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Rings which have (m, n)-flat injective modules

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Abstract. A ring R is said to be a left $IF-(m,n)$ ring if every injective left R-module is (m,n)-flat. In this paper, several characterizations of left $IF-(m,n)$ rings are investigated, some conditions under which R is left $IF-(m,n)$ are given. Furthermore, conditions under which a left $IF-1$ ring (i.e., $IF-(1,1)$ ring) is a field, a regular ring and a semisimple ring are studied respectively.

1. Introduction

Throughout R is an associative ring with identity and all modules are unitary. The character module $Hom_Z(M, Q/Z)$ of a module M will be denoted by M^* and the injective hull of M by $E(M)$.

Let m, n be two fixed positive integers. Let G an abelian group. We write $G^n(G_n)$ for the set of all formal n-dimensional row(column) vectors over G. An R-module M is said to be n-generated if it has a generating set of cardinality at most n[6]. A left R-module M is called (m,n)-injective if for every n-generated submodule K of the left R-module R^m, each R-homomorphism from K to M can be extended to R^m[3]. Note that $(1,n)$-injective modules are also called n-injective in [10],[12]. The ring R is left (m,n)-injective if $_RR$ is (m,n)-injective. It is obvious that $_RM$ is FP-injective if and only if $_RM$ is (m,n)-injective for all positive integers m and n, and $_RM$ is P-injective if and only if $_RM$ is $(1,1)$-injective. We also recall that a module M is FP-injective if and only if M is absolutely pure, M is flat if and only if M^* is FP-injective. A submodule U' of a module $_RU$ is pure in U if and only if the canonical

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map $V \otimes_R V \to V \otimes_R U$ is monic for every finitely presented module V_R. Indicated by these results we have following definition.

Definition 1.1. Let m, n be two fixed positive integers.

(1) A right R-module V is said to be (m,n)-presented if there exists an exact sequence of right R-modules $0 \to K \to R^m \to V \to 0$ with n-generated K.

(2) Given a left R-module U, a submodule U' of U is called (m,n)-pure in U if for every (m,n)-presented right R-module V, the canonical map $V \otimes_R U' \to V \otimes_R U$ is monic.

(3) A left R-module M is (m,n)-flat if for every n-generated submodule K of the right R-module R^m, the canonical map $K \otimes_R M \to R^m \otimes_R M$ is monic.

(4) A module M is called $(\aleph_0, n)$-injective ($(\aleph_0, n)$-flat) if M is (k,n)-injective ((k,n)-flat) for all positive integers k. Module M is called $(m, \aleph_0)$-injective if M is (m,k)-injective for all positive integers k. A submodule U' of a module U is said to be $(\aleph_0, n)$-pure ($(m, \aleph_0)$-pure) in U if U' is (k,n)-pure ((m,k)-pure) in U for all positive integers k.

An exact sequence $0 \to U' \to U \to M \to 0$ is called (m,n)-pure if U' is (m,n)-pure in U, and a module L is called (m,n)-pure projective if L is projective with respect to every (m,n)-pure exact sequence.

Using a standard technique we can prove the following theorem.

Theorem 1.2. (1) An exact sequence of left R-modules $0 \to U' \to U \to M \to 0$ is (m,n)-pure if and only if every (n,m)-presented module is projective with respect to this exact sequence if and only if $(U')^m \bigcap KU = KU'$ for every n-generated submodule K of the right R-module R^m.

(2) A module M is (m,n)-injective if and only if M is (n,m)-pure in every module containing M if and only if M is (n,m)-pure in $E(M)$.

(3) A module M is (m,n)-flat if and only if M^* is (m,n)-injective if and only if every exact sequence $0 \to U' \to U \to M \to 0$ is (m,n)-pure. Every $(n, \aleph_0)$-pure submodule of a (m,n)-flat module is (m,n)-flat.

(4) If $U' \leq U$ and U is (m,n)-flat, then U/U' is (m,n)-flat if and only if U' is (m,n)-pure in U.

Recall that a ring R is called left IF ring if every injective left R-module is flat [4],[9]. In this paper we shall generalize this notion and introduce left IF-(m,n) rings providing several characterizations of such rings. We also discuss left IF-$(1,1)$ rings in a number of special cases.

2. Results

In this section, m and n will be two fixed positive integers. We start with the following

Definition 2.1. A ring R is called left IF-(m,n) ring if every injective left R-module is (m,n)-flat.

