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On intersections of normal subgroups in free groups

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Abstract. Let N_1 (respectively N_2) be a normal closure of a set $R_1 = \{u_i\}$ (respectively $R_2 = \{v_j\}$) of cyclically reduced words of the free group $F(A)$. In the paper we consider geometric conditions on R_1 and R_2 for $N_1 \cap N_2 = [N_1, N_2]$. In particular, it turns out that if a presentation $< A \mid R_1, R_2 >$ is aspherical (for example, it satisfies small cancellation conditions $C(p)\&T(q)$ with $1/p + 1/q = 1/2$), then the equality $N_1 \cap N_2 = [N_1, N_2]$ holds.

Introduction

Let $F = F(A)$ be a free group generated by an alphabet A. Let N_1 (respectively N_2) be the normal closure of a set $R_1 = \{u_i\}$ (respectively $R_2 = \{v_j\}$) of cyclically reduced words of F. We will consider non-intersecting symmetrized R_1 and R_2.

It is evident that the inclusion $[N_1, N_2] \subset N_1 \cap N_2$ always holds. But the reverse inclusion does not always hold (the simplest example is $R_1 = \{a_1, {a_1}^{-1}\}$ and $R_2 = \{a_1, a_2, {a_1}^{-1}, {a_2}^{-1}\}$). The aim of this paper is to find necessary and sufficient conditions on R_1 and R_2 for

$$N_1 \cap N_2 = [N_1, N_2]. \quad (1)$$

These conditions are expressed in terms of certain geometric objects called pictures (see, for example [4]).

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In particular, it turns out that if a presentation $< A \mid R_1, R_2 >$ is aspherical (for example, it satisfies small cancellation conditions $C(p)\&T(q)$ with $1/p + 1/q = 1/2$ ([4], theorem 2.2)), then the equality $N_1 \cap N_2 = [N_1, N_2]$ holds.

The paper is divided into three sections, each of which is further subdivided. In the first section we give main definitions, prove some results concerning relations between them, formulate the main result of the paper (Theorem 1) and prove corollaries of it. The second section is devoted to the proof of Theorem 1. In the third section we prove some simple corollaries of Theorem 1 in the case of free products.

It should be noted that a geometric test of the equality (1) obtained in Theorem 1 is hard to verify, but its corollaries are useful. The following question seems to be open: if the equality (1) holds if and only if there exist sets of words $\tilde{R}_1$ and $\tilde{R}_2$ such that
(i) $\tilde{R}_1^F = N_1$ and $\tilde{R}_2^F = N_2$;
(ii) the presentation $G =< A \mid \tilde{R}_1 \cup \tilde{R}_2 >$ is strictly $(\tilde{R}_1, \tilde{R}_2)$-separable (see Definition 1 below).

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1. Formulation and corollaries of Theorem 1.

In the beginning we give some definitions and recall the definition of pictures.

1.1 Main definitions. Relations between definitions.

Let N be the normal closure of a set R of cyclically reduced words of the free group $F(A)$.

A picture P over presentation $G =< A \mid R >$ on a surface S (see details in [4]) is a finite collection of "vertices" $V_1, ..., V_n \in S$, together with a finite collection of simple pairwise disjoint connected oriented "edges" $E_1, ..., E_m \in S \setminus \{\{V_1, ..., V_n\} \cup \partial S\}$, labelled by letters of A (in [4], vertices are called discs and edges are called arcs). But these edges need not all connect two vertices. An edge may connect a vertex to a vertex (possibly coincident), a vertex to ∂S, or ∂S to ∂S. Moreover, some edges need have no endpoints at all, but be circles disjoint from the rest of P, such edges are called edges-circles. If an edge connects two vertices, then one is the start of the edge, the other one is the end of the edge.

For each vertex V of P consider a circle C of a small radius with center at V and a point p on C not lying on any edge of P. The labels of edges intersected by C starting at p form a word $r \in R$. Changing of the disposition of p on C and the direction of moving around C, we can read any cyclic permutation of r and r^{-1}.

Below we will consider pictures on S, where S is a sphere (spherical pictures) or a disk (planar pictures).

In the case of a planar picture the labels of edges, intersected by a circle $\bar{C}$ near the boundary of the disk ∂S, starting at a point $\bar{p}$ on $\bar{C}$, form a word W, which will be called *the boundary label* of the picture.

The following result is well-known (use Theorem 11.1 of [1] and dualise):

Lemma 1. *Let W be a non-empty word on the alphabet A. Then W represents the identity of the group $G = F/N$ if and only if there is a planar picture over the presentation $G =< A \mid R >$ with the boundary label W.*

Let P be a picture over $G =< A \mid R >$ and γ be a path on S not passing through any vertex of P. If we travel around γ we encounter a succession of edges. Reading the labels on these edges gives a word called *the word along the path* γ and denoted by $Lab(\gamma)$.

If γ is closed, consider a point p of γ not lying on any edge of P. The word along γ read from p will be denoted by ${Lab_p}^+(\gamma)$ or by ${Lab_p}^-(\gamma)$ (depending as the direction of reading is counterclockwise or not). It is clear that ${Lab_p}^+(\gamma)^{-1} = {Lab_p}^-(\gamma)$. Changing the disposition of p we obtain the same word up to cyclic permutation. Changing the direction of reading we obtain the inverse word. We will denote the word along the path γ by $Lab(\gamma)$ when the disposition of p and the direction of reading are not essential.

By $\mathbf{1}$ denote the identity element of the free group.

A *dipole* in a picture P over $G =< A \mid R >$ is two vertices V_1 and V_2 of P if there is a simple path ψ connecting points p_1 and p_2 lying on circles C_1 and C_2 around these vertices such that the following conditions hold:

(i) $Lab(\psi) = \mathbf{1}$;

(ii) ${Lab_{p_1}}^+(C_1) = {Lab_{p_2}}^-(C_2)$.

A picture over $G =< A \mid R >$ is *reduced* if it does not contain a dipole.

A presentation $G =< A \mid R >$ is *aspherical* if every connected spherical picture over $G =< A \mid R >$ contains a dipole.

Definition 1. *Let R_1 and R_2 be two sets of words in the free group $F(A)$. We say that a presentation $G =< A \mid R_1 \cup R_2 >$ is strictly (R_1, R_2)-separable or satisfies the condition of strict (R_1, R_2)-separability if for every reduced spherical picture P containing both R_1-vertices and R_2-vertices there is a simple closed path γ dividing the sphere into two disks such that the following conditions hold:*

1) *the both disks contain vertices;*

2) *$Lab(\gamma) = \mathbf{1}$.*

Assertion 1. *If in Definition 1 "every reduced spherical picture" is replaced by "every spherical picture", then the class of presentations satisfying strict (R_1, R_2)-separability is not changed.*

Proof. Let P be a non-reduced spherical picture over $G =< A \mid R_1 \cup R_2 >$ containing both R_1-vertices and R_2-vertices. Since P is not reduced, there is a dipole, i.e., there are two vertices V_1 and V_2 that inverse words from $R_1 \cup R_2$ correspond to, and V_1 and V_2 can be connected by a simple path ψ such that $Lab(\psi) = \mathbf{1}$. It is easily seen that the simple closed path γ from Definition 1 may be obtained going around V_1 and V_2 and by-passing near ψ in the both directions. □

Definition 2. *Let R_1 and R_2 be two sets of words in $F(A)$. We say that a presentation $G =< A \mid R_1 \cup R_2 >$ is weakly (R_1, R_2)-separable or satisfies the condition of weak (R_1, R_2)-separability if for every reduced spherical picture P containing both R_1-vertices and R_2-vertices there is a simple closed path γ dividing the sphere into two disks such that the following three conditions hold:*

1) *the both disks contain vertices;*

2) *$Lab(\gamma) \in [N_1, N_2]$;*

3) *if one of the disks contains only R_1-vertices, then the other one contains only R_2-vertices.*

Assertion 2. *If the condition 3) in Definition 2 is omitted, then the class of presentations satisfying only 1) and 2) of Definition 2 will be wider.*

Proof. As a counterexample proving the assertion, one can consider a presentation $G =< A \mid R_1 \cup R_2 >$, where R_1 is a symmetrized set of words $\{a_1, a_3, [a_1, a_2]\}$ and R_2 is a symmetrized set of words $\{a_2, a_3[a_1, a_2]\}$, where $\{a_i\} \in A$. This presentation satisfies the conditions $1), 2)$ of Definition 2, but it does not satisfy the condition 3).

Indeed, let P denote a reduced spherical picture containing both both R_1-vertices and R_2-vertices.

If there is an R_1-vertex labelled by $[a_1, a_2] \in [N_1, N_2]$ in P, then the path γ is obtained going around this vertex.

If, in P, there is not any R_1-vertex labelled by $[a_1, a_2]$ and there is an R_1-vertex labelled by a_3, then the edge starting at this a_3-vertex must have the end at an R_2-vertex labelled by $a_3[a_1, a_2]$, since P is reduced. Thus the path γ is obtained going around this (a_3)-vertex, the edge starting at this a_3-vertex and the $(a_3[a_1, a_2])$-vertex.

If in P there are neither R_1-vertices labelled by $[a_1, a_2]$ nor R_1-vertices labelled by a_3, then P is non-reduced. Hence the conditions $1), 2)$ of Definition 2 hold.

