi - xa) = \\ = q_l x\varphi + q_l\pi b\varphi - xa' - \pi ba' - q_l x\varphi + \frac{q_l}{q_k}xa = \\ = x(\frac{q_l}{q_k}a - a') + \pi(q_l b\varphi - ba')\end{aligned}$$

is an element of M. Now, $[x]_\lambda = I_\lambda$ while $[q_l(b\varphi - ba')]_\lambda \cap I_\lambda$ and $\left[x(\frac{q_l}{q_k}a - a') + \pi(q_l b\varphi - ba')\right]_\lambda \cap I_\lambda$ are both finite. Hence $\frac{q_l}{q_k}a - a' = 0$ and thus it follows from the above that $\pi(q_l b\varphi - \frac{q_l}{q_k}ba) = \frac{q_l}{q_k}\pi(q_k b\varphi - ba) \in M \subseteq_* \widehat{B}$. Since $\frac{q_l}{q_k} \in \mathbb{S}$ this implies $\pi(q_k b\varphi - ba) \in M$ contradicting (++).

Therefore either $y = x$ or $y = z$ satisfies $y\varphi \notin \langle M, yA\rangle_* =: M'$.

Clearly, M' is also an A-module.

It remains to show that M' is $\mathbb{S}$-cotorsion-free. Since $M' \subseteq_* \widehat{B}$ it is torsion-free and reduced. To prove $\mathrm{Hom}(\widehat{R}, M') = 0$ let $\varphi : \widehat{R} \to M'$ be a homomorphism and let $k < \omega$ such that $q_k(1\varphi) \in M + yA$, say

$$q_k(1\varphi) = m + ya (m \in M, a \in A).$$

Moreover, for any $r \in \widehat{R}$, let $k \leq k_r < \omega, m_r \in M, a_r \in A$ such that

$$q_{k_r}(r\varphi) = m_r + ya_r.$$

By the continuity of φ we have $r\varphi = r(1\varphi)$ and thus we deduce

$$0 = q_{k_r}(r\varphi) - q_{k_r}r(1\varphi) = m_r + ya_r - \frac{q_{k_r}}{q_k}r(m + ya),$$

respectively,

$$m_r - \frac{q_{k_r}}{q_k} rm = y(\frac{q_{k_r}}{q_k} ra - a_r).$$

Therefore, since $\left[m_r - \frac{q_{k_r}}{q_k} rm\right]_\lambda \cap I_\lambda$ is finite and $[y]_\lambda \cap I_\lambda$ is infinite for either $y = x$ or $y = x + \pi b$, we conclude that both sides of the above equation equal zero, i.e. $a_r = \frac{q_{k_r}}{q_k} ra$ and $m_r = \frac{q_{k_r}}{q_k} rm$ for each $r \in \widehat{R}$. Hence, by the purity of A in $\widehat{A}$ and of M in $\widehat{B}$ and since $\frac{q_{k_r}}{q_k} \in \mathbb{S}$, we have that $ra \in A$ and $rm \in M$ for all $r \in \widehat{R}$, which implies $a = 0$ and $m = 0$ by the cotorsion-freeness of A and M. Thus $1\varphi = 0$ and so φ is the zero-homomorphism as required. $\square$

We are now ready to construct the desired module.

Construction 1.2.3. Let $(\varphi_\beta)_{\beta<\lambda}$ be a family of canonical homomorphisms as given by Corollary 1.1.7. For any $\beta < \lambda$ let $P_\beta = \mathrm{dom}\varphi_\beta$, i.e. $\varphi_\beta : P_\beta \to \widehat{P_\beta}$.

We inductively define elements $y_\gamma \in \widehat{P_\gamma}$ and pure R-submodules G^β of $\widehat{B}$ such that, for all $\gamma < \beta < \lambda$,

(1) $\|y_\gamma\| = \|P_\gamma\| (= \|\varphi_\gamma\|)$,

(2) $G^\beta = \langle B', y_\gamma A(\gamma < \beta)\rangle_*$, and

(3) G^β is $\mathbb{S}$-cotorsion-free.

Recall that $B' = \bigoplus_{\alpha<\lambda} e_\alpha A \supseteq B$ (see beginning of Subsection 1.1). Also note that the G^βs are then clearly A-modules.

Let $G^0 = B' \subseteq_* \widehat{B} = \widehat{B'}$; obviously B' is $\mathbb{S}$-cotorsion-free since A is, by assumption, and it also satisfies the conditions (1) and (2) since there are no relevant y_γs.

Next let β be a limit ordinal and suppose that G^γ satisfies all the required conditions for any $\gamma < \beta$. We put $G^\beta = \bigcup_{\gamma<\beta} G^\gamma$. Then G^β certainly satisfies (1) and (2). Moreover, G^β is clearly torsion-free and reduced and so it remains to show $\mathrm{Hom}(\widehat{R}, G^\beta) = 0$. So, let $\varphi : \widehat{R} \to G^\beta$ be a homomorphism and let $\delta < \beta$ such that $1\varphi \in G^\delta = \langle B', y_\gamma A(\gamma < \delta)\rangle_*$. Then, for each $r \in \widehat{R}$, we have $[r\varphi] \subseteq [1\varphi]$ and hence $\| [r\varphi]_\lambda \cap [y_\gamma]_\lambda \| < \|y_\gamma\|$ for all $\gamma \geq \delta$ (see Corollary 1.1.7(iii)). Therefore $r\varphi \in G^\delta$ for all $r \in \widehat{R}$, respectively, $\mathrm{Im}\varphi \subseteq G^\delta$ and thus $\varphi = 0$, i.e. G^β is $\mathbb{S}$-cotorsion-free.

It remains to tackle the successor case. Assume G^β is given satisfying all the conditions.

We consider φ_β. Since $\|\varphi_\beta\| \in E \subseteq \lambda^o$ there are $(\alpha_n, \varepsilon_n) \in [\varphi_\beta]$ $(n < \omega)$ depending on β, such that $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ and $\|\varphi_\beta\| =$

$\sup_{n<\omega} \alpha_n$. We put $I = \{(\alpha_n, \varepsilon_n) | n < \omega\}$. Then $\|I_\lambda \cap [g]_\lambda\| < \|\varphi_\beta\|$, respectively $I_\lambda \cap [g]_\lambda$ is finite, for all $g \in G^\beta$ by (1), (2) and condition (iii) in Corollary 1.1.7.

We differentiate two cases.

If $\varphi_\beta : P_\beta \to \widehat{P_\beta}$ satisfies $\mathrm{Im}\varphi_\beta \subseteq G^\beta$ and $\varphi_\beta \notin A$, then we apply the Step Lemma 1.2.2 to I as above, $P = P_\beta \subseteq_* B \subseteq_* G^\beta$ and $M = G^\beta \subseteq_* \widehat{B}$. We deduce the existence of an element $y = y_\beta \in \widehat{P_\beta}$ and of an A-module

$$G^{\beta+1} = \langle G^\beta, y_\beta A \rangle_* = \left\langle B', y_\gamma A(\gamma \leq \beta) \right\rangle_*$$

which is a $\mathbb{S}$-cotorsion-free pure submodule of $\widehat{B}$ such that

$$y_\beta \varphi_\beta \notin G^{\beta+1},$$

where $y_\beta = \sum_{n<\omega} q_n e_{\alpha_n, \varepsilon_n}$ or $y_\beta = \sum_{n<\omega} q_n e_{\alpha_n, \varepsilon_n} + \pi b$ $(\pi \in \widehat{R}, b \in B)$. Hence y_β satisfies (1) and $G^{\beta+1}$ satisfies (2) and (3).

If $\mathrm{Im}\varphi_\beta \nsubseteq G^\beta$ or $\varphi_\beta \in A$, then we put $y_\beta = \sum_{n<\omega} q_n e_{\alpha_n, \varepsilon_n}$ and $G^{\beta+1} = \langle G^\beta, y_\beta A \rangle_*$. Then, also in this case, y_β and $G^{\beta+1}$ satisfy all the required conditions (cf. Step Lemma 1.2.2).

Finally, we define G by

$$G = \bigcup_{\beta<\lambda} G^\beta = \langle B', y_\beta A(\beta < \lambda) \rangle_*.$$

$\square$

It is an immediate consequence from the construction that G is an A-module of cardinality λ which is also an $\mathbb{S}$-cotorsion-free pure submodule of $\widehat{B}$. Next we describe the elements of G.

Lemma 1.2.4. *Let G be as in Construction 1.2.3.*

(a) *The set $\{e_\alpha | \alpha < \lambda\} \cup \{y_\beta | \beta < \lambda\}$ is linearly independent over A, i.e. $\langle B', y_\beta A(\beta < \lambda) \rangle = B' \oplus \bigoplus_{\beta<\lambda} y_\beta A$ is a free A-module.*

(b) *If $g \in G \setminus B'$ then there are a finite non-empty subset N of λ and $k < \omega$ such that $q_k g \in B' \oplus \bigoplus_{\beta \in N} y_\beta A$ and $[g]_\lambda \cap [y_\beta]_\lambda$ is infinite iff $\beta \in N$. In particular, if $\|g\|$ is a limit ordinal then $\|g\| = \|y_{\max N}\|$.*

Proof. First we show (a). We already know that $\{e_\alpha | \alpha < \lambda\}$ is a linearly independent set since $B' = \bigoplus_{\alpha<\lambda} e_\alpha A$ is a free A-module by definition. Now, it follows from Corollary 1.1.7(iii) that $\| [y_\gamma]_\lambda \cap [y_\beta]_\lambda \| < \|y_\beta\|$, respectively that $[y_\gamma]_\lambda \cap [y_\beta]_\lambda$ is finite, for $\gamma < \beta < \lambda$ since $[y_\beta]_\lambda \subseteq [\varphi_\beta]_\lambda$ and $\|y_\beta\| = \|\varphi_\beta\|$ for all β. Moreover, $[b]_\lambda$ is finite for all $b \in B'$. Therefore the independence follows from the $\mathbb{S}$-torsion-freeness of A together with the fact that $y_\beta \restriction e_{\alpha_n} = q_n$ for all but finitely many $n < \omega$ and for certain $\alpha_n < \lambda$, where $g \restriction e_\alpha = g_\alpha$ for $g = (e_\alpha, g_\alpha)_{\alpha<\lambda} \in \widehat{B} = \widehat{B'} \subseteq \prod_{\alpha<\lambda} e_\alpha \widehat{A} (g_\alpha \in \widehat{A})$.

It remains to show (b). So let $g \in G \backslash B'$. Since $G = \langle B', y_\beta A(\beta < \lambda)\rangle_*$ there is $k < \omega$ such that $q_k g \in \langle B', y_\beta A(\beta < \lambda)\rangle = B' \oplus \bigoplus_{\beta<\lambda} y_\beta A$. Therefore

$$q_k g = b + \sum_{\beta \in N} y_\beta a_\beta \ (b \in B', 0 \neq a_\beta \in A, \emptyset \neq N \subseteq \lambda \text{ finite})$$

is a unique expression (for fixed k); in fact, for a $k' \neq k$ the expression only differs by an $\mathbb{S}$-multiple, i.e. N is unique. Thus the conclusion follows from Corollary 1.1.7(iii) since $[y_\beta]_\lambda \cap [y_{\beta'}]_\lambda$ is finite for $\beta \neq \beta'$ and $[q_k g]_\lambda = [g]_\lambda$. □

Using the above lemma we prove further properties of G.

Lemma 1.2.5. *Let G be as in Construction 1.2.3 and define G_α $(\alpha < \lambda)$ by $G_\alpha := \{g \in G | \|g\| < \alpha, \|g\|_A < \alpha\}$. Then:*

- **(a)** *$G \cap \widehat{P_\beta} \subseteq G^{\beta+1}$ for all $\beta < \lambda$;*
- **(b)** *$\{G_\alpha | \alpha < \lambda\}$ is a λ-filtration of G; and*
- **(c)** *if $\beta < \lambda, \alpha < \lambda$ are ordinals such that $\|\varphi_\beta\| = \alpha$ then $G_\alpha \subseteq G^\beta$.*

Note, we used the upper index $(\beta < \lambda)$ for the construction while we use the lower index $(\alpha < \lambda)$ for the filtration.