Theorem 2.2. For any ring R, the following conditions are equivalent:
(1) R is a left IF-(m,n) ring.
(2) Every FP-injective left R-module is (m,n)-flat.
(3) Every $(\aleph_0, n)$-injective left R-module is (m,n)-flat.
(4) If ${}_RN_1 \leq {}_RN, N_1$ is $(\aleph_0, m)$-injective and N is $(\aleph_0, n)$-injective, then N/N_1 is (m,n)-flat.

Proof. (1)⇒(3). If ${}_RN$ is $(\aleph_0, n)$ -injective then it is $(n, \aleph_0)$-pure in $E(N)$. Thus, from the (m,n)-flatness of $E(N)$, we have that N is (m,n)-flat.

(3)⇒(4). Since N_1 is $(\aleph_0, m)$-injective, so N_1 is $(m, \aleph_0)$-pure in N and hence (m,n)-pure in N. But N is (m,n)-flat, so N/N_1 is (m,n)-flat.

The implications (4)⇒(2)⇒(1) are clear.

Let P and M be left R-modules. There is a natural homomorphism $\sigma = \sigma_{P,M} : Hom_R(P,R) \otimes M \to Hom_R(P,M)$ defined by $\sigma(f \otimes m)(p) = f(p)m$ for $f \in Hom(P,R), m \in M, p \in P$.

Lemma 2.3. Let M be a left R-module. If $\sigma_{P,M}$ is an epimorphism for every (n,m)-presented left R-module P, then M is (m,n)-flat.

Proof. For every (n,m)-presented left R-module P and $u \in Hom_R(P,M)$, suppose $u = \sigma(\sum_{i=1}^{k} f_i \otimes m_i), f_i \in Hom_R(P,R), m_i \in M$. Let $F = R^k$, and define $v : P \to F$ by $v(p) = (f_1(p), \cdots, f_k(p)); w : F \to M$ by $w(r_1, \cdots, r_k)$, then $u = wv$. It follows that P is projective with respect to every exact sequence of left R-modules $0 \to M' \to M'' \to M \to 0$. Consequently, M is (m,n)-flat by Theorem 1.2(1),(3).

Lemma 2.4. Suppose that ${}_RU$ is (m,n)-flat. The following statements are equivalent
(1) ${}_RU^I$ is (m,n)-flat for every set I.
(2) For every n-generated submodule K of the right R-module R^m, the natural homomorphism $\varphi_K : K \otimes_R U^I \to (K \otimes_R U)^I$ defined by $\varphi_K(x \otimes u)(\alpha) = x \otimes u(\alpha), \alpha \in I, u \in U^I, x \in K$, is an isomorphism.

Proof. (1)⇒(2). Assume (1) holds. Then the exact sequence $0 \to K \to R^m \to R^m/K \to 0$ induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \otimes U^I & \longrightarrow & R^m \otimes U^I & \longrightarrow & (R^m/K) \otimes U^I & \longrightarrow & 0 \\
 & & \Big\downarrow \phi_K & & \Big\downarrow \phi_{R^m} & & \Big\downarrow \phi_{R^m/K} & & \\
0 & \longrightarrow & (K \otimes U)^I & \longrightarrow & (R^m \otimes U)^I & \longrightarrow & (R^m/K \otimes U)^I & \longrightarrow & 0
\end{array}$$

which has exact rows, since ${}_RU$ and ${}_RU^I$ are (m, n)-flat. The maps φ_{R^m} and $\varphi_{R^m/K}$ are isomorphisms since R^m and R^m/K are finitely presented. Hence φ_K is an isomorphism.

(2)⇒(1). For every n-generated submodule K of R^m we have the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & K \otimes U^I & \longrightarrow & R^m \otimes U^I \\
 & & \Big\downarrow \phi_K & & \Big\downarrow \phi_{R^m} \\
0 & \longrightarrow & (K \otimes U)^I & \longrightarrow & (R^m \otimes U)^I
\end{array}$$

which has exact bottom row, and φ_K and $\varphi_{R^m/K}$ are isomorphism by (2). Hence the top row is also exact, and thus U^I is (m, n)-flat.

Theorem 2.5. For any ring R the following conditions are equivalent

(1) R is left IF-(m, n).

(2) The injective hull of every finitely presented left R-module is (m, n)-flat.

(3) Every (n, m)-presented left R-module is a submodule of a free module.

(4) For every free right R-module F, F^* is (m, n)-flat.

(5) For every n-generated submodule K of the left R-module R^m, the map $\varphi_k : K \otimes_R (R^*)^I \to (K \otimes_R R^*)^I$ is an isomorphism.

(6) For every m-generated submodule K of the left R-module R^n there exist finite elements $\beta_1, \beta_2, \cdots, \beta_p$ in R^n such that $K = l_{R^n}\{\beta_1, \beta_2, \cdots, \beta_p\}$.