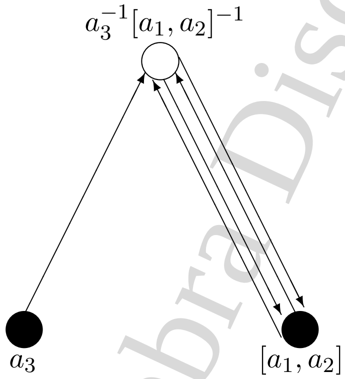

Fig. 1

But the presentation does not satisfy the condition 3), which is shown by the picture containing only three vertices: an (a_3)-vertex, an $([a_1, a_2])$-vertex and an $(a_3[a_1, a_2])$-vertex (see Fig.1), since a_3, $a_3[a_1, a_2] \notin [N_1, N_2]$. $\square$

Assertion 3. *If a presentation $G =< A \mid R_1 \cup R_2 >$ is strictly (R_1, R_2)-separable, then it is weakly (R_1, R_2)-separable.*

Proof. Let P be a reduced spherical picture over strictly (R_1, R_2)-separable $G =< A \mid R_1 \cup R_2 >$ containing both R_1-vertices and R_2-vertices. It is sufficient to find a simple closed path γ dividing the sphere into two disks such that

1) the both disks contain vertices;

2) $Lab(\gamma) = \mathbf{1}$;

3) one of the disks contains only R_1-vertices and the other one contains only R_2-vertices.

It follows from the condition of strict (R_1, R_2)-separability that there is a simple closed path γ_1 in P dividing the sphere into two disks such that the following conditions hold:

1) the both disks contain vertices;

2) $Lab(\gamma_1) = \mathbf{1}$.

If at least one of these disks both contains R_1-vertices and R_2-vertices, then it follows from the condition of strict (R_1, R_2)-separability that there is another simple closed path γ_2 in P non-crossing the path γ_1 and dividing such disk into two subdisks such that the following conditions hold:

1) the both subdisks contain vertices;

2) $Lab(\gamma_2) = \mathbf{1}$.

We continue in this fashion to obtain a finite set of simple closed paths $\Gamma = \{\gamma_i\}$ satisfying the following conditions:

1) these paths are pairwise disjoint;

2) $Lab(\gamma) = \mathbf{1}$ for each $\gamma_i \in \Gamma$;

3) the union of the paths $\gamma_i \in \Gamma$ divides the sphere into parts $\{D_k\}$ each of which contains only R_1-vertices or only R_2-vertices.

Each path of Γ separates one part of $\{D_k\}$ from another one. If any path $\gamma_i \in \Gamma$ separates one part of $\{D_k\}$ from another one so that the both parts contain either only R_1-vertices or only R_2-vertices, then we will remove this path γ_i from Γ. We thus get that each path of Γ separates a part of $\{D_k\}$ containing only R_1-vertices from another part containing only R_2-vertices.

Below we will transform each not simply connected part D_i of $\{D_k\}$ in order to decrease the number of paths in Γ. Since D_i is not simply connected, there are at least two paths γ_i and γ_j of Γ bounding the part D_i. Join the paths as follows. Fix a point a_i^+ on the path γ_i and a point a_j^+ on γ_j so that a_i^+ and a_j^+ do not belong to any edge of the picture P. It is clear that the points a_i^+ and a_j^+ may be joined by a simple path $\psi_{(i,j)}^+$ so that

1) the whole path $\psi_{(i,j)}^+$ lies in D_i;

2) $\psi_{(i,j)}^+$ does not pass through any vertex of P;

3) $\psi^+_{(i,j)}$ does not intersect the paths of Γ except γ_i and γ_j at the points a_i^+ and a_j^+.

It is possible to draw a path $\psi^-_{(i,j)}$ through points $a_i^- \in \gamma_i$ and $a_j^- \in \gamma_j$ close to the path $\psi^+_{(i,j)}$, in a parallel way with the properties similar to $\psi^+_{(i,j)}$ so that $\psi^-_{(i,j)}$ intersects the same edges of P as $\psi^+_{(i,j)}$ does, and that $\psi^+_{(i,j)}$ and $\psi^-_{(i,j)}$ are disjoint. Besides $\psi^-_{(i,j)}$ can be drawn so that the segment $[a_i^+, a_i^-]$ of γ_i and the segment $[a_j^+, a_j^-]$ of γ_j do not intersect the edges of P. Removing these segments gives a new simple closed path $\gamma_{ij} = (\gamma_i/[a_i^+, a_i^-]) * \psi^+_{(i,j)} * (\gamma_j/[a_j^+, a_j^-]) * \psi^-_{(i,j)}$ such that $Lab(\gamma_{ij}) = \mathbf{1}$. Replacing γ_i and γ_j by γ_{ij} in Γ gives a set of paths satisfying the same properties $1), 2), 3)$. Besides the number of paths in the resulting set of paths becomes less than in the original set.

Consequently after a finite number similar transformations, all parts from $\{D_k\}$ become simply connected. Therefore the resulting set Γ contains just one path, which is the desired path γ. □

Assertion 4. *The condition of weak (R_1, R_2)-separability is not equivalent to the condition of strict (R_1, R_2)-separability.*

Proof. As an example proving the assertion one can consider a presentation $G =< A \mid R_1 \cup R_2 >$, where R_1 is a symmetrized set of words $\{a_1, [a_1, a_2]\}$ and $R_2 = \{a_2, {a_2}^{-1}\}$, where $\{a_i\} \in A$.

It follows from Theorem 1 and Corollary 5 (see below) that this presentation is weakly (R_1, R_2)-separable since $N_1 \cap N_2 = [N_1, N_2]$.

Let us show that the existence of a spherical picture P containing only an $[a_1, a_2]$-vertex and two a_2-vertices contradicts the condition of strict (R_1, R_2)-separability.

Indeed, suppose that there is a simple closed path γ dividing the sphere into two disks. Then one of the disks must contain only one vertex. Consequently $Lab(\gamma)$ must be equal to the label of this vertex, which is not equal to the identity element in the free group. This contradicts the definition of strict (R_1, R_2)-separability. □

Assertion 5. *If every spherical picture over a presentation $G =< A \mid R_1 \cup R_2 >$ containing both R_1-vertices and R_2-vertices is not reduced (this condition will be called (R_1, R_2)-asphericity), then the presentation is strictly (R_1, R_2)-separable.*

Proof. is similar to the proof of Assertion 1. □

Definition 3. *Let R_1 and R_2 be two sets of words in $F(A)$. We say that a presentation $G =< A \mid R_1 \cup R_2 >$ is (R_1, R_2)-separable or satisfies the condition of (R_1, R_2)-separability if for every spherical picture*

P containing both R_1-vertices and R_2-vertices there is a simple closed path γ dividing the sphere into two disks such that the following three conditions hold:

1) *the both disks contain vertices;*

2) *$Lab(\gamma) \in [N_1, N_2]$;*

3) *one of the disks contains only R_1-vertices and the other one contains only R_2-vertices.*

1.2 Formulation of Theorem 1 and corollaries from it.

Theorem 1. *A presentation $G =< A \mid R_1 \cup R_2 >$ is weakly (R_1, R_2)-separable if and only if $N_1 \cap N_2 = [N_1, N_2]$.*

Corollary 1. *The conditions of weak (R_1, R_2)-separability and (R_1, R_2)-separability are equivalent.*

Proof. It is easy to see that if a presentation is (R_1, R_2)-separable, then it is weakly (R_1, R_2)-separable.

It remains to prove the converse statement. Let P be a spherical picture over a weakly (R_1, R_2)-separable presentation $G =< A \mid R_1 \cup R_2 >$ containing both R_1-vertices and R_2-vertices. It is evident that there is a simple closed path γ not passing through any vertex of P and dividing the sphere into two disks one of which contains only R_1-vertices and the other one contains only R_2-vertices. Hence $Lab(\gamma) \in N_1 \cap N_2$. Since $G =< A \mid R_1 \cup R_2 >$ is weakly (R_1, R_2)-separable, Theorem 1 leads to $Lab(\gamma) \in [N_1, N_2]$. Consequently γ is desired. □

Corollary 2. *The conditions of weak (R_1, R_2)-separability and weak (R_2, R_1)-separability are equivalent. Moreover we get the equivalent condition if in Definition* 2 *the item* 3) *is replaced by*
$3'$) *if one disk contains both R_1- both R_1- and R_2-vertices, then the other one both R_1- and R_2-vertices.*

Corollary 3. *Let a presentation $G =< A \mid R_1 \cup R_2 >$ satisfy one of the following conditions:*

(i) *strict (R_1, R_2)-separability;*

(ii) *(R_1, R_2)-asphericity;*

(iii) *asphericity.*

Then $N_1 \cap N_2 = [N_1, N_2]$.

Proof. (i) It follows directly from Theorem 1 and Assertion 3.
(ii) It follows directly from (i) and Assertion 4.
(iii) It follows from (ii). □

Corollary 4. *Let $\{R_1, R_2\}$ be a set of words satisfying one of the following small cancellation conditions: either $C(6)$, or $C(4)\&T(4)$, or $C(3)\&T(6)$. Then $N_1 \cap N_2 = [N_1, N_2]$.*

Proof. According to [4] (see also [3]), every small cancellation condition described above is sufficient to asphericity of $G =< A \mid R_1 \cup R_2 >$. □

Corollary 5. *Suppose that:*

1) *an alphabet $A = X \sqcup Y \sqcup Z$;*

2) *R_1 is an arbitrary set of words on X and R_2 is an arbitrary set of words on Y.*

Then $N_1 \cap N_2 = [N_1, N_2]$.

