On power structures

Ivica Bošnjak and Rozália Madarász

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ABSTRACT. In general, power structure of a structure $\mathcal{A}$ (with universe A) is an appropriate structure defined on the power set $\mathcal{P}(A)$. There are lot of papers concerning this topic in which power structures appear explicitly or implicitly. The aim of this paper is to give an overview of the results that are interesting from the universal-algebraic point of view.

1. Introduction

There are three basic types of power constructions in universal algebra. The definitions and a short summary of the most important results are given in sections 2-4.

In the first type of power construction we "lift" an operation f on a set A to the operation f^+ on the power set $\mathcal{P}(A)$ (see section 2). This construction is the natural generalization of the multiplication of cosets in group theory. Some questions related to this construction are studied in papers: [7], [24], [28], [31], [34] - [36], [41], [45], [48], [55], [60] - [61], [68], [72]-[75], [77], [79].

A more general construction, i.e. the powering of relational structures (see section 3), was introduced by Jónsson and Tarski in [44]. This construction has proved to be very useful in various areas of algebraic logic and the theory of non-classical logics. A very comprehensive overview concerning this type of power constructions can be found in [32].

There are several different ways to lift a relation from a base set to its power set and they are considered in section 4. A general definition

of n-ary power relation was given by Whitney ([82], see also [37]). This type of construction is widely used in theoretical computer science, in the context of powerdomains (for an overview see [52] and [67]).

A successful attempt to give a general view on power structures and to present common ideas used by different authors was made by Chris Brink in [10]. In [11] he puts power structures in a universal-algebraic context, thus giving rise to a number of interesting questions and results. Some of the questions were answered in [3]-[6], [12], [33], [49]-[50]. The main results from these papers are presented in sections 5-7.

2. Power algebra of an algebra

Let $\mathcal{A} = \langle A, \cdot \rangle$ be a groupoid and $X, Y \subseteq A$. The most natural way to extend the operation $\cdot$ from A to subsets of A is the following:

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

This leads us to the general definition of a power operation and a power algebra:

Definition 1. *Let* $f : A^n \to A$. *We define* $f^+ : \mathcal{P}(A)^n \to \mathcal{P}(A)$ *in the following way:*

$$f^+(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n) \mid x_1 \in X_1, \ldots, x_n \in X_n\}.$$

If $\mathcal{A} = \langle A, \{f \mid f \in \mathcal{F}\} \rangle$ *is an algebra, the* **power algebra** $\mathcal{P}(\mathcal{A})$ *is defined as:*

$$\mathcal{P}(\mathcal{A}) = \langle \mathcal{P}(A), \{f^+ \mid f \in \mathcal{F}\} \rangle.$$

Sometimes, it is convenient to exclude the empty set from the universe of a power algebra. The power algebra defined in this way is denoted by $\mathcal{P}_+(\mathcal{A})$.

The first appearance of this concept was probably in the context of group theory. Namely, if G is a group and N is a normal subgroup of G, the multiplication of cosets in the factor group G/N is defined as: $xN \cdot yN = xyN$ for every $x, y \in G$. It is easy to see that this is only a special case of Definition 1. As any subset of a group was referred to as a complex, power algebras are sometimes called **complex algebras**. Also, some authors (usually in semigroup theory) use the term **global of an algebra**. Beside group theory and semigroup theory, power operations are implicitly used in some other fields. For instance, the set of ideals of a distributive lattice L again forms a lattice, and meets and joins in the new lattice are precisely the power operations of meets and joins in

L. In the formal language theory the product of two languages is simply
the power operation of concatenation of words.

The first question that naturally arises here is *what properties of a
structure remain invariant under power construction*? For example, if an
algebra $\mathcal{A}$ satisfies some identity, under what conditions the correspond-
ing power algebra will satisfy the same identity? The complete answer
to this question was given by Gautam (1957).

Theorem 1. *(Gautam [31]) Let t_1, t_2 be different terms. The following
conditions are equivalent.*

(1) *Each individual variable in the identity $t_1 \approx t_2$ occurs precisely
    once on each side of the identity.*
(2) *For every algebra $\mathcal{A}$, if the identity $t_1 \approx t_2$ is valid in $\mathcal{A}$, it remains
    valid in $\mathcal{P}(\mathcal{A})$.*

Of course, one can ask a more general question: if V is a variety of
algebras, what identities hold in the variety generated by $\{\mathcal{P}(\mathcal{A}) \mid \mathcal{A} \in
V\}$? It is convenient to introduce the following notation. If V is a variety
of algebras, by $\mathcal{P}(V)$ we denote the variety generated by $\{\mathcal{P}(\mathcal{A}) \mid \mathcal{A} \in V\}$.
Similarly, we denote by $\mathcal{P}_+(V)$ the variety generated by power algebras
whose universe does not include the empty set. We say that a variety
V is $\mathcal{P}$-**closed** ($\mathcal{P}_+$-**closed**) if $V = \mathcal{P}(V)$ ($V = \mathcal{P}_+(V)$). An identity is
said to be **linear** if each variable occurs at most once on each side of the
identity. An identity is **regular** if the same set of variables occurs on
both side of the identity. Gautam's theorem could be now formulated in
the following way: A variety defined by a single identity is $\mathcal{P}$-closed if
and only if this identity is linear and regular. In general case, we have
the following theorem.

