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RESEARCH ARTICLE

On the representation of a number as a sum of the k-th powers in an arithmetic progression

N.S. Prosyanyuk

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Abstract. In this paper we obtain the asymptotic formula for a natural $n \leqslant x$ which representate as a sum of two non-negative k-th powers in an arithmetic progression.

In the works of C. Hooley [1], E. Kratzël [2] W. Müller and W. Nowak ([3],[4]) have obtained asymptotic formulaes for a sum

$$\sum_{n \leqslant x} r_k^a(n), \quad (a = 1, 2),$$

where $r_k(n)$ is the number of representations of the positive integer n as a sum of two non-negative k-th powers.

In 1986 E. Bombieri and H. Íwaniec [5] developed entirely new ideas to estimate exponential sums and then M. Huxley ([6],[7]) this method (which he christened "discrete Hardy-Littlwood method") to derive significant improvements in the problem on the number of lattice points in planal domains.

In the present paper, we apply the result of Huxley in order to obtain an asymptotic formula for

$$\sum_{\substack{n \equiv l (\mathrm{mod}\ q) \\ n \leqslant x}} r_k(n).$$

For $k = 2$ P. Varbanets [8] obtained an asymptotic formula which is non-trivial if $q \leqslant x^{\frac{2}{3} - \varepsilon}$.

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Notation. We write $[x]$ for the integer part of the real number x;
$\Psi(x) = x - [x] - \frac{1}{2}, \quad e_q(x) = \exp\left(\frac{2\pi i x}{q}\right)$;
$(a_1, \ldots, a_m)$ denote greatest common divisor of $a_1, a_2, \ldots, a_m$;
$\Gamma(u)$ is gamma function.

The symbol $O(z)$ stands for any term whose absolute value as at most $B \cdot z$, where B is a dimensionless constant.

The Vinogradov symbol $Z_1 \ll Z_2$ means $Z_1 = O(Z_2)$. We use ε for a positive exponent which may be taken arbitrarily close to zero.

We prove the following theorem:

Theorem. *For natural numbers l and q, $1 \leqslant l < q$, $(l, q) = 1$, we have for $k \geqslant 3$*

$$\sum_{\substack{n \equiv l (\text{mod } q) \\ n \leqslant x}} r_k(n) = a_0(k) \frac{I(l,q)}{q^2} x^{\frac{2}{k}} +$$

$$+ a_1(k) \left(\frac{x}{q^k}\right)^{\frac{1}{k} - \frac{1}{k^2}} \sum_{m=1}^{\infty} \frac{\Omega(m,l,q)}{m^{1+\frac{1}{k}}} \sin\left(2\pi m x^{\frac{1}{k}} - \frac{\pi}{2k}\right) +$$

$$+ \frac{x^{\frac{1}{k}}}{q} \sum_{\substack{l_1^k + l_2^k \equiv l (q) \\ 0 < l_1, l_2 < q}} \left(\Psi\left(\frac{l_1}{q}\right) + \Psi\left(\frac{l_2}{q}\right) + 1\right) -$$

$$- \left(\frac{x}{2q^{2k}}\right)^{\frac{1}{k}} \sum_{\substack{l_1^k + l_2^k \equiv l (q) \\ 0 < l_1, l_2 < q}} (l_1 + l_2) + O\left(x^{46/73k} \cdot q^{27/73} \cdot \log^3 x\right),$$

where

$$a_0(k) = \frac{\Gamma^2\left(\frac{1}{k}\right)}{2 \cdot k \cdot \Gamma\left(\frac{2}{k}\right)},$$

$$a_1(k) = \frac{2}{\pi} \left(\frac{k}{2\pi}\right)^{\frac{1}{k}} \Gamma\left(1 + \frac{1}{k}\right),$$

$$\Omega(m,l,q) = \sum_{l_1^k + l_2^k \equiv l (q)} e_q(m l_1)$$

$$I(l,q) = \sum_{\substack{l_1, l_2 (\text{mod } q) \\ l_1^k + l_2^k \equiv l (q)}} 1 \text{ *uniformly in* } l, q \leqslant x^{\frac{1}{k} - \varepsilon}.$$

In order this result obtain we use the following propositions.

Lemma 1. *Let $0 \leqslant \alpha, \beta < 1$. Then for $X \to \infty$ we have*

$$\sum_{\substack{m,n\geqslant 0\\ (m+\alpha)^k+(n+\beta)^k\leqslant X}} 1 =$$

$$= a_0(k)X^{\frac{2}{k}} + a_1(k)\sum_{n=1}^{\infty}\frac{\cos 2\pi n\alpha + \cos 2\pi n\beta}{n^{1+\frac{1}{k}}} sin\Big(2\pi n X^{\frac{1}{k}} - \frac{\pi}{2k}\Big) -$$

$$- \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\alpha<m\leqslant X^{\frac{1}{k}}-\alpha} \Psi\left(\left(X-(m+\alpha)^k\right)^{\frac{1}{k}}\right) -$$

$$- \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\beta<n\leqslant X^{\frac{1}{k}}-\beta} \Psi\left(\left(X-(n+\beta)^k\right)^{\frac{1}{k}}\right) +$$

$$+ X^{\frac{1}{k}}(\Psi(\alpha)+\Psi(\beta)+1) - \left(\frac{X}{2}\right)^{\frac{1}{k}}(\alpha+\beta) + O(1).$$

