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Tiled orders over discrete valuation rings, finite Markov chains and partially ordered sets. II

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Abstract. The main concept of this part of the paper is that of a reduced exponent matrix and its quiver, which is strongly connected and simply laced. We give the description of quivers of reduced Gorenstein exponent matrices whose number s of vertices is at most 7. For $2 \leqslant s \leqslant 5$ we have that all adjacency matrices of such quivers are multiples of doubly stochastic matrices. We prove that for any permutation σ on n letters without fixed elements there exists a reduced Gorenstein tiled order Λ with $\sigma(\mathcal{E}) = \sigma$. We show that for any positive integer k there exists a Gorenstein tiled order Λ_k with $in\Lambda_k = k$. The adjacency matrix of any cyclic Gorenstein order Λ is a linear combination of powers of a permutation matrix P_σ with non-negative coefficients, where $\sigma = \sigma(\Lambda)$. If A is a noetherian prime semiperfect semidistributive ring of a finite global dimension, then $Q(A)$ be a strongly connected simply laced quiver which has no loops.

1. Introduction

This is the second part of a work whose first part is [2]. We use terminology, definitions and results given in [2]. All rings are associative with $1 \neq 0$. The terms "artinian", "noetherian", etc. will refer to both sides of a ring, in particular, an "artinian ring" means a right artinian ring which is also left artinian.

Let $\mathcal{O}$ be a complete discrete valuation ring with the field of fractions K, A a finite-dimensional separable K-algebra and Λ a completely

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decomposable order in A, i.e. each local idempotent $e \in \Lambda$ is a local idempotent of A (see [24]). Then Λ is a finite direct product of tiled orders if and only if Λ is the intersection of its maximal overrings.

A semiperfect semidistributive ring shall be called an $SPSD$-ring. According to the "Decomposition theorem for noetherian semiprime $SPSD$-rings" (see [25], [2], Theorem 3.8), the tiled orders over (non-necessarily commutative) discrete valuation rings are exactly the noetherian (but not artinian) prime $SPSD$-rings. Since no other orders are considered in the paper, by a tiled order we shall always mean a tiled order Λ over a discrete valuation ring $\mathcal{O}$, and write $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda)$ is the exponent matrix.

If $\mathcal{E}(\Lambda)$ is a $(0,1)$-matrix, then Λ is a $(0,1)$-order. The next condition is often used in the theory of orders (see [10], [11], [32], [33], [40]): for a given order Λ there exists a maximal order Γ such that

$$rad\,\Gamma \subset \Lambda \subset \Gamma.$$

It is easy to see that if a tiled order Λ satisfies the above condition is tiled then Λ is necessarily a $(0,1)$-order.

The main concept of this part of the work is that of a reduced exponent matrix and its quiver. Note that exponent matrices appeared first in the study of completely decomposable orders (see [26], [42]) and were used for the investigation of semimaximal rings of finite type (see [43], [44]).

In Section 2 we introduce the notion of equivalence for reduced exponent matrices and show that the quivers of equivalent exponent matrices can be obtained from each other by a renumeration of vertices. We observe that the quiver of a reduced exponent matrix is strongly connected and simply laced. A strongly connected and simply laced quiver Q is called **admissible** if there exists a reduced exponent matrix $\mathcal{E}$ with $Q(\mathcal{E}) = Q$.

We give the list of admissible quivers without loops for $2 \leqslant s \leqslant 4$, where s is the number of vertices of $Q(\mathcal{E})$. The number of these quivers for $s = 2$ is 1; for $s = 3$ is 2 and for $s = 4$ is 11 (Section 3). It is shown in [38] and [39] that if we remove all loops from the admissible quivers with $2 \leqslant s \leqslant 4$ then we obtain all strongly connected quivers from [16] (Appendix 2, Digraph diagrams).

Sections 4 and 5 are devoted to the description of quivers of reduced Gorenstein exponent matrices whose number of vertices is at most 7. For $2 \leqslant s \leqslant 5$ we have that all adjacency matrices of such quivers are multiples of doubly stochastic matrices. In Section 6 we prove that for any permutation σ on n letters without fixed elements there exists a reduced

Gorenstein exponent matrix $\mathcal{E}$ with $\sigma(\mathcal{E}) = \sigma$. We also show that the Cayley table of the elementary abelian 2-group of order 2^k is a reduced Gorenstein exponent matrix and its index equals $k+1$. Therefore, for any positive integer k there exists a Gorenstein tiled order Γ_{k-1} such that $in\Gamma_{k-1} = k$ (Section 7).

With respect to the results of Sections 6 and 7, it is natural to ask the following question. Suppose that a Latin square $\mathcal{E}$ [21] defined on $S = \{0, 1, \ldots, n-1\}$ is an exponent matrix which is doubly symmetric, that is $\mathcal{E}$ is symmetric with respect to the main diagonal and is also symmetric with respect to the secondary diagonal. Suppose also that the first row of $\mathcal{E}$ is $\{0\,1\,2\,\ldots\,n-1\}$.

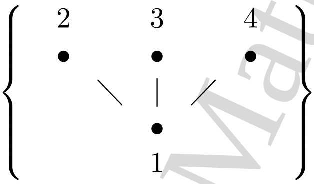
Is it true that $\mathcal{E}$ is necessarily the Cayley table of an elementary abelian 2-group?

The main result of Section 8 is Theorem 8.7. Observe the following fact (Proposition 8.8). *The adjacency matrix of any cyclic Gorenstein order Λ is a linear combination of powers of a permutation matrix P_σ with non-negative coefficients, where $\sigma = \sigma(\Lambda)$.*

The last section deals with the global dimension of tiled orders. R.B. Tarsy conjectured in [37] that the upper bound $gl.dim\,\Lambda \leqslant n-1$ holds for any tiled order in $M_n(\mathcal{D})$, where $\mathcal{D}$ is a division ring of fractions of a discrete valuation ring $\mathcal{O}$. V.A. Jategaonkar (see [19], [20]) proved Tarsy's conjecture for triangular orders and for n =2,3,4, E. Kirkman and J. Kuzmanovich did it for $(0,1)$-orders (see [27]). Observe also that Tarsy's conjecture is true for tiled orders of width 2 (see [5]). In [12] H. Fujita proved Tarsy's conjecture for n =5 and constructed a counterexample to this conjecture. More precisely, he showed that there exists a tiled order Λ_n in $M_n(\mathcal{D})$ with $gl.dim\,\Lambda_n = n$ for $n \geqslant 6$ (see [12], Example 2.5). Tiled orders having large global dimension were considered in [35], [18] and [13]. In what follows we shall often refer to a tiled order Λ and its ideals, indicating only their exponent matrices. Using the approach, proposed by K.W.Roggenkamp in [32] (see also [27], Proposition 2.4), we prove that if a poset P_Λ associated with a $(0,1)$-order Λ has a unique maximal element or unique minimal element, then $gl.dim\,\Lambda < \infty$. If P_Λ is disconnected, then $gl.dim\,\Lambda = \infty$ (see [10]). For the order Ω_n (see [37]) we construct a chain of the projective idempotent ideals $I_1 \subset I_2 \subset \ldots \subset I_{n-1} \subset \Omega_n$ such that the quotient ring Ω_n/I_1 is quasi-hereditary (see [4], [7], [30]).

We use the example by V. Dlab and C. Ringel ([7], p. 283) for a construction of a tiled order Λ of width 2 with $gl.dim\,\Lambda = 4$ such that $Q(\Lambda)$ has five vertices.

We show that all tiled orders of finite global dimension whose quiver has at most four vertices have width 2, except the order F_4 for which

$\mathcal{E}(F_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Note that $P_{F_4} =$ .

In general case $P_{F_n} = \{1, 2, \ldots, n\}$, where $1 \leqslant i$ for $i = 2, \ldots, n$ and the elements $2, 3, \ldots, n$ are pairwise non-comparable. It is easy to see that $gl.dim\, F_n = 2$ for any n.

In the case s=4, using the Drozd-Kirichenko rejection lemma (see [9], [17]), we construct the sequence of tiled orders: $\Omega_4 \subset \mathcal{E}_4 \subset \mathcal{E}_3 \subset \mathcal{E}_2 \subset H_4$ (see Section 3). These orders and F_4 exhaust all orders Λ with $gl.dim\, \Lambda < \infty$ for s=4 (see [12]).

In conclusion observe the following fact given in the last section: If A is a noetherian prime semiperfect semidistributive ring of a finite global dimension, then $Q(A)$ is a strongly connected simply laced quiver which has no loops (see also [41] and [6]).

2. Exponent matrices

Denote by $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over the ring of integers $\mathbb{Z}$. Let $\mathcal{E} \in M_n(\mathbb{Z})$.

Definition 2.1. *We call a matrix $\mathcal{E} = (\alpha_{ij})$, an* **exponent matrix** *if $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 0$ for $i = 1, \ldots, n$. These relations are called* **ring inequalities***. An exponent matrix $\mathcal{E}$ is called* **reduced** *if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$.*

Definition 2.2. *We shall call two exponents matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$* **equivalent** *if they can be obtained from each other by transformations of the following two types :*

(1) subtracting an integer from the i-th row with simultaneous adding it to the i-th column;

(2) simultaneous interchanging of two rows and the equally numbered columns.

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Set $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \ldots, n$, and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1 \leqslant k \leqslant n} (\beta_{ik} + \beta_{kj})$. Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is a $(0,1)$-matrix. By Theorem 4.11 and Corollary 5.3 [2] we have the following assertion.

Theorem A. *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

Definition 2.3. *The quiver $Q(\mathcal{E})$ shall be called the* **quiver of the reduced exponent matrix** *$\mathcal{E}$.*

Definition 2.4. *A strongly connected simply laced quiver shall be called* **admissible** *if it is a quiver of a reduced exponent matrix.*

Definition 2.5. *A reduced exponent matrix* $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ *shall be called* **Gorenstein** *if there exists a permutation* σ *of* $\{1, 2, \ldots, n\}$ *such that* $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ *for* $i, k = 1, \ldots, n$.

The permutation σ is denoted by $\sigma(\mathcal{E})$. Notice that $\sigma(\mathcal{E})$ for a reduced Gorenstein exponent matrix $\mathcal{E}$ has no cycles of length 1.

Definition 2.6. *The* **index** *(in* $\mathcal{E}$*) of a reduced exponent matrix* $\mathcal{E}$ *is the maximal real eigenvalue of the adjacency matrix* $[Q(\mathcal{E})]$ *of* $Q(\mathcal{E})$.

Example 1. The Cayley table of the Klein four-group $(2) \times (2)$ can be written in such form:

$$K = K(4) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

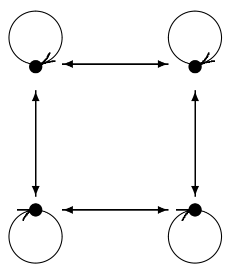
Then $K(4)$ is a reduced Gorenstein exponent matrix with permutation $\sigma = \sigma(K(4)) = (14)(23)$. Obviously,

$$K^{(2)} = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{bmatrix}$$

and

$$[Q(K)] = K^{(2)} - K^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = 3 \cdot P_1,$$

where P_1 is a doubly stochastic matrix, and $Q(K)$ is



Obviously, $in\, K = 3$.

Definition 2.7. (see [31], p. 140). *A quasigroup Q which satisfies the identity $(xu)(vy) = (xv)(uy)$ for $x, y, u, v \in Q$ is called* **entropic**.

Example 2. ([31], p. 141, V. 2.2.1. Example).

Let $Q(5) = \{0, 1, 2, 3, 4\}$ is the quasigroup with the following Cayley table

0	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

It is clearly, that $Q(5)$ is an entropic quasigroup. The Cayley table

$$\mathcal{E}(5) = \begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

of $Q(5)$ is a reduced Gorenstein exponent matrix with $\sigma(\mathcal{E}(5)) = (12345)$.

Obviously,

$$[Q(\mathcal{E}(5))] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = 2P_2,$$

where P_2 is a doubly stochastic matrix, and $in\,\mathcal{E}(5) = 2$.

Definition 2.8. *A reduced Gorenstein exponent matrix $\mathcal{E}$ is called* **cyclic** *if $\sigma(\mathcal{E})$ is a cycle.*

Theorem B. *Let $\mathcal{E}$ be a cyclic reduced Gorenstein exponent matrix. Then $[Q(\mathcal{E})] = \lambda P$, where λ is a positive integer and P is a doubly stochastic matrix.*

The proof follows from ([34], Theorem 3.4).

Example 3. With any finite partially ordered set (poset) P we relate a reduced exponent $(0, 1)$-matrix $\mathcal{E}_P = (\lambda_{ij})$ by the following way: $\lambda_{ij} = 0 \Leftrightarrow i \leqslant j$, otherwise $\lambda_{ij} = 1$.

It is easy to see that $\mathcal{E}_P$ is indeed reduced exponent matrix.

The definition for a diagram $Q(P)$ of a finite poset see in [15] and [2].