Proof. Without loss of generality, one can assume that R_1 and R_2 are symmetrized.

By Corollary 3 it is sufficient to show that $G =< A \mid R_1 \cup R_2 >$ is strictly (R_1, R_2)-separable.

Let P be a reduced spherical picture containing both both R_1-vertices and R_2-vertices.

If there is an edge-circle in P dividing the sphere into two disks each of which contains vertices, then Corollary 5 is proved (the simple closed path can be drawn near this edge-circle).

If there is an edge-circle in P dividing the sphere into two disks one of which does not contain vertices, then this edge-circle can be removed from P.

Therefore we can suppose that there is no edge-circle, hence each edge connects a vertex to a vertex.

Since R_1 and R_2 are written on the disjoint alphabets X and Y, we conclude that if any edge starts at an R_1-vertex, then it ends at an R_1-vertex (similarly for R_2-vertices).

For an R_1-vertex, consider a connected component of P containing this vertex. All vertices of this component are R_1-vertices. Since P also contains R_2-vertices, the complement of this component also contains vertices and falls into connected components. Among all these connected components there is at least one, which can be covered by a domain homeomorphic to a disk such that this domain does not intersect the other connected components. Then the boundary of the domain forms the desired simple closed path γ. $Lab(\gamma) = \mathbf{1}$, since no edge of P intersects γ, and the statement follows. □

2. Proof of Theorem 1.

In the beginning we give several definitions, which we will use in the proof of Theorem 1.

2.1 Additional definitions.

1) *Picture P with equator Equ. Subpictures of P.*
Let P be a picture on the sphere S^2 over a presentation $G =< A \mid R_1 \cup R_2 >$. Fix a simple closed path (denoted by Equ) on S^2 not passing through any vertex of P and dividing S^2 into two parts so that one part does not contain R_1-vertices and the other one does not contain R_2-vertices. The path Equ is called an *equator*.

$Lab(Equ)$ is denoted by W or W^{-1} (the sign depends on the direction of reading). By the choice of the equator, it follows that $W \in N_1 \cap N_2$.

Remark 1. The start point on Equ is not fixed, i.e., the equatorial label W is considered as a cyclic word (if W belongs to some normal subgroup, then all its cyclic permutations belong to the same subgroup). For simplicity of notation all cyclic permutations of W will be denoted again by W.

In the sequel, P denotes a picture with a fixed equator Equ.

Let $P = P' \sqcup P''$ be a disjoint union of two spherical pictures. P' (respectively, P'') will be called *a subpicture* of P. P' (respectively, P'') may be both connected and disconnected.

2) Boundaries of vertices. North and south vertices.
Every vertex is labelled by a word of R_1 (R_1-word) or R_2 (R_2-word). Consider a small disk with centre at a vertex such that the word along its boundary coincides with the label of the vertex. The boundary of the disk will be called *the boundary of the vertex*.

Equ divides the sphere into two hemispheres: so called north and south. A vertex is called *north* (respectively, *south*) if it lies in the north (respectively, south) hemisphere.

Suppose that R_1-words correspond to the north vertices (so called R_1-vertices), R_2-words correspond to the south vertices (so called R_2-vertices). Each word can be read along the boundary of the corresponding vertex starting at some point on the boundary and choosing the direction of reading.

3) Admissible moves.
In the proof of Theorem 1 we will transform a picture P with an equator Equ on S^2. A move (i.e., a transformation) is called *admissible* if, after the move, the word W along Equ is replaced by a word W' equal to

W to within an element of $[N_1, N_2]$ (for simplicity of notation, we will use the same letter W for the notation W'). Moreover admissible moves preserve the subdivision of P by Equ into the north and south vertices.

4) Map.
Each domain $U \subset S^2$ homeomorphic to a square

$$\{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\},$$

together with vertices and parts of edges lying in it is called *a map*.

5) Components.
Each spherical subpicture of P is called *a component* if it contains vertices. A component is called *reduced* (respectively, *non-reduced*) if the corresponding subpicture is reduced (respectively, non-reduced). A component is called north (respectively, south) if the corresponding subpicture contains only north (respectively, only south) vertices.

South and north components are called *uniform*. A component is called *mixed* if the corresponding subpicture contains both south and north vertices. Note that P is a component of itself.

6) States. Territories of states. Boundaries of states.
A component is called a *state* if it is uniform and can be covered by a closed domain homeomorphic to a disk so that the domain does not intersect the other components of P. The state is called *north* (respectively, *south*) if the corresponding uniform component is north (respectively, south).

The domain mentioned above is called a *territory* of the state. The boundary of the territory is called the *boundary of the state*. Note that the boundary of each state is homeomorphic to a circle. Without loss of generality we can assume that the boundary of each state intersects Equ in a finite set of points the number of which can not be decreased by changing the territory.

7) Pieces of equator Equ.
Let T be a state. Equ intersects its boundary $2m$ times and is divided into $2m$ connected parts by these intersection points. Those of parts which lie on the territory of T are called *pieces of the equator*.

8) Regions of states.
Let T be a north state. Equ divides the territory of T into connected parts, which are called *regions north* or *south* depending on what hemisphere they belong to. A north region of a north state is called *regular* if its boundary consists exactly of two parts: a connected part belonging to the boundary of T and a piece of the equator. Respectively, a north region of a north state is called *irregular* if it is not regular. Similarly, one can define regions and regular regions of south states by replacing the word "north" by "south".

9) σ*-state.*
A north state is called a north σ*-state* if all its north regions are regular. (South σ-states are defined similarly by replacing the word "north" by "south".)

10) σ*-picture.*
A picture with a fixed equator is called σ-picture if it can be reduced to a picture containing only σ-states by a finite number of admissible moves.

2.2 Proof of Theorem 1.

The proof of a statement that weak (R_1, R_2)-separability implies $N_1 \cap N_2 = [N_1, N_2]$ is similar to the proof of Corollary 1.

Therefore it remains to prove the converse statement. Let a presentation $G =< A \mid R_1 \cup R_2 >$ satisfy weak (R_1, R_2)-separability.

Since the inclusion $[N_1, N_2] \subset N_1 \cap N_2$ always holds, it is sufficient to prove the reverse inclusion.

Let W be an arbitrary word of the intersection $N_1 \cap N_2$. Then there are two representations:

$$W = \prod_k g_k r_{1,k} {g_k}^{-1}, \qquad \text{since } W \in N_1; \tag{2}$$

and

$$W^{-1} = \prod_l h_l r_{2,l} {h_l}^{-1}, \qquad \text{since } W \in N_2,, \tag{3}$$

where g_k, h_l are words of F, each $r_{1,k}$ belongs to R_1, and each $r_{2,l}$ belongs to R_2. To show that $W \in [N_1, N_2]$, let us construct two planar pictures. The word W in the form (2) is written on the boundary of the first picture which contains only R_1-vertices. The word W^{-1} in the form (3) is written on the boundary of the second picture which contains only R_2-vertices. Pasting together the planar pictures by their boundaries gives a picture P on S^2 with a fixed equator Equ. $Lab(Equ)$ is equal to W or W^{-1} depending on the direction of moving along Equ.

Now the proof of Theorem 1 follows from the following Propositions 1 and 2, which will be proved in the subsections 2.4 and 2.5 below:

Proposition 1. *Let a picture P with a fixed equator Equ be over a presentation $G =< A \mid R_1 \cup R_2 >$. If the presentation satisfies weak (R_1, R_2)-separability, then P is a σ-picture.*

Proposition 2. *Let a picture P with a fixed equator Equ be over a presentation $G =< A \mid R_1 \cup R_2 >$. If P is a σ-picture, then the word W along Equ can be reduced to the identity element in the free group by a finite number of admissible moves.*

Indeed, by Propositions 1, 2, there is a finite sequence of admissible moves, which reduces the equatorial label W in P to the identity element in the free group. Since admissible moves replace W by words equal to W to within elements of $[N_1, N_2]$, we have that $W \in [N_1, N_2]$, as claimed. □

2.3 Some admissible moves. Auxiliary lemmas.

In the subsection 2.3 we describe admissible moves, which will be used in the proof of Propositions 1 and 2.

1) Isotopy.

An isotopy of P is defined by replacing P by a picture $F_1(P)$, where $F_t : S^2 \times [0,1] \to S^2 \times [0,1]$ is a continuous isotopy of the sphere S^2 such that

(i) F_t leaves fixed all vertices, i.e., $F_t(V_i) = V_i$ for $t \in [0,1]$ and for each vertex V_i;

(ii) for each t and each edge E_j the intersection of an edge $F_t(E_j)$ and Equ consists of a finite number of points, moreover the edge $F_1(E_j)$ intersects Equ transversally at every intersection point.

It is evident that an isotopy of P is an admissible move because either it corresponds to an insertion or a cancellation of pairs of inverse letters in the equatorial label W (see Fig.2) or it does not change W at all.

