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Flows in graphs and the homology of free categories

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Abstract. We study the $R-$module of generalized flows in a graph with coefficients in the $R-$representation of the graph over a ring R with 1 and show that this $R-$module is isomorphic to the first derived functor of the colimit. We generalize Kirchhoff's laws and build an exact sequence for calculating the $R-$module of flows in the union of graphs.

1. Preliminaries

Let $\Gamma = (A \overset{s}{\underset{t}{\rightrightarrows}} V)$ be a directed graph, R a ring with 1, $\{F(v)\}_{v\in V}$ a family of left $R-$modules with a family of $R-$homomorphisms $\{F(\gamma) : F(s(\gamma)) \to F(t(\gamma))\}_{\gamma\in A}$. *A (generalized) flow on* Γ *with coefficients in* F is a family $\{f_\gamma\}_{\gamma\in A}$ of $f_\gamma \in F(s(\gamma))$ such that almost all of f_γ are zeros and for each $v \in V$ the following equality holds

$$\sum_{s(\gamma)=v} f_\gamma = \sum_{t(\gamma)=v} F(\gamma)(f_\gamma)$$

If $F(v) = \mathbb{Z}$ for all $v \in V$ where $\mathbb{Z}$ is the additive group of integers and if $F(\gamma)$ are the identity homomorphisms $id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ for all $\gamma \in A$ then we have the (ordinary) *integer* flows.

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The purpose of this work is to applicate the homology theory of small categories (in the sense of [2, Application 2], [3], [7]) for the study of the $R-$module of generalized flows in the directed graphs with $R-$homomorphisms $F(\gamma)$ on the edges $\gamma \in A$. Flows with intensifications and flows with delays are the examples of such generalized flows. Our approach to the flows in graphs is entirely distinguished with the theory described in Wagner's work [8] where it was considered the functorial sequence of groups of flows in undirected graphs and proved that this sequence is the invariant of graphs.

We show that the $R-$module of flows is isomorphic to the first homology of the free category generated by Γ with coefficients in the functor corresponding to F. This generalize that the abelian group of integer flows is isomorphic the first integer homology group of the graph. This allows us to applicate the homological methods in the study of the networks. We calculate the $R-$module of flows in the union of graphs and generalize Kirchhoff's laws.

Let $\mathbf{C}$ be a small category, R a ring with identity. Denote by Mod_R the category of left R-modules and R-homomorphisms, $Mod_R^{\mathbf{C}}$ the category of functors $\mathbf{C} \to Mod_R$, $\mathrm{colim}^{\mathbf{C}} : Mod_R^{\mathbf{C}} \to Mod_R$ the colimit functor. The category $Mod_R^{\mathbf{C}}$ has enough projectives [3]. The functor $\mathrm{colim}^{\mathbf{C}}$ is right exact. Hence for every integer $n \geqslant 0$ it is defined the n-th left derived functor $\mathrm{colim}_n^{\mathbf{C}} : Mod_R^{\mathbf{C}} \to Mod_R$ of the colimit. Let $F : \mathbf{C} \to Mod_R$ be a functor. For arbitrary family $\{a_i\}_{i\in I}$ we will say that *almost all* a_i *are zeros* if there exists a finite subset $J \subseteq I$ such that $a_i = 0$ for all $i \in I \setminus J$. Denote $C_n(\mathbf{C}, F) = \sum\limits_{c_0\to\ldots\to c_n} F(c_0)$ and write elements of $C_n(\mathbf{C}, F)$ as sums $\sum\limits_{c_0\to\ldots\to c_n} f_{c_0\to\ldots\to c_n}[c_0 \to \ldots \to c_n]$ with $f_{c_0\to\ldots\to c_n} \in F(c_0)$ where almost all $f_{c_0\to\ldots\to c_n}$ are zeros. For every $c_0 \to \ldots \to c_{n+1}$ and $f \in F(c_0)$ we let

$$d_n(f[c_0 \to \ldots \to c_{n+1}]) = F(c_0 \to c_1)(f)[c_1 \to \ldots \to c_{n+1}] + \\ + \sum_{i=1}^{n+1} (-1)^i f[c_0 \to \ldots \to \hat{c}_i \to \ldots \to c_{n+1}]$$

and define homomorphisms $d_n : C_{n+1}(\mathbf{C}, F) \to C_n(\mathbf{C}, F)$ by

$$d_n\left(\sum_{c_0\to\ldots\to c_{n+1}} f_{c_0\to\ldots\to c_{n+1}}[c_0 \to \ldots \to c_{n+1}]\right) = \\ = \sum_{c_0\to\ldots\to c_{n+1}} d_n\left(f_{c_0\to\ldots\to c_{n+1}}[c_0 \to \ldots \to c_{n+1}]\right).$$