Theorem 2. *(Grätzer, Lakser, [36]) Let V be a variety of algebras.*

(a) *The identities satisfied by $\mathcal{P}_+(V)$ are precisely those identities re-
    sulting through identification of variables from the linear identities
    true in V.*
(b) *The identities satisfied by $\mathcal{P}(V)$ are precisely those regular iden-
    tities resulting through identification of variables from the linear
    identities true in V.*

There is another subject that has attracted attention of algebraists,
especially semigroup theorists. If K is a class of algebras, and $\mathcal{A}$ and
$\mathcal{B}$ are isomorphic algebras from K, then $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ are obviously
isomorphic. It is natural to ask whether the converse is true, i.e. is it
true that for any $\mathcal{A}, \mathcal{B}$ from K it holds:

$$\mathcal{P}(\mathcal{A}) \cong \mathcal{P}(\mathcal{B}) \Rightarrow \mathcal{A} \cong \mathcal{B} ?$$

If the class K has this property, we say that K is a **globally determined class**. The first result in this direction comes from 1967, when T. Tamura and J. Shafer proved that the class of groups is globally determined. It follows immediately from this result that rings are globally determined. Later, Tamura proved that some other important classes of semigroups, such as completely simple semigroups, completely 0-simple semigroups, left (right) zero semigroups and rectangular bands are also globally determined. Nevertheless, the class of all semigroups is not globally determined, which was proved by E. M. Mogiljanskaja. In particular, involutive semigroups are not globally determined ([24]). An important result was obtained by Y. Kobayashi ([45]). He proved that semilattices are globally determined, which implies the same result for lattices and Boolean algebras. Unary algebras have been also investigated in this context. A. Drapal ([28]) proved that finite partial monounary algebras are globally determined, while the class of all monounary algebras is not globally determined. There are also classes of finite algebras which are not globally determined. Namely, E. Lukács ([48]) proved that for every finite group G, the class of finite G-algebras is globally determined if and only if G is a cyclic group.

3. Power (complex) algebras of relational structures

A more general construction, i.e. the powering of relational structures, was introduced by Jónsson and Tarski ([44]). They applied this construction to extend the well-known Stone representation theorem for Boolean algebras to normal Boolean algebras with operators. In short, a Boolean algebra with operators is a Boolean algebra with some additional operations which are additive (distributive on all its arguments, with respect to Boolean addition). Normality means that the result of the operation is 0 whenever at least one of the arguments is 0. Above mentioned representation theorem of Jónsson and Tarski says that every normal Boolean algebra with operators is isomorphic to a subalgebra of the full power (complex) algebra of some relational structure. To obtain this result, it was necessary to incorporate the usual set-theoretic operations into the definition of power algebra of a relational structure.

Definition 2. *If $R \subseteq A^{n+1}$ then the power operation $R^{\uparrow} : \mathcal{P}(A)^n \to \mathcal{P}(A)$ is defined in the following way:*

$$R^{\uparrow}(X_1, \ldots, X_n) = \{y \in A \mid (\forall i \leq n)(\exists x_i \in X_i)\, (x_1, \ldots, x_n, y) \in R\}.$$

*If $\mathcal{A} = \langle A, \mathcal{R} \rangle$ is a relational structure, then the **full complex algebra** $\mathcal{A}^+$ is defined as $\mathcal{A}^+ = \langle \mathcal{P}(A), \cup, \cap, ^-, \{R^{\uparrow} \mid R \in \mathcal{R}\} \rangle$. Every subalgebra of a full complex algebra will be called **a complex algebra over** $\mathcal{A}$.*

This construction is more general than the first one (see Definition 1) in the following sense: if f is an n-ary operation on A and $G_f \subseteq A^{n+1}$ is the so-called graph of an operation f, i.e.

$$G_f = \{(a_1, \ldots, a_n, f(a_1, \ldots, a_n)) \mid a_i \in A, 1 \le i \le n\},$$

then $G_f^{\uparrow} = f^+$.

This construction has proved to be very useful in various areas of algebraic logic. Let us mention an example. **Relation algebras** are Boolean algebras with three additional operators (a binary, a unary and a constant) which were equationally defined by Tarski in 1941. They arose by abstraction from concrete algebras of binary relations (see for example [43], or [51]). Namely, the set of all binary relations on a set X is a Boolean set algebra with natural (non-Boolean) operations of relational composition $R \circ S$, inversion R^{-1} and the diagonal relation $\Delta = \{(x, x) \mid x \in X\}$. The class RRA of so-called representable relation algebras consists of those algebras which are embeddable in a product of algebras of binary relations as above. Tarski thought in the beginning that the five defining axioms of relation algebras that he proposed, exactly determine the class RRA. In fact, RRA is a variety, but it turned out that it is not finitely axiomatizable. The Jónsson-Tarski theory represents the class of relation algebras RA as complex algebras of an elementary class of some relational structures, so-called poly-groups. In 1964 R. Lyndon obtained the first example of a non-representable relation algebra as the complex algebra of a poly-group (although he did not used that terminology). Later Jónsson constructed non-representable relation algebras using non-Arguesian projective plane. Developing his ideas, Lyndon associated a relation algebra to an arbitrary projective plane in which each line contains at least 4 points and showed that under certain conditions this relation algebra is not representable. Using similar methods, Maddux (1978) constructed a relation algebra associated to an arbitrary modular lattice with 0. Both Lyndon's and Maddux's algebras can be viewed as some complex algebras.