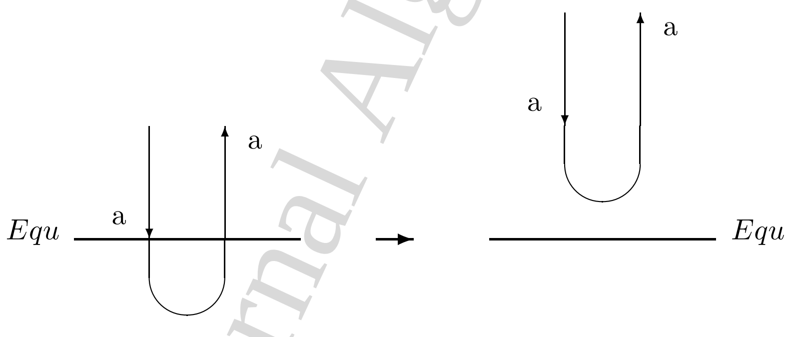

Fig. 2

2) *Bridge moves.*

Assume that a map U contains only two edges

$$\{x = -1/2, -1 < y < 1\} \text{ and } \{x = 1/2, -1 < y < 1\},$$

which are contrariwise oriented and labelled by the same letter. A move of P is called *a bridge move* (see also in [3]) if it does not change P out of the map U and change P in U as is shown on Fig 3.

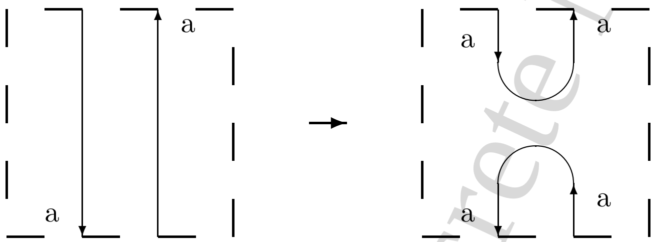

Fig. 3

The bridge move is an admissible move because either it corresponds to an insertion or a cancellation of pairs of inverse letters in the equatorial label W or it does not change W at all.

3) *Removing components and edge-circles of P not intersecting Equ.* If a component (in particular, a state) of P does not intersect Equ, then it does not contribute to the equatorial label W. Therefore such component can be removed. Similarly one can remove edge-circles not intersecting Equ.

4) *Removing superfluous loops.*
Assume that Equ intersects any edge in two successive points which divide Equ into two parts such that one of these two parts does not intersect edges. Such part of the edge is called *a superfluous loop*. It is evident that superfluous loops do not contribute to the equatorial label W (considered as an element of the free group). Therefore superfluous loops can be removed (see Fig. 2).

(This move is a special case of isotopy (see the move 1)) or a composition of the admissible moves 2) and 3).)

5) *Uniting σ-states.*
Let T_1 and T_2 be two distinguished north σ-states (the move of south states is similar). Assume that there are points p_1 on the boundary of T_1 and p_2 on the boundary of T_2 so that

(i) the points p_1 and p_2 lie in the south hemisphere;

(ii) it is possible to connect the points p_1 and p_2 by a simple path η which does not intersect the territory of any state and lies in the south hemisphere as a whole.

Then the territories of the σ-states T_1 and T_2 can be united in one by adding a small neighborhood of the path η (see Fig.4).

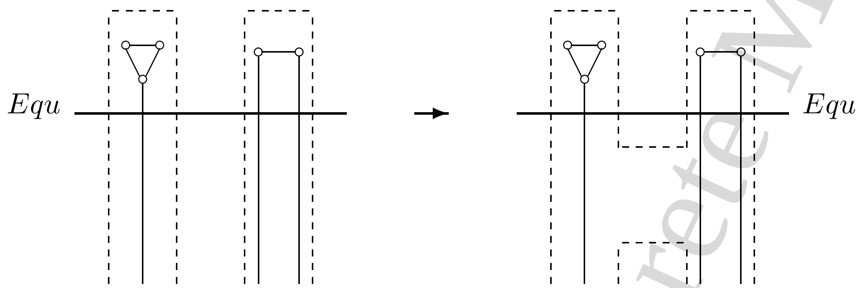

Fig. 4

It is clear that the united territory is homeomorphic to a disc again. The σ-states T_1 and T_2 lying in the united territory form one state, which is a σ-state again.

Remark 2. By a deformation of territories of σ-states, we can exclude the case when the territory of any σ-state successively intersects Equ in two pieces I_1 and I_2 so that a part of Equ between I_1 and I_2 does not intersect the territory of any state.

6) *Pasting a map with a commutator subpicture in P.*
In notation of the subsection 2.5, assume that there are two regular regions: one of them belongs to a north state T_1, the other one belongs to a south state T_2. By Lemma 2 below, the north region contains a picture with a word $w_1 \in N_1$ written along a piece I_2 of the equator, the south region contains a picture with a word $w_2 \in N_2$ written along a piece I_3 of the equator.

We choose a map M_1 so that the north region is contained in $\{0 \leq y \leq 1\}$, $I_2 \subset \{y = 0\}$, and the intersection of T_1 and $\{-1 < y < 0, -1 < x < 1\}$ contains only parts of edges given in coordinates (x, y) by $\{x = x'_1, -1 < y < 0\}, ..., \{x = x'_{n'}, -1 < y < 0\}$.

Similarly we choose a map M_2 so that the south region is contained in $\{0 \geq y \geq -1\}$, $I_3 \subset \{y = 0\}$, and the intersection of T_2 and $\{0 < y < 1, -1 < x < 1\}$ contains only parts of edges given in coordinates (x, y) by $\{x = x_1, 0 < y < 1\}, ..., \{x = x_n, 0 < y < 1\}$.

For the map M_1 (respectively, for M_2) we construct a mirror-like map M'_1 (respectively, M'_2) by reflecting M_1 (respectively, M_2) with respect to the axis $\{x = 0\}$ and by changing the orientations of the edges. The map M'_1 (respectively, M'_2) contains the picture with the word ${w_1}^{-1}$

(respectively, ${w_2}^{-1}$) written along the piece of the equator. The piece of the equator in the map M_1' (respectively, M_2') will be denoted by I_2' (respectively, I_3').

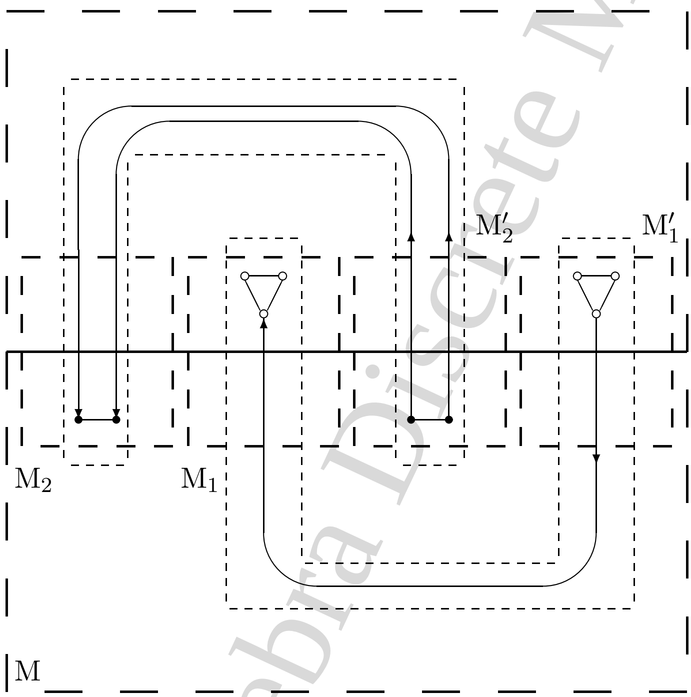

Fig. 5

A map M containing a picture for the word $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$ is constructed as follows:

(i) the equator passes through $\{y = 0\}$;

(ii) a part corresponding to the south hemisphere is $\{0 \geq y \geq -1\}$ and a part corresponding to the north hemisphere is $\{0 \leq y \leq 1\}$;

(iii) the map M_2 is disposed in the rectangle $\{-1/5 < y < 1/5, -4/5 < x < -3/5\}$; the map M_1 is disposed in $\{-1/5 < y < 1/5, -2/5 < x < -1/5\}$; the map M_2' is in $\{-1/5 < y < 1/5, 1/5 < x < 2/5\}$; the map M_1' is in $\{-1/5 < y < 1/5, 3/5 < x < 4/5\}$;

(iv) in the part corresponding to the south hemisphere, we extend the edges of the map M_1 to join them with the corresponding edges of M_1'; the union of the pictures of these maps and the joining edges forms a north σ-state which will be denoted by T_1^{comm};

(v) in the part corresponding to the north hemisphere, we extend the edges of the map M_2 to join them with the corresponding edges of M_2'; the union of the pictures of these maps and the joining edges forms a south σ-state which will be denoted by T_2^{comm}. (See Fig. 5)

It is clear that the word along the equator of the map M is $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$.

A small map M_s in P containing nothing but a part $\{y = 0\}$ of Equ is replaced by the constructed map M. This move is admissible because it corresponds to an insertion of the commutator $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$ of the elements from N_1 and N_2 in the equatorial label W.