Here

$$[c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} \hat{c}_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}] =$$

$$\begin{cases} [c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} c_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} c_{i+1} \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}], & if\ 0 < i < n+1, \\ [c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} c_n], & if\ i = n+1. \end{cases}$$

It is well-known [2, Application 2] that R-modules $\mathrm{colim}_n^{\mathbf{C}} F$ are isomorphic to homologies of the complex

$$0 \xleftarrow{d_{-1}} C_0(\mathbf{C}, F) \xleftarrow{d_0} C_1(\mathbf{C}, F) \xleftarrow{d_1} \cdots \xleftarrow{d_{n-1}} C_n(\mathbf{C}, F) \xleftarrow{d_n} \cdots$$

in the sense of $\mathrm{colim}_n^{\mathbf{C}} F \cong Kerd_{n-1}/Imd_n$, $n \geqslant 0$.

If $\alpha \circ \beta = id$ in $\mathbf{C}$ implies $\alpha = id$ and $\beta = id$, then $\mathbf{C}$ is called to be *without retractions*. If $\mathbf{C}$ is a small category without retractions, then the complex $\{C_n(\mathbf{C}, F), d_n\}$ includes the subcomplex $\{C_n^+(\mathbf{C}, F), d_n\}$ where

$$C_n^+(\mathbf{C}, F) = \sum_{c_0 \underset{\neq}{\to} \cdots \underset{\neq}{\to} c_n} F(c_0)$$

is the submodule in which sequences $c_0 \to \cdots \to c_n$ do not contain identity morphisms if $n > 0$, with $C_0^+(\mathbf{C}, F) = C_0(\mathbf{C}, F)$. As the dual affirmation [5, Proposition 2.2] we can prove the following.

Lemma 1.1. *Let $\mathbf{C}$ be a small category without retractions, R a ring with identity. Then for each functor $F : \mathbf{C} \to Mod_R$ the R-modules $\mathrm{colim}_n^{\mathbf{C}} F$ are isomorphic to n-th homology modules of the complex $\{C_n^+(\mathbf{C}, F), d_n\}$.*

Let $\Delta_{\mathbf{C}}\mathbb{Z}$ be a functor from a small category to the category Ab of Abelian groups and homomorphisms which assign to every $c \in \mathbf{C}$ the group of integers $\mathbb{Z}$ and to every $\alpha \in Mor\mathbf{C}$ the identity homomorphism $id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$. We denote by $H_n(\mathbf{C})$ the groups $\mathrm{colim}_n^{\mathbf{C}} \Delta_{\mathbf{C}}\mathbb{Z}$ for all $n \geqslant 0$. Let $S : \mathbf{C} \to \mathbf{D}$ be a functor between small categories, d an object in $\mathbf{D}$. The *comma-category* d/S is defined as the following category:

Objects of d/S are pairs (c, α) with $c \in Ob\mathbf{C}$ and $\alpha \in \mathbf{D}(d, S(c))$, morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ in d/S consist of the triples $(\beta \in \mathbf{C}(c_1, c_2), \alpha_1, \alpha_2)$ satisfying $S(\beta) \circ \alpha_1 = \alpha_2$. A functor $S : \mathbf{C} \to \mathbf{D}$ is called *strong cofinal* if $H_n(d/S) \cong H_n(pt)$ for all $n \geqslant 0$. Here $pt = \{*\}$ is the discrete category with one object, thus S is strong cofinal if and only if for all $d \in Ob\mathbf{D}$ the groups $H_n(d/S)$ are zeros for all $n > 0$ and d/S are connected.

By Oberst's Theorem [7, Theorem 2.3] if S is strong cofinal, then for every functor $F : \mathbf{D} \to Mod_R$ the canonical homomorphisms $\mathrm{colim}_n^{\mathbf{D}} F \to \mathrm{colim}_n^{\mathbf{C}} (F \circ S)$ are isomorphisms.

Let $\mathbf{C}$ be a small category. Each object $a \in Ob\mathbf{C}$ will be considered as the identity id_a, so $Ob\mathbf{C} \subseteq Mor\mathbf{C}$. The *factorization category* [1] $\mathcal{F}\mathbf{C}$ is the category such that $Ob\mathcal{F}\mathbf{C} = Mor\mathbf{C}$ and for every pair $f, g \in Mor\mathbf{C}$ the set $\mathcal{F}\mathbf{C}(f, g)$ of morphisms consists of the pairs (α, β), with $\alpha, \beta \in Mor\mathbf{C}$, for which $\beta \circ f \circ \alpha = g$.