Another discipline where this construction appears is **the theory of non-classical logics**. The set theoretic semantics for any well-behaved logic is based on certain relation structures called Kripke frames. On the other side, such a logic is characterized by its so-called Lindenbaum-Tarski algebra and the corresponding variety. The power construction discussed in this section plays a key role in connecting the logic, its Kripke semantics and its algebraic semantics. Jónsson's and Tarski's work on Boolean algebras with operators and complex algebras came to the attention of logicians during the early seventies. Before that time these theories had developed almost independently.

A very comprehensive overview concerning this type of power constructions can be found in [32]. In this paper attention is focused on varieties whose members can be represented as complex algebras of relational structures. A recent paper on this topic, [40], concerns the problem of axiomatization of the class $\{\mathcal{A}^+ \mid \mathcal{A} \in V\}$, for a given variety V. For that purpose the authors use some "two-player games" and translate the existence of a winning strategy for one player into a set of first-order axioms.

4. Power relation of a relation

There are several approaches to the problem of lifting a relation from a base set to its power set. The early papers mostly deal with special (binary) relations, such as set-theoretical inclusion or partial orderings (see for example [8], [9], [81]). For instance, in theoretical computer science authors use three different ways to construct a power relation of a partial order. A general definition of n-ary power relations was given by Whitney in [82], see also [37].

Definition 3. *(a) If $R \subseteq A^n$, we define $R_k^{\uparrow} : \mathcal{P}(A)^{n-1} \to \mathcal{P}(A)$ as*

$$
\begin{aligned}
R_k^{\uparrow}(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n) = \\
= \{x_k \mid (\exists x_1 \in X_1) \ldots (\exists x_{k-1} \in X_{k-1}) \\
(\exists x_{k+1} \in X_{k+1}) \ldots (\exists x_n \in X_n) \\
(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \in R\},
\end{aligned}
$$

for all $X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n \subseteq A$.

(b) If $R \subseteq A^n$, we define $R^+ \subseteq \mathcal{P}(A)^n$ in the following way: $(X_1, X_2, \ldots, X_n) \in R^+$ if

$$X_1 \subseteq R_1^{\uparrow}(X_2, X_3, \ldots, X_n)$$

$$X_2 \subseteq R_2^{\uparrow}(X_1, X_3, \ldots, X_n)$$

$$\vdots$$

$$X_n \subseteq R_n^{\uparrow}(X_1, X_2, \ldots, X_{n-1})$$

for all $X_1, \ldots, X_n \subseteq A$.

(c) Let $\mathcal{A} = \langle A, \mathcal{R} \rangle$ be a relational structure. The corresponding **power structure** *is $\mathcal{P}(\mathcal{A}) = \langle \mathcal{P}(A), \{R^+ \mid R \in \mathcal{R}\} \rangle$.*

In particular, if R is a binary relation on A then R^+ is a binary relation on $\mathcal{P}(A)$ such that

$$X R^+ Y \iff (\forall x \in X)(\exists y \in Y)\, xRy \;\&\; (\forall y \in Y)(\exists x \in X)\, xRy.$$

In the binary case this definition is a generalization of Definition 1. Namely, if $f : A \to A$ and $G_f = \{(a, f(a)) \mid a \in A\}$ is the graph of f, then $G_f^+ = G_{f^+}$. For operations of arity greater than 1, if we exclude the empty set from the universe of a power algebra, it holds $G_{f^+} \subseteq G_f^+$.

The basic properties of this power construction with respect to the usual operations on a set of binary relations are given in the following theorem (see [11]).

Theorem 3. *Let $R, S \subseteq A^2$. Then it holds:*

  (a) $R \subseteq S \Rightarrow R^+ \subseteq S^+$;
  (b) $(R \cap S)^+ \subseteq R^+ \cap S^+$;
  (c) $R^+ \cup S^+ \subseteq (R \cup S)^+$;
  (d) $(R \circ S)^+ = R^+ \circ S^+$;
  (e) $(R^+)^{-1} = (R^{-1})^+$.

Some important properties of relations remain invariant under the ascent to power relations (reflexivity, transitivity, symmetry), but some do not. This raises the general question of *characterizing those first-order properties which do remain invariant.* The paper of Grätzer and Whitney [37] provides the first result in this direction. This paper concerns structures in general, where a structure means a set endowed by some relations and operations (of possibly infinite arity). A **variety of structures** is a class axiomatized by a set of atomic formulas in the relevant first-order language.

Theorem 4. *(Grätzer-Whitney [37]) A variety of structures is closed under power structures iff it is axiomatizable by regular linear atomic formulas.*

This theorem does not solve the general problem stated above, which still remains open.

In [10] there are a number of results concerning the connection between power relations, power algebras and quotient algebras, but we will pay attention to this subject later in section 5.

Power graphs are another topic that has been studied in this context. Namely, any graph can be considered as a relational structure with one binary relation. Thus a power graph of a graph is simply the power structure defined according to Definition 3. The notion of a power graph

is explicitly defined in [1]. In this paper the authors give some results about automorphisms of power graphs and formulate the problem of global determinism of graphs. Analogously to the case of algebras (section 2), a class K of graphs is globally determined if for every $G_1, G_2 \in K$ it holds:

$$\mathcal{P}(G_1) \cong \mathcal{P}(G_2) \Rightarrow G_1 \cong G_2.$$

It is known that the class of (all) graphs is not globally determined. It follows from the result of Drapal ([28]) about monounary algebras, mentioned in section 2. I. Bošnjak in his Ph.D. thesis [6] obtained some results on global determinism of classes of finite graphs. He proved that the classes of finite trees, finite tournaments and some classes of graphs with loops are globally determined. Many questions are yet to be answered here. For instance, we do not know whether the class of all finite graphs is globally determined. Bošnjak also made an attempt to connect the problem of global determinism of graphs with the same problem for algebras. Namely, there are some well-known constructions that associate a special algebra to every graph from a certain class. For example, to every tournament we can associate a commutative, idempotent groupoid. Such groupoids are studied in different contexts (see [16], [22] [23], [42]). It is proved in [6] that this class of groupoids is globally determined.