Lemma 2. *Let T be a state in P. Let I be an arbitrary piece of the equator belonging to the territory of T. Then $Lab(I) \in N_1$ if T is north, and $Lab(I) \in N_2$ if T is south.*

Proof. Let T be north (the proof in the case of a south state is similar). The piece I divides the territory of T into two parts T' and T''. We consider one of them denoted by T'. T' may contain only north vertices (i.e., R_1-vertices) and edges labelled by letters of the alphabet A. Hence T' contains a planar picture over $< A \mid R_1 >$. By Lemma 1, a word along the boundary of T' belongs to N_1. Since the edges intersect the boundary of T' only in a part which coincides with I, Lemma 2 follows. □

Lemma 3. *Let T be a state in P. If Equ intersects the boundary of T exactly two times, then the word along the piece of the equator lying inside the territory of T is equal to the identity element in the free group.*

Proof. Assume that T is north (the proof in the case of a south state is similar). The given piece of the equator divides the territory of T into two regions: north and south. Moreover all vertices lie in the north region. Hence each edge intersects Equ even times. Clearly, there is an edge a part of which forms a superfluous loop in the south hemisphere. Removing superfluous loops (see the admissible move 4)) gives rise to the case when no edge of T intersects Equ. Therefore the original word along the piece of the equator lying inside the territory of T was equal to the identity element in the free group. □

2.4 The proof of Proposition 1.

The proof of Proposition 1 will be divided into several steps (lemmas). In Step 1 we will show that the picture P with the fixed equator Equ can be divided into a finite number of uniform components. In Step 2 the uniform components will be transformed to states and edges-circles not belonging to the states. In Step 3 we will get rid of the edges-circles not belonging to the states. In Step 4 the states will be divided into σ-states. In all steps we will use only a finite number of admissible moves. Therefore we will get that P is a σ-picture.

All pictures obtained from P will be denoted by P again for simplicity of notation.

Step 1. *Reducing P to a picture containing only uniform components.*
In Step 1 we will use the following two admissible moves.
Operation A: Transformations of reduced mixed components.
Let the presentation $G = < A \mid R_1 \cup R_2 >$ be weakly (R_1, R_2)-separable. By K denote a reduced mixed component.

Since K is a reduced spherical picture, the condition of weak (R_1, R_2)-separability leads to the existence of a simple closed path γ dividing the sphere into two parts so that

1) the both parts contains vertices;

2) $U = Lab(\gamma) \in [N_1, N_2]$;

3) if one part contains only north vertices then the other one contains only south vertices.

By the property 3) of γ, the following three cases are possible.

The first case: the path γ divides K into two parts one of which contains only south vertices, the other one contains only north vertices. The second case: γ divides K into two parts one of which contains only south vertices, the other one contains both south and north vertices. In these two cases we can assume that some segment ψ of the path γ lies on Equ. The complement of ψ to γ will be denoted by $\neg\psi$. One of the endpoints of ψ will be denoted by p.

The third case: the path γ divides K into two parts each of which contains both south and north vertices. Consequently, γ is intersected by Equ and divided by Equ into segments among which there are segments lying in the north hemisphere wholly. Fix one of them. By ψ denote its connected part not intersecting Equ. By p denote one of the endpoints of ψ. By $\neg\psi$ denote the complement of ψ to γ.

In each of these three cases one can assume that all edges intersecting the path γ intersect it in the segment ψ, because otherwise all edges intersecting $\neg\psi$ can be moved by isotopy (the admissible move 1)) to ψ along the path γ in the direction of the point p starting successively at the nearest to p edge.

In each of these three cases we select a map M on S^2 with the following properties: the map M contains the segment ψ of γ (coinciding with the part of Equ in the first two cases) and parts of edges intersecting ψ: more precisely,

1) $\{y = 0\}$ is the segment ψ of the path γ;

2) $\{x = x_1\}, ..., \{x = x_n\}$ correspond to the edges, which intersect the segment ψ;

3) in the first two cases the rectangle $\{-1 < x < 1, y < 0\}$ belongs to the south hemisphere and the rectangle $\{-1 < x < 1, y > 0\}$ belongs to the north hemisphere; in the third case the both rectangles belong to the north hemisphere. (An example for the third case is shown on Fig. 6)

A new map M' will be constructed as follows. Since $[N_1, N_2] \subset N_1 \cap N_2$, the word U along ψ belongs to the both groups N_1 and N_2. We construct planar pictures P_1 and P_2 with the boundary labels respectively U and U^{-1}. Moreover in the first two cases P_1 is constructed over $< A \mid R_2 >$ (using south vertices) and P_2 is constructed over $< A \mid R_1 >$ (using north vertices); in the third case the both pictures P_1 and P_2 are constructed over $< A \mid R_1 >$ (using north vertices). Then these pictures are disposed on the new map M' as follows:
1) P_1 lies in the rectangle $\{-1 < y < -1/2, -1 < x < 1\}$; and the edges corresponding to the boundary of P_1 start at $(x_1, -1)$,...,$(x_n, -1)$;
2) P_2 lies in the rectangle $\{1/2 < y < 1, -1 < x < 1\}$; and the edges corresponding to the boundary of P_2 start at $(x_1, 1)$,...,$(x_n, 1)$;
3) $\{y = 0\}$ is the segment ψ of γ (coinciding with the part of Equ in the first two cases).

The old map M is cut out from P and replaced by the new one M'. (An example for the third case is shown on Fig. 7.) By this replacing of maps the equatorial label W is changed by the commutator from $[N_1, N_2]$ in the first two cases and it is not changed in the third case. After such replacing all R_1-vertices lie in the north hemisphere and all R_2-vertices lie in the south one. Therefore this move of P is admissible.

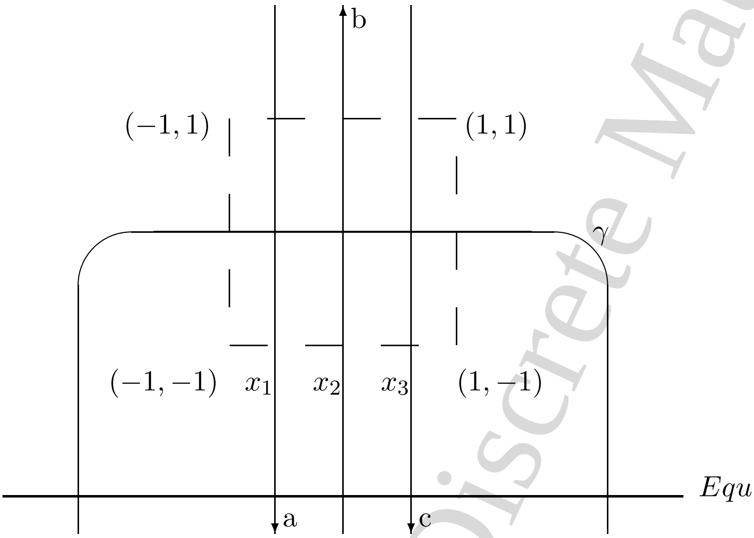

Fig. 6

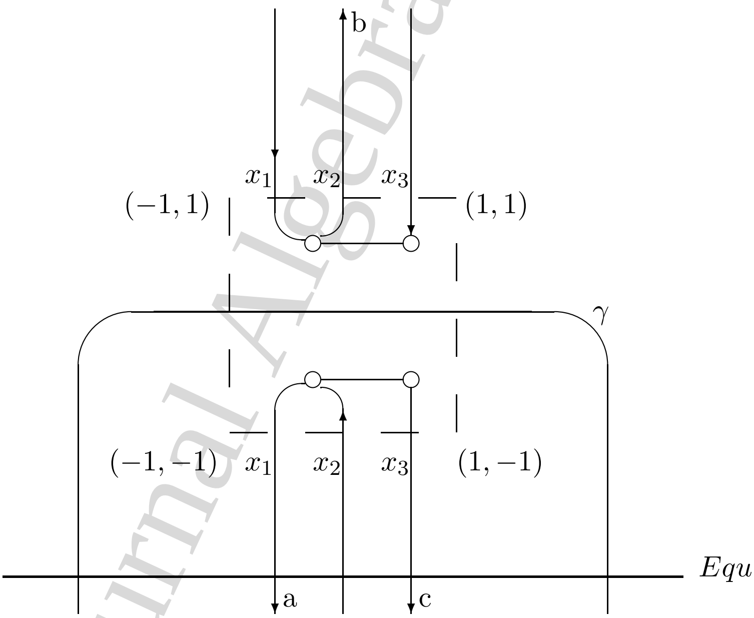

Fig. 7

By such replacing of maps the component K falls into two components K_1 and K_2 separated from each other by the path γ. In the first case each of the components K_i is uniform; in the second case one of the components (let it be denoted by K_1) is mixed, the other one (K_2) is south uniform; in the third case each of the components K_i is mixed.

Remark 3. The components K_i can be non-reduced. The number of south vertices in each mixed component K_i (K_1 in the second case; K_1 and K_2 in the third case) is strictly less than the number of south vertices in the original component K, since by Operation A we added only north vertices to obtain the mixed components.

Operation B: Transformations of non-reduced mixed components.
Let K be a non-reduced component. Then there is a dipole in K, i.e., there are two vertices V' and V'' satisfying the following conditions:

(i) there is a simple path ψ joining some points p_1 and p_2, which lie on the boundaries C_1 and C_2 of these vertices, so that $Lab(\psi) = \mathbf{1}$;

(ii) $Lab_{p_1}{}^{+}(C_1) = Lab_{p_2}{}^{-}(C_2)$.

Remark 4. The both vertices of a dipole can be either north or south, since the sets R_1 and R_2 are mutually disjoint.