Proof. (1)⇒(2) is trivial.

(2)⇒(3). Let u be the injection of a given (n, m)-presented module ${}_RP$ into $E(P)$, and let $0 \to K \to F \xrightarrow{w} E(P) \to 0$ be an exact sequence with F being free. Then there exists $v : P \to F$ such that $u = wv$, which implies that v is monic, and (3) holds.

(3)⇒(4). Let F_R be any free R-module, then F^* is injective. Let ${}_RP$ be (n, m)-presented and ${}_RL$ is a finitely generated free module containing P. The following diagram is commutative

$$\begin{array}{ccc} Hom(L,R)\otimes F^* & \xrightarrow{\sigma_L} & Hom(L,F^*) \\ \downarrow{\alpha} & & \downarrow{\beta} \\ Hom(P,R)\otimes F^* & \xrightarrow{\sigma_P} & Hom(P,F^*) \end{array}$$

Since F^* is injective, β is an epimorphism. Besides, σ_L is an isomorphism since L is finitely generated and free. Therefore σ_P is epic, and thus F^* is (m,n)-flat by Lemma 2.3.

(4)⇒(1). Let E be any injective left R-module. There is a free module F and an epimorphism $F \to E^*$, which give a monomorphism $E^{**} \to F^*$. Since F^* is (m,n)-flat and $E \subseteq E^{**}$, then E is a direct summand of F^* and hence E is (m,n)-flat.

(4)⇔(5). Follows from Lemma 2.4.

(3)⇒(6). Since K is a m-generated submodule of the left R-module R^n, by (3), R^n/K can be embeded in R^p for some positive integer p. Let $f : R^n/K \to R^p$ be a monomorphism and suppose that $f(\overline{e_i}) = (a_{i1}, a_{i2}, \cdots, a_{ip})$, where e_i is the element in R^n with 1 in the ith position and $0'$s in all other positions, $i = 1, 2, \cdots, n$. Write $\beta_j = (a_{1j}, a_{2j}, \cdots, a_{nj})^T$, $j = 1, 2, \cdots, p$, then $K = l_{R^n}\{\beta_1, \beta_2, \cdots, \beta_p\}$.

(6)⇒(3). Let $K = l_{R^n}\{\beta_1, \beta_2, \cdots, \beta_p\}$, where $\beta_j = (a_{1j}, a_{2j}, \cdots, a_{nj})^T \in R^n$, $j = 1, 2, \cdots, p$. Write $\alpha_i = (a_{i1}, a_{i2}, \cdots, a_{ip})$, $i = 1, 2, \cdots, n$, and define $f : R^n/K \to R^p$ by

$$(r_1, r_2, \cdots, r_n) + K \longmapsto (r_1, r_2, \cdots, r_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

then f is a left R-monomorphism.

We call a ring R right (m,n)-coherent if every n-generated submodule of R_R^m is finitely presented. A straightforward modification of Chase's proof of [2, Theorem 2.1] shows that R is right (m,n)-coherent if and only if the direct product of any family of (m,n)-flat left R-modules is (m,n)-flat.

Corollary 2.6. If R is right (m,n)-coherent then R is left IF-(m,n) if and only if ${}_R(R^*)$ is (m,n)-flat.

Proof. Follows from Theorem 2.5(4).

Corollary 2.7. If $_RR$ is a cogenerator and R is right (m, n)-coherent then R is left IF-(m, n).

Proof. Since R is right (m, n)-coherent, $_RR^I$ is (m, n)-flat for every set I. Since $_RR$ is a cogenerator, every left R-module can be imbedded in an (m, n)-flat module. Consequently, every injective left R-module is a direct summand of an (m, n)-flat module and hence is (m, n)-flat.

Corollary 2.8. If R is a left IF-(m, n) ring then R is right (m, n)-injective.

Proof. Let K be any m-generated submodule of the left R-module R^n. Since R is left IF-(m, n), $K = l_{R^n}\{\beta_1, \beta_2, \cdots, \beta_p\}$ for some $\beta_1, \beta_2, \cdots, \beta_p$ in R_n by Theorem 2.5. This implies $l_{R^n} r_{R^n}(K) = K$, and thus R is right (m, n)-injective by [3,Theorem 2.4].

Corollary 2.9 [9,Theorem 3.3] A left IF-ring is right FP-injective.