Denote by P_{max} (resp. P_{min}) the set of the maximal (resp. minimal) elements of P and by $P_{max} \times P_{min}$ their Cartesian product.

From ([2], Theorem 6.12) we have

Theorem C. *The quiver $Q(\mathcal{E}_P)$ can be obtained from the diagram $Q(P)$ by adding the arrows σ_{ij} for all $(p_i, p_j) \in P_{max} \times P_{min}$.*

Recall that if X and Y are finite posets and $X \oplus Y$ is their ordinal sum (see [1], pp. 198-199), then $\mathcal{E}_{X\oplus Y} = \begin{pmatrix} \mathcal{E}_X & 0_{m\times n} \\ U_{n\times m} & \mathcal{E}_Y \end{pmatrix}$, where m (resp. n) is a number of elements in X (resp. in Y); $0_{m\times n}$ is the zero $m \times n$-matrix and $U_{n\times m}$ is an $n \times m$-matrix, whose every entry is 1. We write $U_{n\times n} = U_n$ and $0_{n\times n} = 0_n$.

Proposition 2.9. *Suppose that $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are exponent matrices and Θ is obtained from $\mathcal{E}$ by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$. If $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.*

Proof. We have

$$\theta_{ij} = \begin{cases} \alpha_{ij}, & \text{if } i \neq l, j \neq l, \\ 0, & \text{for } i = l, j = l, \\ \alpha_{lj} - t, & \text{if } i = l, j \neq l, \\ \alpha_{il} + t, & \text{if } i \neq l, j = l, \end{cases}$$

where t is an integer. It can be directly checked that if $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ for some i, j, k, then $\theta_{ij} + \theta_{jk} = \theta_{ik}$. Since these transformations are invertible, the converse transformations have similar form. So the equality $\theta_{ij} + \theta_{jk} = \theta_{ik}$ implies $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$. Therefore, $\theta_{ij} + \theta_{jk} = \theta_{ik}$ if and only if $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$.

Denote $\Theta^{(1)} = (\mu_{ij})$ and $\Theta^{(2)} = (\nu_{ij})$.

The equalities $\gamma_{ij} = \beta_{ij}$, $\nu_{ij} = \mu_{ij}$ or inequalities $\gamma_{ij} > \beta_{ij}$, $\nu_{ij} > \mu_{ij}$ simultaneously hold for the entries of the matrices $(\beta_{ij}) = \mathcal{E}_{(1)}$, $(\mu_{ij}) = \Theta^{(1)}$, $(\gamma_{ij}) = \mathcal{E}^{(2)}$, $(\nu_{ij}) = \Theta^{(2)}$. Therefore, $\mathcal{E}^{(2)} - \mathcal{E}^{(1)} = \Theta^{(2)} - \Theta^{(1)}$ and $[Q(\mathcal{E})] = [Q(\Theta)]$.

Suppose that $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, i.e., $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$. Whence, $\theta_{ij} + \theta_{j\sigma(i)} = \theta_{i\sigma(i)}$. This means that the matrix Θ is also Gorenstein with the same permutation $\sigma(\mathcal{E})$. □

Let τ be a permutation which determines simultaneous transpositions of rows and columns of the reduced exponent matrix $\mathcal{E}$ under transformations of the second type. Then $\theta_{ij} = \alpha_{\tau(i)\tau(j)}$ and $\Theta = P_\tau^T \Theta P_\tau$, where

$P_\tau = \sum_{i=1}^{n} e_{i\tau(i)}$ is the permutation matrix, and P_τ^T stands for the transposed matrix of P_τ. Since $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ and $\alpha_{ij} = \theta_{\tau^{-1}(i)\tau^{-1}(j)}$, we have $\theta_{\tau^{-1}(i)k} + \theta_{k\tau^{-1}(\sigma(i))} = \theta_{\tau^{-1}(i)\tau^{-1}(\sigma(i))}$. Hence the permutation π of Θ satisfies $\pi(\tau^{-1}(i)) = \tau^{-1}(\sigma(i))$ for all i. Whence, $\pi = \tau^{-1}\sigma\tau$.

Since

$$\mu_{ij} = \beta_{\tau(i)\tau(j)}, \quad \nu_{ij} = \min_k(\mu_{ik} + \mu_{kj}) = \min_l(\beta_{\tau(i)l} + \beta_{l\tau(j)}) = \gamma_{\tau(i)\tau(j)},$$

it follows that,

$$\tilde{q}_{ij} = \nu_{ij} - \mu_{ij} = \gamma_{\tau(i)\tau(j)} - \beta_{\tau(i)\tau(j)} = q_{\tau(i)\tau(j)},$$

where $[\tilde{Q}] = (\tilde{q}_{ij})$ is the adjacency matrix of the quiver $\tilde{Q}$ of Θ. We proved the following.

Proposition 2.10. *Under transformations of the second type the adjacency matrix $[\tilde{Q}]$ of $Q(\Theta)$ changes according to the formula: $[\tilde{Q}] = P_\tau^T[Q]P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If $\mathcal{E}$ is Gorenstein then Θ is also Gorenstein and for the new permutation π we have: $\pi = \tau^{-1}\sigma\tau$, i.e., $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.*

Note that the type of the permutation does not change under transformations of the second type. Therefore, in order to describe the reduced Gorenstein exponent matrices, one needs to examine matrices with different types of permutations. Further, to simplify calculations we can assume that a row or a column of $\mathcal{E}$ is zero. It can always be obtained by transformations of the first type and entries of new exponent matrix will be non-negative integers.

Indeed, let $\mathcal{E}(\Lambda) = (\alpha_{ij})$ be an exponent matrix. Subtracting α_{i1} from the entries of the i-th row and adding this number to the entries of the i-th column, we obtain the matrix

$$\begin{pmatrix} 0 & \alpha_{21}+\alpha_{12} & \alpha_{31}+\alpha_{13} & \cdots & \alpha_{s1}+\alpha_{1s} \\ 0 & 0 & \alpha_{23}+\alpha_{31}-\alpha_{21} & \cdots & \alpha_{2s}+\alpha_{s1}-\alpha_{21} \\ 0 & \alpha_{32}+\alpha_{21}-\alpha_{31} & 0 & \cdots & \alpha_{3s}+\alpha_{s1}-\alpha_{31} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \alpha_{s2}+\alpha_{21}-\alpha_{s1} & \alpha_{s3}+\alpha_{31}-\alpha_{s1} & \cdots & 0 \end{pmatrix}.$$

Since ring inequalities hold for the entries of the matrix $\mathcal{E} = (\alpha_{ij})$, it follows that, $\alpha_{ij} + \alpha_{j1} - \alpha_{i1} \geqslant 0$, i.e., after these transformations we have a matrix with non-negative elements.

3. Adjacency matrices of admissible quivers without loops (cases $s = 2, 3, 4$)

In [16] (Appendix 2, Digraph diagrams) there is a list of simply laced digraphs without loops for $s \leqslant 4$ (s is the number of vertices Q). The number of such quivers for $s = 2$ is 3; for $s = 3$ is 16; for $s = 4$ is 218. Using this list, it is easy to see that the number of strongly connected quivers among them for $s = 2$ is 1; for $s = 3$ is 5; for $s = 4$ is 83.

We will give the list of admissible quivers without loops for $2 \leqslant s \leqslant 4$.

The number of these quivers for $s = 2$ is 1; for $s = 3$ is 2 and for $s = 4$ is 11.

We use such notations:

$$H_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix},$$

$$F_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix},$$

$\Omega_s = (\omega_{ij})$, where $\omega_{ij} = 0$ for $i \leqslant j$ and $\omega_{ij} = i - j$ for $i \geqslant j$; $H_s, F_s, \Omega_n \in M_s(\mathbb{Z})$.

Proposition 3.1. *There exists only one admissible quiver without loops for $s = 2$ which is $C_2 = Q(H_2)$ and the two admissible quivers without loops $C_3 = Q(H_3)$ and $Q(\Omega_3)$ for $s = 3$.*

In what follows we assume that exponent matrices are reduced and their first rows are zero.

Proof. Let $s = 2$. Then $\mathcal{E} = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$, $\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ and $\mathcal{E}^{(2)} = \begin{pmatrix} (2, \alpha) & (1, \alpha) \\ \alpha + 1 & (2, \alpha) \end{pmatrix}$, where $(\alpha_1, \dots, \alpha_t) = \min(\alpha_1, \dots, \alpha_t)$. So $[Q(\mathcal{E})] = \begin{pmatrix} (1, \alpha - 1) & 1 \\ 1 & (1, \alpha - 1) \end{pmatrix}$ and $Q(\mathcal{E})$ is either C_2 for $\alpha = 1$ or $\mathcal{L}C_2$ for $\alpha \geqslant 2$ (C_n is a simple cycle with n vertices, $[\mathcal{L}C_n] = [C_n] + E_n$, E_n is the identity $n \times n$ matrix).

Let $s = 3$. Then $\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & \delta \\ \beta & \gamma & 0 \end{pmatrix}$, $\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & \delta \\ \beta & \gamma & 1 \end{pmatrix}$ and

$$\mathcal{E}^{(2)} = \begin{pmatrix} (2,\alpha,\beta) & (1,\gamma) & (1,\delta) \\ (\alpha+1,\beta+\delta) & (2,\alpha,\gamma+\delta) & (\alpha,\delta+1) \\ (\beta+1,\alpha+\gamma) & (\beta,\gamma+1) & (2,\beta,\gamma+\delta) \end{pmatrix}.$$

Obviously, one can suppose $1 \leqslant \alpha \leqslant \beta$. Then $\alpha = 1$. Really, if $\alpha \geqslant 2$ we have a loop in the first vertex. If $\beta = 1$ we have either $\mathcal{E} \sim H_3$ or $\mathcal{E} \sim F_3$. Obviously, $\Omega_3 \sim F_3$. If $\beta = 2$, then $\mathcal{E} \sim \Omega_3$. □

Let $s = 4$. As above we obtain the admissible quivers without loops, listed below. Notation $\mathcal{E} \sim \Theta$ means equivalence of these matrices by transformations of the first type.

We put $d = d(\mathcal{E}) = \sum\limits_{i,j=1}^{n} \alpha_{ij}$ for an exponent matrix $\mathcal{E} = (\alpha_{ij})$. Obviously, $d(\mathcal{E}) = d(\Theta)$ for equivalent reduced exponent matrices $\mathcal{E}$ and Θ.

It is convenient for us to place the first six exponent matrices in the following sequence:

(1) $d = \mathbf{6}$, $\mathcal{E}_1 = H_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_1)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;

(2) $d = \mathbf{7}$, $\mathcal{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_2)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

and $\mathcal{E}_2 \sim \Theta_2$, where $\Theta_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;

(3) $d = \mathbf{8}$, $\mathcal{E}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_3)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

and $\mathcal{E}_3 \sim \Theta_3$, where $\Theta_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;

(4) $d = \mathbf{9}$, $\mathcal{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_4)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

and $\mathcal{E}_4 \sim \Theta_4$, where $\Theta_4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$;

(5) $d = \mathbf{10}$, $\mathcal{E}_5 = \Omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\Omega_4)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$;

(6) $d = \mathbf{9}$, $\mathcal{E}_6 = F_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, $[Q(F_4)] = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;

(7) $d = \mathbf{8}$, $\mathcal{E}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_7)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$,

and $\mathcal{E}_7 \sim \Theta_7$, where $\Theta_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;

(8) $d = \mathbf{10}$, $\mathcal{E}_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_8)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$;

(9) $d = \mathbf{10}$, $\mathcal{E}_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_9)] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$,

$\sigma(\mathcal{E}_9) = (1423)$;

(10) $d = \mathbf{11}$, $\mathcal{E}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_{10})] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$;

(11) $d = \mathbf{11}$, $\mathcal{E}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$, $[Q(\mathcal{E}_{11})] = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

4. Quivers of reduced Gorenstein exponent matrices (cases $s = 2, 3, 4, 5$)

In Sections 4 and 5 we shall assume that the first column of a reduced Gorenstein exponent matrix $\mathcal{E}$ is zero.

Denote $J_s^-(0) = e_{21} + \ldots + e_{ss-1}$ – the lower nilpotent Jordan block and

$$T_{\alpha,s} = \begin{bmatrix} 0 & \alpha & \ldots & \alpha \\ 0 & 0 & \ldots & \alpha \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & o \end{bmatrix},$$

where α is a natural integer, $T_{\alpha,s} \in M_s(\mathbb{Z})$. $T_{\alpha,s}$ is a reduced Gorenstein exponent matrix with $\sigma(T_{\alpha,s}) = (12\ldots s)$.

Proposition 4.1.

$$[Q(T_{1,s})] = J_s^-(0) + e_{1s};$$

$$[Q(T_{\alpha,s})] = E_s + J_s^-(0) + e_{1s},$$

where E_s is an identity $s \times s$-matrix.

Proof is obviously.

Let s=2. Then any reduced exponent matrix $\mathcal{E} \in M_2(\mathbb{Z})$ is equivalent to $T_{\alpha,2}$.