Evidently, it is possible to surround V' and V'' by a simple closed path $\gamma\prime$ passing along ψ such that $Lab(\gamma\prime) = \mathbf{1}$. Therefore if $\gamma\prime$ intersects edges then among them there are two edges intersecting $\gamma\prime$ successively and labelled by inverse letters. These edges can be removed from $\gamma\prime$ by bridge moves. It is easily seen that no edge intersects $\gamma\prime$ after a finite number of bridge moves.

Thus the component K falls into two components K_1 and K_2 not connected with each other. The component K_1 contains only the dipole of V' and V'' and the edges joining them together with some edges-circles. Hence K_1 is uniform. The component K_2 may be either uniform or mixed, either reduced or non-reduced.

Remark 5. Operation B does not increase the number of vertices. Therefore the number of vertices in each K_i is strictly less than the number of vertices in the original component K. In particular, the number of south vertices in K_2 is not more than the corresponding number in the original component K.

Lemma 4. *Let a presentation $G = < A \mid R_1 \cup R_2 >$ be weakly (R_1, R_2)-separable. Then a picture P with a fixed equator Equ falls into a finite number of uniform components by a finite number of Operations A and B (being admissible moves).*

Proof. Let P consist of n components $\{K_i\}$. In particular, n may be equal to one. Using the admissible move 3) we have that all components $\{K_i\}$ intersect Equ.

To each mixed component K_l of P assign a number s_l equal to the number of south vertices in K_l. Let s be the maximum of $\{s_l\}$.

The proof of Lemma 4 will be by induction on s.

If $s = 0$, then all components $\{K_i\}$ are uniform, hence there is nothing to prove.

Let $s > 0$. In this case we transform each mixed component $K \in \{K_i\}$ containing s south vertices as follows.

a) If K is not reduced, then by Operation B, the component K falls into two components K'_1 and K'_2, at least one of which (say K'_1) contains only a dipole and possibly edges-circles. The component K'_1 is uniform. The component K'_2 contains s'_2 south vertices, where $s'_2 \leq s$. If K'_2 is mixed and non-reduced, then Operation B applies to it again. By Remark 5, after applying a finite number of Operations B, the component K falls into uniform components $\{K'_i\}$ containing dipoles and a reduced component K'' containing s'' south vertices, where $s'' \leq s$. If K'' is mixed and $s'' = s$, then we transform it in the way described in the item b) below.

b) If K is reduced, then it follows from weak (R_1, R_2)-separability that Operation A can be applied to K. By Operation A, the component K falls into two components K''_1 and K''_2. If any of K''_j is mixed, then by Remark 3 it contains s''_j south vertices, where $s''_j < s$.

Thus after a finite number of Operations A and B the picture P falls into a finite number of uniform components, since s and n are finite. □

Step 2. *Reducing a picture P containing only uniform components to a picture containing only states and edges-circles not belonging to the states.*
Let $\{K_i\}$ be a subdivision of P into uniform components.

Fix a point on S^2 lying neither on any edge and any vertex of P nor on Equ. Assume that this point lies in the north hemisphere. We will call it *the north pole*.

Assume that a component $K \in \{K_i\}$ is such that some edges of K form a simple closed loop η. The loop η divides the sphere into two parts homeomorphic to disks: a disk containing the north pole will be called *exterior* and the other disk will be called *interior*.

If there is another component of $\{K_i\}$ lying in the interior disk with respect to the loop η, then this component is called *interior* for K. If the interior disk of η contains at least one interior component for K but it does not contain other closed loops of K with the same property, then η is called *minimal*.

The intersection of all exterior disks for all components contains the north pole and is called *the absolute exterior*.

In the step 2 we will use the following two admissible moves.
Operation C: Uniting a component and its interior components belonging to the same hemisphere.
Let η be a minimal loop of a component K. Assume that all interior components being interior with respect to η belong to the same hemisphere as the component K does. Then we unite these interior components and K in one component which will be denoted again by K.
Operation D: Separating a component and its interior components belonging to the other hemisphere.

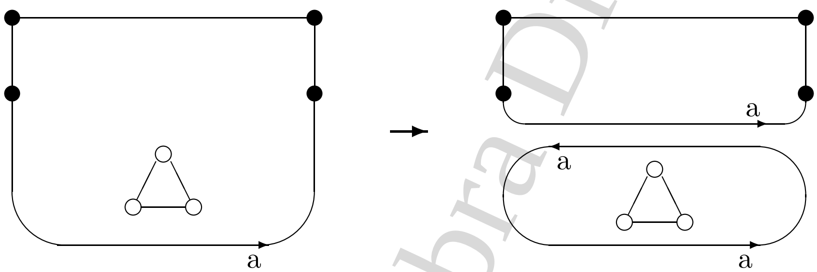

Fig. 8

Let η be a minimal loop of a component K. If at least one of the interior components being interior with respect to η does not belong to the same hemisphere as K does, then we move all interior components of K into the absolute exterior by bridge moves (see Fig. 8).

Remark 6. Operation D strictly decreases the summary number of interior components in P and adds a finite number of edges-circles to P.

Lemma 5. *Let $\{K_i\}$ be a finite subdivision of a picture P into uniform components. Then a finite number of Operation C and D (being admissible moves) gives a subdivision of P into states and edges-circles not belonging to the states.*

Proof. To each component K_i of P assign a number s_i equal to the number of minimal loops in the component K_i. Let s be the maximum of $\{s_i\}$.

For $s = 0$, there is nothing to prove.

Let $s > 0$. We transform each component $K \in \{K_i\}$ containing s minimal loops as follows.

Let η be one of s minimal loops in K. If all interior components interior with respect to η belong to the same hemisphere as K does, then by Operation C the loop η ceases to be minimal, hence the number of minimal loops in K becomes strictly less.

If there are some interior components interior with respect to η such that they belong to the other hemisphere, then by Operation D the number of minimal loops in K becomes strictly less. Operation D increases the number of minimal loops for none of components $\{K_i\}$, but it increases the number of edges-circles (see Remark 6).

Decreasing the maximum number s of minimal loops by admissible moves, we obtain that each component lies in exterior disks for the other components, i.e., P consists only of the states and edges-circles not belonging to the states. This proves Lemma 5. □

Step 3. *Getting rid of edges-circles.*
Assume that P consists not only of states but of some edges-circles not belonging to the states. Each such edge-circle divides the sphere into two parts homeomorphic to disks: an exterior disk containing the north pole (see Step 2) and an interior disk not containing it.

Applying the admissible moves 4) and 3), we can assume that both interior and exterior disks of each edge-circle contain states.

An edge-circle C is called *minimal*, if the interior disk of C does not contain the other edges-circles.

In Step 3 we will use the following two admissible moves.
Operation E: Uniting an edge-circle and its interior states belonging to the same hemisphere.
Let C be a minimal edge-circle such that its interior disk contains states belonging to the same hemisphere only. Then we unite C and the states interior to C in one state of the same hemisphere.

Remark 7. Operation E decreases the number of edges-circles in P. An edge-circle, which was not minimal before Operation E, can become minimal after Operation E.

Operation F: Cutting an edge-circle if its interior states belong to the different hemispheres.
Let C be a minimal edge-circle such that its interior disk contains states belonging to the different hemispheres. By bridge moves, C can be cut into some edges-circles $\{C_i\}$ so that for each C_i its interior states belong only to the same hemisphere (see Fig. 9).

Remark 8. Operation F is a composition of a finite number of bridge moves. New edges-circles $\{C_i\}$ are minimal again. An edge-circle, which was not minimal before Operation F, remains not minimal after Operation F.

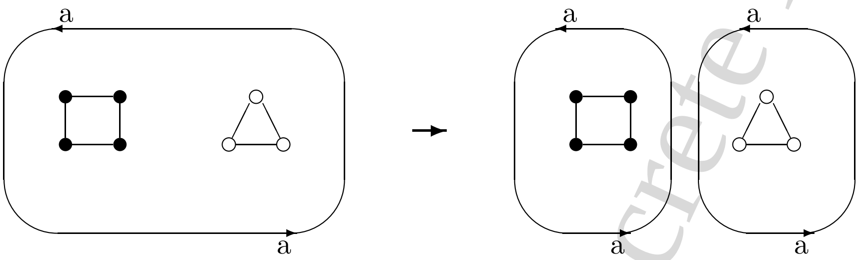

Fig. 9

Lemma 6. *Assume that there are a finite subdivision of P into states $\{T_i\}$ and edges-circles not belonging to the states. Then after a finite number of bridge moves, P will consist of states only.*

Proof. Assume that P contains s edge-circles not belonging to the states.

The proof is by induction on s. For $s = 0$, there is nothing to prove. Let $s > 0$.

Let m be the number of minimal edges-circles ($m > 0$) and n be the number of edges-circles not being minimal among all s edges-circles ($n = s - m$).

If there are states from the different hemispheres interior to any minimal edge-circle C, then by Operation F the edge-circle C can be cut into a finite number of minimal edges-circles $\{C_i\}$ such that for each C_i its interior states belong only to the same hemisphere. Note that Operation F does not change the number n of not minimal edges-circles.

Thus we can obtain that for each minimal edge-circle its interior states are from the same hemisphere. By Operation E, we can unite each minimal edges-circle and its interior states in one state. In addition, the number of the edge-circles not belonging to the states becomes at most n, where $n < s$. □

Step 4. *Reducing a picture P containing only states to a picture containing only σ-states.*
Let P contain only states.