Lemma 1.2. *The functor $s : (\mathcal{F}\mathbf{C})^{op} \to \mathbf{C}$ which assign to any $f \in Ob\mathcal{F}\mathbf{C}$ its domain $s(f)$ and to any morphism (α, β) the morphism α is strong cofinal.*

Proof. Objects of c/s for $c \in Ob\mathbf{C}$ are pairs (x, α) of morphisms $c \xrightarrow{x} s(\alpha) \xrightarrow{\alpha} t(\alpha)$, and morphisms $(x, \alpha) \to (y, \beta)$ are commutative diagrams

$$\begin{array}{ccccc} c & \xrightarrow{x} & s(\alpha) & \xrightarrow{\alpha} & t(\alpha) \\ \downarrow id_c & & \downarrow & & \uparrow \\ c & \xrightarrow{y} & s(\beta) & \xrightarrow{\beta} & t(\beta) \end{array}$$

For every $c \in Ob\mathbf{C}$ the category c/s includes the full subcategory consisting of all the objects (id_c, α). This subcategory is isomorphic to $(c/\mathbf{C})^{op}$. For every object (x, α) there is a morphism $(id_c, \alpha \circ x) \to (x, \alpha)$ such that for each morphism $(id_c, \beta) \to (x, \alpha)$ there exists the unique morphism $(id_c, \beta) \to (id_c, \alpha \circ x)$ for which the following diagram is commutative

$$\begin{array}{ccc} (id_c, \alpha \circ x) & \longrightarrow & (x, \alpha) \\ \uparrow \exists! & & \uparrow \\ (id_c, \beta) & \xrightarrow{id} & (id_c, \beta) \end{array}$$

It follows that there exists a right adjoint functor to the inclusion $(c/\mathbf{C})^{op} \subseteq c/s$. A right adjoint is strong cofinal, hence

$$\mathrm{colim}_n^{c/s} \Delta\mathbb{Z} \cong \mathrm{colim}_n^{(c/\mathbf{C})^{op}} \Delta\mathbb{Z}.$$

But $(c/\mathbf{C})^{op}$ has a terminal object. Thus $H_n(c/s) \cong H_n(pt)$ for all $n \geqslant 0$. $\square$

2. Generalized Flows

By a *(directed) graph* we mean a pair of sets (A, V) and a pair of functions $A \overset{s}{\underset{t}{\rightrightarrows}} V$. The elements of A are called *arrows*, V is the set of *vertexes*, $s(\alpha)$ and $t(\alpha)$ are called the *source* and the *target* of $\alpha \in A$ respectively.

Let $\Gamma = (A, V, s, t)$ be a graph, R a ring with identity. A *R-representation* of Γ is a family of R-modules $\{F(v)\}_{v \in V}$ with a family of homomorphisms $\{F(\alpha) : F(s(\alpha)) \to F(t(\alpha))\}_{\alpha \in A}$. A *path* in Γ from $u \in V$ to

$v \in V$ is an arbitrary word $\alpha_1\alpha_2\cdots\alpha_n$ with $t(\alpha_1) = v$ and $s(\alpha_n) = u$ such that $s(\alpha_i) = t(\alpha_{i+1})$ for all $1 \leqslant i \leqslant n-1$. For each vertex $v \in V$ define id_v as the empty path from v to v. Objects of the *category of paths in* Γ are vertexes $v \in V$, and morphisms are paths in Γ with the composition law

$$\alpha_1\cdots\alpha_n \circ \beta_1\cdots\beta_m = \alpha_1\cdots\alpha_n\beta_1\cdots\beta_m,$$

for $s(\alpha_n) = t(\beta_1)$. Let $W\Gamma$ be the category of paths in Γ. For every R-representation F there is the unique functor $\tilde{F} : W\Gamma \to Mod_R$ such that $\tilde{F}|_\Gamma = F$. This functor is defined by $\tilde{F}(\alpha_1\cdots\alpha_n) = F(\alpha_1)\cdots F(\alpha_n)$ if $n > 0$, and $\tilde{F}(id_v) = id_{F(v)}$.

Definition 2.1. *Let* Γ *be a graph,* F *a* R*-representation of* Γ*. A* flow in Γ with coefficients in F *is a family* $\{f_\gamma\}_{\gamma\in A}$ *of elements* $f_\gamma \in F(s(\gamma))$ *such that almost all of* f_γ *are zeros and for each* $v \in V$ *the following equality holds:*

$$\sum_{s(\gamma)=v} f_\gamma = \sum_{t(\gamma)=v} F(\gamma)(f_\gamma)$$

We denote by $\Phi(\Gamma; F)$ the R-module of flows in Γ with coefficients in F. We have the following exact sequence:

$$0 \to \Phi(\Gamma; F) \to \sum_{\gamma\in A} F(s(\gamma)) \xrightarrow{d} \sum_{v\in V} F(v) \to \operatorname{colim}^{W\Gamma}\tilde{F} \to 0, \qquad (1)$$

where $d(\sum_{\gamma\in A} f_\gamma[\gamma])_v = \sum_{t(\gamma)=v} F(\gamma)(f_\gamma) - \sum_{s(\gamma)=v} f_\gamma$.