Power graphs have been recently utilized in theoretical computer science. In [46] and [47] power graphs are treated as models of concurrent systems and compared with some other models.

Theoretical computer science gave raise to the study of two alternative notions of power relation (for an overview see [52] and [67]; see also [14] and [83]). The first one is associated with so called **Hoare powerdomain** and the second one with so called **Smyth powerdomain**. Generalizations of these two power relations are given in the following definition.

Definition 4. *([10], [11]) Let $X, Y \subseteq A^2$. We define power relations $R^\rightarrow$ and $R^\leftarrow$ in the following way:*

$$XR^\rightarrow Y \iff (\forall x \in X)(\exists y \in Y)\, xRy;$$

$$XR^\leftarrow Y \iff (\forall y \in Y)(\exists x \in X)\, xRy.$$

Note that $R^+ = R^\rightarrow \cap R^\leftarrow$.

These three power relations are cornerstones of the theory of powerdomains. Domains are certain countably-algebraic complete partial

orders that play an important role when assigning a *denotation* to a deterministic program. Namely, if we have a deterministic program, the denotation of this program is taken to be a continuous mapping $D \to D$, where $(D, \sqsubseteq)$ is a countably-algebraic complete partial order. Then the meaning of a non-deterministic program can be viewed as a continuous mapping $D \to \mathcal{P}(D)$. So it is natural to try to turn $\mathcal{P}(D)$ into a domain and for this purpose a power order is used. If $\sqsubseteq$ is the base order on D, then $\sqsubseteq^{\rightarrow}$ is associated with the so called Hoare powerdomain, $\sqsubseteq^{\leftarrow}$ with Smyth powerdomain and $\sqsubseteq^{+}$ with Plotkin powerdomain.

5. Power algebras and quotient algebras

Power algebras are closely related to quotient algebras. Namely, the base set of the quotient algebra $\mathcal{A}/\theta$ is included in the base set of the power algebra $\mathcal{P}(\mathcal{A})$. But $\mathcal{A}/\theta$ is not always a subalgebra of $\mathcal{P}(\mathcal{A})$. Also, it is natural to ask what is the relation between the power algebra $\mathcal{P}(\mathcal{A})$ and the power algebra of a quotient algebra $\mathcal{P}(\mathcal{A}/\theta)$. These and similar questions are investigated in [11], and in this section, we present some results from that paper.

Definition 5. *([11]) Let $\mathcal{A} = \langle A, F \rangle$ be an algebra, $R \subseteq A^2$, and f is an n-ary operation from F. We say that R **preserves the arguments of operation** f if for all $z, y_1, \ldots, y_n \in A$ it holds: If $zRf(y_1, \ldots, y_n)$, then there exist $x_1, \ldots, x_n$ from A such that $x_i R y_i$ for all $i \in \{1, \ldots, n\}$ and $z = f(x_1, \ldots, x_n)$. If R preserves the arguments of all the operations from F and is compatible with all the operations from F, then we say that R **preserves the structure of algebra** $\mathcal{A}$.*

Theorem 5. *([11]) Let $\mathcal{A} = \langle A, F \rangle$ be an algebra and $\theta \in Con\mathcal{A}$. Then A/θ is a subalgebra of $\mathcal{P}(\mathcal{A})$ if and only if θ preserves the arguments of all the operations from F.*

As we have already mentioned, by lifting an equivalence relation to the power set, we obtain also an equivalence relation. Moreover, the following theorem holds:

Theorem 6. *([11]) Let $\mathcal{A} = \langle A, F \rangle$ be an algebra and $\theta \in Con\mathcal{A}$. Then $\theta^{+} \in Con\mathcal{P}(\mathcal{A})$.*

It is interesting that powering (of algebras and relations) in some sense commutes with forming quotient algebras. More precisely:

Theorem 7. *([11]) Let $\mathcal{A} = \langle A, F \rangle$ be an algebra and $\theta \in Con\mathcal{A}$. Then*

$$\mathcal{P}(\mathcal{A}/\theta) \cong \mathcal{P}(\mathcal{A})/\theta^{+}.$$

The following three theorems are "power versions" of well known isomorphism theorems.

Theorem 8. *([11]) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ is an epimorphism. Then it holds*

$$\mathcal{P}(\mathcal{A})/(ker\varphi)^+ \cong \mathcal{P}(\mathcal{B}).$$

If $\mathcal{B}$ is a subalgebra of an algebra $\mathcal{A}$, then $[B]^\theta$ is defined as $[B]^\theta = \bigcup \{b/\theta \mid b \in B\}$. The subalgebra of $\mathcal{A}$ with the base set $[B]^\theta$ is denoted by $[\mathcal{B}]^\theta$.