For s=3, $\sigma(\mathcal{E}) = (123)$ and it follows from Theorem 3.14 ([2] that

$$\begin{cases} \alpha_{13} + \alpha_{32} = \alpha_{12}, \\ \alpha_{21} + \alpha_{13} = \alpha_{23}, \\ \alpha_{32} + \alpha_{21} = \alpha_{31}. \end{cases}$$

Since $\alpha_{21} = \alpha_{31} = 0$, then $\alpha_{32} = 0$ and $\alpha_{12} = \alpha_{13} = \alpha_{23}$. Set $\alpha_{12} = \alpha$. Then, we have $\mathcal{E} = \begin{pmatrix} 0 & \alpha & \alpha \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} = T_{\alpha,3}$.

Proposition 4.2. *For s=2 every reduced tiled order is cyclic Gorenstein and for s=3 every reduced Gorenstein tiled order is cyclic.*

For s=4 we have two permutations (1234) and (12)(34). If σ = (1234), using Theorem 3.14 [2], we obtain the following system of equations:

$$\begin{cases} \alpha_{13} + \alpha_{32} = \alpha_{14} + \alpha_{42} = \alpha_{12}, \\ \alpha_{21} + \alpha_{13} = \alpha_{24} + \alpha_{43} = \alpha_{23}, \\ \alpha_{31} + \alpha_{14} = \alpha_{32} + \alpha_{24} = \alpha_{34}, \\ \alpha_{42} + \alpha_{21} = \alpha_{43} + \alpha_{31} = \alpha_{41}. \end{cases}$$

The conditions $\alpha_{21} = \alpha_{31} = \alpha_{41} = 0$ and $\alpha_{42} = \alpha_{43} = 0$ imply that $\alpha_{34} = \alpha_{14} = \alpha_{12}$, $\alpha_{24} = \alpha_{23} = \alpha_{13}$, $\alpha_{32} = \alpha_{12} - \alpha_{13}$.

Put $\alpha_{12} = \alpha$, $\alpha_{13} = \beta$. Whence, $\alpha_{32} = \alpha - \beta$.

Since $\mathcal{E}$ is a reduced order, we have $\alpha - \beta \geqslant 0$, $\alpha > 0$, $\beta > 0$.

Hence, $\mathcal{E}_{\alpha,\beta} = \begin{pmatrix} 0 & \alpha & \beta & \alpha \\ 0 & 0 & \beta & \beta \\ 0 & \alpha-\beta & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We have

$$[Q(\mathcal{E}_{\alpha,\beta})] = \begin{pmatrix} (1,\beta-1) & 0 & (1,\alpha-\beta) & 1 \\ 1 & (1,\beta-1) & 0 & (1,\alpha-\beta) \\ (1,\alpha-\beta) & 1 & (1,\beta-1) & 0 \\ 0 & (1,\alpha-\beta) & 1 & (1,\beta-1) \end{pmatrix}.$$

Then $Q(\mathcal{E}_{1,1})$ is the simple cycle C_4. Obviously, $\mathcal{E}_{\alpha,\alpha} = T_{\alpha,4}$ and for $\alpha \geqslant 2$ $Q(\mathcal{E}_{\alpha,\alpha}) = \mathcal{L}C_4$. For $\beta = 1$ we have $\alpha \geqslant 2$ and

$$[Q(\mathcal{E}_{\alpha,1})] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

General case: $\alpha > \beta > 1$.

$$[Q(\mathcal{E}_{\alpha,\beta})] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

If $\sigma = (12)(34)$, then

$$\mathcal{E}(\mathcal{E}_{\gamma,\delta}) = \begin{bmatrix} 0 & \gamma+\delta & \gamma & \delta \\ 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & \delta \\ 0 & \gamma & \gamma & 0 \end{bmatrix}.$$

Obviously, $[Q(\mathcal{E}_{1,\delta})] = [Q(\mathcal{E}_{\gamma,1})]$ and $[Q(\mathcal{E}_{1,\delta})] = \begin{bmatrix} O & U_2 \\ U_2 & O \end{bmatrix}$.

In general: if $\gamma \geqslant 2$ and $\delta \geqslant 2$ we have $[Q(\mathcal{E}_{\gamma,\delta})] = \begin{bmatrix} E_2 & U_2 \\ U_2 & E_2 \end{bmatrix}$.

Let s=5. In this case we have two permutations: $\sigma = (12345)$ and $\sigma = (123)(45)$.

Suppose $\sigma = (12345)$ and take a reduced Gorenstein exponent matrix $\mathcal{E}$ with σ, then we come to the following system of linear equations for elements of $\mathcal{E}$:

$$\begin{cases} \alpha_{13}+\alpha_{32}=\alpha_{14}+\alpha_{42}=\alpha_{15}+\alpha_{52}=\alpha_{12}, \\ \alpha_{21}+\alpha_{13}=\alpha_{24}+\alpha_{43}=\alpha_{25}+\alpha_{53}=\alpha_{23}, \\ \alpha_{31}+\alpha_{14}=\alpha_{32}+\alpha_{24}=\alpha_{35}+\alpha_{54}=\alpha_{34}, \\ \alpha_{41}+\alpha_{15}=\alpha_{42}+\alpha_{25}=\alpha_{43}+\alpha_{35}=\alpha_{45}, \\ \alpha_{52}+\alpha_{21}=\alpha_{53}+\alpha_{31}=\alpha_{54}+\alpha_{41}=\alpha_{51} \end{cases}.$$

Hence, putting $\alpha_{12}=\alpha$ and $\alpha_{13}=\beta$, we obtain the following exponent matrix:

$$\mathcal{E}_{\alpha,\beta}=\begin{pmatrix} 0 & \alpha & \beta & \beta & \alpha \\ 0 & 0 & \beta & 2\beta-\alpha & \beta \\ 0 & \alpha-\beta & 0 & \beta & \beta \\ 0 & \alpha-\beta & \alpha-\beta & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha-\beta\geqslant 0$, $2\beta-\alpha\geqslant 0$, $\alpha\geqslant 1$, $\beta\geqslant 1$.

$$[Q(\mathcal{E}_{\alpha,\beta})]=\begin{pmatrix} (1,\beta-1) & 0 & (1,\alpha-\beta) & (1,\alpha-\beta) & (1,2\beta-\alpha) \\ (1,2\beta-\alpha) & (1,\beta-1) & 0 & (1,\alpha-\beta) & (1,\alpha-\beta) \\ (1,\alpha-\beta) & (1,2\beta-\alpha) & (1,\beta-1) & 0 & (1,\alpha-\beta) \\ (1,\alpha-\beta) & (1,\alpha-\beta) & (1,2\beta-\alpha) & (1,\beta-1) & 0 \\ 0 & (1,\alpha-\beta) & (1,\alpha-\beta) & (1,2\beta-\alpha) & (1,\beta-1) \end{pmatrix}$$

If $\sigma=(123)(45)$ we obtain analogously:

$$\mathcal{E}_{\alpha}=\begin{pmatrix} 0 & 3\alpha & 3\alpha & 2\alpha & 2\alpha \\ 0 & 0 & 3\alpha & \alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 2\alpha & 0 & 2\alpha \\ 0 & \alpha & 2\alpha & 2\alpha & 0 \end{pmatrix}$$

and

$$[Q(\mathcal{E}_{\alpha})]=\begin{pmatrix} [Q(T_{3\alpha,3})] & U_{3\times 2} \\ U_{2\times 3} & E_2 \end{pmatrix}=\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

and $in\,\mathcal{E}_{\alpha}=4$.

Proposition 4.3. *An adjacency matrix of the quiver of a reduced Gorenstein exponent matrix G for $s=2,3,4,5$ has a form $[Q(G)]=\lambda S$, where S is doubly stochastic matrix.*

5. Quivers of reduced Gorenstein exponent matrices (cases $s = 6, 7$)

Let $s = 6$. Then there are four types of permutations.

(a) $\sigma = (123456)$.

$$\mathcal{E} = \begin{pmatrix} 0 & \alpha+\beta+\gamma & \beta+\gamma & \alpha+\gamma & \beta+\gamma & \alpha+\beta+\gamma \\ 0 & 0 & \beta+\gamma & \gamma & \gamma & \beta+\gamma \\ 0 & \alpha & 0 & \alpha+\gamma & \gamma & \alpha+\gamma \\ 0 & \beta & \beta & 0 & \beta+\gamma & \beta+\gamma \\ 0 & \alpha & \beta & \alpha & 0 & \alpha+\beta+\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\beta+\gamma \geqslant 1$, $\alpha+\gamma \geqslant 1$.

Then

$$[Q] = \begin{pmatrix} \delta & 0 & (1,\alpha,\beta) & (1,\beta) & (1,\alpha,\gamma) & (1,\gamma) \\ (1,\gamma) & \delta & 0 & (1,\alpha,\beta) & (1,\beta) & (1,\alpha,\gamma) \\ (1,\alpha,\gamma) & (1,\gamma) & \delta & 0 & (1,\alpha,\beta) & (1,\beta) \\ (1,\beta) & (1,\alpha,\gamma) & (1,\gamma) & \delta & 0 & (1,\alpha,\beta) \\ (1,\alpha,\beta) & (1,\beta) & (1,\alpha,\gamma) & (1,\gamma) & \delta & 0 \\ 0 & (1,\alpha,\beta) & (1,\beta) & (1,\alpha,\gamma) & (1,\gamma) & \delta \end{pmatrix},$$

where $\delta = (2, \beta+\gamma, \gamma+\alpha) - 1$.

(b) $\sigma = (123)(456)$.

$$\mathcal{E} = \begin{pmatrix} 0 & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\gamma & \alpha+\beta & \beta+\gamma \\ 0 & 0 & \alpha+\beta+\gamma & \gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & \alpha+\beta & 0 & \alpha+\beta & \beta \\ 0 & \gamma & \beta+\gamma & \gamma & 0 & \beta+\gamma \\ 0 & \alpha & \alpha+\gamma & \gamma+\alpha & \alpha & 0 \end{pmatrix},$$

where $\alpha+\beta \geqslant 1$, $\beta+\gamma \geqslant 1$, $\gamma+\alpha \geqslant 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$. Put $\delta = (2, \alpha+\beta, \beta+\gamma, \gamma+\alpha)$. Then

$$[Q] = \begin{pmatrix} \delta-1 & 0 & (1,\alpha,\beta,\gamma) & (1,\beta) & (1,\gamma) & (1,\alpha) \\ (1,\alpha,\beta,\gamma) & \delta-1 & 0 & (1,\alpha) & (1,\beta) & (1,\gamma) \\ 0 & (1,\alpha,\beta,\gamma) & \delta-1 & (1,\gamma) & (1,\alpha) & (1,\beta) \\ (1,\beta) & (1,\alpha) & (1,\gamma) & \delta-1 & 0 & (1,\alpha,\beta,\gamma) \\ (1,\gamma) & (1,\beta) & (1,\alpha) & (1,\alpha,\beta,\gamma) & \delta-1 & 0 \\ (1,\alpha) & (1,\gamma) & (1,\beta) & 0 & (1,\alpha,\beta,\gamma) & \delta-1 \end{pmatrix}.$$

(c) $\sigma = (1234)(56)$.

$$\mathcal{E} = \begin{pmatrix} 0 & \alpha+\beta+\gamma & 2\gamma & \alpha+\beta+\gamma & \alpha+\gamma & \beta+\gamma \\ 0 & 0 & 2\gamma & 2\gamma & \gamma & \gamma \\ 0 & \alpha+\beta-\gamma & 0 & \alpha+\beta+\gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & \gamma & \beta+\gamma & 0 & \beta+\gamma \\ 0 & \alpha & \gamma & \alpha+\gamma & \alpha+\gamma & 0 \end{pmatrix},$$

where $\gamma \geqslant 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta \geqslant \gamma$.

Then

$$[Q] = \begin{pmatrix} \delta & 0 & \rho & 1 & (1,\beta) & (1,\alpha) \\ 1 & \delta & 0 & \rho & (1,\alpha) & (1,\beta) \\ \rho & 1 & \delta & 0 & (1,\beta) & (1,\alpha) \\ 0 & \rho & 1 & \delta & (1,\alpha) & (1,\beta) \\ (1,\beta) & (1,\alpha) & (1,\beta) & (1,\alpha) & \delta & 0 \\ (1,\alpha) & (1,\beta) & (1,\alpha) & (1,\beta) & 0 & \delta \end{pmatrix},$$

where $\rho = (1,\alpha,\beta,\alpha+\beta-\gamma)$, $\delta = (\beta+\gamma,\gamma+\alpha)-1$.