On this step we will apply the following admissible move.
Operation G. Crushing a state into σ-states.
Let $\mathcal{R}$ be an irregular north region of a north state T (the move of an irregular south region of a south state is similar). The boundary of the

region $\mathcal{R}$ contains at least two connected pieces of the equator. We fix any of them and denote it by I. The following transformations are performed near I. A map M is chosen in $\mathcal{R}$ so that it contains I together with edges intersecting Equ at points of I: more precisely, $\{y = -1\}$ coincides with I; $\{x = x_1\}, ..., \{x = x_n\}$ correspond to the edges intersecting Equ in I; $\{x = -1\}$ and $\{x = 1\}$ coincide with parts of the boundary of T. (See an example on Fig. 10.)

The construction of a new map M' is similar to Step 1. Namely, by Lemma 2, $U = Lab(I) \in N_1$. Two planar pictures P_1 and P_2 with the boundary labels respectively U and U^{-1} are constructed and disposed on M' as follows:
1) P_1 lies in the rectangle $\{-1 < y < -1/2, -1 < x < 1\}$ and all edges intersecting the boundary of P_1 start at $(x_1, -1)$,...,$(x_n, -1)$;
2) P_2 lies in the rectangle $\{1/2 < y < 1, -1 < x < 1\}$ and all edges intersecting the boundary of P_2 start at $(x_1, 1)$,...,$(x_n, 1)$;
3) the territory of T divides into parts $\{y \leq -1/2\}$ and $\{y \geq 1/2\}$;
4) $\{y = -1\}$ is the considered piece I of the equator.

The map M is cut out from P and replaced by the new map M' (see Fig.11).

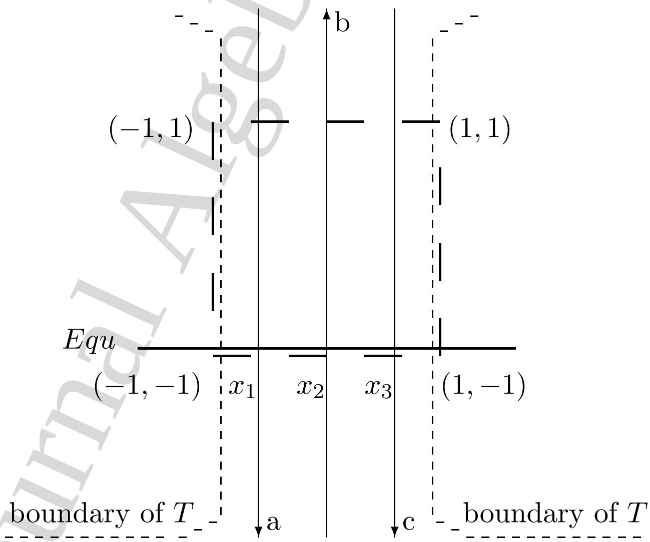

Fig. 10

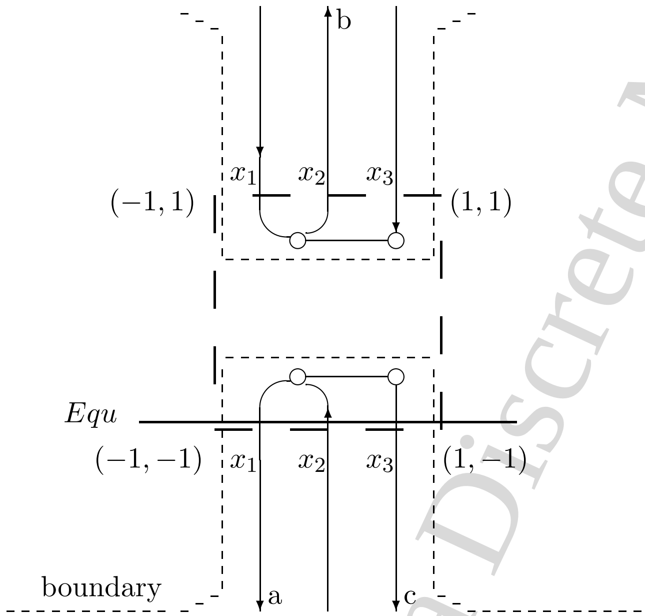

Fig. 11

This move does not change the equatorial label W, hence it is admissible.

Remark 9. By this move the number of states increases by one and the irregular region $\mathcal{R}$ divides into two parts: one is a regular region and the other one is an irregular region such that the number of pieces of the equator belonging to its boundary is strictly less than the number of pieces belonging to the boundary of the original $\mathcal{R}$.

Lemma 7. *Let a picture P contain only a finite number of states. Then P can be reduced to a picture containing only σ-states by a finite number of Operations G (being admissible moves).*

Proof. If all north regions of all north states and all south regions of all south states are regular, then there is nothing to prove.

Otherwise, to each irregular region $\mathcal{R}_i$ assign the number s_i of pieces of the equator lying on the boundary of $\mathcal{R}_i$. Let s be the maximum of $\{s_i\}$ $(s > 1)$.

Operation G applied to each irregular region decreases the number s by one. Therefore all states become σ-states after a finite number of Operations G. □

Thus Proposition 1 follows from Lemmas 4 - 7. □

2.5 Proof of Proposition 2.

By admissible moves, P is reduced to a picture containing only σ-states.

We assume that P consists only of two σ-states: a north σ-state T_1 and a south σ-state T_2, since we can always unite all north σ-states with each other and all south σ-states with each other (see the admissible move 5)).

Equ can be divided into arcs so that each of these arcs is intersected by one σ-state only. We fix the direction of moving along Equ (let it be from the west to the east) and renumber these arcs $J_1, J_2, ..., J_k$ successively. We may assume that an arc J_i is intersected by the north σ-state for even i and by the south σ-state for odd i. An arc will be called south (respectively, north), if it is intersected only by the south σ-state (respectively only by the north σ-state). According to Remark 2 (see the admissible move 5)), each arc J_i contains precisely one piece I_i of the equator belonging to the territory of any σ-state (this piece I_i is a part of the boundary of a regular region of the corresponding σ-state)

The proof of Proposition 2 is by induction on k equal to the number of arcs.

If $k \leq 2$, then Proposition 2 follows from Lemma 3.

Let $k > 2$. Then we do the following admissible move.

We consider the second north arc J_2 and the third south arc J_3. By Lemma 2, a word w_1 along the piece I_2 of the north arc J_2 belongs to N_1 and a word w_2 along the piece I_3 of the south arc J_3 belongs to N_2.

Construct a map M containing a subpicture with the part $\{y = 0\}$ of the equator corresponding to the word $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$. Cut out a small map M_s between I_1 and I_2 such that M_s contains nothing but a small part $\{y = 0\}$ of Equ. Paste M in place of M_s (see the admissible move 6)). This adds two new σ-states: north one T_1^{comm} and south one T_2^{comm}. (See an example on Fig. 12) This move corresponds to the insertion of the commutator $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$ of $w_1 \in N_1$ and $w_2 \in N_2$ in the word $W = ...w_1 w_2...$, and replacing it by $W' = ...(w_2 w_1 {w_2}^{-1} {w_1}^{-1}) w_1 w_2....$

By the admissible move 5), we unite the north σ-states T_1 and T_1^{comm} in one σ-state T_1^n, after that we use Remark 2. Let I_2^n be a piece of the equator in the north σ-state T_1^n containing the piece of the equator I_2 of the original σ-state T_1 and the piece of the equator I_2' of the original σ-state T_1^{comm} (the notation are from the description of the admissible move 6)). Since $Lab(I_2^n) = \mathbf{1}$, after a finite number of bridge moves, no edge of T_1^n intersects I_2^n (see the admissible move 2)).

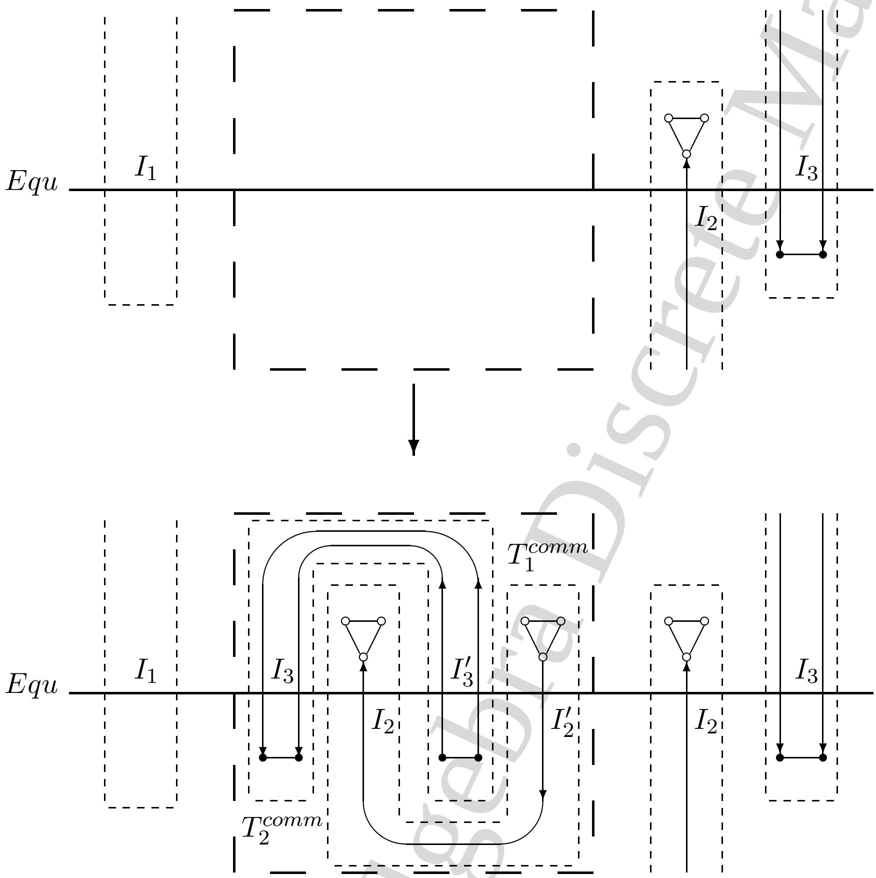

Fig. 12

If a regular region (the boundary of which contains the piece I_2^n) contains a subpicture not intersecting Equ, then this subpicture is removed (see the admissible move 3)). Deformation of the territory of T_1^n gives that the territory of T_1^n does not intersect Equ by the piece I_2^n.