Theorem 9. *([11]) Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$, $\theta \in Con\mathcal{A}$, $\theta_1 = \theta \bigcap B^2$, $\theta_2 = \theta \bigcap ([B]^\theta)^2$. Then*

$$\mathcal{P}(\mathcal{B})/\theta_1^+ \cong [\mathcal{P}(\mathcal{B})]^{\theta^+}/\theta_2^+.$$

Theorem 10. *([11]) Let ψ and θ be congruence relations on algebra $\mathcal{A}$ and $\psi \subseteq \theta$. Then*

$$\mathcal{P}(\mathcal{A}/\psi) \Big/ (\theta/\psi)^+ \cong \mathcal{P}(\mathcal{A})/\theta^+.$$

6. Good quotient relations

In [11], the author made an attempt to generalize the results quoted in the previous section to relations on an algebra which are not necessarily congruence relations. This problem is also studied in [3], [4], [5], [12], [49], [50], [33].

It is well known that the notion of a quotient algebra $\mathcal{A}/R$ is defined for any universal algebra $\mathcal{A} = \langle A, F \rangle$ and any congruence R on $\mathcal{A}$ in the following way:

$$\mathcal{A}/R = \langle A/R, \{\lceil f \rceil \mid f \in F\} \rangle,$$

where A/R is the corresponding *quotient set* (i.e. the set of all equivalence classes $a/R = \{b \in A \mid bRa\}, a \in A$), and for any $f \in F$ of arity n, operation $\lceil f \rceil : (A/R)^n \to A/R$ is defined by

$$\lceil f \rceil(a_1/R, \dots, a_n/R) = f(a_1, \dots, a_n)/R. \tag{1}$$

Of course, if $R \subseteq A^2$ is not a congruence on $\mathcal{A} = \langle A, F \rangle$, then operations $\lceil f \rceil$ ($f \in F$) are not necessarily well defined by (1).

Definition 6. *Let R be an arbitrary binary relation on A.*

   (a) *For any $a \in A$ we define $a/R = \{b \mid bRa\}$. The corresponding* **generalized quotient set** *is $A/R = \{a/R \mid a \in A\}$.*

(b) *Relation $\varepsilon(R) \subseteq A^2$ is defined by:*

$$(a, b) \in \varepsilon(R) \iff a/R = b/R.$$

It is easy to see that definition of $\ulcorner f \urcorner$ in (1) is "good" (for every $f \in F$) if and only if $\varepsilon(R)$ is a congruence on $\mathcal{A}$.

Definition 7. *([33], [49]) Let $\mathcal{A} = \langle A, F \rangle$ be an algebra and $R \subseteq A^2$.*

(a) *We call R a **good (quotient) relation** on $\mathcal{A}$ if $\varepsilon(R)$ is a congruence on $\mathcal{A}$. We denote the set of all good relations on $\mathcal{A}$ by $G(\mathcal{A})$.*
(b) *If R is a good quotient relation on $\mathcal{A}$, the corresponding **generalized quotient algebra** $\mathcal{A}/R$ is*

$$\mathcal{A}/R = \langle A/R, \{\ulcorner f \urcorner \mid f \in F\}\rangle,$$

where operations $\ulcorner f \urcorner$ ($f \in F$) are defined by (1).

Examples

- Any congruence on an algebra is good. In fact, an equivalence relation is good if and only if it is a congruence.
- Every partial order on A is a good relation on every algebra with the base set A. Thus every non-trivial algebra has good relations which are not congruences.
- Every compatible quasi-order (i.e. a reflexive and transitive relation which is compatible) is a good relation.
- Every structure-preserving relation on an algebra (see Definition 5) is a good relation.
- If A is a set with more than 2 elements, then for any $n \geq 1$, there is a relation $R \subseteq A^2$ and an operation $f : A^n \to A$ such that R is neither reflexive, nor symmetric, nor transitive nor compatible on $\mathcal{A} = \langle A, f \rangle$, but R is a good quotient relation on $\mathcal{A}$.

In order to find out when the partially ordered set $\mathcal{G}(\mathcal{A}) = \langle G(\mathcal{A}), \subseteq \rangle$ is a lattice, we describe the good relations in the following way:

Definition 8. *Let $R \subseteq A^2$ and $\varphi : A/R \to \mathcal{P}(A)$. The relation $R_\varphi \subseteq A^2$ is defined as*

$$aR_\varphi b \iff a \in \varphi(b/R).$$

Lemma 1. *Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$. Then R is a good quotient relation if and only if there is a congruence θ on $\mathcal{A}$ and an injection $\varphi : A/\theta \to \mathcal{P}(A)$ such that $R = \theta_\varphi$.*

Theorem 11. *([4]) Let $\mathcal{A}$ be an algebra. The following conditions are equivalent:*

(1) $\mathcal{G}(\mathcal{A})$ is a lattice;

(2) $EqA = Con\mathcal{A}$.

(3) $G(\mathcal{A}) = \mathcal{P}(A^2)$.

It is still an open problem to find necessary and sufficient conditions for an ordered set $\langle S, \leq \rangle$ to be isomorphic to $\mathcal{G}(\mathcal{A})$, for some algebra $\mathcal{A}$.

It can be easily proved that the following "extended" homomorphism theorem holds for good quotient relations.

Theorem 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of the same type. Then $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ if and only if there is a good relation R on $\mathcal{A}$ such that $\mathcal{B} \cong \mathcal{A}/R$.*

Not surprisingly, the well known isomorphism theorems cannot be generalized to the whole class of good relations. So, it is natural to search for smaller classes of good relations for which some generalized versions of the isomorphism theorems could be proved. The following types of good relations proved to be suitable.

Definition 9. *Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$.*

*(a) ([12]) We call R a **quasi-equivalence** on A if for all $x,y \in A$*

$$x/R = y/R \iff xRy \ \& \ yRx.$$

We denote the set of all quasi-equivalences on A by $QEqA$.