According to [4], a right R-module N is called T-finitely generated (resp. H-finitely generated) if N contains a finitely generated submodule N_0 such that $N/N_0 \otimes R^* = 0 (resp. Hom_R(N/N_0, R) = 0)$. A right R-module M is called T-finitely presented (resp. H-finitely presented) if there exists a finitely generated free module F and an epimorphism of F onto M with T-finitely generated (resp. H-finitely generated) kernel. We call a ring R right $T-(m, n)$-coherent (resp. $H-(m, n)$-coherent) if every n-generated submodule K of R_R^m is T-finitely presented (resp. H-finitely presented).

Theorem 2.10. Suppose that $_R(R^*)$ is flat. The following statements are equivalent

(1) R is left IF-(m, n).

(2) R is right T-(m, n)-coherent.

(3) R is right H-(m, n)-coherent.

Proof. (1)$\Rightarrow$(2). Follows from Theorem 2.5(5) and [5,Lemma 1.2].

(2)$\Leftrightarrow$(3). Follows from [4,Lemma 3,4].

We call a ring R left IF-n ring if every injective left R-module is n-flat (i.e. $(1, n)$-flat).

Remark 2.11. In general, the classes of left IF-(m, n) rings are different for different pairs (m, n). For example, let K be a field and L a proper subfield of K such that $\rho : K \to L$ is an isomorphism. Let $K[x, \rho]$ be the ring of twisted left polynomials over K, where $xk = \rho(k)x$ for all $k \in K$. Set $R = K[x, \rho]/(x^2)$. It is readily verified that the only proper left ideal of R is $Rx = Kx$. Thus R is left artinian and so satisfies the ascending chain condition on annihilator right ideals. It is easy to check that $Rx = l(x)$, so R is a left IF-1 ring by Theorem 2.5. As in [11, Example 1], we can show that R is not QF, and thus R is not right 2-injective by [11,Corollary 3]. Therefore, R is not a left IF-2 ring by Corollary 2.8.

We characterize left IF-1 rings in some special cases.

Theorem 2.12. Let R be a left IF-1 ring.

(1) R is a field if and only if R is a domain.

(2) R is a regular ring if and only if every principally left ideal is flat.

(3) R is a semisimple ring if and only if R is a semiprime left Goldie ring.

Proof. We only need to prove the sufficiency.

(1) Over a domain, 1-flat modules are torsion free [7, Theorem,3.3]. Thus every injective module, being 1-flat, is torsion free. Hence for every essential ideal I, R/I is torsion free, and so $I = R$. This means that R is semisimple, and hence is a field.

(2) If every principally left ideal is flat then submodules of 1-flat left R-modules are 1-flat by [12, §5, (f)]. We conclude that every left R-module is 1-flat since R is a left IF-1 ring. In particular, R/Ra is 1-flat for each $a \in R$. It follows that Ra is $(1, 1)$-pure in R, and thus $aR \cap Ra = aRa$. Consequently, $a \in aRa$ and R is regular.

(3) Since R is left $IF - 1$, then R_R is P-injective and hence divisible. This implies that every regular element of R has a left inverse and thus is invertible. Observe that in a semiprime Goldie ring, every essential left ideal has a regular element [8,Theorem 3.9], which is invertible in our case. Therefore R has no proper essential left ideals and hence, R is semisimple.

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References

[1] F. W. Anderson and K. R. Fuller, Rings and categories of modules, Springer-verlag, 1974,210-211.

[2] S. U. Chase, Direct products of modules, Trans.Amer. Math. Soc.97(1960), 457-473.

[3] J. L. Chen, N. Q. Ding, Y. L. Li and Y. Q. Zhou, On (m,n)-injectivity of modules, Comm. Algebra 29(12), 2001, 5589-5603.

[4] R. R. Colby, Rings which have flat injective modules, J. Algebra. 35(1975), 239-252.

[5] R. R. Colby and E. A. Rutter, π-flat and π-projective modules, Arch. Math. 22, 1971, 246-251.

[6] D. E. Dobbs, On n-flat modules over a Commutative ring, Bull. Austral. Math. Soc. 43, 1991, 491-498.

[7] D. E. Dobbs, On the criteria of D. D. Anderson for invertible and flat ideals, Canad. Math. Bull. 29(1), 1986, 26-32.

[8] A. W. Goldie, Semi-prime rings with maximum condition, Proc. London Math. Soc. 10(3), 1960.

[9] S. Jain, Flat and FP-injectivity, Proc. Amer. Math. Soc. 41(2),1973,437-442.

[10] W. K. Nicholson and M. F. Yousif, Principally injective rings, J. Algebra. 174(1995), 77-93.

[11] E.A.Rutter, Rings with the principal extension property, Comm. Algebra, 3(3),1975,203-212.

[12] A. Shamsuddin, n-injective and n-flat modules, Comm. Algebra. 29(5), 2001, 2039-2050.

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