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RESEARCH ARTICLE

N – real fields

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Abstract. A field F is n-real if -1 is not the sum of n squares in F. It is shown that a field F is m-real if and only if rank (AA^t) = rank (A) for every $n \times m$ matrix A with entries from F. An n-real field F is n-real closed if every proper algebraic extension of F is not n-real. It is shown that if a 3-real field F is 2-real closed, then F is a real closed field. For F a quadratic extension of the field of rational numbers, the greatest integer n such that F is n-real is determined.

A field F is formally real if -1 is not a sum of squares in F, or equivalently if 0 is not a sum of squares in F with non-zero summand. The study of these fields was initiated by Artin and Schreier, [1]. Many results on vector spaces over a subfield of the field of real numbers remain valid if the field of scalars is formally real; e.g. many results on real quadratic forms. For finite dimensional vector spaces the following weaker condition often suffices:

Definition. *Let n be a positive integer, and let $\nu = (a_1, \ldots, a_n)$, $\omega = (b_1, \ldots, b_n) \in F^n$. The scalar product $\nu \cdot \omega = a_1b_1 + \ldots + a_nb_n$. A field F is n-*real *if for every non zero-vector $\nu \in F^n$ the scalar product $\nu \cdot \nu \neq 0$.*

Clearly every field is 1-real. For $n > 1$, a field F is n-real if and only if -1 is not the sum of $n-1$ squares, and F is a formally real field if and only if F is n-real for every positive integer n. If F is n-real then F is m-real for every $m < n$.

Example. The field $\mathbb{Q}(\sqrt{-5})$ is 2-real but not 3-real.

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Theorem 1. *A field F is m-real if and only if for every positive integer n, and every $n \times m$ matrix A with entries from F, the rank $r(AA^t) = r(A)$.*

Proof. Suppose that F is m-real, and let $r(A) = k$. Performing a Gram-Schmidt like process on k linearly independent rows of A, with the scalar product replacing the inner product, yields orthogonal vectors $\nu_1, ..., \nu_k \in F^m$. The $n \times m$ matrix B with first k rows $\nu_1, ..., \nu_k$, and with remaining $n - k$ rows the 0-vector, can be obtained by performing a series of elementary row operations on A. Therefore there exists an $n \times n$ matrix C with entries from F such that $B = CA$. The matrix BB^t is diagonal with first k diagonal entries non-zero, and remaining entries 0. Therefore $k = r(BB^t) \leq r(AA^t) \leq r(A) = k$, and so $r(AA^t) = r(A)$. If F is not m-real then there exists a non-zero vector $\nu \in F^m$ such that $\nu \cdot \nu = 0$. For any positive integer n let A be the $n \times m$ matrix all of whose rows are ν. Then $r(A) = 1$, but $r(AA^t) = 0$. □

If every proper algebraic extension of a formally real field F is not formally real, then F is said to be a real closed field. The obvious parallel concept for n-real fields is:

Definition. *An n-real field F is an n-real closed field if every proper algebraic extension of F is not n-real.*

A simple Zorn's Lemma argument yields:

Lemma 2. *Every n-real field, $n > 1$, is contained in an n-real closed field.*

Lemma 3. *Let F be a 2-real closed field and let $a \in F$. Then either a or $-a$ is a square in F.*

Proof. If a is not a square in F then $F(\sqrt{a})$ is not 2-real, so there exist $b, c \in F$ such that $(b + c\sqrt{a})^2 = -1$, i.e., $b^2 + c^2 a + 2bc\sqrt{a} = -1$. Since $\sqrt{a} \notin F$, and F is 2-real it follows that $b = 0$, and $-a = (c^{-1})^2$. □

Recall [4], p. 271, that a field F is ordered if there exists a subset P of F such that $F = P \bigcup \{0\} \bigcup -P$ is a disjoint union, and $a + b$, $ab \in P$ for all $a, b \in P$. A well known result of Artin-Schreier is that a field is formally real if and only if it is ordered.

Corollary 4. *Let F be a 2-real closed field. If F is 3-real then F is real closed.*

Proof. Let F be a 2-real closed, 3-real field. It suffices to show that F is ordered. Let P be the set of non-zero squares in F. It follows from

Lemma 3 that $F = P \bigcup \{0\} \bigcup -P$. If $a \in P \bigcap -P$, then there exist nonzero elements $b, c \in F$ such that $a = b^2 = -c^2$, and so $b^2 + c^2 = 0$, a contradiction. Therefore the above union is a disjoint union. Let $a, b \in P$. Clearly $ab \in P$, so it suffices to show that $a + b \in P$. If not, then by Lemma 3 there exists $c \in F$ such that $a + b = -c^2$. Since $a, b \in P$ there exist $a_1, b_1 \in F$ such that $a = a_1^2$, and $b = b_1^2$. Therefore $a_1^2 + b_1^2 + c^2 = 0$ contradicting the fact that F is 3-real. □

If F is a real closed field, and $f(x) \in F[x]$ is a polynomial of odd degree, then $f(x)$ has a root in F; see [8], p. 226, Theorem 2. An almost identical argument yields:

Theorem 5. *If F is an n-real closed field, $n > 1$, and $f(x) \in F[x]$ is a polynomial of odd degree, then $f(x)$ has a root in F.*

Let F be a field of prime characteristic p. Since 0 is the sum of p copies of 1^2 it follows that F is not formally real. The following known number theory result yields that properties of p determine completely whether or not F is n-real for every positive integer n.