(d) $\sigma = (12)(34)(56)$.

$$\mathcal{E} = \begin{pmatrix} 0 & \alpha+\gamma+\xi+\chi & \gamma+\xi & \alpha+\chi & \alpha+\gamma+\chi-\beta & \beta+\xi \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha+\chi & 0 & \alpha+\chi & \alpha & \beta \\ 0 & \gamma+\xi & \gamma+\xi & 0 & \gamma & \beta+\xi-\chi \\ 0 & \beta+\xi & \xi & \chi & 0 & \beta+\xi \\ 0 & \alpha+\gamma+\chi-\beta & \gamma+\chi-\beta & \alpha+\chi-\beta & \alpha+\gamma+\chi-\beta & 0 \end{pmatrix},$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\xi \geqslant 0$, $\chi \geqslant 0$, $\alpha+\chi \geqslant 1$, $\beta+\xi \geqslant 1$, $\alpha+\xi \geqslant 1$, $\alpha+\gamma+\chi-\beta \geqslant 1$, $\gamma+\chi \geqslant 1$, $\gamma+\xi \geqslant 1$, $\alpha+\chi-\beta \geqslant 0$, $\gamma+\chi-\beta \geqslant 0$, $\beta+\xi-\chi \geqslant 0$.

$\Delta_1 = (2,\alpha+\gamma+\chi-\beta,\gamma+\xi,\alpha+\chi,\beta+\xi)$, $\Delta_2 = (2,\gamma+\xi,\alpha+\chi,\alpha+\xi,\gamma+\chi)$, $\Delta_3 = (2,\alpha+\xi,\gamma+\chi,\beta+\xi,\alpha+\gamma+\chi-\beta)$.

Then $[Q] = \begin{pmatrix} V & W \end{pmatrix}$, where

$$V = \begin{pmatrix} \Delta_1-1 & 0 & (1,\chi,\alpha+\chi-\beta) \\ 0 & \Delta_1-1 & (1,\xi,\gamma+\chi-\beta) \\ (1,\alpha,\beta) & (1,\gamma,\beta+\xi-\chi) & \Delta_2-1 \\ (1,\gamma,\beta+\xi-\chi) & (1,\alpha,\beta) & 0 \\ (1,\xi,\chi) & (1,\alpha+\chi-\beta,\gamma+\chi-\beta) & (1,\beta,\gamma) \\ (1,\alpha+\chi-\beta,\gamma+\chi-\beta) & (1,\xi,\chi) & (1,\alpha,\beta+\xi-\chi) \end{pmatrix},$$

$$W = \begin{pmatrix} (1,\xi,\gamma+\chi-\beta) & (1,\beta,\beta+\xi-\chi) & (1,\alpha,\gamma) \\ (1,\chi,\alpha+\chi-\beta) & (1,\alpha,\gamma) & (1,\beta,\beta+\xi-\chi) \\ 0 & (1,\chi,\gamma+\chi-\beta) & (1,\xi,\alpha+\chi-\beta) \\ \Delta_2-1 & (1,\xi,\alpha+\chi-\beta) & 1,\chi,\gamma+\chi-\beta) \\ (1,\alpha,\beta+\xi-\chi) & \Delta_3-1 & 0 \\ (1,\beta,\gamma) & 0 & \Delta_3-1 \end{pmatrix}.$$

Let $s = 7$. There exist four types of permutations.

(a) $\sigma = (1234567)$.

$$\mathcal{E} = \begin{pmatrix} 0 & \alpha+\beta+\gamma & \beta+\gamma & \gamma+\alpha & \gamma+\alpha & \beta+\gamma & \alpha+\beta+\gamma \\ 0 & 0 & \beta+\gamma & \gamma & \alpha+\gamma-\beta & \gamma & \beta+\gamma \\ 0 & \alpha & 0 & \gamma+\alpha & \alpha+\gamma-\beta & \alpha+\gamma-\beta & \gamma+\alpha \\ 0 & \beta & \beta & 0 & \gamma+\alpha & \gamma & \gamma+\alpha \\ 0 & \beta & 2\beta-\alpha & \beta & 0 & \beta+\gamma & \beta+\gamma \\ 0 & \alpha & \beta & \beta & \alpha & 0 & \alpha+\beta+\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\beta+\gamma \geqslant 1$, $2\beta \geqslant \alpha$, $\alpha+\gamma \geqslant \beta$, $\gamma+\alpha \geqslant 1$.

$$[Q] = \begin{pmatrix} \delta & 0 & (1,\alpha,\beta) & \mu & (1,\beta,\gamma) & \theta & \nu \\ \nu & \delta & 0 & (1,\alpha,\beta) & \mu & (1,\beta,\gamma) & \theta \\ \theta & \nu & \delta & 0 & (1,\alpha,\beta) & \mu & (1,\beta,\gamma) \\ (1,\beta,\gamma) & \theta & \nu & \delta & 0 & (1,\alpha,\beta) & \mu \\ \mu & (1,\beta,\gamma) & \theta & \nu & \delta & 0 & (1,\alpha,\beta) \\ (1,\alpha,\beta) & \mu & (1,\beta,\gamma) & \theta & \nu & \delta & 0 \\ 0 & (1,\alpha,\beta) & \mu & (1,\beta,\gamma) & \theta & \nu & \delta \end{pmatrix},$$

where $\delta = (2,\beta+\gamma,\gamma+\alpha)-1$, $\nu = (1,\gamma,\alpha+\gamma-\beta)$, $\theta = (1,\alpha,\alpha+\gamma-\beta)$, $\mu = (1,\beta,2\beta-\alpha)$.

(b) Let $\sigma = (12345)(67)$.

$$\mathcal{E} = \begin{pmatrix} 0 & 2\gamma+\beta & 3\gamma-\beta & 3\gamma-\beta & 2\gamma+\beta & 2\gamma & 2\gamma \\ 0 & 0 & 3\gamma-\beta & 4\gamma-3\beta & 3\gamma-\beta & 2\gamma-\beta & 2\gamma-\beta \\ 0 & 2\beta-\gamma & 0 & 3\gamma-\beta & 3\gamma-\beta & \gamma & \gamma \\ 0 & 2\beta-\gamma & 2\beta-\gamma & 0 & 2\gamma+\beta & \beta & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & \gamma & 2\gamma-\beta & 2\gamma & 0 & 2\gamma \\ 0 & \beta & \gamma & 2\gamma-\beta & 2\gamma & 2\gamma & 0 \end{pmatrix},$$

where $\beta \geqslant 1$, $\gamma \geqslant 1$, $2\beta \geqslant \gamma$, $4\gamma \geqslant 3\beta$.

$$[Q] = \begin{pmatrix} 1 & 0 & (1,2\beta-\gamma) & (1,2\beta-\gamma) & (1,4\gamma-3\beta) & 1 & 1 \\ (1,4\gamma-3\beta) & 1 & 0 & (1,2\beta-\gamma) & (1,2\beta-\gamma) & 1 & 1 \\ (1,2\beta-\gamma) & (1,4\gamma-3\beta) & 1 & 0 & (1,2\beta-\gamma) & 1 & 1 \\ (1,2\beta-\gamma) & (1,2\beta-\gamma) & (1,4\gamma-3\beta) & 1 & 0 & 1 & 1 \\ 0 & (1,2\beta-\gamma) & (1,2\beta-\gamma) & (1,4\gamma-3\beta) & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(c) $\sigma = (1234)(567)$.

$$\mathcal{E} = \begin{pmatrix} 0 & 4\gamma-\alpha & 2\alpha & 4\gamma-\alpha & 2\gamma & 2\gamma & 2\gamma \\ 0 & 0 & 2\alpha & 2\alpha & \alpha & \alpha & \alpha \\ 0 & 4\gamma-3\alpha & 0 & 4\gamma-\alpha & 2\gamma-\alpha & 2\gamma-\alpha & 2\gamma-\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\gamma-\alpha & \alpha & 2\gamma & 0 & 2\gamma & \gamma \\ 0 & 2\gamma-\alpha & \alpha & 2\gamma & \gamma & 0 & 2\gamma \\ 0 & 2\gamma-\alpha & \alpha & 2\gamma & 2\gamma & \gamma & 0 \end{pmatrix},$$

where $\alpha \geqslant 1$, $\gamma \geqslant 1$, $4\gamma - 3\alpha \geqslant 0$.

$$[Q] = \begin{pmatrix} 1 & 0 & (1,4\gamma-3\alpha) & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & (1,4\gamma-3\alpha) & 1 & 1 & 1 \\ (1,4\gamma-3\alpha) & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & (1,4\gamma-3\alpha) & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

(d) $\sigma = (123)(45)(67)$.

$$\mathcal{E} = \begin{pmatrix} 0 & 3\alpha & 3\alpha & 2\alpha & 2\alpha & 2\alpha & 2\alpha \\ 0 & 0 & 3\alpha & \alpha & \alpha & \alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 2\alpha & 0 & 2\alpha & 2\alpha-\beta & 2\alpha-\gamma \\ 0 & \alpha & 2\alpha & 2\alpha & 0 & 2\alpha-\gamma & 2\alpha-\beta \\ 0 & \alpha & 2\alpha & \gamma & \beta & 0 & 2\alpha \\ 0 & \alpha & 2\alpha & \beta & \gamma & 2\alpha & 0 \end{pmatrix},$$

where $\alpha \geqslant 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $2\alpha - \beta \geqslant 0$, $2\alpha - \gamma \geqslant 0$, $2\alpha - \gamma + \beta \geqslant 1$, $2\alpha - \beta + \gamma \geqslant 1$.

$$[Q] = \begin{pmatrix} 1 & 0 & 1 & (1,\beta,\gamma) & (1,\beta,\gamma) & \omega & \omega \\ 1 & 1 & 0 & (1,\beta,\gamma) & (1,\beta,\gamma) & \omega & \omega \\ 0 & 1 & 1 & (1,\beta,\gamma) & (1,\beta,\gamma) & \omega & \omega \\ \omega & \omega & \omega & \delta & 0 & (1,\beta) & (1,\gamma) \\ \omega & \omega & \omega & 0 & \delta & (1,\gamma) & (1,\beta) \\ (1,\beta,\gamma) & (1,\beta,\gamma) & (1,\beta,\gamma) & (1,2\alpha-\gamma) & (1,2\alpha-\beta) & \delta & 0 \\ (1,\beta,\gamma) & (1,\beta,\gamma) & (1,\beta,\gamma) & (1,2\alpha-\beta) & (1,2\alpha-\gamma) & 0 & \delta \end{pmatrix},$$

where $\delta = (2, 2\alpha-\beta+\gamma, 2\alpha-\gamma+\beta) - 1$, $\omega = (1, 2\alpha-\beta, 2\alpha-\gamma)$.

6. Gorenstein orders and entropic quasigroups

For the Cayley table

$$\mathcal{E}(n) = \begin{bmatrix} 0 & n-1 & n-2 & \dots & 2 & 1 \\ 1 & 0 & n-1 & \dots & 3 & 2 \\ 2 & 1 & 0 & \dots & 4 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & n-4 & \dots & 0 & n-1 \\ n-1 & n-2 & n-3 & \dots & 1 & 0 \end{bmatrix}$$

of the entropic quasigroup $Q(n)$, we have $[Q(\mathcal{E}(n))] = E_n + J_n^-(0) + e_{1n}$, where $J_n^-(0) = e_{21} + \ldots + e_{nn-1}$ is the lower nilpotent Jordan block.

The next definition is given in ([34], Section IV).

Definition 6.1. *A finite quasigroup Q defined on the set $S = \{0, 1, \ldots, n-1\}$ is called Gorenstein if its Cayley table $C(Q) = (\alpha_{ij})$ has a zero main diagonal and there exists a permutation $\sigma : i \to \sigma(i)$ for $i = 1, \ldots, n$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i = 1, \ldots, n$.*

If σ is a cycle then G is a cyclic Gorenstein quasigroup.

Proposition 6.2. *The quasigroup $Q(n)$ is Gorenstein with permutation $\sigma = (12 \ldots n)$, i.e. $Q(n)$ is a cyclic Gorenstein quasigroup.*

Proof. It's obvious. □

Remark. *Note, that a reduced tiled order Λ is Gorenstein if and only if its exponent matrix $\mathcal{E}(\Lambda)$ is Gorenstein.*

Theorem 6.3. *For any permutation $\sigma \in S_n$ without fixed elements there exists a Gorenstein reduced tiled order Λ with permutation $\sigma(\Lambda) = \sigma$.*

Proof. Suppose that σ has no cycles of length 1 and decomposes into a product of non-intersecting cycles $\sigma = \sigma_1 \cdots \sigma_k$, where σ_i has length m_i. Denote by t the least common multiple of the numbers $m_1-1, \ldots, m_k-1$.

Consider the matrix

$$\mathcal{E}(m_1,\ldots,m_s)=\begin{pmatrix} t_1\mathcal{E}(m_1) & tU_{m_1\times m_2} & tU_{m_1\times m_3} & \ldots & tU_{m_1\times m_k}\\ 0 & t_2\mathcal{E}(m_2) & tU_{m_2\times m_3} & \ldots & tU_{m_2\times m_k}\\ 0 & 0 & t_3\mathcal{E}(m_3) & \ldots & tU_{m_3\times m_k}\\ \ldots & \ldots & \ldots & \ddots & \ldots\\ 0 & 0 & 0 & \ldots & t_k\mathcal{E}(m_k)\end{pmatrix},$$

where $t_j=\frac{t}{m_j-1}$, $U_{m_i\times m_j}$ is an $m_i\times m_j$ - matrix whose entries equal 1; $\mathcal{E}(m)=(\varepsilon_{ij})$, $\varepsilon_{ij}=\begin{cases} i-j, & \text{if } i\geqslant j;\\ i-j+m, & \text{if } i<j.\end{cases}$

Let us remark that $\varepsilon_{ij}+\varepsilon_{j\sigma(i)}=\varepsilon_{i\sigma(i)}=m-1$ for all i,j.