Example 2.2. Let $\Gamma = (A, V, s, t)$ be a (directed) graph. If $F(v) = \mathbb{Z}$ for all $v \in V$ and if $F(\alpha) = id_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$ then $\tilde{F} = \Delta_{W\Gamma}\mathbb{Z}$. It is well known that $\Phi(\Gamma, F)$ is isomorphic to the integer group homology $H_1(\Gamma, \mathbb{Z})$.

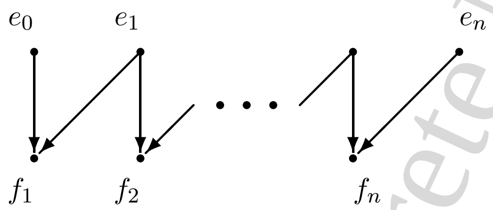
If $F(v) = \mathbb{Z}$ for all $v \in V$ and if $F(\alpha) : \mathbb{Z} \to \mathbb{Z}$ act by $z \mapsto p_\alpha z$ for some family of $p_\alpha \in \mathbb{Z}$ then we have *flows with intensifications.*

Let $F(v) = \mathbb{Z}^\mathbb{Z}$ be the abelian group of functions $f : \mathbb{Z} \to \mathbb{Z}$ for all $v \in V$. If $F(\alpha) : \mathbb{Z}^\mathbb{Z} \to \mathbb{Z}^\mathbb{Z}$ act by $F(\alpha)(f)(t) = f(t - t_\alpha)$ for some family of $t_\alpha \in \mathbb{Z}$ then $\Phi(\Gamma; F)$ consists of *flows with delays.*

Let $S : \mathbf{C} \to \mathbf{D}$ be a functor between small categories, $d \in Ob\mathbf{D}$ an object. We denote by S/d the category which objects are pairs (c, α) with $c \in Ob\mathbf{C}$, $\alpha \in \mathbf{D}(S(c), d)$); morphisms $(c_1, \alpha_1) \to (c_2, \alpha_2)$ in S/d are triples $(\beta \in \mathbf{C}(c_1, c_2), \alpha_1, \alpha_2)$ satisfying $\alpha_2 \circ S(\beta) = \alpha_1$. It is clear that $d/(S^{op}) \cong (S/d)^{op}$. Denote $\mathcal{F}W\Gamma = \mathcal{F}(W\Gamma)$.

Lemma 2.3. *Let* $\Gamma = (A, V, s, t)$ *be a graph,* Γ' *the full subcategory of* $\mathcal{F}W\Gamma$ *such that* $Ob\Gamma' = A \coprod V$*. Then* Γ'^{op} *is strong cofinal in* $(\mathcal{F}W\Gamma)^{op}$*.*

Proof. We denote by S the inclusion $\Gamma' \subseteq \mathcal{F}W\Gamma$. The objects of S/w are pairs $(x,(\alpha,\beta))$ where α and β are paths in Γ satisfying $\beta\circ x\circ\alpha = w$ with either $x = id$ or $x \in A$. Hence for $w = (v_0 \xrightarrow{\alpha_1} v_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} v_n)$ the category S/w is the following:



The category S/w

with the objects

$$e_0 = (id_{v_0},(id_{v_0},\alpha_n\cdots\alpha_1)), f_1 = (\alpha_1,(id_{v_0},\alpha_n\cdots\alpha_2)),$$
$$e_1 = (id_{v_1},(\alpha_1,\alpha_n\cdots\alpha_2)),\cdots,$$
$$f_n = (\alpha_n,(\alpha_{n-1}\cdots\alpha_1, id_{v_n})), e_n = (id_{v_n},(\alpha_n\cdots\alpha_1, id_{v_n})),$$

where all morphisms excluding $e_{i-1}\to f_i$ and $e_i \to f_i$, for $i\in\{1,2,\cdots,n\}$, are identities.

But w/S^{op} is isomorphic to $(S/w)^{op}$, hence $H_n(w/S^{op}) = 0$ for $n > 0$ and w/S^{op} is connected. □

Lemma 2.4. *Let $\Gamma = (A,V,s,t)$ be a graph, $F : (\mathcal{F}W\Gamma)^{op} \to Mod_R$ a functor. Then $\operatorname{colim}_1^{(\mathcal{F}W\Gamma)^{op}} F$ is isomorphic to the submodule in $\sum_{\gamma\in A} F(\gamma)$ consisting of families $\{g_\gamma\}_{\gamma\in A}$ for which the following equality holds for every $v\in V$:*

$$\sum_{v=s(\gamma)} F(v \stackrel{(id,\gamma)}{\to} \gamma)g_\gamma = \sum_{v=t(\gamma)} F(v \stackrel{(\gamma,id)}{\to} \gamma)g_\gamma \qquad (2)$$

Here γ runs the set A.