Proof. First we show (a). Let $g \in G \cap \widehat{P_\beta}$ for some $\beta < \lambda$. Since $G^0 = B' \subseteq G^{\beta+1}$ we assume $g \in G \setminus B'$. Then, by Lemma 1.2.4, $q_k g \in B' \oplus \bigoplus_{\gamma \in N} y_\gamma A$ for some finite $N \subseteq \lambda, k < \omega$ such that $[g]_\lambda \cap [y_\gamma]_\lambda$ is infinite for $\gamma \in N$.

Since $g \in \widehat{P_\beta}$ we also have $[g]_\lambda \subseteq [P_\beta]_\lambda$ $\left(= \left[\widehat{P_\beta}\right]_\lambda\right)$.

If $\|g\| < \|P_\beta\|$ then $N \subseteq \beta$ by Corollary 1.1.7(ii) and thus $g \in G^\beta \subseteq G^{\beta+1}$.

Otherwise, if $\|g\| = \|P_\beta\| (\in \lambda^o)$ then $\|g\| = \|y_{\gamma_*}\| = \|\varphi_{\gamma_*}\|$ for $\gamma_* = \max N$ and $[g]_\lambda \cap [y_{\gamma_*}]_\lambda \subseteq [\varphi_\beta]_\lambda \cap [\varphi_{\gamma_*}]_\lambda$ is infinite. Hence $\beta = \gamma_*$ by condition (iii) of Corollary 1.1.7 and so $g \in G^{\beta+1}$ as required.

Condition (b) is obvious.

To see (c) let $\beta < \lambda, \alpha < \lambda$ with $\|\varphi_\beta\| = \alpha$ and let $g \in G_\alpha$. If $g \in B'$ we are finished. Otherwise, by Lemma 1.2.4, we have $q_k g \in B' \oplus \bigoplus_{\gamma \in N} y_\gamma A$ $(N \subseteq \lambda \text{ finite}, k < \omega)$ with $[g]_\lambda \cap [y_\gamma]_\lambda$ is infinite for $\gamma \in N$. This implies $\|\varphi_\gamma\| = \|y_\gamma\| \leq \|g\| < \alpha = \|\varphi_\beta\|$ for all $\gamma \in N$ and thus $N \subseteq \beta$ by Corollary 1.1.7(ii), i.e. $g \in G^\beta$, which finishes the proof. □

Finally, we are ready to prove the main theorem of this subsection, i.e. the realization theorem.

Proof of Theorem 1.2.1. Let G be the A-module as constructed in 1.2.3. We already know that G is an $\mathbb{S}$-cotorsion-free R-module of cardinality λ. It remains to show $\mathrm{End}_R G = A$.

Obviously, $A \subseteq \mathrm{End}_R G$. Conversely, suppose there exists $\psi \in \mathrm{End}_R G \setminus A$. Let $\psi' = \psi \restriction B$, then $\psi' \notin A$ since ψ is uniquely determined by ψ' ($B \subseteq_* G \subseteq_* \widehat{B}$).

Let $I = \{(\alpha_n, \varepsilon_n) | n < \omega\} \subseteq \lambda \times \rho$ such that $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ and $I_\lambda \cap [g]_\lambda$ is finite for all $g \in G$. Note, the existence of I can be easily arranged, e.g. let $E \subsetneq \lambda^o, \alpha \in \lambda^o \setminus E$, $\varepsilon_n \in \rho$ $(n < \omega)$ arbitrary and $(\alpha_n)_{n<\omega}$ any ladder on α.

By the Step Lemma 1.2.2 there exists an element y of $\widehat{B}$ such that $y\psi \notin \langle G, yA\rangle_* = G'$. By the Strong Black Box (Corollary 1.1.7) the set

$$E' = \{\alpha \in E | \exists \beta < \lambda : \|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi' \subseteq \psi, [y] \subseteq [\varphi_\beta]\}$$

is stationary since $|[y]| \leq \aleph_0 \leq \kappa$. Note, $[y] \subseteq [\varphi_\beta]$ implies $y \in \widehat{P_\beta}$. Moreover, let $C = \{\alpha < \lambda | G_\alpha \psi \subseteq G_\alpha\}$. Then C is a cub since $\{G_\alpha | \alpha < \lambda\}$ is a λ-filtration of G by Lemma 1.2.5(b).

Now let $\alpha \in E' \cap C$ $(\neq \emptyset)$. Then $G_\alpha \psi \subseteq G_\alpha$ and there exists an ordinal $\beta < \lambda$ such that $\|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi$ and $y \in \widehat{P_\beta}$. The first property implies $G_\alpha \subseteq G^\beta$ by Lemma 1.2.5(c) and the latter properties imply $\varphi_\beta \notin A$.

Moreover, $P_\beta \subseteq B$ with $\|P_\beta\|_A \leq \|P_\beta\| = \alpha$ and hence P_β, and so also $(P_\beta)\psi$ are contained in $G_\alpha \subseteq G^\beta$.

Therefore $\varphi_\beta : P_\beta \to G^\beta$ with $\varphi_\beta \notin A$ and thus it follows from the Construction 1.2.3 that $y_\beta \varphi_\beta \notin G^{\beta+1}$. On the other hand, it follows from Lemma 1.2.5(a) that $y_\beta \varphi_\beta = y_\beta \psi \in G \cap \widehat{P_\beta} \subseteq G^{\beta+1}$ – a contradiction.

So we have shown that no such ψ exists and this means $\mathrm{End}_R G = A$ as required. $\square$

We would like to mention that one can also show, using standard arguments, that G is an $\aleph_1$-free A-module.

We finish this section with pointing out that the constructions and proofs in this section can be simplified for $|A| \leq \kappa$. In this case we may work directly with $B = \bigoplus_{\alpha<\lambda} e_\alpha A$ and with $P = \bigoplus_{\alpha \in I} e_\alpha A$ as canonical summand provided $|I| \leq \kappa$ and $(I \cap \mu)h_\alpha = I \cap \mathrm{Im} h_\alpha$ $(\alpha \in I)$ (cf. Definition 1.1.1). The definition of the equivalence relation on the set $\mathfrak{C}$ of all canonical homomorphisms has to be adjusted: φ, φ' are of the same type if $[\varphi] \cap \mu = [\varphi'] \cap \mu$ and there is an order-isomorphism $f : [\varphi] \to [\varphi']$ such that $(e_\alpha a)\varphi \bar{f} = (e_\alpha a)\bar{f}\varphi' = (e_{\alpha f} a)\varphi'$ for all $a \in A, \alpha \in [\varphi]$

(see Definition 1.1.3). All other adjustments are obvious (see also §2 for comparison).

Note, the simplifications we can achieve in this way are due to the fact that the support function maps into λ rather than into $\lambda \times \rho$. In fact, for $|A| \leq \kappa$, we only need to assume that A is $\mathbb{S}$-cotorsion-free, i.e. no "ρ", respectively "F", is needed here (see beginning of §1).

§2. Existence of $E(R)$-Algebras

Throughout this second section let, as before, R be a commutative ring with 1 and let $\mathbb{S}$ be a countable multiplicatively closed subset of R containing no units except 1 such that R is ($\mathbb{S}$-)cotorsion-free. We refer the reader to §1 for the definition of $\mathbb{S}$-cotorsion-free. Here we additionally assume that R^+ is torsion-free (as an abelian group).

Also as before, we fix an enumeration $\mathbb{S} = \{s_n | n < \omega\}$ of $\mathbb{S}$ with $s_0 = 1$ and define a divisor chain $(q_n)_{n<\omega}$ by $q_n = s_0 \cdot \ldots \cdot s_n$ to describe the $\mathbb{S}$-adic topology of an R-module M by $\{q_n M | n < \omega\}$ as a basis of neighbourhoods of zero.

In this section we show that, given infinite cardinals κ, μ, λ satisfying $\kappa \geq |R|, \mu^\kappa = \mu, \lambda = \mu^+$, there exists an $E(R)$-algebra $\widetilde{R}$ of cardinality λ which is also an $\mathbb{S}$-cotorsion-free R-module. Recall, an R-algebra A is an $E(R)$*-algebra* if it satisfies $\operatorname{End}_R(A_R) = A$. The $E(R)$-algebra $\widetilde{R}$ shall be constructed in Subsection 2.2 using a suitable, yet another, version of the Strong Black Box. This desired version will be introduced in the first subsection.

For a proof of the same result using the General Black Box we refer to [DMV].

2.1. The Black Box Theorem

In this subsection we shall formulate the needed version of the Strong Black Box which will only be slightly different to the one given in Subsection 1.1. Hence we shall only outline the proofs. Nevertheless, we will include all necessary definitions and results following the same pattern as in 1.1.

Let $R, \mathbb{S}$ as well as κ, μ, λ be as above.

As usual, we formulate the parameters of the Black Box with respect to a free R-module B and its $\mathbb{S}$-adic completion $\widehat{B}$. In the present case, however, B is also a ring, namely a polynomial ring over R. Let $B = R[X_\alpha | \alpha < \lambda]$ be the polynomial ring in the commuting variables X_α and let $\mathfrak{M}$ be the set of all monomials including the trivial monomial 1. Then $B = \bigoplus_{m \in \mathfrak{M}} Rm$.

For any $g = (g_m m)_{m\in\mathfrak{M}} \in \widehat{B} \subseteq \prod_{m\in\mathfrak{M}} \widehat{R}m$ we define the *support* of g by

$$[g] = \{m \in \mathfrak{M} | g_m \neq 0\}$$

and the support of $M \subseteq \widehat{B}$ by $[M] = \bigcup_{g\in M} [g]$; note $|[g]| \leq \aleph_0$ for all $g \in \widehat{B}$. Moreover, we define the X-*support* of g by

$$[g]_X = \{\alpha \in \lambda | X_\alpha \text{ occurs in some } m \in [g]\} \subseteq \lambda.$$

Next we define a *norm*, as before, by $\|\{\alpha\}\| = \alpha + 1$ $(\alpha \in \lambda)$, $\|M\| = \sup_{\alpha\in M} \|\{\alpha\}\|$ $(M \subseteq \lambda)$ and $\|g\| = \| [g]_X \|$ $(g \in \widehat{B})$, i.e. $\|g\| = \min\{\beta \in \lambda | [g]_X \subseteq \beta\}$. Note, $[g]_X \subseteq \beta$ holds if and only if g is an element of $\widehat{B_\beta}$ where $B_\beta := R[X_\alpha | \alpha < \beta]$. As before, for a subset M of $\widehat{B}$ the above definitions extend naturally.

Again, we need to say what we mean by a canonical homomorphism. For this we fix bijections $h_\alpha : \mu \to \alpha$ for all α with $\mu \leq \alpha < \lambda$ where we put $h_\mu = \mathrm{id}_\mu$. For technical reasons we also put $h_\alpha = h_\mu$ for $\alpha < \mu$.

Definition 2.1.1. *Let the bijections h_α $(\alpha < \lambda)$ be as above.*

We define P to be a canonical subalgebra *of B if $P = R[X_\alpha | \alpha \in I]$ for some $I \subseteq \lambda$ with $|I| \leq \kappa$ such that* $(I \cap \mu)\, h_\alpha = I \cap \mathrm{Im} h_\alpha$ *for all $\alpha \in I$.*

Accordingly, an R-module homomorphism $\varphi : P \to \widehat{B}$ is said to be a canonical homomorphism *if P is a canonical subalgebra of B and* $\mathrm{Im}\varphi \subseteq \widehat{P}$*; we put $[\varphi] = [P]$, $[\varphi]_X = [P]_X$ and $\|\varphi\| = \|P\|$.*

Let $\mathfrak{C}$ denote the set of all canonical homomorphisms; clearly $|\mathfrak{C}| = \lambda$ (as in §1).