Evidently, $\mathcal{E}(m_1,\ldots,m_s)$ is the exponent matrix with permutation $\pi(A)=(123\ldots m_1)(m_1+1\ldots m_1+m_2)\cdots(m_1+m_2+\cdots+m_{k-1}+1\ldots m_1+m_2+\cdots+m_{k-1}+m_k)$.

Since the permutations σ and π have the same type, these permutations are conjugate, i. e., there exists a permutation τ such that $\sigma=\tau^{-1}\pi(A)\tau$.

Consequently, by Propositions 2.9 and 2.10, the matrix $P_\tau^T\mathcal{E}(m_1,\ldots,m_s)P_\tau$ is the exponent matrix of a Gorenstein reduced tiled order Λ with permutation $\sigma(\Lambda)=\sigma$. □

Example. (B.V. Novikov). The matrix

$$C(\mathcal{L}_{12})=\begin{pmatrix} 0&1&2&3&4&5&6&7&8&9&10&11\\ 1&0&5&2&3&4&7&8&9&6&11&10\\ 2&5&0&4&1&3&8&10&7&11&6&9\\ 3&2&4&0&5&1&10&6&11&7&9&8\\ 4&3&1&5&0&2&9&11&6&10&8&7\\ 5&4&3&1&2&0&11&9&10&8&7&6\\ 6&7&8&10&9&11&0&2&1&3&4&5\\ 7&8&10&6&11&9&2&0&5&1&3&4\\ 8&9&7&11&6&10&1&5&0&4&2&3\\ 9&6&11&7&10&8&3&1&4&0&5&2\\ 10&11&6&9&8&7&4&3&2&5&0&1\\ 11&10&9&8&7&6&5&4&3&2&1&0 \end{pmatrix}$$

is the Cayley table of a Gorenstein quasigroup $\mathcal{L}_{12}$ with the permutation

$$\sigma=\begin{pmatrix} 1&2&3&4&5&6&7&8&9&10&11&12\\ 12&11&10&9&8&7&6&5&4&3&2&1\end{pmatrix}.$$

The rings inequalities do not hold:

$$\alpha_{17} + \alpha_{79} = 7 < \alpha_{19} = 8.$$

Obviously, $\mathcal{L}_{12}$ is an abelian loop of period 2 and 0 is the neutral element of $\mathcal{L}_{12}$.

All subgroups in $\mathcal{L}_{12}$ are elementary abelian 2-groups. There are no subgroups of order 8 in $\mathcal{L}_{12}$. The loop $\mathcal{L}_{12}$ contains the following Klein subgroups: $< 0, 1, 10, 11 >$; $< 0, 2, 9, 11 >$; $< 0, 3, 8, 1 >$; $< 0, 4, 7, 11 >$; $< 0, 5, 6, 11, >$. There are no subgroups different from these in $\mathcal{L}_{12}$. Denote by * the binary operation in $\mathcal{L}_{12}$.

We have: $(1 * 2) * 2 = 5 * 2 = 3$ and $1 * (2 * 2) = 1 * 0 = 1$. Then $1 \neq 3$ and $\mathcal{L}_{12}$ is not diassociative and, consequently, it is not Moufang.

The subgroup $K =< 0, 1 >$ is normal in $\mathcal{L}_{12}$. Then the quotient loop $\mathcal{L}_{12}/K$ has the following Cayley table:

$$\mathcal{L}_{12}/K = \begin{array}{|cccccc|} \hline \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \bar{1} & \bar{0} & \bar{5} & \bar{2} & \bar{3} & \bar{4} \\ \bar{2} & \bar{5} & \bar{0} & \bar{4} & \bar{1} & \bar{3} \\ \bar{3} & \bar{2} & \bar{4} & \bar{0} & \bar{5} & \bar{1} \\ \bar{4} & \bar{3} & \bar{1} & \bar{5} & \bar{0} & \bar{2} \\ \bar{5} & \bar{4} & \bar{3} & \bar{1} & \bar{2} & \bar{0} \\ \hline \end{array}.$$

The loop $\mathcal{L}_{12}/K$ is simple.

7. Cayley tables of elementary abelian 2-groups

Put $G_0 = \{e\}$. Denote by $\Gamma_0 = \{\mathcal{O}, \mathcal{E}(\Gamma_0)\}$ a Gorenstein tiled order with exponent matrix $\mathcal{E}(\Gamma_0) = (0)$ and $\pi\mathcal{O} = \mathcal{O}\pi$ is a unique maximal ideal of $\mathcal{O}$.

The matrix $\mathcal{E}(\Gamma_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Cayley table of the cyclic group G_1 of order 2 and also the exponent matrix of a Gorenstein tiled order Γ_1 with permutation $\sigma(\Gamma_1) = (12)$.

Clearly, the Cayley table of the Klein four-group $(2) \times (2)$ can be written as

$$\mathcal{E}(\Gamma_2) = \begin{pmatrix} \mathcal{E}(\Gamma_1) & \mathcal{E}(\Gamma_1) + 2U_2 \\ \mathcal{E}(\Gamma_1) + 2U_2 & \mathcal{E}(\Gamma_1) \end{pmatrix}.$$

Consider

$$\mathcal{E}(\Gamma_k) = \begin{pmatrix} \mathcal{E}(\Gamma_{k-1}) & \mathcal{E}(\Gamma_{k-1}) + 2^{k-1}U_{2^{k-1}} \\ \mathcal{E}(\Gamma_{k-1}) + 2^{k-1}U_{2^{k-1}} & \mathcal{E}(\Gamma_{k-1}) \end{pmatrix}.$$

Proposition 7.1. *$\mathcal{E}(\Gamma_k)$ is the exponent matrix of a tiled order Γ_k, where*

$$\Gamma_k = \begin{pmatrix} \Gamma_{k-1} & \pi^{2^{k-1}}\Gamma_{k-1} \\ \pi^{2^{k-1}}\Gamma_{k-1} & \Gamma_{k-1} \end{pmatrix}.$$

Proof. Proof by induction on k easily yields that Γ_k is a tiled order. □

Let $G = H\times < g >$ be a finite Abelian group, $H = \{h_1, \dots, h_n\}$, $g^2 = e$. We shall consider the Cayley table of H as the matrix $C(H) = (h_{ij})$ with entries in H, where $h_{ij} = h_i h_j$. The following proposition is obvious.

Proposition 7.2. *The Cayley table of G is*

$$C(G) = \begin{pmatrix} C(H) & gC(H) \\ gC(H) & C(H) \end{pmatrix}.$$

Proposition 7.3. *$\mathcal{E}(\Gamma_k)$ is the Cayley table of the elementary abelian group G_k of order 2^k.*

Proof. The proof goes by induction on k. The basis of induction have been already done. If $\mathcal{E}(\Gamma_{k-1})$ is the Cayley table of G_{k-1}, then, by Proposition 7.2, $\mathcal{E}(\Gamma_k)$ is the Cayley table of G_k. □

Proposition 7.4. *The tiled order Γ_k is Gorenstein with permutation* $\sigma(\Gamma_k) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^k-1 & 2^k \\ 2^k & 2^k-1 & 2^k-2 & \dots & 2 & 1 \end{pmatrix}$.

Proof. It is obvious for $k = 1$. Suppose that Γ_k is Gorenstein with exponent matrix $\mathcal{E}(\Gamma_k) = (\alpha^k_{ij})$ ($i, j = 1, 2, \dots, 2^k$) and $\sigma(\Gamma_k) = \sigma_k$, where $\sigma_k(i) = 2^k + 1 - i$. Then $\alpha^k_{ij} + \alpha^k_{j\sigma_k(i)} = \alpha^k_{i\sigma_k(i)}$ for all $i, j = 1, 2, \dots, 2^k$. Since

$$\alpha^{k+1}_{2^k+i,j} = \alpha^{k+1}_{i,2^k+j} = \alpha^k_{ij} + 2^k, \ \ \alpha^{k+1}_{2^k+i,2^k+j} = \alpha^{k+1}_{ij} = \alpha^k_{ij}$$

for all $i, j = 1, 2, \dots, 2^k$ and

$$(\alpha^k_{ij} + 2^k) + \alpha^k_{j\sigma_k(i)} = (\alpha^k_{ij} + \alpha^k_{j\sigma_k(i)}) + 2^k = \alpha^k_{i\sigma_k(i)} + 2^k,$$

we obtain that

$$\alpha^{k+1}_{ij} + \alpha^{k+1}_{j,2^k+\sigma_k(i)} = \alpha^{k+1}_{i,2^k+\sigma_k(i)}, \quad \alpha^{k+1}_{i,2^k+j} + \alpha^{k+1}_{2^k+j,2^k+\sigma_k(i)} = \alpha^{k+1}_{i,2^k+\sigma_k(i)},$$

$$\alpha^{k+1}_{2^k+i,2^k+j} + \alpha^{k+1}_{2^k+j,\sigma_k(i)} = \alpha^{k+1}_{2^k+i,\sigma_k(i)}, \quad \alpha^{k+1}_{2^k+i,j} + \alpha^{k+1}_{j\sigma_k(i)} = \alpha^{k+1}_{2^k+i,\sigma_k(i)},$$

$i, j = 1, 2, \dots, 2^k$. Putting $\sigma_{k+1}(i) = 2^k + \sigma_k(i)$, $\sigma_{k+1}(2^k + i) = \sigma_k(i)$, we have $\alpha^{k+1}_{pq} + \alpha^{k+1}_{q\sigma_{k+1}(p)} = \alpha^{k+1}_{p\sigma_{k+1}(p)}$ for all $p, q = 1, 2, \dots, 2^{k+1}$, i.e., Γ_{k+1} is Gorenstein with permutation $\sigma(\Gamma_{k+1}) = \sigma_{k+1}$, where $\sigma_{k+1}(i) = 2^{k+1} + 1 - i$. □

Theorem 7.5. [34]. *The Cayley table of a finite group G is the exponent matrix of a reduced Gorenstein tiled order if and only if $G = G_k = (2) \times \ldots \times (2)$.*

We calculate the adjacency matrix of the quiver $Q(\Gamma_k)$.

Let $R_k = rad\ \Gamma_k$ be the Jacobson radical of Γ_k and $\mathcal{E}(\Gamma_k) = (\alpha_{ij}^k)$, $\mathcal{E}(R_k) = (r_{ij}^k)$, $\mathcal{E}(R_k^2) = (\beta_{ij}^k)$. We have

$$R_k = \begin{pmatrix} R_{k-1} & \pi^{2^{k-1}}\Gamma_{k-1} \\ \pi^{2^{k-1}}\Gamma_{k-1} & R_{k-1} \end{pmatrix},$$

$$R_k^2 = \begin{pmatrix} R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} & \pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} \\ \pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} & R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} \end{pmatrix}.$$

Since $r_{ij}^{k-1} \leqslant 2^{k-1}$, then $\beta_{ij}^{k-1} \leqslant 2^k \leqslant 2^k + \alpha_{ij}^{k-1}$. Therefore, $R_{k-1}^2 + \pi^{2^k}\Gamma_{k-1} = R_{k-1}^2$.

The equality $(radA)A = A(radA) = radA$ holds for any tiled order A. Consequently, $\pi^{2^{k-1}}R_{k-1}\Gamma_{k-1} = \pi^{2^{k-1}}R_{k-1}$. Since $\mathcal{E}(\pi^{2^{k-1}}R_{k-1}) - \mathcal{E}(\pi^{2^{k-1}}\Gamma_{k-1}) = (2^{k-1} + \mathcal{E}(R_{k-1}) - (2^{k-1} + \mathcal{E}(\Gamma_{k-1})) = E$, we obtain

$$\mathcal{E}(R_k^2) - \mathcal{E}(R_k) = \begin{pmatrix} \mathcal{E}(R_{k-1}^2) - \mathcal{E}(R_{k-1}) & E \\ E & \mathcal{E}(R_{k-1}^2) - \mathcal{E}(R_{k-1}) \end{pmatrix}.$$

Whence,

$$[Q(\Gamma_k)] = \begin{bmatrix} [Q(\Gamma_{k-1})] & E \\ E & [Q(\Gamma_{k-1})] \end{bmatrix}.$$

We compute the characteristic polynomial $\chi_{k+1}(x) = \chi_{[Q(\Gamma_{k+1}]}(x)$.

$$\chi_{k+1}(x) = |xE - [Q(\Gamma_{k+1})]| = \begin{vmatrix} xE - [Q(\Gamma_k)] & -E \\ -E & xE - [Q(\Gamma_k)] \end{vmatrix} =$$

$$= \begin{vmatrix} xE - [Q(\Gamma_k)] - E & 0 \\ -E & xE - [Q(\Gamma_k)] + E \end{vmatrix} =$$

$$= |(x-1)E - [Q(\Gamma_k)]| \cdot |(x+1)E - [Q(\Gamma_k)]|$$

Therefore,

$$\chi_{k+1}(x) = \chi_k(x-1) \cdot \chi_k(x+1). \qquad (*)$$

Since $\chi_1(x) = \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = x(x-2)$,

then $\chi_2(x) = (x-3)(x-1)(x-1)(x+1) = (x-3)(x-1)^2(x+1)$, $\chi_3(x) = (x-4)(x-2)^2x(x-2)x^2(x+2) = (x-4)(x-2)^3x^3(x+2)$.