Proof. We have by Lemma 2.3 from the Oberst theorem that $\operatorname{colim}_n^{(\mathcal{F}W\Gamma)^{op}} F$ is isomorphic to $\operatorname{colim}_n^{\Gamma'^{op}} F|_{\Gamma'^{op}}$. The category Γ'^{op} has no retractions except identities.

Hence, by Lemma 1.1 R-modules $\operatorname{colim}_n^{(\mathcal{F}W\Gamma)^{op}} F$ are isomorphic to the homology groups of the chain complex

$$0 \to C_1^+(\Gamma'^{op}, F) \to C_0^+(\Gamma'^{op}, F) \to 0$$

which is equal to

$$0 \to \sum_{id \neq (v\to\gamma) \in Mor\Gamma'} F(\gamma) \xrightarrow{d} \sum_{\gamma\in A} F(\gamma) \oplus \sum_{v \in V} F(v) \to 0$$

with

$$d(\sum_{v\to\gamma} f_{v\to\gamma}[v\to\gamma]) = \sum_{v\to\gamma} d(f_{v\to\gamma}[v\to\gamma]) =$$
$$= \sum_{v\to\gamma} (F(v\to\gamma)(f_{v\to\gamma})[v] - f_{v\to\gamma}[\gamma]).$$

(Here we consider the homology of the category which is opposite to Γ', in particular $f_{v\to\gamma} \in F(\gamma)$ and a homomorphism $F(v\to\gamma)$ acts from $F(\gamma)$ into $F(v)$.) Therefore $\mathrm{colim}_1^{\Gamma'^{op}} F$ is isomorphic to the R-module of families $f_{v\to\gamma} \in F(\gamma)$ satisfying $f_{s(\gamma)\to\gamma} + f_{t(\gamma)\to\gamma} = 0$ for each $\gamma \in A$, and $\sum\limits_{v\to\gamma} F(v\to\gamma)(f_{v\to\gamma}) = 0$ for each $v \in V$.

We denote $g_\gamma = f_{s(\gamma)\to\gamma}$ for a such family. Then $f_{t(\gamma)\to\gamma} = -g_\gamma$, and $\sum\limits_{v=s(\gamma)} F(v\to\gamma)(g_\gamma) = \sum\limits_{v=t(\gamma)} F(v\to\gamma)(g_\gamma)$. Thus $\mathrm{colim}_1^{\mathcal{F}W\Gamma^{op}} F$ is isomorphic to the submodule of $\sum\limits_{\gamma\in A} F(\gamma)$ consisting of $\{g_\gamma\}_{\gamma\in A}$ for which the equation (2) holds. □

Theorem 2.5. *Let $\Gamma = (A, V, s, t)$ be a graph, R a ring with identity, F a R-representation of Γ. Then $\Phi(\Gamma; F) \cong \mathrm{colim}_1^{W\Gamma} \tilde{F}$.*

Proof. By Lemma 1.2 the functor $s : (\mathcal{F}\mathbf{C})^{op} \to \mathbf{C}$ is strong cofinal for an arbitrary small category. Hence $\mathrm{colim}_1^{W\Gamma} \tilde{F} \cong \mathrm{colim}_1^{\mathcal{F}W\Gamma^{op}} (\tilde{F} \circ s)$. The substitution of $\tilde{F} \circ s$ instead F in Lemma 2.4 leads to the concluding that $\mathrm{colim}_1^{W\Gamma} \tilde{F}$ is isomorphic to the submodule of $\sum\limits_{\gamma\in A} F(\gamma)$ which consists of families $f_\gamma \in F(s(\gamma))$ satisfying $\sum\limits_{v=s(\gamma)} f_\gamma = \sum\limits_{v=t(\gamma)} F(\gamma)(f_\gamma)$, for each $v \in V$. □

Example 2.6. Let $\Gamma = (A, V, s, t)$ be the graph with $A = \{\gamma\}$ consists of the one arrow, $V = \{v\}$, where $s(\gamma) = t(\gamma) = v$. Then the category $W\Gamma$ is isomorphic the free monoid generated by one element. For every functor $\tilde{F} : W\Gamma \to Mod_R$ the $R-$module $\Phi(\Gamma; F)$ contains $f \in F(v)$ for which $F(\gamma)(f) = f$. Hence, $H_1(W\Gamma, \tilde{F})$ is isomorphic the submodule of all fixed elements of the action $F(\gamma) : F(v) \to F(v)$.

