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RESEARCH ARTICLE

A note on maximal ideals in ordered semigroups

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Abstract. In commutative rings having an identity element, every maximal ideal is a prime ideal, but the converse statement does not hold, in general. According to the present note, similar results for ordered semigroups and semigroups -without order- also hold. In fact, we prove that in commutative ordered semigroups with identity each maximal ideal is a prime ideal, the converse statement does not hold, in general.

There is an important class of ideals of rings which are prime, namely, the maximal ideals. In fact, in a commutative ring with identity every maximal ideal is a prime ideal. On the other hand, there are rings possesing a nontrivial prime ideal which is not maximal (cf. e.g. [1]). Similar results for ordered semigroups, also for semigroups -without order- also hold.

If $(S, ., \leq)$ is an ordered semigroup, a non-empty subset I of S is called a left (resp. right) ideal of S if 1) $SI \subseteq I$ (resp. $IS \subseteq I$) and 2) $a \in I$, $S \ni b \leq a$ implies $b \in I$ [2]. If $(S, .)$ is a semigroup, a left (resp. right) ideal of S is a non-empty subset I of S such that $SI \subseteq I$ (resp. $IS \subseteq I$). If S is a semigroup or an ordered semigroup and I both a left and a right ideal of S, then it is called an ideal of S. An ideal I of a semigroup (resp. ordered semigroup) S is called prime if $a, b \in S$ such that $ab \in I$ implies $a \in I$ or $b \in I$. Equivalent Definition: $A, B \subseteq S$ such that $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ [2]. An ideal M of a semigroup or an ordered semigroup S is called proper if $M \neq S$ [3]. A proper ideal M of a semigroup or an ordered semigroup S is called maximal if there exists no ideal T of S such that $M \subset T \subset S$, equivalently, if for each ideal T of S such that $M \subseteq T$, we have $T = M$ or $T = S$ (cf. also [2]). If S is an ordered semigroup and $H \subseteq S$, we denote

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

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If S is an ordered semigroup (resp. semigroup) and $\emptyset \neq A \subseteq S$, we denote by $\mathcal{I}(A)$ the ideal of S generated by A i.e. the smallest -under inclusion relation- ideal of S containing A. For an ordered semigroup S, we have $\mathcal{I}(A) = (A \cup SA \cup AS \cup SAS]$ (cf. [2]). For a semigroup S, we have $\mathcal{I}(A) = A \cup SA \cup AS \cup SAS$.

Let $\{(S_i, \circ_i, \leq_i) \mid i \in I\}$ be a non-empty family of ordered semigroups. The cartesian product $\prod\limits_{i\in I} S_i$ with the multiplication "$*$" and the order "$\preceq$" on $\prod\limits_{i\in I} S_i$ defined by

$$* : \prod_{i\in I} S_i \times \prod_{i\in I} S_i \to \prod_{i\in I} S_i \mid ((x_i)_{i\in I}, (y_i)_{i\in I}) \to (x_i)_{i\in I} * (y_i)_{i\in I} \quad \text{where}$$

$$(x_i)_{i\in I} * (y_i)_{i\in I} := (x_i \circ_i y_i)_{i\in I}$$

$$\preceq := \left\{((x_i)_{i\in I}, (y_i)_{i\in I}) \in \prod_{i\in I} S_i \times \prod_{i\in I} S_i \mid x_i \leq_i y_i \;\forall\, i \in I\right\}$$

is an ordered semigroup.
In the following we consider the $\prod\limits_{i\in I} S_i$ as the ordered semigroup with the multiplication and the order defined above.

Lemma 1. *Let $\{(S_i, \circ_i, \leq_i) \mid i \in I\}$ be a family of ordered semigroups. If J_i is an ideal of S_i for every $i \in I$, then the set $\prod\limits_{i\in I} J_i$ is an ideal of $\prod\limits_{i\in I} S_i$.*

Proof. 1) $\emptyset \neq \prod\limits_{i\in I} J_i \subseteq \prod\limits_{i\in I} S_i$ (since $J_i \neq \emptyset \;\forall\, i \in I$).
2) $\prod\limits_{i\in I} S_i * \prod\limits_{i\in I} J_i \subseteq \prod\limits_{i\in I} J_i$. In fact:
Let $(x_i)_{i\in I} \in \prod\limits_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod\limits_{i\in I} J_i$. Since $x_i \in S_i$ and $y_i \in J_i$ for every $i \in I$, we have $x_i \circ_i y_i \in S_i \circ_i J_i \subseteq J_i$ for every $i \in I$. Then we have

$$(x_i)_{i\in I} * (y_i)_{i\in I} := (x_i \circ_i y_i)_{i\in I} \in \prod_{i\in I} J_i$$