We are now ready to formulate the desired version of the Strong Black Box:

Strong Black Box Theorem 2.1.2. *Let κ, μ, λ be as before and let $E \subseteq \lambda^o$ be a stationary subset of λ.*

Then there exists a family $\mathfrak{C}^*$ of canonical homomorphisms with the following properties:

(1) *If $\varphi \in \mathfrak{C}^*$ then $\|\varphi\| \in E$.*

(2) *If φ, φ' are two different elements of $\mathfrak{C}^*$ of the same norm α then $\| [\varphi]_X \cap [\varphi']_X \| < \alpha$.*

(3) Prediction: *For any R-homomorphism $\psi : B \to \widehat{B}$ and for any subset I of λ with $|I| \leq \kappa$ the set*

$$\{\alpha \in E | \exists \varphi \in \mathfrak{C}^* : \|\varphi\| = \alpha, \varphi \subseteq \psi, I \subseteq [\varphi]_X\}$$

is stationary.

Note that, although the above theorem reads exactly like the Strong Black Box Theorem in §1, the definition of a canonical homomorphism is slightly different to Definition 1.1.1. As mentioned before, we will not give all the details of the proof (again). However, we do state all used definitions and results, even when they coincide with their counterpart in §1.

We begin by adjusting the definition of the equivalence relation on $\mathfrak{C}$:

Definition 2.1.3. *Canonical homomorphisms φ, φ' are said to be* equivalent*, or* of the same type *(notation: $\varphi \equiv \varphi'$), if*

$$[\varphi]_X \cap \mu = [\varphi']_X \cap \mu$$

and there exists an order-isomorphism $f : [\varphi]_X \to [\varphi']_X$ such that

$$(x\bar{f})\varphi' = (x\varphi)\bar{f} \text{ for all } x \in \mathrm{dom}\varphi$$

where

$$\bar{f} : \widehat{\mathrm{dom}\varphi} \to \widehat{\mathrm{dom}\varphi'}$$

is the unique extension of the R-homomorphism defined by

$$\left(X^{k_0}_{\alpha_0} \cdots X^{k_n}_{\alpha_n}\right) \bar{f} = X^{k_0}_{\alpha_0 f} \cdots X^{k_n}_{\alpha_n f} (\alpha_0, \ldots, \alpha_n \in [\varphi]_X).$$

Note, $f : [\varphi]_X \to [\varphi']_X$ is unique since $[\varphi]_X, [\varphi']_X$ are well ordered. Thus, if $\varphi \equiv \varphi'$ and $[\varphi]_X = [\varphi']_X$ then $f = \mathrm{id}$ and so $\varphi = \varphi'$.

As in §1 it is easy to see that there are at most μ different types (equivalence classes) in $(\mathfrak{C}, \equiv)$.

Next we recall the definition of an admissible sequence and of all other related notions:

Definition 2.1.4. *Let $\varphi_0 \subset \varphi_1 \subset \ldots \subset \varphi_n \subset \ldots$ $(n < \omega)$ be an increasing sequence of canonical homomorphisms.*

Then $(\varphi_n)_{n<\omega}$ is said to be admissible *if*

$$[\varphi_0]_X \cap \mu = [\varphi_n]_X \cap \mu \text{ and } \|\varphi_n\| < \|\varphi_{n+1}\|$$

for all $n < \omega$.

Also, we say that $(\varphi_n)_{n<\omega}$ is admissible for a sequence $(\beta_n)_{n<\omega}$ of ordinals *in λ (or $(\varphi_n, \beta_n)_{n<\omega}$* is admissible*), if $(\varphi_n)_{n<\omega}$ is admissible satisfying*

$$\|\varphi_n\| \leq \beta_n < \|\varphi_{n+1}\| \text{ and } [\varphi_n]_X = [\varphi_{n+1}]_X \cap \beta_n$$

for all $n < \omega$.

Moreover, two admissible sequences $(\varphi_n)_{n<\omega}$, $(\varphi'_n)_{n<\omega}$ are said to be equivalent*, or* of the same type*, if $\varphi_n \equiv \varphi'_n$ for all $n < \omega$.*

Note, if $(\varphi_n)_{n<\omega}$ is admissible then $\varphi = \bigcup_{n<\omega} \varphi_n$ is an element of $\mathfrak{C}$ with $\|\varphi\| \in \lambda^o$.

Let $\mathfrak{T}$ denote the set of all possible types of admissible sequences of canonical homomorphisms; clearly, $|\mathfrak{T}| \le \mu^{\aleph_0} = \mu$.

If $(\varphi_n)_{n<\omega}$, respectively $(\varphi_n, \beta_n)_{n<\omega}$, is admissible of type τ, then we also use the notion τ-admissible. Moreover, if $\tau = (\tau_n)_{n<\omega} \in \mathfrak{T}$ and $(\varphi_n)_{n<k}$ $(k < \omega)$ is a finite increasing sequence of canonical homomorphisms satisfying $\varphi_n \in \tau_n$ and $\|\varphi_n\| < \|\varphi_{n+1}\|$ for all $n < k$, then we shall also speak of $(\varphi_n)_{n<k}$ to be of type τ, keeping in mind that such a finite sequence could belong to different types in $\mathfrak{T}$.

We are now ready to show the following result which is, as before, the "main ingredient" for the proof of the Strong Black Box Theorem 2.1.2. Because of this importance we do include a sketch of the proof.

Proposition 2.1.5. *Let $\psi : B \to \widehat{B}$ be an R-homomorphism, $I \subseteq \lambda$ a set of cardinality at most κ and $\mathfrak{K} = \mathfrak{K}_{\psi,I} = \{\varphi \in \mathfrak{C} | \varphi \subseteq \psi, I \subseteq [\varphi]_X\}$.*

Then there exists a type $\tau \in \mathfrak{T}$ such that

$$\exists \varphi_0 \in \mathfrak{K}\ \forall \beta_0 \ge \|\varphi_0\| \dots \exists \varphi_n \in \mathfrak{K}\ \forall \beta_n \ge \|\varphi_n\| \dots$$

with $(\varphi_n, \beta_n)_{n<\omega}$ being τ-admissible.

Proof. Suppose, for contradiction, that the conclusion fails. Then, since the above formula is of "finite character", we have for any type $\tau \in \mathfrak{T}$,

$$\forall \varphi_0 \in \mathfrak{K}\ \exists \beta_0(\tau, \varphi_0) \ge \|\varphi_0\| \dots \forall \varphi_n \in \mathfrak{K}\ \exists \beta_n(\tau, \varphi_0, \dots, \varphi_n) \ge \|\varphi_n\| \dots$$

with $(\varphi_n, \beta_n)_{n<\omega}$ not being τ-admissible.

In the following we fix ordinals $\beta_n(\tau, \varphi_0, \dots, \varphi_n)$ as above $(\tau \in \mathfrak{T}, \varphi_i \in \mathfrak{K}, i \le n < \omega)$.

We define C to be the set of all $\alpha < \lambda$ such that $B_\alpha \psi \subseteq \widehat{B_\alpha}$ (recall: $B_\alpha = R[X_\beta | \beta < \alpha]$) and $\beta_n(\tau, \varphi_0, \dots, \varphi_n) \le \alpha$ for each type $\tau \in \mathfrak{T}$ and for any finite sequence $(\varphi_0, \dots, \varphi_n)$ of elements of $\mathfrak{K}$ with $\|\varphi_i\| \le \alpha$ (iff $[\varphi_i]_X \subseteq \alpha$). Then C is an unbounded set (cf. proof of Proposition 1.1.5).

Now we choose an increasing sequence $\alpha_0 < \alpha_1 < \dots < \alpha_n < \dots$ in C with $\alpha_0 \ge \mu, \|I\|$ and put $\alpha = \sup_{n<\omega} \alpha_n$. Note that $B_\alpha \psi \subseteq \widehat{B_\alpha}$. Using these α_ns we inductively define subsets I_n of $\alpha = [B_\alpha]_X$ $(n < \omega)$ of cardinality at most κ by:

$$I_0 = I \cup \{\alpha_n | n < \omega\}$$

and

$$I_{n+1} = I_n \cup \overline{I_n}^{\psi} \cup \overline{I_n}^{h}$$

where

$$\overline{I_n}^{\psi} = [(R[X_\beta|\beta \in I_n])\,\psi]_X,$$
$$\overline{I_n}^{h} = \bigcup_{\beta\in I_n}\Big((I_n \cap \mu)\,h_\beta \cup (I_n \cap \mathrm{Im}h_\beta)\,h_\beta^{-1}\Big).$$

We put $I^* = \bigcup_{n<\omega} I_n$ and $P = R[X_\beta|\beta \in I^*]$. It is easy to check that P is a canonical subalgebra satisfying $\|P\| = \alpha$ and $P\psi \subseteq \widehat{P}$. Hence $\varphi = \psi \restriction P$ is a canonical homomorphism with $I \subseteq [\varphi]_X$, i.e. $\varphi \in \mathfrak{K}$.

Finally, we put $\varphi_n = \varphi \restriction (P \cap B_{\alpha_n})$. Using the same arguments as in the proof of Proposition 1.1.5 we deduce that $(\varphi_n, \beta_n)_{n<\omega}$ is a τ-admissible sequence for some type τ and for $\beta_n = \beta_n(\tau, \varphi_0, \ldots, \varphi_n)$. This contradiction finishes the proof. □

We have now provided all necessary definitions and results to prove the main theorem of this subsection. We also use Lemma 1.1.6 again, which has nothing to do with the special setting and hence it does not need to be adjusted.

Proof of the Strong Black Box Theorem 2.1.2. Exactly as in the proof of Theorem 1.1.2, we decompose the given stationary set E into $|\mathfrak{T}| \leq \mu$ pairwise disjoint stationary subsets, $E = \bigcup_{\tau\in\mathfrak{T}} E_\tau$, and, for each $\tau \in \mathfrak{T}$, we choose a ladder system $\{\eta_\alpha|\alpha \in E_\tau\}$ such that the set $\{\alpha \in E_\tau|\mathrm{Im}\eta_\alpha \subseteq C\}$ is stationary for any cub C (cf. Lemma 1.1.6).

Also as in 1.1.2, we define $\mathfrak{C}^* = \bigcup_{\alpha\in E}\mathfrak{C}_\alpha$ where, for each $\alpha \in E_\tau$, the set $\mathfrak{C}_\alpha$ consists of all canonical homomorphisms φ such that $\|\varphi\| = \alpha$ and $\varphi = \bigcup_{n<\omega}\varphi_n$ for some τ-admissible sequence $(\varphi_n)_{n<\omega}$ with $[\varphi_n]_X = [\varphi]_X \cap \eta_\alpha(n)$ $(n < \omega)$. Note, for $\varphi, \varphi' \in \mathfrak{C}_\alpha$ with $\mathrm{dom}\varphi = \mathrm{dom}\varphi'$ (iff $[\varphi]_X = [\varphi']_X$) we deduce $\varphi = \varphi'$.

Now, condition (1) is obviously satisfied. Condition (2) follows from

$$[\varphi]_X = \bigcup_{n<\omega}([\varphi]_X \cap \alpha_n) = \bigcup_{n<\omega}([\varphi]_X \cap \mu)\,h_{\alpha_n} =$$

$$\bigcup_{n<\omega}([\varphi']_X \cap \mu)\,h_{\alpha_n} = \bigcup_{n<\omega}([\varphi']_X \cap \alpha_n) = [\varphi']_X$$

for $\mu \leq \alpha_n \in [\varphi]_X \cap [\varphi']_X$ with $\sup_{n<\omega}\alpha_n = \|\varphi\| = \|\varphi'\|$ (cf. 1.1.2).