By the admissible move 5), we unite the south σ-states T_2 and T_2^{comm} in one σ-state T_2^n. By Remark 2, the piece I_3^n of the south σ-state T_2^n contains the piece I_3 of the σ-state T_2 and the piece I_3' of the σ-state T_2^{comm}. By similar transformations as above, we obtain that the σ-state T_2^n does not intersect Equ by the piece I_3^n.

Note that as a result of the transformations described above the equatorial label $W' = ...(w_2 w_1 {w_2}^{-1} {w_1}^{-1}) w_1 w_2...$ is reduced to the form $W' = ...w_2 w_1....$

Thus again P consists of the two σ-states T_1^n and T_2^n, which intersect Equ by the same pieces as the σ-states T_1 and T_2 did, but T_1^n and T_2^n

intersect Equ in other order. More precisely, the south pieces of the equator I_1 and I_2 turn out to be side by side (similarly the north pieces I_2 and I_4 turn out to be side by side). Therefore by Remark 2, the number of arcs in the subdivision of Equ becomes strictly less and the number of states and vertices does not increase.

By induction, we get a subdivision of Equ into two arcs, i.e., the case when the equatorial label W is equal to the identity element in the free group, which is the desired conclusion. □

3. Some corollaries in the case of free products.

Let $G_1 =< X \mid S_1 >$ and $G_2 =< Y \mid S_2 >$ represent some groups and $G = G_1 * G_2$ be a free product of them. We will assume that the sets S_1 and S_2 are symmetrized. We consider the normal closure N_1 of a set of elements $R_1 = \{u_i\}$ and the normal closure N_2 of a set of elements $R_2 = \{v_j\}$ in G. We assume that R_1 and R_2 are mutually disjoint and symmetrized.

Assertion 6. *Let a presentation $< X \cup Y \mid S_1 \cup S_2 \cup R_1 \cup R_2 >$ be weakly (T_1, T_2)-separable, where T_1 and T_2 are defined in one of the following ways:*
(i) $T_1 = R_1 \cup S_1$, $T_2 = R_2 \cup S_2$;
(ii) $T_1 = R_1 \cup S_1 \cup S_2$, $T_2 = R_2$.
Then $N_1 \cap N_2 = [N_1, N_2]$.

Proof. Let us show that Assertion 6 follows from the similar statement in the case of free groups.

Let $N_1{}' =< T_1 >^F$ and $N_2{}' =< T_2 >^F$ denote the normal closures of the sets of words respectively T_1 and T_2 in the free group $F = F(X) * F(Y)$. By Theorem 1, weak (T_1, T_2)-separability leads to

$$N_1{}' \cap N_2{}' = [N_1{}', N_2{}']. \qquad (*)$$

Consider the canonical homomorphism $\psi : F \longrightarrow G = F/ < S_1 \cup S_2 >^F$. Let us show that the following equalities hold:
1) $\psi(N_1{}') = N_1$;
2) $\psi(N_2{}') = N_2$;
3) $\psi([N_1{}', N_2{}']) = [N_1, N_2]$;
4) $\psi(N_1{}' \cap N_2{}') = N_1 \cap N_2$.

The first two equalities are immediate. It follows from them that the third equality and the inclusion $\psi(N_1{}' \cap N_2{}') \subset N_1 \cap N_2$ hold. Therefore it remains to prove only the inclusion $N_1 \cap N_2 \subset \psi(N_1{}' \cap N_2{}')$.

Let $c \in N_1 \cap N_2$ be an arbitrary element and $\bar{n}_1 \in N_1{}'$ and $\bar{n}_2 \in N_2{}'$ be some of its preimages. We have $\bar{n}_1 = n * \bar{n}_2$, where $n \in Ker\, \psi$. If n

is equal to the identity element in the free group, then there is nothing to prove. Otherwise, we will look for such preimages of c that n will be equal to the identity element in the free group.

In the case (ii) the element $n^{-1} * \bar{n}_1$ belongs to $N_1{}'$. Thus in this case preimages of c can be chosen so that n will be equal to the identity element in the free group.

Consider the case (i). The element n can be represented as a product of elements $s_{1,i} \in < S_1 >^F$ and $s_{2,j} \in < S_2 >^F$. There are two possibilities: the first factor belongs either to $< S_1 >^F$ or to $< S_2 >^F$. We will show that it is possible to replace $\bar{n}_1$ by $\tilde{n}_1$ and $\bar{n}_2$ by $\tilde{n}_2$ in each of these cases so that $\psi(\bar{n}_1) = \psi(\tilde{n}_1)$, $\psi(\bar{n}_2) = \psi(\tilde{n}_2)$, and the number of factors in the factorization of the product $\tilde{n}_1 * \tilde{n}_2^{-1}$ will be strictly less than one in the factorization of $\bar{n}_1 * \bar{n}_2^{-1}$.

If $s_{2,1} \in < S_2 >^F$ is the first factor of n, that is $n = \bar{n}_1 * \bar{n}_2^{-1} = s_{2,1} * g$, then we conjugate it by the element $s_{2,1} \in Ker\psi$: $(s_{2,1}{}^{-1} * \bar{n}_1 * s_{2,1}) * (s_{2,1}{}^{-1} * \bar{n}_2^{-1} * s_{2,1}) = g * s_{2,1}$ and denote the conjugated elements $\bar{n}_1$ and $\bar{n}_2$ by $\bar{n}_1$ and $\bar{n}_2$ again. We have that the number of factors in the factorization of n does not increase.

Therefore we can assume that $s_{1,1} \in < S_1 >^F \subset \ker\psi$ is the first factor of n. We multiply n by $s_{1,1}{}^{-1}$ on the left and denote the product $s_{1,1}{}^{-1} * \bar{n}_1$ again by $\bar{n}_1$. Note that the new $\bar{n}_1$ also belongs to $N_1{}'$ and the number of factors in the factorization of n decreases.

If the last factor of n is $s_{2,k} \in < S_2 >^F \subset \ker\psi$, then we multiply n by $s_{2,k}{}^{-1}$ on the right and denote the product $\bar{n}_2 * s_{2,k}{}^{-1}$ again by $\bar{n}_2$. We have that the new $\bar{n}_2 \in N_2{}'$ and the number of factors in the factorization of n decreases.

Repeating as above, we can reduce n to the identity element in the free group.

It follows now from (3), (4), $(*)$ that $N_1 \cap N_2 = [N_1, N_2]$. $\square$

Corollary 6. *If R_1 belongs to G_1, and R_2 belongs to G_2, then $N_1 \cap N_2 = [N_1, N_2]$.*

Proof. It follows from Corollary 5, Theorem 1, and (i) of Assertion 6. $\square$

Corollary 7. *Under the notation of Assertion* 6*, let a presentation $< X \cup Y \mid T_1 \cup T_2 >$ satisfy one of the following conditions:*
(i) strict (T_1, T_2)-separability;
(ii)(T_1, T_2)-asphericity;
(iii) asphericity.

Then $N_1 \cap N_2 = [N_1, N_2]$.

Proof. It follows from Corollary 3, Theorem 1, and Assertion 6. $\square$

Corollary 8. *Let a set* $\{S_1, S_2, R_1, R_2\}$ *satisfy one of the following small cancellation conditions: either* $C(6)$*, or* $C(4)\&T(4)$*, or* $C(3)\&T(6)$*.*

Then $N_1 \cap N_2 = [N_1, N_2]$*.*

Proof. It follows from Corollary 4, Theorem 1, and Assertion 6. □

References

[1] A.Yu. Ol'shanskij. *Geometry of defining relations in groups*, Moscow, "Nauka", 1989 (in Russian).

[2] Ol'shanskij A.Yu. *Geometry of defining relations in groups*, Mathematics and Its Applications. Soviet Series, 70, Dordrecht atc.: Kluwer Academic Publishers, 1991.

[3] R.S.Lindon, P.E.Schupp. *Combinatorial group theory,* Springer-Verlag, Berlin - Heidelberg - NewYork, 1977.

[4] W.A. Bogley, S.J. Pride. *Aspherical relative presentations,* Proc. Edinburgh Math. Soc., vol. **35** (ser. II), part 1, 1-40, 1992.

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