Proposition 7.6. $\chi_m(x) = \prod_{i=0}^{m}(x-m-1+2i)^{C_m^i}$.

Proof. We shall prove this proposition by induction on m. The basis of induction is clear. Suppose that the formula is true for $m = k$. Then, by formula (*), we obtain

$$\chi_{k+1}(x) = \prod_{i=0}^{k}(x-k-2+2i)^{C_k^i} \cdot \prod_{j=0}^{k}(x-k+2j)^{C_k^j} =$$

$$= (x-k-2)\prod_{i=1}^{k}(x-k-2+2i)^{C_k^i} \cdot \prod_{j=0}^{k-1}(x-k+2j)^{C_k^j}(x+k) =$$

$$= (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_k^{i+1}} \cdot \prod_{j=0}^{k-1}(x-k+2j)^{C_k^j}(x+k) =$$

$$= (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_k^i+C_k^{i+1}}(x+k).$$

Since $C_k^i + C_k^{i+1} = C_{k+1}^{i+1}$, we obtain $\chi_{k+1}(x) = (x-k-2)\prod_{i=0}^{k-1}(x-k+2i)^{C_{k+1}^{i+1}}(x+k) == (x-k-2)\prod_{j=1}^{k}(x-k+2(j-1))^{C_{k+1}^j}(x+k) = \prod_{j=0}^{k+1}(x-(k+1)-1+2j)^{C_{k+1}^j}$.

By induction on k, it is easy to prove that $\sum_{i=1}^{2^k} q_{ij}(\Gamma_k) = k+1$, $\sum_{j=1}^{2^k} q_{ij}(\Gamma_k) = k+1$.Thus, $[Q(\Gamma_k)] = (k+1)P_k$, where P_k is a doubly stochastic matrix. □

8. Exponent matrices of reduced cyclic Gorenstein orders

Lemma 8.1. *Let Λ be a cyclic reduced Gorenstein tiled order with exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ and permutation $\sigma(\Lambda) = (12\ldots n)$. If $\alpha_{i1} = 0$ for all $i = 1,\ldots,n$, then $\alpha_{1j} = \alpha_{1,n+2-j}$ for $1 < j \leqslant n$.*

Proof. By Corollary 1 of Lemma 3.3 [34], $\alpha_{ij} + \alpha_{ji} = \alpha_{\sigma^m(i)}\alpha_{\sigma^m(j)} + \alpha_{\sigma^m(j)}\alpha_{\sigma^m(i)}$ for any positive integer m. For $i = 1$, we have

$$\alpha_{1j} + \alpha_{j1} = \alpha_{\sigma^m(1)}\alpha_{\sigma^m(j)} + \alpha_{\sigma^m(j)}\alpha_{\sigma^m(1)}.$$

Since σ is cyclic, then there exists m such that $\sigma^m(j) = 1$. Since $\sigma^m(j) \equiv j + m \pmod{n}$, we have $j + m = n + 1$. Whence, $m = n + 1 - j$. Hence, $\sigma^m(1) = 1 + m = n + 2 - j$ and $\alpha_{1j} + \alpha_{j1} = \alpha_{n+2-j,1} + \alpha_{1,n+2-j}$. Since $\alpha_{j1} = \alpha_{n+2-j,1} = 0$, we obtain $\alpha_{1j} = \alpha_{1,n+2-j}$. □

Proposition 8.2. *Let $\alpha_{12}, \alpha_{13}, \dots, \alpha_{1n}$ be an arbitrary set of real numbers; then there exists a unique matrix (α_{ij}) such that these numbers are the entries of its first row, $\alpha_{kk} = \alpha_{k1} = 0$, and $\alpha_{ik} + \alpha_{ki+1} = \alpha_{ii+1}$ for all $k = 1, \dots, n; i = 1, \dots, n-1$.*

Proof. Put $\alpha_{k1} = 0$. Other entries α_{km} of (α_{ij}) will be obtained from the system of linear equations $\alpha_{ik} + \alpha_{ki+1} = \alpha_{ii+1} \quad k = 1, \dots, n; i = 1, \dots, n-1$.

We have, $\alpha_{kk+1} = \alpha_{k1} + \alpha_{1k+1} = \alpha_{1k+1}$ for $k < n$. It follows from the equality $\alpha_{1k} + \alpha_{k2} = \alpha_{12}$ that, $\alpha_{k2} = \alpha_{12} - \alpha_{1k}$. Since $\alpha_{k2} + \alpha_{2k+1} = \alpha_{kk+1} = \alpha_{1k+1}$, we obtain $\alpha_{2k+1} = \alpha_{1k+1} - \alpha_{k2} = \alpha_{1k+1} + \alpha_{1k} - \alpha_{12}$ or $\alpha_{2j} = \alpha_{1j} + \alpha_{1j-1} - \alpha_{12}$ for $j > 1$.

Also, $\alpha_{2k} + \alpha_{k3} = \alpha_{23} = \alpha_{13}$. Whence, $\alpha_{k3} = \alpha_{13} - \alpha_{2k} = \alpha_{13} + \alpha_{12} - \alpha_{1k} - \alpha_{1k-1}$ for $k > 1$. It follows from the equality $\alpha_{k3} + \alpha_{3k+1} = \alpha_{kk+1} = \alpha_{1k+1}$ that, $\alpha_{3k+1} = \alpha_{1k+1} - \alpha_{k3} = \alpha_{1k+1} + \alpha_{1k} + \alpha_{1k-1} - \alpha_{13} - \alpha_{12}$ or $\alpha_{3j} = \alpha_{1j} + \alpha_{1j-1} + \alpha_{1j-2} - \alpha_{13} - \alpha_{12}$ for $j > 2$.

Continuing in the same way, we successively obtain unknown entries of (α_{ij}). In the general case,

$$\alpha_{km} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1k-j} \text{ for } k > m-2; \tag{8.1}$$

$$\alpha_{km} = \sum_{j=0}^{k-1} \alpha_{1m-j} - \sum_{j=2}^{k} \alpha_{1j} \text{ for } m > k-1. \tag{8.2}$$

Thus we have the matrix (α_{ij}) with the entries

$$\alpha_{km} = \begin{cases} 0, & \text{for } m = 1, \\ \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1k-j}, & \text{if } k \geqslant m > 1, \\ \sum_{j=0}^{k-1} \alpha_{1m-j} - \sum_{j=2}^{k} \alpha_{1j}, & \text{if } 1 < k < m, \\ \alpha_{1m}, & \text{for } k = 1. \end{cases} \tag{8.3}$$

Clearly, $\alpha_{ik} + \alpha_{ki+1} = \alpha_{ii+1}$ for all $k = 1, \dots, n; i = 1, \dots, n-1$. Furthermore,

$$\alpha_{kk} = \sum_{j=2}^{k} \alpha_{1j} - \sum_{j=0}^{k-2} \alpha_{1k-j} = 0 \text{ for } k = 1, \dots, n.$$

□

Corollary 8.3. *Let (α_{ij}) be a matrix whose entries satisfy (3), and $\alpha_{1j} = \alpha_{1,n+2-j}$ for $j = 2,\ldots,n$. Then $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ for all $i,j = 1,\ldots,n$, where $\sigma = (12\ldots n)$.*

Since $\alpha_{1j} = \alpha_{1,n+2-j}$, then

$$\alpha_{nm} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1n-j} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{i=2}^{m} \alpha_{1,n+2-i} = 0$$

and $\alpha_{nm} + \alpha_{m1} = \alpha_{n1} = 0$ for all m. Thus, the entries of (α_{ij}) satisfy the condition $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ for all $i,j = 1,\ldots,n$, where $\sigma = (12\ldots n)$.

Proposition 8.4. *Let (α_{ij}) be a matrix whose entries satisfy (3). Then, for any three pairwise different i,j,k, there exist p,q such that $\alpha_{ij}+\alpha_{jk}-\alpha_{ik} = \alpha_{pq}$.*

Proof. We transform the equalities (1)-(2):

$$\begin{aligned}
\alpha_{km} &= \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1k-j} = \sum_{t=2}^{m} \alpha_{1t} - \sum_{t=k-m+2}^{k} \alpha_{1t}, \quad \text{for } k \geqslant m > 1,\\
\alpha_{km} &= \sum_{j=0}^{k-1} \alpha_{1m-j} - \sum_{j=2}^{k} \alpha_{1j} = \sum_{t=m-k+1}^{m} \alpha_{1t} - \sum_{t=2}^{k} \alpha_{1t}, \quad \text{for } m \geqslant k > 1.
\end{aligned}$$

Put $S_{ijk} = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}$.

For $min(i,j,k) > 1$, we consider 6 cases.

Case 1: If $i > j > k$, then

$$S_{ijk} = \left(\sum_{t=2}^{j} \alpha_{1t} - \sum_{t=i-j+2}^{i} \alpha_{1t}\right) + \\ + \left(\sum_{t=2}^{k} \alpha_{1t} - \sum_{t=j-k+2}^{j} \alpha_{1t}\right) - \left(\sum_{t=2}^{k} \alpha_{1t} - \sum_{t=i-k+2}^{i} \alpha_{1t}\right).$$

Here the third and the fifth sums are the same. In addition, the first and the forth sums contain the same summands, the second and the sixth sums possess identical summands, too. Simplifying, we obtain:

$$S_{ijk} = \sum_{t=2}^{j-k+1} \alpha_{1t} - \sum_{t=i-j+2}^{i-k+1} \alpha_{1t}.$$

But $i-j+2 = (i-k+1)-(j-k+1)+2$. Note also that $i-k+1 > j-k+1$. Therefore, in this case, $S_{ijk} = \alpha_{i-k+1,j-k+1}$.

Similarly, in the other five cases, we obtain $\alpha_{ij} + \alpha_{jk} - \alpha_{ik} = \alpha_{pq}$, where:
Case 2: $p = k - j,\ q = i - j + 1$ for $i > k > j$;
Case 3: $p = i - k + 1,\ q = j - k + 1$ for $j > i > k$;
Case 4: $p = j - i,\ q = k - i$ for $j > k > i$;
Case 5: $p = k - j,\ q = i - j + 1$ for $k > i > j$;
Case 6: $p = j - i,\ q = k - i$ if $k > j > i$.

If $min(i, j, k) = 1$, then $p = i,\ q = j$ for $k = 1$; $p = k - 1,\ q = i$ for $j = 1$; $p = j - 1,\ q = k - 1$ for $i = 1$.

Thus we have

$$\alpha_{pq} = \begin{cases} \alpha_{i-k+1,j-k+1}, & \text{if } min(i,j,k) = k, \\ \alpha_{k-j,i-j+1}, & \text{if } min(i,j,k) = j, \\ \alpha_{j-i,k-i}, & \text{if } min(i,j,k) = i. \end{cases}$$

If at the least two indices are same, then $S_{ijj} = 0$, $S_{iik} = 0$, $S_{iji} = \alpha_{ij} + \alpha_{ji} > 0$. □

Corollary 8.5. *Suppose that the entries of (α_{ij}) are non-negative, satisfy equalities (3), and $\alpha_{1j} = \alpha_{1,n+2-j}$ for all $2 \leqslant j \leqslant n$. Then (α_{ij}) is exponent matrix of a cyclic Gorenstein order with the permutation $\sigma(\Lambda) = (12\dots n)$.*

By Proposition 8.4, the entries of (α_{ij}) satisfy the ring inequalities. Using formula (1), we obtain $\alpha_{ii} = 0$ for all i. Therefore, by Corollary 8.3, (α_{ij}) is the exponent matrix of a Gorenstein tiled order Λ with permutation $\sigma(\Lambda) = (12\dots n)$.

Proposition 8.6. *Let (α_{ij}) be the exponent matrix of a cyclic reduced Gorenstein tiled order Λ with permutation $\sigma(\Lambda) = (12\dots n)$ and $\alpha_{i1} = 0$ for $i = 1, 2, \dots, n$. Then (α_{ij}) is symmetric with respect to the secondary diagonal.*

Proof. Evidently, any matrix (α_{ij}) is symmetric with respect to the secondary diagonal iff $\alpha_{ij} = \alpha_{n+1-j,n+1-i}$ for all i, j.