3. The First Kirchhoff Law

Let $\Gamma = (A, V, s, t)$ be a graph, F a R-representation of Γ. Elements of $\sum\limits_{v\in V} F(v)$ and $\sum\limits_{\gamma\in A} F(s(\gamma))$ are called 0-*chains* and 1-*chains* respectively.

Let $\varepsilon : \sum\limits_{v\in V} F(v) \to \operatorname{colim}^{W\Gamma}\tilde{F}$ be the canonical R-homomorphism. It follows from the exact sequence (1) that the equation $df = \varphi$ has a solution for $\varphi \in \sum\limits_{v\in V} F(v)$ if and only if $\varepsilon(\varphi) = 0$. Hence $\operatorname{colim}^{W\Gamma}\tilde{F}$ can be interpreted as the R-module of "obstructions". Denote it by $\Phi_0(\Gamma; F)$. We have the exact sequence

$$0 \to \Phi(\Gamma; F) \overset{\subseteq}{\to} \sum_{\gamma\in A} F(s(\gamma)) \overset{d}{\longrightarrow} \sum_{v\in V} F(v) \overset{\varepsilon}{\to} \Phi_0(\Gamma; F) \to 0$$

with $\Phi(\Gamma; F) \cong \operatorname{colim}_1^{W\Gamma}\tilde{F}$ and $\Phi_0(\Gamma; F) \cong \operatorname{colim}^{W\Gamma}\tilde{F}$.

A *network* (Γ, E, F) consists of the following data:

1) a graph $\Gamma = (A, V, s, t)$;

2) an arbitrary subset $E \subseteq V$ which elements are called *external*;

3) a R-representation F of Γ.

We say that the 1-chain $\{f_\gamma\}_{\gamma\in A}$ *satisfies to the first Kirchhoff law* if

$$d(\{f_\gamma\}_{\gamma\in A})_v = 0, \quad \forall v \notin E.$$

Let $\Phi(\Gamma, E; F)$ be a R-module of all 1-chains satisfying to the first Kirchhoff law in the network (Γ, E, F). For $E = \emptyset$ an 1-chain satisfies to the first Kirchhoff law if and only if it is a flow in Γ with coefficients in F. Thus, $\Phi(\Gamma, \emptyset; F) = \Phi(\Gamma; F)$.

A vertex $v \in V$ is called to be *attractive* if there are not arrows with $s(\gamma) = v$. Let $E \subseteq V$ be any subset such that all $e \in E$ are attractive vertexes, F_E the R-representation of Γ with $F_E(v) = 0$ for all $v \notin E$, and $F_E(v) = F(v)$ for all $v \in E$. We have by Theorem 2.5 the following

Lemma 3.1. *Let (Γ, E, F) be a network. If all vertexes in E are attractive then $\Phi(\Gamma, E; F) \cong \operatorname{colim}_1^{W\Gamma}(\widetilde{F/F_E})$.*

To a description the R-module of 1-chains satisfying to the first Kirchhoff law in any network (Γ, E, F) we add to the graph Γ the vertex $*$ and the arrows γ_e for all $e \in E$ with $s(\gamma_e) = e$ and $t(\gamma_e) = *$. Denote by $\Gamma \cup_E pt$ the obtained graph. Let $F \oplus_E 0$ be the R-representation of $\Gamma \cup_E pt$ such that $(F \oplus_E 0)|_\Gamma = F$ and $(F \oplus_E 0)(*) = 0$.

Theorem 3.2. *For any network (Γ, E, F) the R-module $\Phi(\Gamma, E; F)$ is isomorphic to $\operatorname{colim}_1^{W(\Gamma\cup_E pt)}(\widetilde{F \oplus_E 0})$.*

Proof. Consider the network $(\Gamma \cup_E pt, pt, F \oplus_E 0)$. The vertex $*$ is attractive in $\Gamma \cup_E pt$. It follows from the previous lemma that $\Phi(\Gamma \cup_E pt, pt; F \oplus_E 0) \cong \mathrm{colim}_1^{W(\Gamma \cup_E pt)}(\widetilde{F \oplus_E 0})$. But $\Phi(\Gamma \cup_E pt, pt; F \oplus_E 0) = \Phi(\Gamma, E; F)$. □

4. Flows in the Union of Graphs

Let $(I, \leqslant)$ be a partially ordered set. A covering $X = \bigcup\limits_{i \in I} X_i$ of a set is called to be *locally filtered* if $i < j$ in I implies $X_i \subseteq X_j$ and if for each $x \in X_i \cap X_j$ there exists $k \in I$ such that $k < i$, $k < j$, and $x \in X_k \subseteq X_i \cap X_j$.