Finally, the proof of condition (3) is the same as the corresponding part of the proof of the Strong Black Box Theorem 1.1.2 using Proposition 2.1.5 instead of Proposition 1.1.5. □

As in §1 we also present an "enumerated" version of the Strong Black Box Theorem. For the proof we refer to the proof of Corollary 1.1.7.

Corollary 2.1.6. *Let the assumptions be the same as in the Strong Black Box Theorem 2.1.2.*

Then there exists a family $(\varphi_\beta)_{\beta<\lambda}$ *of canonical homomorphism such that*

(i) $\|\varphi_\beta\| \in E$ *for all* $\beta < \lambda$;

(ii) $\|\varphi_\gamma\| \leq \|\varphi_\beta\|$ *for all* $\gamma \leq \beta < \lambda$;

(iii) $\| [\varphi_\gamma]_X \cap [\varphi_\beta]_X \| < \|\varphi_\beta\|$ *for all* $\gamma < \beta < \lambda$;

(iv) Prediction: *For any* R*-homomorphism* $\psi : B \to \widehat{B}$ *and for any subset* I *of* λ *with* $|I| \leq \kappa$ *the set*

$$\{\alpha \in E | \exists \beta < \lambda : \|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi, I \subseteq [\varphi_\beta]_X\}$$

is stationary. □

2.2. Constructing $E(R)$-algebras

In this subsection we shall apply the Strong Black Box as given in Corollary 2.1.6 to prove the following theorem:

Theorem 2.2.1. *Let* $R, \mathbb{S}$ *and* κ, μ, λ *be as before.*

Then there exists an $E(R)$*-algebra* $\widetilde{R}$ *of cardinality* λ *which is also an* $\mathbb{S}$*-cotorsion-free* R*-module.*

Before we construct the desired $E(R)$-algebra we need:

Step Lemma 2.2.2. *Let* $P = R[X_\alpha | \alpha \in I^*]$ *for some* $I^* \subseteq \lambda$ *and let* M *be an* R*-subalgebra of* $\widehat{B}$ *with* $P \subseteq_* M \subseteq_* \widehat{B}$ *which is an* $\mathbb{S}$*-cotorsion-free* R*-module and a torsion-free abelian group.*

Also suppose that there is a set $I = \{\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots (n < \omega)\} \subseteq I^* = [P]_X$ *such that* $I \cap [g]_X$ *is finite for all* $g \in M$.

Moreover, let $\varphi : P \to M$ *be such an* R*-homomorphism which is not multiplication by an element of* M.

Then there exists an element y *of* $\widehat{P}$ *such that* $y\varphi \notin M' = (M[y])_*$ *where '*$*$*' denotes the* $\mathbb{S}$*-purification in* $\widehat{B}$*,* $M[y]$ *denotes the* R*-subalgebra of* $\widehat{B}$ *generated by* M, y *and* φ *is identified with its unique extension* $\varphi : \widehat{P} \to \widehat{M}$.

Moreover, M' *is again an* R*-subalgebra of* $\widehat{B}$ *which is also an* $\mathbb{S}$*-cotorsion-free* R*-module with* $M \subseteq_* M' \subseteq_* \widehat{B}$ *and a torsion-free abelian group.*

(The element y *can be chosen to be either* $y = x$ *or* $y = x + \pi b$ *for some suitable* $\pi \in \widehat{R}, b \in P$ *and for* $x = \sum_{n<\omega} q_n X_{\alpha_n}$*.)*

Note, it is straightforward that the purification of an R-subalgebra of $\widehat{B}$ is also an R-subalgebra.

Proof. Let the assumptions be as above. Either $x = \sum_{n<\omega} q_n X_{\alpha_n}$ satisfies $x\varphi \notin (M[x])_*$ or not. In the latter case there are $k, n < \omega, r_i \in M (i \leq n)$ such that

$$q_k x\varphi = \sum_{i\leq n} r_i x^i. \qquad (+)$$

Note, since φ is not multiplication by an element of M, also $q_k\varphi \notin M$ since $M \subseteq_* \widehat{B}$ is $\mathbb{S}$-torsion-free.

We differentiate two cases.

First assume $n \leq 1$. By the above $P(q_k\varphi - r_1) \neq 0$ and hence there exists an element b of P such that $0 \neq b(q_k\varphi - r_1) = q_k b\varphi - br_1 \in M$. By the cotorsion-freeness of M there is $\pi \in \widehat{R}$ with

$$\pi(q_k b\varphi - br_1) \notin M. \qquad (++)$$

Let $z = x + \pi b$ and suppose $z\varphi \in (M[z])_*$. Then there are $n' < \omega, k \leq l < \omega, t_i \in M(i \leq n')$ such that

$$q_l z\varphi = \sum_{i\leq n'} t_i z^i.$$

Using $(+)$ we obtain that

$$q_l \pi b\varphi = q_l z\varphi - q_l x\varphi = \sum_{i\leq n'} t_i (x + \pi b)^i - \frac{q_l}{q_k}(r_0 + r_1 x).$$

Since $[\pi b] \subseteq [b], [q_l \pi b\varphi] \subseteq [b\varphi]$ and $\{X^i_{\alpha_n} | n < \omega\} \subseteq [x^i]$, respectively $[x^i]_X = I$, we deduce $n' = 1$ and $t_1 = \frac{q_l}{q_k} r_1$ by the assumption on I. Therefore

$$q_l \pi b\varphi = t_0 - \frac{q_l}{q_k} r_0 + \frac{q_l}{q_k} r_1 \pi b$$

and so

$$\frac{q_l}{q_k}\pi(q_k b\varphi - r_1 b) = t_0 - \frac{q_l}{q_k} r_0 \in M \subseteq_* \widehat{B} \quad (\frac{q_l}{q_k} \in \mathbb{S}),$$

respectively

$$\pi(q_k b\varphi - r_1 b) \in M$$

contradicting $(++)$.

Now suppose $n > 1$ in $(+)$. We may assume that $r_n \neq 0$ and so $0 \neq nr_n \in M$ by the torsion-freeness of M^+. Thus there is $\pi \in \widehat{R}$ with

$$\pi n r_n \notin M. \qquad (+++)$$

Let $z = x + \pi$ (i.e. $b = 1 \in R \subseteq P \subseteq M$) and suppose

$$q_l z\varphi = \sum_{i \leq n'} t_i z^i$$

for some $n' < \omega, k \leq l < \omega, t_i \in M (i \leq n')$. Using (+) we obtain

$$q_l \pi\varphi = q_l z\varphi - q_l x\varphi = \sum_{i \leq n'} t_i z^i - \frac{q_l}{q_k} \sum_{i \leq n} r_i x^i.$$

Comparing the supports again we deduce $n' = n$, $t_n = \frac{q_l}{q_k} r_n$ and $t_{n-1} + t_n \pi n = \frac{q_l}{q_k} r_{n-1}$ and so

$$\frac{q_l}{q_k} r_n \pi n = \frac{q_l}{q_k} r_{n-1} - t_{n-1} \in M \subseteq_* \widehat{B},$$

respectively

$$r_n \pi n \in M,$$

contradicting $(+++)$.

Therefore, in both cases, either $y = x$ or $y = z$ satisfies $y\varphi \notin M' = (M[y])_*$.

The remaining properties of M' can be shown using support arguments and the assumptions on M (cf. proof of Lemma 1.2.2). $\square$

We are now ready to construct the desired $E(R)$-algebra.

Construction 2.2.3. Let $(\varphi_\beta)_{\beta<\lambda}$ be a family of canonical homomorphisms as given by Corollary 2.1.6. For any $\beta < \lambda$ let $P_\beta = \mathrm{dom}\varphi_\beta = R[X_\alpha | \alpha \in [\varphi_\beta]_X]$.

We inductively define elements $y_\gamma \in \widehat{P_\gamma}$ and R-subalgebras R^β of $\widehat{B}$ such that, for all $\gamma < \beta < \lambda$,

(1) $\|y_\gamma\| = \|P_\gamma\|$ $(= \|\varphi_\gamma\|)$,

(2) $R^\beta = (B[y_\gamma | \gamma < \beta])_*$,

(3) R^β is an $\mathbb{S}$-cotorsion-free R-module, and

(4) R^β is torsion-free as abelian group.

Let $R^0 = B = R[X_\alpha | \alpha < \lambda]$; clearly B satisfies (2) and also $B = \bigoplus_{m \in \mathfrak{M}} Rm \subseteq_* \widehat{B}$ is an $\mathbb{S}$-cotorsion-free R-module and a torsion-free abelian group since R is, by assumption. Note, condition (1) is not relevant in this case.

Next let β be a limit ordinal and suppose that R^γ satisfies all the required conditions for any $\gamma < \beta$. We put $R^\beta = \bigcup_{\gamma<\beta} R^\gamma$. Then R^β

certainly satisfies (1), (2) and (4). Moreover, it is easy to check that R^β is $\mathbb{S}$-cotorsion-free (cf. Construction 1.2.3).

It remains to tackle the successor case. Suppose R^β is given satisfying all the conditions.

We consider φ_β. Since $\|\varphi_\beta\| \in \lambda^o$ there are ordinals $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots (n < \omega)$ in $[\varphi_\beta]_X$ such that $\|\varphi_\beta\| = \sup_{n<\omega} \alpha_n$. We put $I = \{\alpha_n | n < \omega\}$. Then $\|I \cap [g]_X\| < \|\varphi_\beta\|$, respectively $I \cap [g]_X$ is finite, for all $g \in R^\beta$ by (1), (2) and condition (iii) in Corollary 2.1.6.

We differentiate two cases:

If $\varphi_\beta : P_\beta \to \widehat{P_\beta}$ satisfies $\mathrm{Im}\varphi_\beta \subseteq R^\beta$ and $\varphi_\beta \notin R^\beta$ then we apply the Step Lemma 2.2.2 to I as above, $P = P_\beta$ and $M = R^\beta \subseteq_* \widehat{B}$. We thus deduce the existence of an element $y = y_\beta \in \widehat{P_\beta}$ and of a pure $\mathbb{S}$-cotorsion-free R-submodule $R^{\beta+1}$ of $\widehat{B}$ with

$$R^{\beta+1} = \left(R^\beta[y_\beta]\right)_* = (B[y_\gamma|\gamma \leq \beta])_*$$

which is also an R-algebra such that

$$y_\beta \varphi_\beta \notin R^{\beta+1}$$

and $R^{\beta+1}$ is a torsion-free abelian group, where $y_\beta = \sum_{n<\omega} q_n X_{\alpha_n}$ or $y_\beta = \sum_{n<\omega} q_n X_{\alpha_n} + \pi b$ ($\pi \in \widehat{R}, b \in P_\beta$). Therefore y_β satisfies (1) and $R^{\beta+1}$ satisfies (2), (3) and (4).

If $\mathrm{Im}\varphi_\beta \nsubseteq R^\beta$ or $\varphi_\beta \in R^\beta$ then we put $y_\beta = \sum_{n<\omega} q_n X_{\alpha_n}$ and $R^{\beta+1} = \left(R^\beta[y_\beta]\right)_*$. Then, also in this case, y_β and $R^{\beta+1}$ satisfy all the required conditions.

Finally we define $\widetilde{R}$ by

$$\widetilde{R} = \bigcup_{\beta<\lambda} R^\beta = (B[y_\beta|\beta < \lambda])_* .$$

□

It is an immediate consequence from the construction that $\widetilde{R}$ is an R-subalgebra of cardinality λ which is also a pure $\mathbb{S}$-cotorsion-free R-submodule of $\widehat{B}$ and a torsion-free abelian group.

Next we describe the elements of $\widetilde{R}$.