Proof. For $k \in N$: we get

$$\sum_{\substack{m,n\geq 0\\ (m+\alpha)^k+(n+\beta)^k\leq X}} 1 = \sum_{(m+\alpha)^k\leq X} 1 + \sum_{(n+\beta)^k\leq X} 1 + \sum_{\substack{m,n\geq 1\\ (m+\alpha)^k+(n+\beta)^k\leq X}} 1 =$$

$$= 2X^{\frac{1}{k}} - \alpha - \beta - 1 - \Psi\left(X^{\frac{1}{k}}-\alpha\right) - \Psi\left(X^{\frac{1}{k}}-\beta\right) +$$

$$+ \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}<m+\alpha\leq X^{\frac{1}{k}}} \left[\left(X-(m+\alpha)^k\right)^{\frac{1}{k}}\right] + \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}<n+\beta\leq X^{\frac{1}{k}}} \left[\left(X-(n+\beta)^k\right)^{\frac{1}{k}}\right] +$$

$$+ \sum_{(m+\alpha)\leq\left(\frac{X}{2}\right)^{\frac{1}{k}}} \sum_{(n+\beta)\leq\left(\frac{X}{2}\right)^{\frac{1}{k}}} 1 = 2X^{\frac{1}{k}} - \alpha - \beta - 1 - \Psi\left(X^{\frac{1}{k}}-\alpha\right) - \Psi\left(X^{\frac{1}{k}}-\beta\right) +$$

$$+ \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\alpha<m\leq X^{\frac{1}{k}}-\alpha} \left(X-(m+\alpha)^k\right)^{\frac{1}{k}} + \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\beta<n\leq X^{\frac{1}{k}}-\beta} \left(X-(n+\beta)^k\right)^{\frac{1}{k}} -$$

$$- X^{\frac{1}{k}} + \left(\frac{X}{2}\right)^{\frac{1}{k}} - \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\alpha<m\leq X^{\frac{1}{k}}-\alpha} \Psi\left(\left(X-(m+\alpha)^k\right)^{\frac{1}{k}}\right) -$$

$$- \sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\beta<n\leq X^{\frac{1}{k}}-\beta} \Psi\left(\left(X-(n+\beta)^k\right)^{\frac{1}{k}}\right) +$$

$$+ \left(\left(\frac{X}{2}\right)^{\frac{1}{k}} - \alpha - \frac{1}{2} - \Psi\left(\left(\frac{X}{2}\right)^{\frac{1}{k}}-\alpha\right)\right)\left(\left(\frac{X}{2}\right)^{\frac{1}{k}} - \beta - \frac{1}{2} - \Psi\left(\left(\frac{X}{2}\right)^{\frac{1}{k}}-\alpha\right)\right)$$

Now, by using Euler summation formula we infer after simple calculation

$$\sum_{\substack{m,n\geqslant 0\\ (m+\alpha)^k+(n+\beta)^k\leqslant X}} 1 =$$

$$= a_0(k)X^{\frac{2}{k}} + X^{\frac{1}{k}}(1+\Psi(\alpha)+\Psi(\beta)) - \left(\frac{X}{2}\right)^{\frac{1}{k}}(\alpha+\beta)+$$

$$+ a_1(k)X^{\frac{1}{k}-\frac{1}{k^2}}\sum_{n=1}^{\infty}(\cos 2\pi n\alpha+\cos 2\pi n\beta)\frac{\sin\left(2\pi nX^{\frac{1}{k}}-\frac{\pi}{2k}\right)}{n^{1+\frac{1}{k}}}-$$

$$-\sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\alpha<m\leqslant X^{\frac{1}{k}}-\alpha}\Psi\left(\left(X-(m+\alpha)^k\right)^{\frac{1}{k}}\right)-$$

$$-\sum_{\left(\frac{X}{2}\right)^{\frac{1}{k}}-\beta<n\leqslant X^{\frac{1}{k}}-\beta}\Psi\left(\left(X-(n+\beta)^k\right)^{\frac{1}{k}}\right)+O(1).$$

□

Lemma 2 (see [7], Theorem 3). *Let $F(x)$ be a real function with three continuous derivatives for $M \leqslant x \leqslant 2M$. Let $C \geqslant 1$ be a constant. Suppose that for $M \leqslant x \leqslant 2M$ either Case 1 or Case 2 holds:*

Case 1. The derivatives $F''(x)$ and $F'''(x)$ are non-zero and $M \leqslant CT^{1/2}$.