We examine the difference $\alpha_{km} - \alpha_{n+1-m,n+1-k}$. Suppose $k > m$, then $n + 1 - m > n + 1 - k$ and

$$\alpha_{km} - \alpha_{n+1-m,n+1-k} =$$
$$= \left(\sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1k-j}\right) - \left(\sum_{j=2}^{n+1-k} \alpha_{1j} - \sum_{j=0}^{n+1-k-2} \alpha_{1,n+1-m-j}\right) =$$
$$= \left(\sum_{j=2}^{m} \alpha_{1j} - \sum_{t=k-m+2}^{k} \alpha_{1t}\right) - \left(\sum_{j=2}^{n+1-k} \alpha_{1j} - \sum_{l=k-m+2}^{n+1-m} \alpha_{1l}\right).$$

If $m < n+1-k$, then $k < n+1-m$ and

$$\alpha_{km} - \alpha_{n+1-m,n+1-k} = -\left(\sum_{j=m+1}^{n+1-k} \alpha_{1j} - \sum_{l=k+1}^{n+1-m} \alpha_{1t}\right).$$

Since, by Lemma 8.1, $\alpha_{1j} = \alpha_{1,n+2-j}$, we see that

$$\sum_{j=m+1}^{n+1-k} \alpha_{1j} = \sum_{j=m+1}^{n+1-k} \alpha_{1,n+2-j} = \sum_{p=k+1}^{n+1-m} \alpha_{1p}$$

and $\alpha_{km} - \alpha_{n+1-m,n+1-k} = 0$.

Suppose $m > n+1-k$. Whence, $k > n+1-m$ and

$$\alpha_{km} - \alpha_{n+1-m,n+1-k} = \sum_{j=n+2-k}^{m} \alpha_{1j} - \sum_{j=n+2-m}^{k} \alpha_{1j}.$$

The application of Lemma 8.1 again yields $\alpha_{km} - \alpha_{n+1-m,n+1-k} = 0$. Thus, $\alpha_{km} = \alpha_{n+1-m,n+1-k}$ for $k > m$.

Similarly, we obtain this equality if $k < m$.

For $m = n+1-k$, the equality is trivial.

Thus the matrix (α_{ij}) is symmetric with respect to the secondary diagonal. □

Combining this proposition with Corollaries 8.3, 8.5, we obtain the following theorem which describes cyclic reduced Gorenstein exponent matrices.

Theorem 8.7. *Any cyclic reduced Gorenstein tiled order is isomorphic to a reduced order Λ with permutation $\sigma(\Lambda) = (1\ 2\ \dots\ n)$ such that the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ of Λ has the following properties:*
1. All entries of (α_{ij}) are expressed by formulas (3) with $\left[\frac{n}{2}\right]$ positive integral parameters $\alpha_{12}, \dots, \alpha_{1,[\frac{n}{2}]+1}$.
2. $\alpha_{1j} = \alpha_{1,n+2-j}$ for all j.
3. The matrix (α_{ij}) is symmetric with respect to the secondary diagonal.

Conversely, every non-negative integral matrix (α_{ij}) with properties 1-3 is the exponent matrix of some cyclic reduced Gorenstein tiled order with permutation $\sigma(\Lambda) = (1\ 2\ \dots\ n)$ if $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$.

Recall that the adjacency matrix of the quiver of any tiled order Λ with the Jacobson radical R is calculated by the formula

$$[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R).$$

If Λ is a cyclic Gorenstein order, then $[Q(\Lambda)] = \lambda P$, where P is a doubly stochastic matrix ([34], Theorem 3.4.). Moreover, from the chain equalities [34]

$$q_{ij} = q_{\sigma(i)\sigma(j)} = \dots = q_{\sigma^{n-1}(i)\sigma^{n-1}(j)}$$

it follows that $[Q(\Lambda)]$ contains at the most n different entries. Therefore it is sufficient to compute the entries of any row or column. Suppose that an exponent matrix $\mathcal{E}(\Lambda)$ satisfies the properties 1–3 of Theorem 8.7. Then

$$\gamma_{i1} = \min_k(\beta_{ik} + \beta_{k1}) - \beta_{i1} = \begin{cases} min(1, \min\limits_{k\neq i,1} \alpha_{ik}), & \text{if } i \neq 1, \\ min(2, \min\limits_{k\neq 1} \alpha_{1k}) - 1, & \text{if } i = 1. \end{cases} \tag{8.4}$$

We obtain the other entries of $[Q(\Lambda)]$ from the following chain of the equalities

$$q_{i1} = q_{\sigma(i)\,2} = q_{\sigma^2(i)\,3} = \dots = q_{\sigma^{n-1}(i)\,n}. \tag{8.5}$$

Since $\sigma^{k-1}(i)k = \sigma^{k-1}(i)\sigma^{n+1-i}(\sigma^{k-1}(i))$, we have

$$[Q] = \sum_{i=1}^{n} q_{i1} P_{\sigma^{n+1-i}}, \tag{8.6}$$

where $P_\sigma = \sum\limits_{i=1}^{n} e_{i\sigma(i)}$ is a permutation matrix, e_{ij} are matrix units.

Proposition 8.8. *The adjacency matrix of any cyclic Gorenstein order is a linear combination of powers of a permutation matrix P_σ with non-negative coefficients.*

Proof. The proof follows from the formula (8.6) and the equality $P_{\sigma^k} = (P_\sigma)^k, \quad k = 0, 1, \dots, n-1.$ □

As an example, we compute the exponent matrix and the adjacency matrix of a cyclic Gorenstein order for $n = 6$. Taking into account condition 1 from Theorem 8.7, we see that the matrix $\mathcal{E}(\Lambda)$ depends on three natural parameters. Put $\alpha_{12} = \alpha, \alpha_{13} = \beta, \alpha_{14} = \gamma$. Then, by condition 2 of Theorem 8.7, $\alpha_{15} = \alpha_{13} = \beta, \alpha_{16} = \alpha_{12} = \alpha$. Using formula (3), we obtain the other entries of the matrix. Thus, the exponent matrix has the form

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & \alpha & \beta & \gamma & \beta & \alpha \\ 0 & 0 & \beta & \beta+\gamma-\alpha & \beta+\gamma-\alpha & \beta \\ 0 & \alpha-\beta & 0 & \gamma & \beta+\gamma-\alpha & \gamma \\ 0 & \alpha-\gamma & \alpha-\gamma & 0 & \beta & \beta \\ 0 & \alpha-\beta & \alpha-\gamma & \alpha-\beta & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the entries of $\mathcal{E}(\Lambda)$ are non-negative (see Section 2), then $\alpha-\beta \geqslant 0$, $\alpha-\gamma \geqslant 0$, $\beta+\gamma-\alpha \geqslant 0$. Consequently, the integral parameters α, β, γ satisfy the system of inequalities $\alpha, \beta, \gamma \geqslant 1, \alpha \geqslant \beta, \alpha \geqslant \gamma, \beta+\gamma \geqslant \alpha$. We calculate, by (8.4), the entries of the first row of the adjacency matrix:

$$\begin{aligned}
q_{11} &= min(2, \min_{k\neq 1} \alpha_{1k}) - 1 = min(2, \beta, \gamma) - 1 = min(1, \beta - 1, \gamma - 1);\\
q_{21} &= min(1, \min_{k\neq 1,2} \alpha_{2k}) = min(1, \beta, \gamma + \beta - \alpha) = min(1, \gamma + \beta - \alpha);\\
q_{31} &= min(1, \min_{k\neq 1,3} \alpha_{3k}) = min(1, \alpha - \beta, \gamma, \gamma + \beta - \alpha) =\\
&= min(1, \alpha - \beta, \gamma + \beta - \alpha);\\
q_{41} &= min(1, \min_{k\neq 1,4} \alpha_{4k}) = min(1, \alpha - \gamma, \beta) = min(1, \alpha - \gamma);\\
q_{51} &= min(1, \min_{k\neq 1,5} \alpha_{5k}) = min(1, \alpha - \beta, \alpha - \gamma, \alpha) =\\
&= min(1, \alpha - \beta, \alpha - \gamma);\\
q_{61} &= min(1, \min_{k\neq 1,6} \alpha_{6k}) = min(1, 0) = 0.
\end{aligned}$$

Further, by (8.5), we obtain the other entries of the adjacency matrix. Thus

$$[Q] = \begin{pmatrix} q_{11} & q_{61} & q_{51} & q_{41} & q_{31} & q_{21} \\ q_{21} & q_{11} & q_{61} & q_{51} & q_{41} & q_{31} \\ q_{31} & q_{21} & q_{11} & q_{61} & q_{51} & q_{41} \\ q_{41} & q_{31} & q_{21} & q_{11} & q_{61} & q_{51} \\ q_{51} & q_{41} & q_{31} & q_{21} & q_{11} & q_{61} \\ q_{61} & q_{51} & q_{41} & q_{31} & q_{21} & q_{11} \end{pmatrix},$$

where q_{i1} are obtained above.

We recall that any doubly stochastic matrix is a linear combination of permutation matrices with non-negative coefficients

$$P = \sum_{\tau \in S_n} t_\tau P_\tau, \text{ where } t_\tau \geqslant 0.$$

In the general case, there are at the most $n!$ summands.

The definition of quiver $Q(B)$ for $B \in M_n(R)$ see in ([2], Section 5).

Proposition 8.9. *Let S be a doubly stochastic matrix. Then a quiver $Q(S)$ is a disjoint union of strongly connected quivers.*

Proof. Let $S \in M_n(R)$ be a doubly stochastic matrix. Suppose that the quiver $Q(S)$ is connected but non-strongly connected. Then there exists a permutational matrix P_τ such that $P_\tau^T S P_\tau = \begin{pmatrix} S_1 & X \\ 0 & S_2 \end{pmatrix}$.

The matrix $P_\tau^T S P_\tau$ is also doubly stochastic as a product of the doubly stochastic matrices. Therefore S_1 and S_2 are stochastic matrices.

Let $S_1^T \in M_n(R)$ and $S_2 \in M_{n-m}(R)$, and $m \geqslant 1$. Denote by $\Sigma(Y)$ the sum of all elements of an arbitrary matrix $Y \in M_n(R)$. Obviously, $\Sigma(P_\tau^T S P_\tau) = \Sigma(S_1) + \Sigma(S_2) + \Sigma(X)$. For any stochastic matrix $S \in M_n(R)$ and S^T, the equality $\Sigma(S) = \Sigma(S^T) = n$ holds. This sum does not change under a simultaneous transposition of rows and columns. Hence, $\Sigma(P_\tau^T S P_\tau) = n$. Clearly, S_1 and S_2 are stochastic matrices. Consequently, $n = m + n - m + \Sigma(X)$. Whence, $\Sigma(X) = 0$ and $X = 0$. Thus, the doubly stochastic matrix S is permutationally decomposable. This completes the proof. □

9. Global dimension of tiled orders

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a tiled order over a discrete valuation ring $\mathcal{O}$ and $M_n(\mathcal{D})$ its classical ring of fractions being $\mathcal{D}$ the classical division ring of fractions of $\mathcal{O}$, $\mathcal{E}(\Lambda) = (\alpha_{ij})$. Write $\mathcal{E}(\Lambda)^T = (\alpha_{ji})$ and $\Lambda^T = \{\mathcal{O},\, \mathcal{E}(\Lambda)^T\}$.

Proposition 9.1. *Λ^T is a tiled order and Λ is anti-isomorphic to Λ^T.*

Proof is obvious.

Proposition 9.2. *$gl.dim\,\Lambda = gl.dim\,\Lambda^T$.*

Proof. The proof follows from the equality $gl.dim\,\Lambda^T = l.gl.dim\,\Lambda$ and from [29], Theorem 20, which asserts that $l.gl.dim\,\Lambda = r.gl.dim\,\Lambda$ if Λ is both right and left noetherian. □

The definitions of the poset $\mathcal{M}(\Lambda) = \mathcal{M}_r(\Lambda)$ and of the width $w(\Lambda)$ of a tiled order Λ can be found in ([2], Section 3).

The following two theorems are proved in [5].

Theorem 9.3. *Let Λ be a tiled order in $M_n(\mathcal{D})$ and $w(\Lambda) \leqslant 2$. If $gl.dim\,\Lambda < \infty$ then $gl.dim\,\Lambda \leqslant n-1$.*

Theorem 9.4. *Let Λ be a tiled order and $w(\Lambda) \leqslant 2$. If $gl.dim\,\Lambda = k < \infty$, then for any $m\,(1 \leqslant m \leqslant k)$ there exists an idempotent $e \in \Lambda$ such that $gl.dim\, e\Lambda e = m$.*

Example. [37]. The tiled order $\Lambda_n = \{\mathcal{O},\, \mathcal{E}(\Lambda_n)\}$, where

$$\mathcal{E}_n = \mathcal{E}(\Lambda_n) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & 0 & 0 & 0 \\ 2 & 2 & 2 & \dots & 1 & 0 & 0 \\ 2 & 2 & 2 & \dots & 2 & 1 & 0 \end{pmatrix}$$

is the $n \times n$-matrix, is a triangular tiled order of width two and $gl.dim\,\Lambda_n = n-1$. This follows from [37] and [43].