*(b) ([5]) We call R a **quasi-congruence** on $\mathcal{A}$ if R is a good quasi-equivalence. The set of all quasi-congruences on $\mathcal{A}$ is denoted by $QCon\mathcal{A}$.*

*(c) ([12]) We call R a **B-quasi-congruence** on $\mathcal{A}$ if R is a compatible quasi-equivalence. The set of all B-quasi-congruences on $\mathcal{A}$ is denoted by $BQCon\mathcal{A}$.*

*(d) ([5]) We call R a **two-side quasi-congruence** on $\mathcal{A}$ if for all $x,y,z \in A$*

$$xRy \ \& \ yRx \ \& \ xRz \Rightarrow yRz.$$

The set of all two-side quasi-congruences on $\mathcal{A}$ is denoted by $TSQCon\mathcal{A}$.

Examples

- Every reflexive and transitive relation on A is a quasi-equivalence on A.

- Every reflexive and anti-symmetric relation on A is a two-side quasi-congruence on arbitrary algebra $\mathcal{A}$ with the base set A.

In general, $TSQCon\mathcal{A}$ and $BQCon\mathcal{A}$ are proper subsets of $QCon\mathcal{A}$, and they are incomparable with each other.

Definition 10. *Let $\mathcal{A}$ be an algebra, B a subuniverse of $\mathcal{A}$, $R \subseteq A^2$. We define $B^R \subseteq A$ as*

$$B^R = \{a \in A \mid (\exists b \in B) \, a/R = b/R\}.$$

Lemma 2. *Let $\mathcal{A}$ be an algebra and $\mathcal{B}$ a subalgebra of $\mathcal{A}$. Then*

(a) *If $R \in QCon\mathcal{A}$ then $R \bigcap B^2 \in QCon\mathcal{B}$.*
(b) *If $R \in G(\mathcal{A})$ then B^R is a subuniverse of $\mathcal{A}$.*

If $R \in G(\mathcal{A})$ and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then the subalgebra of $\mathcal{A}$ with the universe B^R is denoted by $\mathcal{B}^R$.

Theorem 13. *([5]) Let $\mathcal{B}$ be a subalgebra of an algebra $\mathcal{A}$ and $R \in QCon\mathcal{A}$. If $R_1 = R \bigcap B^2$ and $R_2 = R \bigcap (B^R)^2$, then*

$$\mathcal{B}/R_1 \cong \mathcal{B}^R/R_2.$$

Definition 11. *Let $\mathcal{A}$ be an algebra, $R, S \in TSQCon\mathcal{A}$, $R \subseteq S$. Relation $S/R \subseteq A/R \times A/R$ is defined as*

$$(a/R, b/R) \in S/R \Leftrightarrow (a, b) \in S.$$

Lemma 3. *Let $\mathcal{A}$ be an algebra, $R, S \in TSQCon\mathcal{A}$, $R \subseteq S$. Then $S/R \in TSQCon(\mathcal{A}/R)$.*

Theorem 14. *([5]) Let $\mathcal{A}$ be an algebra, $R, S \in TSQCon\mathcal{A}$, $R \subseteq S$. Then*

$$\mathcal{A}/R \Big/ S/R \cong \mathcal{A}/S.$$

In [12] it is proved that for any algebra $\mathcal{A}$, the set $BQCon\mathcal{A}$ is closed under arbitrary intersections. Hence $\langle BQCon\mathcal{A}, \subseteq \rangle$ is a complete lattice. It is easy to see that the same is true for the sets $QEqA$, $QCon\mathcal{A}$ and $TSQCon\mathcal{A}$.

Theorem 15. *([5]) Let $\mathcal{A}$ be an arbitrary algebra with the carrier set A. Then $\langle QEqA, \subseteq \rangle$, $\langle QCon\mathcal{A}, \subseteq \rangle$, $\langle BQCon\mathcal{A}, \subseteq \rangle$, $\langle TSQCon\mathcal{A}, \subseteq \rangle$ are algebraic lattices.*

It is proved in [12] that $Con\mathcal{A}$ is a sublattice of $BQCon\mathcal{A}$. $Con\mathcal{A}$ is also a sublattice of $QEqA$, $QCon\mathcal{A}$ and $TSQCon\mathcal{A}$, which is a consequence of the following theorem.

Theorem 16. *([5]) Let Σ be a set of quasi-congruences of some algebra $\mathcal{A}$ such that $Con\mathcal{A} \subseteq \Sigma$ and Σ is closed under (finite) intersections. If $\langle \Sigma, \subseteq \rangle$ is a lattice, then $\langle Con\mathcal{A}, \subseteq \rangle$ is a sublattice of $\langle \Sigma, \subseteq \rangle$.*

Corollary 1. *([5]) For any algebra $\mathcal{A}$, $\langle Con\mathcal{A}, \subseteq \rangle$ is a sublattice of the following lattices: $\langle QCon\mathcal{A}, \subseteq \rangle$, $\langle BQCon\mathcal{A}, \subseteq \rangle$, $\langle TSQCon\mathcal{A}, \subseteq \rangle$.*

Corollary 2. *([5]) For any algebra $\mathcal{A} = \langle A, F \rangle$, $\langle Con\mathcal{A}, \subseteq \rangle$ is a sublattice of $\langle QEqA, \subseteq \rangle$.*

The two corollaries above covers all the cases when one of the lattices of quasi-congruences is a sublattice of another. Namely, lattices $QCon\mathcal{A}$, $BQCon\mathcal{A}$, $TSQCon\mathcal{A}$ are not necessarily sublattices of $QEqA$. Also, lattices $BQCon\mathcal{A}$ and $TSQCon\mathcal{A}$ may not be sublattices of $QCon\mathcal{A}$.