Case 2. The derivatives $F'(x)$ and $F''(x)$ and the expression $F'(x)\cdot F'''(x)-3(F''(x))^2$ are non-zero, and $M \geqslant C^{-1}\cdot T^{1/2}$.

Let S be sum

$$S=\sum_{M\leqslant m\leqslant M_1<2M}\Psi\left(\frac{T}{M}F\left(\frac{m}{M}\right)\right).$$

Then in both cases $S \ll T^{23/73}\cdot(\log T)^{315/146}$ for $T^{63/146}\cdot(\log T)^{63/242}\leqslant M\leqslant T^{87/146}\cdot(\log T)^{-63/292}$.

Lemma 3. *Let Let $0\leqslant l<q$, $(l,q)=1$. Then*

$$\sum_{\substack{l_1 l_2 (\mathrm{mod}\ q)\\ l_1^k+l_2^k\equiv l (\mathrm{mod}\ q)}} e_q(m_1l_1+m_2l_2) \ll q^{\frac{1}{2}}(m_1,m_2,q)^{\frac{1}{2}}\tau(q).$$

In order this estimation we use Bombieri's theorem on exponential sums in finite fields (see [5]).

Now we are a position to prove our theorem.

Let in Lemma 1 $\alpha = \frac{l_1}{q}$, $\beta = \frac{l_2}{q}$, where $l_1, l_2 \bmod q$ and $l_1^k + l_2^k \equiv l(\bmod q)$. Then we have

$$\sum_{\substack{n\equiv l(\bmod q)\\ n\leqslant x}} r_k(n) = \sum_{\substack{l_1 l_2 (\bmod q)\\ l_1^k+l_2^k\equiv l(\bmod q)}} \sum_{\substack{m,n\geqslant 0\\ (m+\frac{l_1}{q})^k+(n+\frac{l_2}{q})^k\leqslant X}} 1 =$$

$$= a_0(k)\frac{I(l,q)}{q^2}x^{\frac{2}{k}} + \sum_{l_1^k+l_2^k\equiv l(q)} \left(\Psi\left(\frac{l_1}{q}\right) + \Psi\left(\frac{l_2}{q}\right) + 1\right)\frac{x^{\frac{1}{k}}}{q} -$$

$$-\left(\frac{x}{2q^k}\right)^{\frac{1}{k}} \frac{1}{q} \sum_{l_1^k+l_2^k\equiv l(q)} (l_1+l_2) +$$

$$+ a_1(k)\sum_{m=1}^{\infty}\left(\frac{x}{q^k}\right)^{\frac{1}{k}-\frac{1}{k^2}} \sum_{l_1^k+l_2^k\equiv l(q)} e_q(ml_1)\frac{\sin\left(2\pi m\frac{x^{\frac{1}{k}}}{q} - \frac{\pi}{2k}\right)}{m^{1+\frac{1}{k}}} -$$

$$-\sum_{l_1^k+l_2^k\equiv l(q)} \left(\sum_{X_1<n_1\leqslant X_2} \Psi\left(\left(X - \left(n_1+\frac{l_1}{q}\right)^k\right)^{\frac{1}{k}}\right) + \right.$$

$$\left. + \sum_{Y_1<n_2\leqslant Y_2} \Psi\left(\left(X - \left(n_2+\frac{l_2}{q}\right)^k\right)^{\frac{1}{k}}\right)\right) + O(I(l,q)),$$

where $X = \frac{x}{q^k}$, $X_1 = \left(\frac{X}{2}\right)^{\frac{1}{k}} - \frac{l_1}{q}$, $Y_1 = \left(\frac{X}{2}\right)^{\frac{1}{k}} - \frac{l_2}{q}$, $X_2 = X - \frac{l_1}{q}$, $Y_2 = X - \frac{l_2}{q}$.

The last two sum can be estimated by using lemma 2 and the method W. Müller and W. Nowak which they applied for the proof of theorem 1 (see [4]).

Corollary. *For* $q \ll x^{\frac{1}{k}-\varepsilon}$

$$\sum_{\substack{n\equiv l(\bmod q)\\ n\leqslant x}} r_k(n) = a_0(k)\frac{I(l,q)}{q^2}x^{\frac{2}{k}} - \left(\frac{x}{2}\right)^{\frac{1}{k}}\frac{1}{q^2}\sum_{l_1^k+l_2^k\equiv l(q)} (l_1+l_2) +$$

$$+ O\left(\frac{x^{\frac{1}{k}-\frac{1}{k^2}}}{q^{\frac{1}{2}-\frac{1}{k}}}\tau(q)\right) + O\left(x^{46/73k}\cdot q^{27/73}\cdot(\log x)^{315/146}\right).$$

This proposition follow from truncating the Fourier series for $\Psi(u)$, lemma 3 and proven theorem.

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Contact information

N.S. Prosyanyuk vul. Varnenskaya 25, kv. 20, Odessa 65070, UKRAINE
E-Mail: varb@te.net.ua

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