Proposition 6. *Let n be a positive integer.*

1) n is the sum of two squares of integers if and only if the prime decomposition of n has no factor of the form q^e, with q a prime satisfying $q \equiv 3 \bmod 4$, and e odd.

2) n is not the sum of three squares of integers if and only if $n = 4^m(8k + 7)$, with m, k non-negative integers.

Proof. See [5], p. 110 Corollary 5.14, and [7], p. 45, Theorem (Gauss). □

Theorem 7. *A field F of prime characteristic p is not 3-real. It is 2-real if and only if $p \equiv 3 \bmod 4$.*

Proof. Since $1^2 + 1^2 \equiv 0 \bmod 2$ it may be assumed that p is odd. If $p \not\equiv 7 \bmod 8$ then p is the sum of three squares of integers by Proposition 6.2, so F is not 3 - real. If $p \equiv 7 \bmod 8$ then $2p \equiv 6 \bmod 8$ so $2p$ is the sum of three squares of integers by Proposition 6.2 which yields that F is not 3-real. The field F is 2-real if and only if -1 is a quadratic nonresidue mod p, which occurs if and only if $p \equiv 3 \bmod 4$. □

A well known result of Lagrange is that every positive integer is the sum of 4 squares of integers. This yields:

Lemma 8. *Let $F = \mathbb{Q}(\sqrt{a})$, $\alpha \in \mathbb{Q}$ be a quadratic extension of the field of rational numbers. If F is not real then F is not 5-real.*

Proof. If F is not real then it may be assumed that α is a negative integer, [5], Theorem 9.20. Since $-\alpha$ is the sum of 4 squares of integers, it follows that F is not 5-real. □

Definition. *Let F be a field which is not formally real. The least positive integer n such that -1 is the sum of n squares in F was called the Stuffe of F by Pfister,* [6]*; it is, of course, the greatest positive integer n such that F is n-real.*

Pfister proved the following:

Proposition 9. *Let n be a positive integer. There exists a field with Stuffe n if and only if $n = 2^k$, with k a non-negative integer.*

Proof. See [6], Satz 4 and Satz 5. □

Lemma 8 and Proposition 9 yield:

Corollary 10. *Let F be a quadratic extension of $\mathbb{Q}$. If F is not real then the Stuffe of F is either* 1*,* 2*, or* 4*.*

Fein, Gordon and Smith proved the following:

Proposition 11. *For m a negative square free integer, -1 is the sum of two squares in $\mathbb{Q}(\sqrt{m})$ if and only if $m \equiv 2$ or $3 \bmod 4$, or $m \equiv 5 \bmod 8$.*

Proof. [3] Theorem 7. □

Since every imaginary quadratic extension of $\mathbb{Q}$ is of the form $\mathbb{Q}(\sqrt{m})$, with m a square free negative integer, Corollary 10 and Proposition 11 completely determine the Stuffe of such extensions as follows:

Theorem 12. *For m a square free negative integer the Stuffe of $\mathbb{Q}(\sqrt{m})$ is :*

1 if $m = -1$,
2 if $m \equiv 2$ or $3 \bmod 4$, or if $m \equiv 5 \bmod 8$, and
4 otherwise.

Example. The Stuffe of $\mathbb{Q}(\sqrt{-7})$ is 4.

If A is a commutative ring and if $a, b \in A$ are both the sum of four squares in A, then an equality of Euler, [5], Lemma 5.3, yields that ab is the sum of four squares in A. The following generalization of Euler's result for fields was proved by Pfister.

Proposition 13. *Let F be a field, and let $n = 2^m$, with m a non-negative integer. If $a, b \in F$ are both the sum of n squares in F then ab is the sum of n squares in F.*

Proof. See [6], Satz 2. □

Corollary 14. *Let F be a field extension of $\mathbb{Q}$. If F is formally real then $a \in F$ is the sum of four squares in F if and only if $a \geq 0$. If F is not formally real, then the Stuffe of F is ≤ 4 if and only if every rational number is the sum of four squares in F.*

Proof. Every positive integer is the sum of four squares of integers, [5], Theorem 5.6. If a non-zero element a in a field E is the sum of n squares in E, then it is readily seen that a^{-1} is the sum of n squares in E. Therefore either Euler's equality, or Proposition 13 yield that every non-negative rational number is the sum of four squares of rational numbers. If a negative rational number a is the sum of four squares in F then $-1 = a(1/|a|)$ is the sum of four squares and the Stuffe of F is ≤ 4. Conversely, if the Stuffe of F is ≤ 4 then every rational number is the sum of four squares in F by Proposition 13. □

The following Proposition combines two results of Cassels:

Proposition 15. *Let F be a field with characteristic $\neq 2$, let $a \in F$, and let x be an indeterminant. Then $x^2 + a$ is the sum of $n > 1$ squares in $F[x]$ if and only if either -1 or a is the sum of $n - 1$ squares in F.*

Proof. See [2], Theorem 2. □

A simple consequence of Proposition 15 is:

Corollary 16. *If F is a non-formally real field with characteristic $\neq 2$, and with Stuffe n, then every element in F is the sum of $n + 1$ squares in F.*

Proof. Let $a \in F$. By Proposition 15, there exist

$$p_i(x) \in F[x], \quad i = 1, ..., n + 1,$$

such that $x^2 + a = \sum_{i=1}^{n+1} p_i(x)^2$, so $a = \sum_{i=1}^{n+1} p_i(0)^2$. □

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