There exist algebras $\mathcal{A} = \langle A, F \rangle$ such that $QEqA = QCon\mathcal{A}$, or $QEqA = BQCon\mathcal{A}$ and it is not difficult to describe them.

Theorem 17. *([5]) Let $\mathcal{A}$ be an algebra and $|A| \geq 3$. The following conditions are equivalent:*

(1) $QEqA = QCon\mathcal{A}$,

(2) $QEqA = BQCon\mathcal{A}$,

(3) $EqvA = Con\mathcal{A}$.

Another problem that might be interesting is to describe algebras $\mathcal{A}$ such that $BQCon\mathcal{A} = Con\mathcal{A}$ (B-quasi-congruence-trivial algebras). Of course, one can easily notice that for any non-trivial algebra $\mathcal{A}$, $QCon\mathcal{A} \neq Con\mathcal{A}$ and $TSQCon\mathcal{A} \neq Con\mathcal{A}$.

Theorem 18. *([5]) Let $\mathcal{A}$ be an algebra which has a Mal'cev term (or a minority term). Then $BQCon\mathcal{A} = Con\mathcal{A}$.*

Corollary 3. *([5]) Let V be a congruence permutable variety. Then for any $\mathcal{A} \in V$, $BQCon\mathcal{A} = Con\mathcal{A}$.*

There are examples which show that there exist B-quasi-congruence-trivial algebras that are not congruence permutable. Also, there exist congruence distributive varieties which are not B-quasi-congruence-trivial.

The following question remains to be answered: is it true that for any algebraic lattice $\mathcal{L}$, there is an algebra $\mathcal{A}$ such that $\mathcal{L} \cong BQCon\mathcal{A}$?

7. Powering of good relations

On the "first level", good relations do not give new quotient algebras (see Theorem 12). But their "inner structures" can be very different and they do not behave in the same way when we "lift" them to power structures. For some good relations R on $\mathcal{A}$, the power relation R^+ is good on $\mathcal{P}(\mathcal{A})$, for some other it is not (similarly for $R^\rightarrow$ and $R^\leftarrow$).

Definition 12. *([12]) Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$.*

(a) *R is a **very good relation** on $\mathcal{A}$ if R^+ is good on $\mathcal{P}(\mathcal{A})$. We denote the set of all very good relations on $\mathcal{A}$ by $G^+(\mathcal{A})$.*

(b) *R is **Hoare good** (or **H-good**) on $\mathcal{A}$ if $R^\rightarrow$ is good on $\mathcal{P}(\mathcal{A})$. we denote the set of all Hoare good relations on $\mathcal{A}$ by $G^\rightarrow(\mathcal{A})$.*

(c) *R is **Smyth good** (or **S-good**) on $\mathcal{A}$ if $R^\leftarrow$ is good on $\mathcal{P}(\mathcal{A})$. We denote the set of all Smyth good relations on $\mathcal{A}$ by $G^\leftarrow(\mathcal{A})$.*

Every congruence is a very good, H-good and S-good relation. Moreover, the following theorem holds.

Theorem 19. *Let $\mathcal{A}$ be an algebra and $R \in EqA$. The following conditions are equivalent:*

(1) $R \in Con\mathcal{A}$;

(2) $R \in G(\mathcal{A})$;

(3) $R \in G^+(\mathcal{A})$;

(4) $R \in G^\rightarrow(\mathcal{A})$;

(5) $R \in G^\leftarrow(\mathcal{A})$.

Compatible quasi-orders are also examples of very good, H-good and S-good relations. In fact, the following theorem holds.

Theorem 20. *([3]) Let $\mathcal{A}$ be an algebra and $R \subseteq A^2$ a quasi-order on A. The following conditions are equivalent:*

(1) R is compatible on $\mathcal{A}$.

(2) R is Hoare good on $\mathcal{A}$.

(3) R is Smyth good on $\mathcal{A}$.

But there are algebras $\mathcal{A}$ and very good partial orders on $\mathcal{A}$ which are not compatible on $\mathcal{A}$.

It is not hard to see that every very good (H-good, S-good) relation is good on $\mathcal{A}$ (but the converse is not true). The following power version of the homomorphism theorem is proved in [12].

Theorem 21. *Let $\mathcal{A} = \langle A, F \rangle$ be an algebra and $R \subseteq A^2$. Then*

(a) *If $R \in G^{\rightarrow}(\mathcal{A})$ then $\mathcal{P}(\mathcal{A})/R^{\rightarrow}$ is a homomorphic image of $\mathcal{P}(\mathcal{A}/R)$.*

(b) *If $R \in G^{\leftarrow}(\mathcal{A})$ then $\mathcal{P}(\mathcal{A})/R^{\leftarrow}$ is a homomorphic image of $\mathcal{P}(\mathcal{A}/R)$.*

(c) *If $R \in G^{+}(\mathcal{A})$ then $\mathcal{P}(\mathcal{A})/R^{+}$ is a homomorphic image of $\mathcal{P}(\mathcal{A}/R)$.*

It is natural to ask what is the relationship between H-good, S-good and very good relations. The full answer to this question (and some related questions) is given in [3].