A graph $\Gamma = (A, V, s, t)$ is called to be *locally filtered covered* by graphs $\Gamma_i = (A_i, V_i, s_i, t_i)$ if $A = \bigcup\limits_{i \in I} A_i$ and $V = \bigcup\limits_{i \in I} V_i$ are locally filtered coverings and the following diagrams are commutative

$$\begin{array}{ccc} A_i & \xrightarrow{\subseteq} & A_j \\ s_i \downarrow & & s_j \downarrow \\ V_i & \xrightarrow{\subseteq} & V_j \end{array} \qquad \begin{array}{ccc} A_i & \xrightarrow{\subseteq} & A_j \\ \downarrow t_i & & \downarrow t_j \\ V_i & \xrightarrow{\subseteq} & V_j \end{array}$$

for all $i \leqslant j$ in I.

Theorem 4.1. *Let $\Gamma = (A, V, s, t)$ be a graph which is locally filtered covered by graphs $\{\Gamma_i\}_{i \in I} = \{A_i, V_i, s_i, t_i\}$, $E \subseteq V$ a subset such that $E = \bigcup\limits_{i \in I} E_i$ is a locally filtered covering by $E_i \subseteq V_i$. Then for each R-representation F of Γ there exists an exact sequence*

$$0 \to \mathrm{colim}_2^I\{\Phi_0(\Gamma_i, E_i; F_i)\} \to \mathrm{colim}^I\{\Phi(\Gamma_i, E_i; F_i)\} \to$$
$$\to \Phi(\Gamma, E; F) \to \mathrm{colim}_1^I\{\Phi_0(\Gamma_i, E_i; F_i)\} \to 0$$

where $F_i = F|_{\Gamma_i}$.

Proof. At first we consider the case $E = \emptyset$. Let $\Gamma'_i \subseteq \Gamma'$ are the categories defined for the graphs $\Gamma_i \subseteq \Gamma$ in Lemma 2.3. The covering $\{\Gamma'_i\}_{i \in I}$ satisfies to the conditions of [4, Corollary 3.2]. The functors $s : (\mathcal{F}W\Gamma)^{op} \to W\Gamma$ and $S^{op} : \Gamma'^{op} \to (\mathcal{F}W\Gamma)^{op}$ are strong cofinal by Lemma 1.2 and Lemma 2.3. Hence, $\mathrm{colim}_1^{W\Gamma}\tilde{F}$ is isomorphic to $\mathrm{colim}_1^{\Gamma'^{op}}(\tilde{F} \circ s \circ S^{op})$. By [4, Corollary 3.2] there is the spectral sequence with $E^2_{p,q} = \mathrm{colim}_p^I\{\mathrm{colim}_q^{\Gamma'^{op}_i}\tilde{F} \circ s \circ S^{op}|_{\Gamma'^{op}_i}\}$ which converges to $\mathrm{colim}_n^{\Gamma'^{op}}\tilde{F} \circ s \circ S^{op}$. The substitution Γ_i instead Γ leads to the functors $s_i : (\mathcal{F}W\Gamma_i)^{op} \to W\Gamma_i$ and $S_i^{op} : \Gamma'^{op}_i \to (\mathcal{F}W\Gamma_i)^{op}$ which are strong cofinal by Lemmas 1.2 and 2.3. It follows from $(\tilde{F} \circ s \circ S^{op})|_{\Gamma'^{op}_i} = \tilde{F}|_{W\Gamma_i} \circ s_i \circ S_i^{op}$

and from the strong confinality of s_i and S_i^{op} that we have the spectral sequence $\mathrm{colim}_p^I\{\mathrm{colim}_q^{W\Gamma_i}\tilde{F}|_{W\Gamma_i}\} \Rightarrow \mathrm{colim}_{p+q}^{W\Gamma}\tilde{F}$. Then the exact sequence of terms of low degree [6, P.332] gives the exact sequence

$$0 \to \mathrm{colim}_2^I\{\Phi_0(\Gamma_i;F_i)\} \to \mathrm{colim}^I\{\Phi(\Gamma_i;F_i)\} \to \\ \to \Phi(\Gamma;F) \to \mathrm{colim}_1^I\{\Phi_0(\Gamma_i;F_i)\} \to 0$$

For $E \neq \emptyset$ we consider the locally filtered covering $\Gamma\cup_E pt = \bigcup\limits_{i\in I}(\Gamma_i \cup_{E_i} pt)$. There is an exact sequence

$$0 \to \mathrm{colim}_2^I\{\Phi_0(\Gamma_i \cup_{E_i} pt; F_i \oplus_{E_i} 0)\} \to \\ \to \mathrm{colim}^I\{\Phi(\Gamma_i \cup_{E_i} pt; F_i \oplus_{E_i} 0)\} \to \Phi(\Gamma \cup_E pt; F \oplus_E 0) \to \\ \to \mathrm{colim}_1^I\{\Phi_0(\Gamma_i \cup_{E_i} pt; F_i \oplus_{E_i} 0)\} \to 0$$