Lemma 2.2.4. *Let $\widetilde{R}$ be as in Construction 2.2.3.*

(a) *The set $\{y_\beta|\beta < \lambda\}$ is linearly independent over B.*

(b) *If $g \in \widetilde{R} \setminus B$ then there are a finite non-empty subset N of λ and $k < \omega$ such that $q_k g \in B[y_\beta|\beta \in N]$ and $[g]_X \cap [y_\beta]_X$ is infinite iff $\beta \in N$. In particular, if $\|g\|$ is a limit ordinal then $\|g\| = \|y_{\max N}\|$.*

Proof. The conclusion of (a) follows from the facts that $[y_\beta]_X$ is infinite for all $\beta < \lambda$ and $[y_\beta]_X \cap [y_{\beta'}]_X$ is finite for all $\beta \neq \beta'$ (see Corollary 2.1.6(iii)).

To see (b) let $g \in \widetilde{R} \setminus B$. Since $\widetilde{R} = (B[y_\beta|\beta < \lambda])_*$ there is $k < \omega$ such that $q_k g \in B[y_\beta|\beta < \lambda]$ and so $q_k g \in B[y_\beta|\beta \in N]$ for some minimal subset $N \subset \lambda$ where $N \neq \emptyset$ is obvious. Using the independence of the y_βs the desired conditions are easily checked (cf. proof of Lemma 1.2.4). $\square$

Using the above lemma we prove further properties of $\widetilde{R}$.

Lemma 2.2.5. *Let $\widetilde{R}$ be as in Construction 2.2.3 and define R_α $(\alpha < \lambda)$ by $R_\alpha := \{g \in \widetilde{R} | \|g\| < \alpha\}$. Then:*

(a) $\widetilde{R} \cap \widehat{P_\beta} \subseteq R^{\beta+1}$ *for all $\beta < \lambda$;*

(b) R_α *is an R-subalgebra of $\widetilde{R}$ for all $\alpha < \lambda$;*

(c) $\{R_\alpha | \alpha < \lambda\}$ *is a λ-filtration of $\widetilde{R}$; and*

(d) *if $\beta < \lambda, \alpha < \lambda$ are ordinals such that $\|\varphi_\beta\| = \alpha$ then $R_\alpha \subseteq R^\beta$.*

Note, we used the upper index $(\beta < \lambda)$ for the construction while we use the lower index $(\alpha < \lambda)$ for the filtration.

Proof. The proof of (a), (c) and (d) is similar to the one of Lemma 1.2.5 using Lemma 2.2.4 instead of Lemma 1.2.4. Condition (b) follows from $R_\alpha = (B'_\alpha[y_\beta | \|\varphi_\beta\| < \alpha])_*$ where $B'_\alpha = R[X_\beta | \beta + 1 < \alpha]$. $\square$

Finally, we are ready to prove the main theorem of this subsection, i.e. the existence of an $E(R)$-algebra.

Proof of Theorem 2.2.1. Let $\widetilde{R}$ be the R-algebra as constructed in 2.2.3. We already know that $\widetilde{R}$ is an $\mathbb{S}$-cotorsion-free R-module of cardinality λ and also that $\widetilde{R}^+$ is torsion-free. It remains to show $\mathrm{End}_R(\widetilde{R}_R) = \widetilde{R}$.

Clearly, $\widetilde{R} \subseteq \mathrm{End}_R(\widetilde{R}_R)$. Conversely, suppose there exists an R-module homomorphism $\psi : \widetilde{R} \to \widetilde{R}$ which is not multiplication by an element of $\widetilde{R}$. Let $\psi' = \psi \restriction B$. Then $\psi' \notin \widetilde{R}$ since ψ is uniquely determined by ψ' $(B \subseteq_* \widetilde{R} \subseteq_* \widehat{B})$.

Let $I = \{\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots \ (n < \omega)\} \subseteq \lambda$ such that $I \cap [g]_X$ is finite for all $g \in \widetilde{R}$. Note, the existence of I can be easily arranged, e.g. let $E \subsetneq \lambda^o, \alpha \in \lambda^o \setminus E$ and $(\alpha_n)_{n<\omega}$ any ladder on α.

By the Step Lemma 2.2.2 there exists an element y of $\widehat{B}$ such that $y\psi \notin \left(\widetilde{R}[y]\right)_*$. By the Strong Black Box (Corollary 2.1.6) the set

$$E' = \{\alpha \in E | \exists \beta < \lambda : \|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi' \subseteq \psi, [y] \subseteq [\varphi_\beta]\}$$

is stationary since $|[y]| \leq \aleph_0 \leq \kappa$. Note, $[y] \subseteq [\varphi_\beta]$ implies $y \in \widehat{P_\beta}$ where $P_\beta = \mathrm{dom}\varphi_\beta$. Moreover, let $C = \{\alpha < \lambda | R_\alpha\psi \subseteq R_\alpha\}$. Then C is a cub since $\{R_\alpha | \alpha < \lambda\}$ is a λ-filtration of $\widetilde{R}$ by Lemma 2.2.5(c).

Now let $\alpha \in E' \cap C$ ($\neq \emptyset$). Then $R_\alpha\psi \subseteq R_\alpha$ and there exists an ordinal $\beta < \lambda$ such that $\|\varphi_\beta\| = \alpha, \varphi_\beta \subseteq \psi$ and $y \in \widehat{P_\beta}$. The first property implies $R_\alpha \subseteq R^\beta$ by Lemma 2.2.5(d) and the latter properties imply $\varphi_\beta \notin \widetilde{R}$, especially $\varphi_\beta \notin R^\beta$.

Moreover, $P_\beta \subseteq B$ with $\|P_\beta\| = \alpha$ and hence P_β, and so also $(P_\beta)\psi$ are contained in $R_\alpha \subseteq R^\beta$.

Therefore $\varphi_\beta : P_\beta \to R^\beta$ with $\varphi_\beta \notin R_\beta$ and thus it follows from the Construction 2.2.3 that $y_\beta\varphi_\beta \notin R^{\beta+1}$. On the other hand, it follows from Lemma 2.2.5(a) that $y_\beta\varphi_\beta = y_\beta\psi \in \widetilde{R} \cap \widehat{P_\beta} \subseteq R^{\beta+1}$ – a contradiction.

So we have shown that no such ψ exists and this means $\mathrm{End}_R(\widetilde{R}_R) = \widetilde{R}$ as required. □

Note, that one could also show, using standard arguments, that $\widetilde{R}$ is an $\aleph_1$-free R-module.

§3. Existence of ultra-cotorsion-free modules

Throughout this last section let, again, R be a commutative ring with 1 and let $\mathbb{S}$ be a countable multiplicatively closed subset of R containing no units except 1 such that R is ($\mathbb{S}$-)cotorsion-free, that is, R is torsion-free and reduced (with respect to $\mathbb{S}$) and satisfies $\mathrm{Hom}_R(\widehat{R}, R) = 0$ where $\widehat{R}$ denotes the $\mathbb{S}$-adic completion of R. Note, cotorsion-freeness for an arbitrary R-module M is defined in the same way (with $\mathrm{Hom}_R(\widehat{R}, M) = 0$). Moreover, we fix an enumeration $\mathbb{S} = \{s_n | n < \omega\}$ of $\mathbb{S}$ with $s_0 = 1$ and define a divisor chain $(q_n)_{n<\omega}$ by $q_n = s_0 \cdot \ldots \cdot s_n$ to describe the $\mathbb{S}$-adic topology of an R-module M by $\{q_n M | n < \omega\}$ as a basis of neighbourhoods of zero.

In this section we show that, given infinite cardinals κ, μ, λ satisfying $\kappa \geq |R|, \mu^\kappa = \mu, \lambda = \mu^+$, there exists an ultra-cotorsion-free R-module G of cardinality λ. We define an R-module M to be *ultra-cotorsion-free* if M is $\mathbb{S}$-cotorsion-free and, for any submodule H of M, if M/H is $\mathbb{S}$-cotorsion-free then $H = M$ or $|H| < |M|$. In particular, if M is ultra-cotorsion-free then M has no non-trivial $\mathbb{S}$-cotorsion-free epimorphic images of smaller cardinality. Note, ultra-cotorsion-free modules have been used in [GSW] to show the abundance of cotorsion theories (cotorsion pairs).

The desired R-module G shall be constructed using a suitable, yet again different, version of the Strong Black Box which will be introduced in the first subsection.

3.1. The Black Box Theorem

In this subsection we shall formulate and prove the needed version of the Strong Black Box in full detail. Comparing the here presented version with the one given in §1 should enable the reader to understand that the Strong Black Box can be formulated (and proven) in rather different settings provided that the cardinalities in question are bounded by μ (number of types), respectively by λ (e.g. number of canonical summands).

Now, let $R, \mathbb{S}$ as well as κ, μ, λ be as above. As usual, we formulate the parameters of the Black Box with respect to a free R-module B and its $\mathbb{S}$-adic completion $\widehat{B}$. Let $B = \bigoplus_{\alpha<\lambda} Re_\alpha$.

For any $g = (g_\alpha e_\alpha)_{\alpha\in\lambda} \in \widehat{B} \subseteq \prod_{\alpha\in\lambda} \widehat{R}e_\alpha$ we define the *support* of g by

$$[g] = \{\alpha \in \lambda | g_\alpha \neq 0\} \subseteq \lambda$$

and the support of a subset M of $\widehat{B}$ by $[M] = \bigcup_{g\in M} [g]$; note $|[g]| \leq \aleph_0$ for all $g \in \widehat{B}$.

Moreover, we define a *norm* on λ, respectively on $\widehat{B}$, by $\|\{\alpha\}\| = \alpha+1$ ($\alpha \in \lambda$), $\|M\| = \sup_{\alpha\in M} \|\{\alpha\}\|$ ($M \subseteq \lambda$) and $\|g\| = \|\,[g]\,\|$ ($g \in \widehat{B}$), i.e. $\|g\| = \min\{\beta \in \lambda |\, [g] \subseteq \beta\}$. Note, $[g] \subseteq \beta$ holds iff $g \in \widehat{B_\beta}$ for $B_\beta = \bigoplus_{\alpha<\beta} Re_\alpha$. As before, for a subset M of $\widehat{B}$ the above definitions extend naturally.

We also need to define canonical summands and other "canonical objects" which shall play a crucial role in the formulation and the proof of the Strong Black Box. Note, in the here presented version we basically want to predict kernels of homomorphisms (i.e. submodules, respectively their elements), not the homomorphisms themselves.

As before, we fix bijections $h_\alpha : \mu \to \alpha$ for all α with $\mu \leq \alpha < \lambda$ where we put $h_\mu = \mathrm{id}_\mu$. For technical reasons we also put $h_\alpha = h_\mu$ for $\alpha < \mu$.

Definition 3.1.1. *Let the bijections h_α ($\alpha < \lambda$) be as above.*

We define P to be a canonical summand *of B if $P = \bigoplus_{\alpha\in I} Re_\alpha$ for some $I \subseteq \lambda$ with $|I| \leq \kappa$ such that $(I \cap \mu)h_\alpha = I \cap \mathrm{Im}h_\alpha$ for all $\alpha \in I$.*

We call (P, v) a canonical pair *if P is a canonical summand of B and v is a pure element of $\widehat{P}$.*

Moreover, an infinite sequence $(v_n)_{n<\omega}$ of pure elements of $\widehat{B}$ satisfying $\|v_n\| < \|v_{n+1}\|$ ($n < \omega$) is said to be a Signac-branch.

A pair $(P, (v_n)_{n<\omega})$ is called a canonical Signac-pair *if P is a canonical summand of B and $(v_n)_{n<\omega}$ is a Signac-branch such that:*

- *$v_n \in \widehat{P}$ for all $n < \omega$; and*

- $\|P\| = \sup_{n<\omega} \|v_n\|$.

Let $\mathfrak{P}$ denote the set of all canonical pairs and let $\mathfrak{C}$ denote the set of all canonical Signac-pairs. Then it is easy to see that $|\mathfrak{P}| = |\mathfrak{C}| = \lambda$.