Theorem 22. *Every Hoare good relation on an algebra is Smyth good.*

Theorem 23. *If a relation is Hoare good and Smyth good on an algebra, then it is also very good.*

Corollary 4. *Every Hoare good relation on an algebra is very good.*

No other positive result can be proved in this direction because there exist:

- quasi-congruences which are neither H-good nor S-good nor very good;
- very good relations which are neither H-good nor S-good nor quasi-congruences;
- S-good relations which are neither H-good nor very good nor quasi-congruences;
- very good and S-good relations which are neither quasi-congruences nor H-good;
- H-good relations which are not quasi-congruences.
- H-good quasi-congruences which are not compatible quasi-orders;
- good relations which are neither quasi-congruences nor H-good nor S-good nor very good;
- very good quasi-congruences which are neither H-good nor S-good;
- very good and S-good quasi-congruences which are not H-good;
- S-good quasi-congruences which are not very good.

It is not a hard problem to construct the listed counter-examples (some of them could be found in [3] and [6]).

It is easy to see from the definition that the set of good relations $G(\mathcal{A})$ is closed under complementation and is not closed under $\cap$ and $\cup$. Also, it can be proved that $G^{\rightarrow}(\mathcal{A})$ and $G^{\leftarrow}(\mathcal{A})$ are not closed under $\cap$ and $\cup$. It is not a surprising fact that both $G^{+}(\mathcal{A})$ and $G^{\rightarrow}(\mathcal{A})$ are not closed under complementation. However, despite these facts, it is proved in [4] that:

Theorem 24. *For any algebra $\mathcal{A}$, the set $G^{\leftarrow}(\mathcal{A})$ is closed under complementation.*

As we have already mentioned, if R is a good relation on $\mathcal{A}$, then R^+, $R^{\rightarrow}$ and $R^{\leftarrow}$ are not necessarily good on the corresponding power algebras. A similar statement can be proved for very good relations. Namely, if R is a very good relation on $\mathcal{A}$, then $R^{\rightarrow}$ ($R^{\leftarrow}$, R^+) may not be very good on $\mathcal{P}(\mathcal{A})$. Also, if R is a Smyth good relation on $\mathcal{A}$, then $R^{\rightarrow}$ and R^+ are not necessarily very good on $\mathcal{P}(\mathcal{A})$.

Not all the answers are negative for Smyth good and Hoare good relations.

Theorem 25. *([4]) Let R be a Smyth good relation on $\mathcal{A}$. Then $R^{\leftarrow}$ is a Smyth good relation on $\mathcal{P}(\mathcal{A})$.*

Theorem 26. *([4]) Let R be a Hoare good relation on $\mathcal{A}$. Then $R^{\rightarrow}$ is a Hoare good relation on $\mathcal{P}(\mathcal{A})$.*

8.   Concluding remarks

Besides the disciplines that are mentioned earlier in the text (such as theory of non-classical logics or theoretical computer science) there is another topic where power constructions implicitly appear. The main concept here is that of hyperoperation (or multioperation, or polyoperation; the terminology varies from author to author). Given a non-empty set A, a **hyperoperation** on A is any mapping $f : A^n \to \mathcal{P}(A)$. When one tries to compose such hyperfunctions, it is necessary to lift them to the power set, i.e. any hyperoperation $f : A^n \to \mathcal{P}(A)$ naturally induces a function $f^{\#} : \mathcal{P}(A)^n \to \mathcal{P}(A)$ in the following way:

$$f^{\#}(X_1, \ldots, X_n) = \cup \{ f(x_1, \ldots, x_n) \mid x_i \in X_i \}.$$

Note that the construction $f^{\#}$ is in fact the construction from Definition 2 of Section 3. Namely, there is a natural correspondence between n-ary hyperfunctions on a set A and $n + 1$-ary relations on A given in the following way:

a) If $f : A^n \to \mathcal{P}(A)$ then $R_f = \{ (x_1, \ldots, x_n, y) \mid y \in f(x_1, \ldots, x_n) \}$.
b) If $R \subseteq A^{n+1}$ then $f_R : A^n \to \mathcal{P}(A)$ such that
   $f_R(x_1, \ldots, x_n) = \{ y \in A \mid (x_1, \ldots, x_n, y) \in R \}$

Now, if $f : A^n \to \mathcal{P}(A)$, then $f^{\#} = R_f^{\uparrow}$.

If A is a non-empty set and H is a family of hyperoperations on A, the ordered pair $\langle A, H \rangle$ is called a **hyperalgebra** (multialgebra, polyalgebra). The study of hyperalgebras started in 1934, when so-called **hypergroups** were introduced in [53]. Since then more then 400 papers have appeared (see [18] and [80] for a bibliography; see [19], [20], [21], [65], [66] for some recent results on hypergroups). Most of the papers on this topic are devoted to special hyperalgebras (hypergroups, hyperrings, hyperfields, vector hyperspaces, boolean hyperalgebras). Although there is a number of papers with universal-algebraic flavour (see for example [13], [17], [27], [29], [30], [38], [39], [58], [59], [62], [63], [64], [78]), there is still no coherent theory for general hyperalgebras in the literature.

At the end, let us mention that the theory of power structures has even some non-mathematical applications. Namely, power algebras are nothing else than special set-valued logic algebras (see [57], [71]), which have applications to the design of optical, biological and electronical circuitry.

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Contact information

I. Bošnjak   Institute of Mathematics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia
E-Mail: `ivb@im.ns.ac.yu`

R. Madarász   Institute of Mathematics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia
E-Mail: `rozi@eunet.yu`