.
3) Let $(y_i)_{i\in I} \in \prod\limits_{i\in I} J_i$ and $\prod\limits_{i\in I} S_i \ni (x_i)_{i\in I} \preceq (y_i)_{i\in I}$. Then $(x_i)_{i\in I} \in$

$\prod_{i\in I} J_i$.
Indeed: Since $y_i \in J_i$, $S_i \ni x_i \leq_i y_i$ and J_i is an ideal of S_i for every $i \in I$, we have $x_i \in J_i$ for every $i \in I$. Then $(x_i)_{i\in I} \in \prod_{i\in I} J_i$. Similarly, the set of $\prod_{i\in I} J_i$ is a right ideal of $\prod_{i\in I} S_i$. □

In the following, we denote by S the closed interval $[0,1]$ of real numbers. The set $S := [0,1]$ with the usual multiplication- order "." and "$\leq$" is an ordered semigroup.

Lemma 2. *If $a \in S$, then the set $I_a := [0,a]$ is an ideal of S.*

Proof. First of all $\emptyset \neq I_a \subseteq S$ (since $a \in [0,a]$). Let $x \in S$, $y \in I_a$. Since $0 \leq x \leq 1$, $0 \leq y \leq a$, we have $0 \leq xy \leq 1a = a$. Then $xy \in I_a$. Let $y \in I_a$ and $S \ni x \leq y$. Since $0 \leq x$, $y \leq a$ and $x \leq y$, we have $0 \leq x \leq a$. Then $x \in I_a$. Similarly, the set I_a is a right ideal of S. □

Theorem. *Let $(S,.,\leq)$ be a commutative ordered semigroup with identity. If M is a maximal ideal of S, then M is a prime ideal of S. The converse statement does not hold, in general.*

Proof. Let e be the identity of S, and M a maximal ideal of S.
Let $a, b \in S$, $ab \in M$, $a \notin M$. Then $b \in M$. In fact:
Since S is commutative, we have

$$\begin{aligned}\mathcal{I}(M\cup\{a\}) = &\quad ((M\cup\{a\})\cup S(M\cup\{a\})\cup(M\cup\{a\})S\cup S(M\cup\{a\})S] \\ = &\quad ((M\cup\{a\})\cup S(M\cup\{a\})\cup S^2(M\cup\{a\})].\end{aligned}$$

Since $M\cup\{a\} = e(M\cup\{a\}) \subsetneq S(M\cup\{a\})$, we have

$$S(M\cup\{a\}) \subseteq S^2(M\cup\{a\}) \subseteq S(M\cup\{a\}),$$

then $S(M\cup\{a\}) = S^2(M\cup\{a\})$.

Hence we have

$$\mathcal{I}(M\cup\{a\}) = (S(M\cup\{a\})]..........(*)$$

On the other hand, $M \subset M\cup\{a\} \subseteq \mathcal{I}(M\cup\{a\})$ (since $a \notin M$). Since $\mathcal{I}(M\cup\{a\})$ is an ideal and M a maximal ideal of S, we have $\mathcal{I}(M\cup\{a\}) = S$, and $e \in (S(M\cup\{a\})]$ by (*). Then there exist $x \in S$ and $y \in M\cup\{a\}$ such that $e \leq xy$. Then $b = eb \leq xyb$. If $y \in M$, then $xyb \in SMS \subseteq M$, and $b \in M$. If $y = a$, then $b \leq x(ab) \in SM \subseteq M$, and $b \in M$.

For the converse statement, we consider the ordered semigroup $S := [0, 1]$ and the ordered semigroup $(S \times S, * \preceq)$ constructed above. The set $(S \times S, *, \preceq)$ is a commutative ordered semigroup and the element $(1, 1)$ is the identity element of $S \times S$. Let

$$T := S \times \{0\}(= [0, 1] \times \{0\}).$$

Clearly S is an ideal of S. By Lemma 2, the set $I_0(= \{0\})$ is an ideal of S. Then, by Lemma 1, the set $T := S \times \{0\}$ is an ideal of $S \times S$.
The set T is a prime ideal of $S \times S$. In fact:
Let $(x, y), (z, w) \in S \times S, (x, y) * (z, w) \in T$. Since $(x, y) * (z, w) := (xz, yw) \in T := S \times \{0\}$, we have $yw = 0$, then $y = 0$ or $w = 0$. Then $(x, y) \in S \times \{0\} := T$ or $(z, w) \in S \times \{0\} := T$.
The set T is not a maximal ideal of $S \times S$. Indeed: By Lemma 2, the set $[0, 1/2] := I_{1/2}$ is an ideal of S. By Lemma 1, the set $S \times [0, 1/2]$ is an ideal of $S \times S$. On the other hand, $T := S \times \{0\} \subset S \times [0, 1/2] \subset S \times S$. □

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