For the application in 3.2 we also need to include further parameters, namely $\mathrm{pr}(B) \times \widehat{R}$, where $\mathrm{pr}(B)$ denotes the set of all pure elements of B. Note, an element g of $\widehat{B}$ is called pure if $g = sg'$ $(s \in \mathbb{S}, g' \in \widehat{B})$ implies $s = 1$. In order to formulate the Strong Black Box according to our needs we define "traps" as follows:

Definition 3.1.2. *Let $(P, (v_n)_{n<\omega})$ be a canonical Signac-pair, b a pure element of B and $\pi \in \widehat{R}$. Then $t = (P, (v_n)_{n<\omega}, b, \pi)$ is said to be a* trap *if $b \in P$ (especially $\|b\| < \|P\|$). We put $[t] = [P]$ and $\|t\| = \|P\|$.*

We are now ready to present the desired version of the Strong Black Box:

Strong Black Box Theorem 3.1.3. *Let κ, μ, λ be as before and let $E \subseteq \lambda^o$ be a stationary subset of λ.*

Then there exists a family $\mathfrak{C}^*$ of traps $t = (P_t, (v_{t,n})_{n<\omega}, b_t, \pi_t)$ with the following properties:

(1) *If $t \in \mathfrak{C}^*$ then $\|t\| \in E$.*

(2) *If t, t' are two different elements of $\mathfrak{C}^*$ with $\|t\| = \|t'\| = \alpha$ then $\| \, [t] \cap [t'] \, \| < \alpha$.*

(3) PREDICTION: *For any set U of pure elements of $\widehat{B}$ of cardinality λ, for any pure element b of B and for any $\pi \in \widehat{R}$ the set*

$$\{\alpha \in E | \exists t \in \mathfrak{C}^* : \|t\| = \alpha, \{v_{t,n} | n < \omega\} \subseteq U, b = b_t, \pi = \pi_t\}$$

is stationary.

To prove the above theorem we need further definitions and other results. We begin with an equivalence relation on $\mathfrak{P}$.

Definition 3.1.4. *Canonical pairs $(P, v), (P', v')$ are said to be* equivalent *or* of the same type *(notation: $(P, v) \equiv (P', v')$) if*

$$[P] \cap \mu = \left[P'\right] \cap \mu$$

and there exists an order-isomorphism $f : [P] \to [P']$ such that

$$v\bar{f} = v'$$

where

$$\bar{f} : \widehat{P} \to \widehat{P'}$$

is the unique extension of the R-homomorphism defined by

$$e_\alpha \bar{f} = e_{\alpha f} \quad (\alpha \in [P]).$$

Note, $f : [P] \to [P']$ is unique since $[P]$, $[P']$ are well ordered. Thus, if $(P, v) \equiv (P', v')$ and $[P] = [P']$ then $f = \mathrm{id}$ and so $(P, v) = (P', v')$.

Obviously, any type in $(\mathfrak{P}, \equiv)$ can be represented by a subset V of μ of cardinality at most κ, an order-type of a set M of cardinality at most κ and a countable sequence $(\alpha_n, r_n)_{n<\omega}$ with $\alpha_n \in M$, $r_n \in R$ (which describes v). Therefore there are at most $\mu^\kappa \cdot \kappa^\kappa \cdot \kappa^{\aleph_0} \cdot |R|^{\aleph_0} = \mu$ different types in $(\mathfrak{P}, \equiv)$.

Next we consider certain infinite sequences of canonical pairs:

Definition 3.1.5. *A sequence $(P_n, v_n)_{n<\omega}$ of canonical pairs is said to be* admissible *if*

$$P_0 \subset P_1 \subset \ldots \subset P_n \subset \ldots \ (n < \omega),$$

$$[P_0] \cap \mu = [P_n] \cap \mu \text{ and } \|P_n\| < \|v_{n+1}\| (\leq \|P_{n+1}\|)$$

for each $n < \omega$.

Also, we say that $(P_n, v_n)_{n<\omega}$ is admissible for a sequence $(\beta_n)_{n<\omega}$ of ordinals *in λ, if $(P_n, v_n)_{n<\omega}$ is admissible such that*

$$\|P_n\| \leq \beta_n < \|P_{n+1}\| \text{ and } [P_n] = [P_{n+1}] \cap \beta_n$$

for all $n < \omega$.

Moreover, two admissible sequences $(P_n, v_n)_{n<\omega}$, $(P'_n, v'_n)_{n<\omega}$ are said to be equivalent *or* of the same type*, if $(P_n, v_n) \equiv (P'_n, v'_n)$ for all $n < \omega$.*

Note, if $(P_n, v_n)_{n<\omega}$ is admissible then $\big(P = \bigcup_{n<\omega} P_n, (v_n)_{n<\omega}\big)$ is a canonical Signac-pair since $\|P\| = \sup_{n<\omega} \|P_n\| = \sup_{n<\omega} \|v_n\|$.

Let $\mathfrak{T}$ denote the set of all possible types of admissible sequences of canonical pairs. It follows immediately from the above definition that any type τ in $\mathfrak{T}$ can be represented by $\tau = (\tau_n)_{n<\omega}$ where the τ_ns are equivalence classes of $(\mathfrak{P}, \equiv)$ with the same underlying subset V of μ. Hence we deduce $|\mathfrak{T}| \leq \mu^{\aleph_0} = \mu$.

If $(P_n, v_n)_{n<\omega}$ is an admissible sequence of type τ, then we also use the notion τ-admissible. Moreover, if $\tau = (\tau_n)_{n<\omega} \in \mathfrak{T}$ and $(P_n, v_n)_{n<k}$ ($k < \omega$) is a finite increasing sequence of canonical pairs satisfying $(P_n, v_n) \in \tau_n$ for all $n < k$, then we shall also speak of $(P_n, v_n)_{n<k}$ to be of type τ, keeping in mind that such a finite sequence could belong to different types in $\mathfrak{T}$.

The next result is the key for proving the Strong Black Box Theorem 3.1.3. Note again, that the kind of formula we use to formulate this result goes under the generic name "Svenonius sentences" (cf. [H, p.112], also §1).

Proposition 3.1.6. *Let U be a set of pure elements of $\widehat{B}$ of cardinality λ, let b be a pure element of B, and let $\mathfrak{K} = \mathfrak{K}_{U,b} = \{(P, v) \in \mathfrak{P} | v \in U, b \in P\}$.*

Then there exists a type $\tau \in \mathfrak{T}$ such that

$$\exists (P_0, v_0) \in \mathfrak{K}\ \forall \beta_0 \geq \|P_0\| \ldots \exists (P_n, v_n) \in \mathfrak{K}\ \forall \beta_n \geq \|P_n\| \ldots$$

with $(P_n, v_n)_{n<\omega}$ being τ-admissible for $(\beta_n)_{n<\omega}$.

Proof. Suppose, for contradiction, that the conclusion fails. Then, for any $\tau \in \mathfrak{T}$, we have:

$$\forall (P_0, v_0) \in \mathfrak{K}\ \exists \beta_0(\tau, P_0, v_0) \geq \|P_0\| \ldots$$

$$\forall (P_n, v_n) \in \mathfrak{K}\ \exists \beta_n(\tau, P_0, v_0, \ldots, P_n, v_n) \geq \|P_n]\| \ldots$$

with $(P_n, v_n)_{n<\omega}$ not being τ-admissible for $(\beta_n)_{n<\omega}$.

In the following we fix ordinals $\beta_n(\tau, P_0, v_0, \ldots, P_n, v_n)$ as above ($\tau \in \mathfrak{T}, (P_i, v_i) \in \mathfrak{K}, i \leq n < \omega$). We define C to be the set of all $\alpha < \lambda$ such that $\beta_n(\tau, P_0, v_0, \ldots, P_n, v_n) \leq \alpha$ for each $\tau \in \mathfrak{T}$ and for any finite sequence $(P_i, v_i)_{i \leq n}$ with $(P_i, v_i) \in \mathfrak{K}$ and $\|P_i\| \leq \alpha$ ($i \leq n$). Then C is unbounded since, starting with an arbitrary $\alpha_0 < \lambda$, one can inductively define ordinals α_k ($k < \kappa^+ \leq \mu$) such that $\alpha = \sup\{\alpha_k | k < \kappa^+\}$ is an element of C (cf. proof of Proposition 1.1.5).

Now, we inductively choose elements v_n of U and ordinals α_n in C such that $\alpha_0 \geq \mu, \|b\|$ and $\|v_n\| \leq \alpha_n < \|v_{n+1}\|$ for all $n < \omega$; this is possible since C is unbounded and $|U| = \lambda$, so the norms of the elements of U also form an unbounded set in λ. We put $\alpha = \sup_{n<\omega} \alpha_n = \sup_{n<\omega} \|v_n\|$ and define subsets I_n of α of cardinality at most κ by:

$$I_0 = [b] \cup \{\alpha_n | n < \omega\} \cup \bigcup_{n<\omega} [v_n]$$

and

$$I_{n+1} = I_n \cup \bigcup_{\beta \in I_n} \left((I_n \cap \mu) h_\beta \cup (I_n \cap \mathrm{Im} h_\beta) h_\beta^{-1} \right).$$

We put $I^* = \bigcup_{n<\omega} I_n$. Then it is easy to check that $P = \bigoplus_{\alpha \in I^*} Re_\alpha$ is a canonical summand of B such that $b \in P$, $v_n \in \widehat{P}$ for all $n < \omega$ and $\|P\| = \|I^*\| = \alpha = \sup_{n<\omega} \|v_n\|$, i.e. $(P, (v_n)_{n<\omega})$ is a canonical Signac-pair.

Finally, let $P_n = P \cap B_{\alpha_n}$ ($B_{\alpha_n} = \bigoplus_{\beta<\alpha_n} Re_\beta$), i.e. $[P_n] = [P] \cap \alpha_n$. Then $b \in P_0$ ($\subseteq P_n$) since $\|b\| \leq \alpha_0$ ($[b] \subseteq \alpha_0$) by the definition of C and $v_n \in \widehat{P_n}$ for any $n < \omega$ since $\|v_n\| \leq \alpha_n$ ($[v_n] \subseteq \alpha_n$) by the choice of the v_ns and α_ns. Hence $(P_n, v_n) \in \mathfrak{K}$ for all $n < \omega$. Moreover, $[P_0] \cap \mu = [P_n] \cap \mu = [P] \cap \mu$ ($n < \omega$) since $\mu \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ ($n < \omega$). Therefore $(P_n, v_n)_{n<\omega}$ is an admissible sequence, say of type $\tau \in \mathfrak{T}$. By the definition of C we also have that $\|P_n\| \leq \beta_n = \beta_n(\tau, P_0, \ldots, P_n) \leq \alpha_n$ for any $n < \omega$ and thus $[P_{n+1}] \cap \beta_n = [P_n] \cap \beta_n = [P_n]$. Hence $(P_n, v_n)_{n<\omega}$ is τ-admissible for $(\beta_n)_{n<\omega}$, contradicting the assumption that it is not for $\beta_n = \beta_n(\tau, P_0, \ldots, P_n)$. Therefore the original conclusion holds and so the proof is finished. $\square$

We are now ready to prove the main theorem of this subsection.

Proof of the Strong Black Box Theorem 3.1.3. Let $\mathrm{pr}(B)$ be the set of all pure elements of B and put $\mathcal{P} = \mathrm{pr}(B) \times \widehat{R}$. Then $|\mathcal{P}| = \lambda \cdot 2^\kappa = \lambda$.

First we decompose the given stationary set E into $|\mathfrak{T}| \leq \mu$ pairwise disjoint stationary subsets, say $E = \bigcup_{\tau\in\mathfrak{T}} E_\tau$. Moreover, for each $\tau \in \mathfrak{T}$, we decompose E_τ into $|\mathcal{P}| = \lambda$ pairwise disjoint stationary subsets: $E_\tau = \bigcup_{p\in\mathcal{P}} E_{\tau,p}$. Note, for $p = (b, \pi) \in \mathcal{P}$ and $\tau \in \mathfrak{T}$, we may assume that $\|b\| < \alpha$ for all $\alpha \in E_{\tau,p}$.