The next proposition is very useful.

Proposition 9.5. *([27], Proposition 2.4). Let Λ be an order, and let e be an idempotent of Λ such that $e\Lambda e$ is a hereditary ring. Then $gl.dim\,(\Lambda/I) \leqslant gl.dim\Lambda \leqslant gl.dim\,(\Lambda/I)+2$.*

We use the example ([7], p. 283) of a serial ring A with $gl.dim\,A = 4$ and the Kupisch series 4,4,3 for a construction of a tiled order Λ of width 2 with $gl.dim\,\Lambda = 4$ and such that $Q(\Lambda)$ has five vertices.

Let

$$H_3(\mathcal{O}) = \{\mathcal{O}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\}$$

and

$$\mathcal{E}(I) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Then the tiled order Δ_5 with

$$\mathcal{E}(\Delta_5) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \end{array}\right)$$

has global dimension 4. This follows from Proposition 9.5.

Indeed, let $e = e_{44} + e_{55}$. Then

$$\mathcal{E}(I) = \mathcal{E}(\Delta_5 e \Delta_5) = \left(\begin{array}{ccc|cc} 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \end{array}\right)$$

and $gl.dim\,\Delta_5 \geqslant gl.dim\,\Lambda/I = 4$. It follows from Theorem 9.3 that $gl.dim\,\Delta_5 = 4$. Let $f = e_{11} + e_{22} + e_{33}$. Then

$$\mathcal{E}(J) = \mathcal{E}(\Delta_5 f \Delta_5) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 \end{array}\right).$$

It is easy to see that $gl.dim\,\Delta_5/J = 2$. Consequently, we have $gl.dim\,\Delta_5 = gl.dim\,\Delta_5/I$ and $gl.dim\,\Delta_5 = gl.dim\,\Delta_5/J + 2$.

It follows from this example that the both equalities in Proposition 9.5 may hold.

Theorem 9.6 ([41], [6]). *If Λ is a tiled order and $gl.dim\,\Lambda < \infty$, then $Q(\Lambda)$ has no loops.*

From [37] we have following theorem.

Theorem 9.7. *If Λ is a tiled order and $Q(\Lambda)$ has at most 3 vertices. Then $gl.dim\,\Lambda$ is finite if and only if $Q(\Lambda)$ has no loops. In this case $w(\Lambda) \leqslant 2$.*

The list of the orders Λ with $gl.dim\,\Lambda < \infty$ and such that $Q(\Lambda)$ has 4 vertices is given in [12]. The first six exponent matrices (1)-(6) from Section 3 exhaust this list.

Proposition 9.8. $w(\Omega_4) = 2$.

Proof. Proof is obvious. □

By [9], [43] Ω_4 has a bijective module. We denote the rejection of P by $\mathcal{E} - P$. Using Drozd-Kirichenko rejection Lemma (see [9], [17]) we have: $\Omega_4 \subset \mathcal{E}_4 = \Omega_4 - P_4 \subset \mathcal{E}_3 = \mathcal{E}_4 - P_4 \subset \mathcal{E}_2 = \mathcal{E}_3 - P_3 \subset H_4 = \mathcal{E}_2 - P_4$.

Recall that

$$\mathcal{E}(F_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Obviously, $w(F_4) = 3$.

Note, that all tiled orders of finite global dimension, whose quivers have at most four vertices are isomorphic to $(0,1)$-orders, except Ω_4. Now we give a list of associated posets P_Λ, where $gl.dim\,\Lambda < \infty$ and Λ is a $(0,1)$-order.

List of posets:

$n = 1$, $P_1 = \{\bullet\}$, $gl.dim\,\Lambda_{P_1} = 1$;

$$n = 2,\ P_2 = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right\},\ gl.dim\,\Lambda_{P_2} = 1;$$

$$n = 3,\ P_3 = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\},\ gl.dim\,\Lambda_{P_3} = 1;$$

$$P_4 = \left\{ \begin{array}{ccccc} \bullet & & & & \bullet \\ & \diagdown & & \diagup & \\ & & \bullet & & \end{array} \right\}, \ gl.dim\,\Lambda_{P_4} = 2;$$

$$n = 4,\ P_5 = \left\{ \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\}, \ gl.dim\,\Lambda_{P_5} = 1;$$

$$P_6 = \left\{ \begin{array}{ccc} \bullet & & \\ | & & \\ \bullet & & \bullet \\ | & \diagup & \\ \bullet & & \end{array} \right\}, \ gl.dim\,\Lambda_{P_6} = 2;$$

$$P_7 = \left\{ \begin{array}{ccccc} \bullet & & & & \bullet \\ & \diagdown & & \diagup & \\ & & \bullet & & \\ & & | & & \\ & & \bullet & & \end{array} \right\}, \ gl.dim\,\Lambda_{P_7} = 2;$$

$$P_8 = \left\{ \begin{array}{ccc} \bullet & & \bullet \\ | & \diagdown & | \\ \bullet & & \bullet \end{array} \right\}, \ gl.dim\,\Lambda_{P_8} = 3;$$

$$P_9 = \left\{ \begin{array}{ccccc} \bullet & & \bullet & & \bullet \\ & \diagdown & | & \diagup & \\ & & \bullet & & \end{array} \right\}, \ gl.dim\,\Lambda_{P_9} = 2.$$

It follows from Proposition 9.2 that if the finite posets P_{Λ_1} and P_{Λ_2}, which are associated with $(0,1)$-orders Λ_1 and Λ_2, are anti-isomorphic, then $gl.dim\,\Lambda_1 = gl.dim\,\Lambda_2$.

Definition and results on semilattices and commutative bands can be seen in [3] Section 1.8.

Proposition 9.9. *If $gl.dim\,\Lambda \leqslant 2$, then $\mathcal{M}(\Lambda)$ is a lower semilattice.*

Proof. It is well known that $gl.dim\,\Lambda = 1$ if and only if $\mathcal{M}(\Lambda)$ is a chain [44]. In this case$\mathcal{M}(\Lambda)$ is a lower semilattice. If $\mathcal{M}(\Lambda)$ is not a chain, let P_i and P_j be non-comparable elements of $\mathcal{M}(\Lambda)$. Then $P_i + P_j = M$ and the projective cover $P(M)$ of M is $P_i \oplus P_j$. Let $\varphi : P(M) \longrightarrow M$. Then $Ker\,\varphi \simeq P_i \cap P_j$ is projective. □

Proposition 9.10. *If a poset P_Λ associated with a $(0,1)$-order Λ has a unique maximal element or unique minimal element, then gl.dim $\Lambda < \infty$.*

Proof. By Proposition 9.2 we can assume that P_Λ has a unique minimal element and

$$\mathcal{E}(\Lambda) = \left(\begin{array}{c|cccc} 0 & 0 & \dots & 0 & 0 \\ \hline 1 & 0 & & & \\ & & & & * \\ \vdots & & & \ddots & \\ & & * & & \\ 1 & & & & 0 \end{array}\right).$$

Let $e = e_{11}$ and $I = \Lambda e \Lambda$. In this case the quotient ring Λ/I is, obviously, an l-hereditary ring or a piecewise domain in sense of [14]. Thus, *gl.dim* Λ/I is finite (see, for example, [8], Ch. XI, exercise 12) and by Proposition 9.5 *gl.dim.* Λ is finite. □

Proposition 9.11. *[10]. If the poset P_Λ associated with a $(0,1)$-order Λ is disconnected then gl.dim $\Lambda = \infty$.*

Proposition 9.12. *The chain*

$$\mathcal{E}(I_1) = \begin{pmatrix} n-1 & n-2 & \dots & 1 & 0 \\ n-1 & n-2 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ n-1 & n-2 & \dots & 1 & 0 \\ n-1 & n-2 & \dots & 1 & 0 \end{pmatrix} \subset$$

$$\subset \mathcal{E}(I_2) = \begin{pmatrix} n-2 & n-3 & \dots & 1 & 0 \\ n-2 & n-3 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & \dots & 1 & 0 \\ n-1 & n-2 & \dots & 1 & 0 \end{pmatrix} \subset \dots \subset \mathcal{E}(I_{n-1}) =$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & \dots & 0 & 0 \\ n-1 & n-2 & \dots & 1 & 0 \end{pmatrix} \subset \mathcal{E}(\Omega_n)$$

is a chain of projective idempotent ideals of Ω_n and the quotient ring Ω_n/I_1 is quasi-hereditary.

Proof. Proof is obvious. □

Proposition 9.13. *The chain*

$$\mathcal{E}(J_1) = \begin{pmatrix} 2 & 2 & \dots & 1 & 0 \\ 2 & 2 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & 1 & 0 \\ 2 & 2 & \dots & 1 & 0 \end{pmatrix} \subset$$

$$\subset \mathcal{E}(J_2) = \begin{pmatrix} 2 & 2 & \dots & 0 & 0 \\ 2 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & 0 & 0 \\ 2 & 2 & \dots & 1 & 0 \end{pmatrix} \subset$$

$$\subset \dots \subset \mathcal{E}(J_{n-1}) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & 0 & 0 \\ 2 & 2 & \dots & 1 & 0 \end{pmatrix} \subset \mathcal{E}(\Lambda_n)$$

is a chain of projective idempotent ideals of Λ_n and the quotient ring Ω_n/J_1 is quasi-hereditary.

Proof is obvious.

Theorem 9.14. *If A is a noetherian prime semiperfect semidistributive ring of a finite global dimension, then $Q(\Lambda)$ is a strongly connected simply laced quiver which has no loops.*

Proof. The proof follows from the Decomposition theorem for noetherian semiprime $SPSD$-rings (see [2], Theorem 3.8 and [25]), ([29], Theorem 16), Theorem 9.7 and ([2], Theorem 4.10). □

Now we shall compute the quiver $Q(\Omega_n)$ and its transition matrix for the reduced exponent matrix Ω_n. We use the formula $[Q(\Omega)_n] = \Omega_n^{(2)} - \Omega_n^{(1)}$. Obviously,

$$\Omega_n^{(2)} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 2 & 1 & 1 & \dots & 0 & 0 \\ 2 & 2 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n-2 & n-3 & n-4 & \dots & 1 & 1 \\ n-1 & n-2 & n-3 & \dots & 2 & 1 \end{pmatrix}$$

and $[Q(\Omega_n)] = J_n^-(0) + J_n^+(0) = Y_n$, where $J_n^+(0) = e_{12} + e_{23} + \ldots + e_{n-1n}$. We have that $in\,\Omega_n = 2\cos\frac{\pi}{n+1}$ and

$$\vec{f} = (\sin\frac{\pi}{n+1}, \sin\frac{2\pi}{n+1}, \dots, \sin\frac{(n-1)\pi}{n+1}, \sin\frac{n\pi}{n+1})$$

is a positive eigen-vector of Y with eigen-value $2\cos\frac{\pi}{n+1}$.

Thus the transition matrix S_n for the quiver $Q(\Omega_n)$ by Theorem 5 ([28], p. 324) is:

$$S_n=\lambda^{-1}Z^{-1}Y_nZ=\frac{1}{2\cos\frac{\pi}{n+1}}\begin{pmatrix} \frac{1}{\sin\frac{\pi}{n+1}} & 0 & 0 & \dots & 0 & 0\\ 0 & \frac{1}{\sin\frac{2\pi}{n+1}} & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \frac{1}{\sin\frac{(n-1)pi}{n+1}} & 0\\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\sin\frac{n\pi}{n+1}} \end{pmatrix}.$$

$$\cdot\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 1 & 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sin\frac{\pi}{n+1} & 0 & 0 & \dots & 0 & 0\\ 0 & \sin\frac{2\pi}{n+1} & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & \sin\frac{(n-1)\pi}{n+1} & 0\\ 0 & 0 & 0 & \dots & 0 & \sin\frac{n\pi}{n+1} \end{pmatrix}=$$

$$=\frac{1}{2\cos\frac{\pi}{n+1}}\begin{pmatrix} 0 & \frac{\sin\frac{2\pi}{n+1}}{\sin\frac{\pi}{n+1}} & 0 & 0 & \dots & 0 & 0 & 0\\ \frac{\sin\frac{\pi}{n+1}}{\sin\frac{2\pi}{n+1}} & 0 & \frac{\sin\frac{3\pi}{n+1}}{\sin\frac{2\pi}{n+1}} & 0 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & 0 & \dots & \frac{\sin\frac{(n-2)\pi}{n+1}}{\sin\frac{(n-1)\pi}{n+1}} & 0 & \frac{\sin\frac{n\pi}{n+1}}{\sin\frac{(n-1)\pi}{n+1}}\\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\sin\frac{(n-1)\pi}{n+1}}{\sin\frac{n\pi}{n+1}} & 0 \end{pmatrix}.$$

The matrix S_n define a random walk on the set $\{1,2,\dots,n\}\subset\mathbb{N}$ (see [22], pp. 26-27).

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