The equalities $\Phi_0(\Gamma\cup_E pt; F\oplus_E 0) = \Phi_0(\Gamma, E; F)$ and $\Phi(\Gamma\cup_E pt; F\oplus_E 0) = \Phi(\Gamma, E; F)$ give looking. □

5. The Second Kirchhoff Law

Let R be a field. The *internal product* on a vector space T is a bilinear map $<,>: T \times T \to R$ such that $< a,b >=< b,a >$ for all $a,b \in T$. A network (Γ, E, F) together with an internal product $<,>$ on $\sum\limits_{\gamma\in A} F(s(\gamma))$ is called to be *Euclidian* if the implication $< f,f >= 0 \Rightarrow f = 0$ is true. We say that an 1-chain $f = \{f_\gamma\}$ of an Euclidian network satisfies to the *second Kirchhoff law* if the linear map $< f, - >: \sum\limits_{\gamma\in A} F(s(\gamma)) \to R$ has zero values on $\Phi(\Gamma, F)$ in the sense that

$$< f, - > |_{\Phi(\Gamma,F)} = 0.$$

Theorem 5.1. *Let (Γ, E, F) be an Euclidian network in which R is a field, Γ a finite graph, and $F(v)$ finite dimensional vector spaces for all $v \in V$. Then for each $\varphi \in \sum\limits_{v\in E} F(v)$ satisfying $\varepsilon(\varphi) = 0$ there is the unique 1-chain $f = \{f_\gamma\}$ such that $df = \varphi$ and $< f, - > |_{\Phi(\Gamma,F)} = 0$.*

Proof. If $df = 0$ then $f \in \Phi(\Gamma, F)$, in this case $< f, - > |_{\Phi(\Gamma,F)} = 0$ implies $< f,f >= 0$ and $f = 0$. Considering f_i, with $i \in \{1,2\}$, for which $df_i = \varphi$ and $< f_i, - > |_{\Phi(\Gamma,F)} = 0$, we obtain $f_1 - f_2 = 0$. Hence, the solution is unique.

Consider a map $\sum\limits_{\gamma\in A} F(s(\gamma)) \xrightarrow{\eta} Mod_R(\Phi(\Gamma, F), R) \oplus \varepsilon^{-1}(0)$ which assign to any $g \in \sum\limits_{\gamma\in A} F(s(\gamma))$ the pair $< g, - > |_{\Phi(\Gamma,F)} \oplus dg$. We have proved that the map η is the injection. The exact sequence (1) gives the exact sequence $0 \to \Phi(\Gamma; F) \to \sum_{\gamma\in A} F(s(\gamma)) \to \varepsilon^{-1}(0) \to 0$. The vector space $\sum\limits_{\gamma\in A} F(s(\gamma))$ has a finite dimension. Hence, its dimension equals the dimension of $\Phi(\Gamma, F) \oplus \varepsilon^{-1}(0)$ and cosequently η is the isomorphism. We let $g = \eta^{-1}(0 \oplus \varphi)$. Then $\eta(g) = 0 \oplus \varphi$. Hence $dg = \varphi$ and $< g, - > |_{\Phi(\Gamma,F)} = 0$. Thus there exists the solution. □

References

[1] Baues H.J., Wirsching G. *Cohomology of small categoryies.* J. Pure Appl. Algebra. 1985. V.38,N 2/3. P. 187–211.

[2] Gabriel P., Zisman M. *Calculus of fractions and homotopy theory.* Berlin-Heidelberg-New York: Springer-Verlag, 1967.

[3] Hilton P. J., Stammbach U. *A Course in Homological Algebra.* Berlin-Heidelberg-New York: Springer-Verlag, 1971.

[4] Khusainov A. A. *Cohomology of small categories with coefficients in an Abelian category with exact products.* Sib. Math. J. 1989. V.30, N 4. P.668–672; translation from Sib. Mat. Zh. 1989. V. 30, N 4. P. 210–215.

[5] Khusainov A. A. *Comparision of the dimensions of a small category.* Sib. Math. J. 1997. V. 38, N 6. P. 1230–1240; translation from Sib. Mat. Zh. 1997. V.38, N 6. P. 1413–1426.

[6] MacLane S. *Homology.* Berlin-Heidelberg-New York: Springer-Verlag, 1975.

[7] Oberst U. *Homology of categories and exactness of direct limits.* Math. Z. 1968. Bd. 107. S. 89–115.

[8] Wagner D. G. *The Algebra of Flows in Graphs.* Adv. Appl. Math. 1998. V.21. P.644–684.

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