For each $\tau \in \mathfrak{T}$ and each $p \in \mathcal{P}$ we choose a ladder system $\{\eta_\alpha | \alpha \in E_{\tau,p}\}$ such that the set $\{\alpha \in E_{\tau,p} | \mathrm{Im}\eta_\alpha \subseteq C\}$ is stationary for any cub C (cf. Lemma 1.1.6).

Let $\tau \in \mathfrak{T}$, $p = (b, \pi) \in \mathcal{P}$ and $\alpha \in E_{\tau,p}$; note $\|b\| < \alpha$. We define $\mathfrak{C}_\alpha$ to be the set of all traps $t = (P, (v_n)_{n<\omega}, b, \pi)$ such that $\|t\| = \|P\| = \alpha$ and $P = \bigcup_{n<\omega} P_n$ for some τ-admissible sequence $(P_n, v_n)_{n<\omega}$ of canonical pairs with $[P_n] = [P] \cap \eta_\alpha(n)$. Note, for t, $t' \in \mathfrak{C}_\alpha$ with $[t] = [t']$ (iff $P = P'$), we clearly deduce $t = t'$ since $b_t = b = b_{t'}$, $\pi_t = \pi = \pi_{t'}$, and $[P_{t,n}] = \left[P_{t',n}\right]$, $(P_{t,n}, v_{t,n}) \equiv (P_{t',n}, v_{t',n})$ imply $(P_{t,n}, v_{t,n}) = (P_{t',n}, v_{t',n})$ for all $n < \omega$ (cf. Definitions 3.1.2 and 3.1.4).

Now, we define $\mathfrak{C}^*$ to be the union of all these $\mathfrak{C}_\alpha$s, i.e. $\mathfrak{C}^* = \bigcup_{\alpha\in E} \mathfrak{C}_\alpha$.

Clearly, condition (1) is satisfied.

To see (2) let t, $t' \in \mathfrak{C}^*$ with $\|t\| = \|t'\| = \alpha$. Then t, $t' \in \mathfrak{C}_\alpha$ and thus $t = (P, (v_n)_{n<\omega}, b, \pi)$, $t' = (P', (v'_n)_{n<\omega}, b', \pi')$ with $b = b'$, $\pi = \pi'$, and $(P, (v_n)_{n<\omega})$, $(P', (v'_n)_{n<\omega})$ are of the same type τ ($\alpha \in E_{\tau,(b,\pi)}$) where $P = \bigcup_{n<\omega} P_n$, $P' = \bigcup_{n<\omega} P'_n$, $[P] \cap \eta_\alpha(n) = [P_n]$, $[P'_n] \cap \eta_\alpha(n) = [P'_n]$.

Suppose, for contradiction, that $\|\,[t] \cap [t']\,\| = \alpha$ (recall: $[t] = [P]$ and $[t'] = [P']$). Then there are $\alpha_n \in [P] \cap [P']$ with $\sup_{n<\omega} \alpha_n = \alpha$ (w.l.o.g. $\alpha_n \geq \mu$). Now, $([P] \cap \mu)h_{\alpha_n} = [P] \cap \alpha_n$ and $([P'] \cap \mu)h_{\alpha_n} = [P'] \cap \alpha_n$ since

P, P' are canonical summands of B (see Definition 3.1.1). Moreover,

$$[P] \cap \mu = [P_0] \cap \mu = \left[P_0'\right] \cap \mu = \left[P'\right] \cap \mu$$

and therefore

$$[P] = \bigcup_{n<\omega} ([P] \cap \alpha_n) = \bigcup_{n<\omega} \left(\left[P'\right] \cap \alpha_n\right) = \left[P'\right].$$

This implies $(P, (v_n)_{n<\omega}) = (P', (v_n')_{n<\omega})$ and so property (2) is proven.

It remains to show (3). To do so let U be a set of pure elements of $\widehat{B}$ of cardinality λ and let $p = (b, \pi) \in \mathcal{P} = \mathrm{pr}(B) \times \widehat{R}$. By Proposition 3.1.6 there is a type $\tau \in \mathfrak{T}$ such that

$$\exists (P_0, v_0) \in \mathfrak{K}\ \forall \beta_0 \geq \|P_0\| \ldots \exists (P_n, v_n) \in \mathfrak{K}\ \forall \beta_n \geq \|P_n\| \ldots$$

with $(P_n, v_n)_{n<\omega}$ is τ-admissible for $(\beta_n)_{n<\omega}$ where $\mathfrak{K} = \{(P, v) \in \mathfrak{P} | v \in U,\ b \in P\}$.

Let C be the set of all ordinals $\alpha < \lambda$ such that $\alpha \geq \mu$, $\alpha \geq \|P_0\|$ and, if $(P_n, v_n)_{n\leq k}$ is a finite part of one of the above τ-admissible sequences for $(\beta_n)_{n\leq k}$ with $\beta_k < \alpha$, then there is $(P_{k+1}, v_{k+1}) \in \mathfrak{K}$ with $[P_{n+1}] \subseteq \alpha$ and $(P_n, v_n)_{n\leq k+1}$ is τ-admissible. Obviously, C is a cub. Therefore $E'_{\tau,p} = \{\alpha \in E_{\tau,p} | \mathrm{Im}\eta_\alpha \subseteq C\}$ is stationary.

In the following let $\alpha \in E'_{\tau,p}$ be fixed, i.e. $\eta_\alpha(n) \in C$ for all $n < \omega$. By the definition of C we have $\|P_0\| \leq \eta_\alpha(0) < \eta_\alpha(1)$ and so there is $(P_1, v_1) \in \mathfrak{K}$ with $\|P_1\| \leq \eta_\alpha(1)$ such that $(P_n, v_n)_{n\leq 1}$ is τ-admissible for $(\eta_\alpha(0), \eta_\alpha(1))$. We proceed like this for each $n < \omega$, i.e. whenever we have a sequence $(P_n, v_n)_{n\leq k}$ which is τ-admissible for $(\eta_\alpha(n))_{n\leq k}$ we can find $(P_{k+1}, v_{k+1}) \in \mathfrak{K}$ with $\|P_{k+1}\| \leq \eta_\alpha(k+1)$ such that $(P_n, v_n)_{n\leq k+1}$ is τ-admissible for $(\eta_\alpha(n))_{n\leq k+1}$. Therefore we obtain an infinite τ-admissible sequence $(P_n, v_n)_{n<\omega}$ with $\|P_n\| \leq \eta_\alpha(n)$ and $[P_{n+1}] \cap \eta_\alpha(n) = [P_n]$. We put $P = \bigcup_{n<\omega} P_n$. Then $\|P\| = \sup_{n<\omega} \|P_n\| = \sup_{n<\omega} \eta_\alpha(n) = \alpha$ and $[P] \cap \eta_\alpha(n) = \bigcup_{i\geq n} ([P_i] \cap \eta_\alpha(n)) = [P_n]$. Hence $(P, (v_n)_{n<\omega}, b, \pi) \in \mathfrak{C}_\alpha$. Since $\alpha \in E'_{\tau,p}$ was arbitrary and $E'_{\tau,p}$ is stationary, the proof is finished. $\square$

We finish this subsection with an "enumerated" version of the Strong Black Box. For the proof we refer the reader to the proof of Corollary 1.1.7.

Corollary 3.1.7. *Let the assumptions be the same as in the Strong Black Box Theorem 3.1.3.*

Then there exists a family $(t_\beta = (P_\beta, (v_{\beta,n})_{n<\omega}, b_\beta, \pi_\beta))_{\beta<\lambda}$ *of traps such that*

(i) $\|t_\beta\| \in E$ *for all* $\beta < \lambda$*;*

(ii) $\|t_\gamma\| \leq \|t_\beta\|$ *for all* $\gamma \leq \beta < \lambda$*;*

(iii) $\| [t_\gamma] \cap [t_\beta] \| < \|t_\beta\|$ *for all* $\gamma < \beta < \lambda$*;*

(iv) Prediction: *For any set* U *of pure elements of* $\widehat{B}$ *of cardinality* λ*, for any pure element* b *of* B *and for any* $\pi \in \widehat{R}$*, the set*

$$\{\alpha \in E | \exists \beta < \lambda : \|t_\beta\| = \alpha, \{v_{\beta,n} | n < \omega\} \subseteq U, b = b_\beta, \pi = \pi_\beta\}$$

is stationary. □

3.2. Constructing ultra-cotorsion-free modules

In this final subsection we shall apply the Strong Black Box as given in Corollary 3.1.3 to prove the following theorem:

Theorem 3.2.1. *Let* R*,* $\mathbb{S}$ *and* κ*,* μ*,* λ *be as before.*
Then there exists an ultra-cotorsion-free R*-module* G *of cardinality* λ*.*

Before we construct the desired module we show:

Step Lemma 3.2.2. *Let* M *be a pure* $\mathbb{S}$*-cotorsion-free submodule of* $\widehat{B}$*, let* b *be a pure element of* $B \cap M$ *and let* $\pi \in \widehat{R}$*. Moreover, let* $v = (v_n)_{n<\omega}$ *be a Signac-branch with* $v_n \in M$ *for all* $n < \omega$ *such that* $\|b\| < \|v\|$ $(= \sup_{n<\omega} \|v_n\|)$ *and* $\| [m] \cap [v] \| < \|v\|$ *for all* $m \in M$*.*
Then $M' = \left\langle M, y = \sum_{n<\omega} q_n v_n + \pi b \right\rangle_*$ *is also* $\mathbb{S}$*-cotorsion-free.*

Proof. Let the assumptions be as above and consider a homomorphism $\varphi : \widehat{R} \to M'$. Since $1\varphi \in M'$ there is $k < \omega$ such that $q_k(1\varphi) \in M + Ry$, say

$$q_k(1\varphi) = m + ry$$

for some $m \in M, r \in R$. Moreover, for any $\rho \in \widehat{R}$, let $k \leq k_\rho < \omega, m_\rho \in M, r_\rho \in R$ such that

$$q_{k_\rho}(\rho\varphi) = m_\rho + r_\rho y.$$

Hence, since $\rho\varphi = \rho(1\varphi)$ by the continuity of φ, we deduce

$$0 = q_{k_\rho}(\rho\varphi) - q_{k_\rho}\rho(1\varphi) = m_\rho + r_\rho y - \frac{q_{k_\rho}}{q_k}\rho(m + ry),$$

respectively

$$m_\rho - \frac{q_{k_\rho}}{q_k}\rho m = (\frac{q_{k_\rho}}{q_k}\rho r - r_\rho)y =: g.$$

From $g = m_\rho - \frac{q_{k_\rho}}{q_k}\rho m$ it follows that $g \in \widehat{R}M$ and thus $\|\,[g] \cap [v]\,\| < \|v\|$ by the assumption. On the other hand, $g = (\frac{q_{k_\rho}}{q_k}\rho r - r_\rho)y$ and so $\|\,[g] \cap [v]\,\| = \|v\| = \|y\|$ unless $r_\rho = \frac{q_{k_\rho}}{q_k}\rho r$, i.e. $g = 0$. Therefore $\frac{q_{k_\rho}}{q_k}\rho r = r_\rho \in R$ and $\frac{q_{k_\rho}}{q_k}\rho m = m_\rho \in M$ for all $\rho \in \widehat{R}$ and so, since $\frac{q_{k_\rho}}{q_k} \in \mathbb{S}$, $\rho r \in R$ and $\rho m \in M$ for all $\rho \in \widehat{R}$. The cotorsion-freeness of R and M now implies $r = 0$, $m = 0$ and thus $1\varphi = 0$, respectively $\varphi = 0$, as required. □

We are now ready to construct the desired module.

Construction 3.2.3. Let $(t_\beta = (P_\beta, (v_{\beta,n})_{n<\omega}, b_\beta, \pi_\beta))_{\beta<\lambda}$ be a family of traps as given by Corollary 3.1.7.

We inductively define elements $y_\gamma \in \widehat{P_\gamma}$ and pure submodules G^β of $\widehat{B}$ such that, for all $\gamma < \beta < \lambda$,

(1) $y_\gamma = 0$ or $\|y_\gamma\| = \|P_\gamma\|$ $(= \|t_\gamma\|)$,

(2) $G^\beta = \langle B, y_\gamma(\gamma < \beta)\rangle_*$, and

(3) G^β is $\mathbb{S}$-cotorsion-free.

First we put $G^0 = B = \bigoplus_{\alpha<\lambda} Re_\alpha$; B is clearly a pure $\mathbb{S}$-cotorsion-free submodule of $\widehat{B}$ satisfying (2). Note, condition (1) is not relevant.

Next let β be a limit ordinal and suppose the G^γs $(\gamma < \beta)$ are given satisfying all the required conditions. We put $G^\beta = \bigcup_{\gamma<\beta} G^\gamma$. Then, obviously, the y_γs and G^β satisfy (1) and (2). Moreover, G^β is $\mathbb{S}$-cotorsion-free since, for a homomorphism $\varphi : \widehat{R} \to G^\beta$, we have $1\varphi \in G^\gamma$ for some $\gamma < \beta$ and $[\rho\varphi] \subseteq [1\varphi]$ for all $\rho \in \widehat{R}$; thus we obtain $\mathrm{Im}\varphi \subseteq G^\gamma$ by (1), (2) and condition (iii) in Corollary 3.1.7.

Now, suppose that G^β is given satisfying the above properties and let $t_\beta = (P_\beta, (v_{\beta,n})_{n<\omega}, b_\beta, \pi_\beta)$ be the trap from the above family.

We differentiate two cases.

If $v_{\beta,n} \in G^\beta$ for all $n < \omega$, then we define $y_\beta \in \widehat{P_\beta}$ by

$$y_\beta = \sum_{n<\omega} q_n v_{\beta,n} + \pi_\beta b_\beta$$

and put

$$G^{\beta+1} = \left\langle G^\beta, y_\beta \right\rangle_* = \langle B, y_\gamma(\gamma \leq \beta)\rangle_* .$$

From the Step Lemma 3.2.2 we know that $G^{\beta+1}$ is also a pure $\mathbb{S}$-cotorsion-free submodule of $\widehat{B}$. Moreover, $y_\beta \neq 0$ satisfies (1) since $\|y_\beta\| = \sup_{n<\omega} \|v_{\beta,n}\|$ and $G^{\beta+1}$ satisfies (2) and (3).

If $v_{\beta,n} \notin G^\beta$ for some $n < \omega$ then we do not extend G^β, i.e. we put $G^{\beta+1} = G^\beta$ $(y_\beta = 0)$. Clearly, the above conditions remain satisfied.

Finally, let

$$G = \bigcup_{\beta<\lambda} G^\beta = \langle B,\ y_\beta(\beta < \lambda)\rangle_*.$$

Note, that $G^\beta \neq G^{\beta+1}$ (i.e. $y_\beta \neq 0$) happens "often" since the prediction in Corollary 3.1.7 can, for example, be applied to the set $U = \mathrm{pr}(B)$ of all pure elements of $B = G^0$. $\square$

It is immediate from the construction that $|G| = \lambda$ and that G is a pure $\mathbb{S}$-cotorsion-free submodule of $\widehat{B}$.

Next we describe the elements of G.

Lemma 3.2.4. *Let G be as in Construction 3.2.3.*

(a) *The set $\{e_\alpha|\alpha < \lambda\} \cup \{y_\beta|\beta < \lambda, y_\beta \neq 0\}$ is linearly independent (over R), i.e. $\langle B,\ y_\beta(\beta < \lambda)\rangle = B \oplus \bigoplus_{\beta<\lambda} Ry_\beta$ is a free R-module.*

(b) *If $g \in G \setminus B$ then there are a finite non-empty subset N of λ and $k < \omega$ such that $q_k g \in B \oplus \bigoplus_{\beta\in N} Ry_\beta$ and $\|\,[g] \cap [y_\beta]\,\| = \|y_\beta\| = \|t_\beta\|$ iff $\beta \in N$. In particular, if $\|g\|$ is a limit ordinal then $\|g\| = \|y_{\max N}\|$.*

Proof. The conclusion of part (a) follows easily from $\|\,[y_\gamma] \cap [y_\beta]\,\| < \|y_\beta\| = \|t_\beta\|$ for $\gamma < \beta < \lambda$ and $y_\beta \neq 0$. Part (b) follows from $q_k g \in B \oplus \bigoplus_{\beta<\lambda} Ry_\beta$ for some $k < \omega$ (cf. proofs of the Lemmas 1.2.4 and 2.2.4). $\square$

Using the above lemma we prove further properties of the module G.

Lemma 3.2.5. *Let G be as in Construction 3.2.3 and define G_α $(\alpha < \lambda)$ by $G_\alpha := \{g \in G | \|g\| < \alpha\}$. Then:*

(a) *$\{G_\alpha|\alpha < \lambda\}$ is a λ-filtration of G;*

(b) *if $\beta < \lambda$, $\alpha < \lambda$ are ordinals such that $\|t_\beta\| = \alpha$ then $G_\alpha \subseteq G^\beta$;*

(c) *if $\alpha \notin E$ then $G_{\alpha+1}/G_\alpha$ is free; and*

(d) *if $\alpha \in E$ and $G_{\alpha+1}/G_\alpha \neq 0$ then $G_{\alpha+1}/G_\alpha$ contains a non-zero S-divisible submodule.*

Proof. First we show (a). Let $\alpha < \lambda$ be arbitrary. Then we clearly have $G_\alpha \subseteq G_{\alpha+1}$. Moreover, $|G_\alpha| \leq \left|\widehat{B_\alpha}\right| \leq |B_\alpha|^{\aleph_0} = (|R| \cdot |\alpha|)^{\aleph_0} \leq \mu^{\aleph_0} = \mu < \lambda$ (recall: $B_\alpha = \bigoplus_{\delta<\alpha} Re_\delta$). It is easy to see that the increasing chain of the G_αs is smooth and that $G = \bigcup_{\alpha<\lambda} G_\alpha$ holds.

To see (b) let $\beta < \lambda$, $\alpha < \lambda$ with $\|t_\beta\| = \alpha$ and let $g \in G_\alpha$. If $g \in B$ we are finished. Otherwise, by Lemma 3.2.4, $q_k g \in B \oplus \bigoplus_{\gamma \in N} R y_\gamma$ for some finite $N \subseteq \lambda$ and $k < \omega$ such that $[g] \cap [y_\gamma]$ is infinite iff $\gamma \in N$. This implies $\|t_\gamma\| = \|y_\gamma\| \leq \|g\| < \alpha = \|t_\beta\|$ for all $\gamma \in N$ and thus $N \subseteq \beta$ by Corollary 3.1.7(ii). Hence $g \in G^\beta \subseteq_* G$ and so (b) is proven.

Next we show (c). Let $\alpha < \lambda$ with $\alpha \notin E$. If α is a limit ordinal then, by Corollary 3.1.7(i) and Lemma 3.2.4, there is no element of norm α in G and so $G_{\alpha+1}/G_\alpha = 0$ in this case.

If $\alpha = \delta + 1$ ($\delta \in \lambda$) then $\|e_\delta\| = \alpha$ and any element $g \in G$ with $\|g\| = \alpha$ can be written as $g = re_\delta + g'$ ($r \in R, g' \in G_\alpha$). Therefore $G_{\alpha+1}/G_\alpha = \langle e_\delta + G_\alpha \rangle \cong R$ in this case. Thus $G_{\alpha+1}/G_\alpha$ is free for $\alpha \notin E$ as required.

Finally we show (d). To do so let $\alpha \in E$ with $G_\alpha \neq G_{\alpha+1}$. Then there is an element $g \in G$ with $\|g\| = \alpha$. It hence follows from Lemma 3.2.4 that there exists $\beta < \lambda$ such that $y_\beta \neq 0$ and $\alpha = \|y_\beta\|$ ($= \|t_\beta\|$). This implies $G^\beta \neq G^{\beta+1}$ and hence $v_{\beta,n} \in G^\beta$ with $\|v_{\beta,n}\| < \alpha$ (see Construction 3.2.3 and Definition 3.1.1 of a Signac-branch). Therefore $v_{\beta,n} \in G_\alpha$ for all $n < \omega$. We also know that $b_\beta \in P_\beta \subseteq G_\alpha$. Thus we deduce $y_\beta \equiv q_n \left(\sum_{k \geq n} \frac{q_k}{q_n} v_{\beta,k} + \pi_\beta^{(n)} b_\beta \right) \mod G_\alpha$ for all $n < \omega$, where $\pi_\beta - q_n \pi_\beta^{(n)} \in R$. So $y_\beta + G_\alpha$ is an S-divisible element of $G_{\alpha+1}/G_\alpha$. $\square$

Note, that not all of the above properties are of importance in the here considered context; we included them for completeness.

Finally we are ready to prove the main theorem of this subsection, i.e. the existence of an ultra-cotorsion-free R-module G.

Proof of Theorem 3.2.1. Let G be the R-module as constructed in 3.2.3. We already know that $|G| = \lambda$ and that $G \subseteq_* \widehat{B}$ is $\mathbb{S}$-cotorsion-free.

It remains to show that G is ultra-cotorsion-free. To do so let H be a submodule of G and let $\psi : G \to G/H$ be the canonical epimorphism. Suppose, for contradiction, that $0 \neq G/H$ is $\mathbb{S}$-cotorsion-free and $|H| = \lambda$ ($= |G|$). Since G/H is $\mathbb{S}$-reduced and $G/B \subseteq_* \widehat{B}/B$ is S-divisible H cannot contain B. Therefore, by the $\mathbb{S}$-torsion-freeness of G/H (iff $H \subseteq_* G$), there is a pure element b of B such that $b \notin H (= \ker \psi)$, i.e. $b\psi \neq 0$. We are going to show that $0 \neq \pi b \psi \in G/H$ for all $\pi \in \widehat{R}$. Let $U = \mathrm{pr}(H)$ and let $\pi \in \widehat{R}$ be arbitrary. Then $|U| = \lambda$ and so, by Corollary 3.1.7(iv), there exists $\beta < \lambda$ such that $\{v_{\beta,n} | n < \omega\} \subseteq U \subseteq H \subseteq G$, $b = b_\beta$ and $\pi = \pi_\beta$. Let $\|P_\beta\| = \alpha$. By the definitions of a Signac-branch and of a trap we then obtain $\{v_{\beta,n} | n < \omega\} \subseteq G_\alpha$ where $G_\alpha \subseteq G^\beta$ by Lemma 3.2.5. Therefore it follows from the Construction 3.2.3 that

$0 \neq y_\beta$ $(= \sum_{n<\omega} q_n v_{\beta,n} + \pi b) \in G$ and so $y_\beta \psi \in G/H$. Identifying ψ with its unique extension to $\widehat{B}$ and using the continuity of ψ we deduce

$$y_\beta \psi = \left(\sum_{n<\omega} q_n v_{\beta,n} \right) \psi + (\pi b)\psi = \sum_{n<\omega} q_n \underbrace{(v_{\beta,n}\psi)}_{=0} + \pi(b\psi) = \pi(b\psi)$$

and thus $\pi(b\psi) \in G/H$. Since $\pi \in \widehat{R}$ was arbitrary and $b\psi \neq 0$, we deduce $0 \neq \pi(b\psi) \in G/H$ for all $\pi \in \widehat{R}$, contradicting the $\mathbb{S}$-cotorsion-freeness of G/H. Therefore such a submodule H does not exist, i.e. for G/H to be $\mathbb{S}$-cotorsion-free we need $G = H$ or $|H| < \lambda$ as required, and so the proof is finished. □

Finally note, that the above R-module can be shown to be $\aleph_1$-free using standard arguments (e.g. see [GSW] or [P]).

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