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Action type geometrical equivalence of representations of groups

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Abstract. In the paper we prove (Theorem 8.1) that *there exists a continuum of non isomorphic simple modules over* KF_2*, where* F_2 *is a free group with* 2 *generators* (compare with [Ca] where a continuum of non isomorphic simple 2-generated groups is constructed). Using this fact we give an example of a non action type logically Noetherian representation (Section 9).

In general, the topic of this paper is the action type algebraic geometry of representations of groups. For every variety of algebras Θ and every algebra $H \in \Theta$ we can consider an algebraic geometry in Θ over H. Algebras in Θ may be many sorted (not necessarily one sorted) algebras. A set of sorts Γ is fixed for each Θ. This theory can be applied to the variety of representations of groups over fixed commutative ring K with unit. We consider a representation as two sorted algebra (V, G), where V is a K-module, and G is a group acting on V. In the action type algebraic geometry of representations of groups algebraic sets are defined by systems of action type equations and equations in the acting group are not considered. This is the special case, which cannot be deduced from the general theory (see Corollary from Proposition 3.5, Corollary 2 from Proposition 4.2 and Remark 5.1). In this paper the following basic notions are introduced: action type geometrical equivalence of two representations, action type quasi-identity in representations, action type quasi-variety of representations, action type Noetherian variety of representations, action type geometrically Noetherian representation, action type logically Noetherian representation. Proposition 6.2, and Corollary from Proposition 6.3 provide examples of action type Noetherian variety of representations and action type geometrically Noetherian representations. In Corollary 2 from Theorem 5.1 the approximation-like criterion for two representations to be action type geometrically equivalent is proved. This criterion is similar to the approximation criterion for two algebras to be geometrically equivalent in regular sense ([PPT]). Theorem 6.2 gives a criterion for a representation to be action type logically Noetherian. This criterion is formulated in terms of an action type quasivariety generated by a representation (compare with [Pl4]). In Corollary 2 from Theorem 7.1 we consider a Birkhoff-like description [Bi] of an action type quasi-variety generated by a class of representations. An example of a non action type logically Noetherian representation allows to build an ultrapower of a non action type logically Noetherian representation, which has the same action type quasi-identities but is not action type geometrically equivalent to the original representation (Corollary from Theorem 9.1). This result is parallel to the corresponding theorem for groups [MR].

Introduction

In this paper we consider the action type algebraic geometry of representations of groups. General references for universal algebraic geometry, i.e. the geometry associated with varieties of algebras are [BMR], [MR], [Pl1–Pl4]. First notions in the algebraic geometry of representations of groups were defined in [Pl5]. We outline them in the introduction and consider in detail in the sequel.

We consider only right side modules and throughout the paper "module" means a "right side module". Let K be a commutative ring with unit, G be a group, V be a K-module, and KG be the group ring over the group G. (V, G) is a representation of the group G if V is a KG-module. This is equivalent to the existence of the group homomorphism $\rho : G \to \mathrm{Aut}_K(V)$ and the ring homomorphism $\rho : KG \to \mathrm{End}_K(V)$ (in the paper the homomorphism of groups $\varphi : G \to H$, and the corresponding ring homomorphism $\varphi : KG \to R$, where $R \supseteq KH$ are denoted by the same letter). The multiplication of elements of the module V by elements of G and KG is denoted by $\circ$ and other similar symbols, and is called the action of the group G (ring KG) on elements of the module V. The variety of representations of groups over the fixed commutative ring K we denote $Rep - K$ ([PV]).

The homomorphism of two representations $(\alpha, \beta) : (V, G) \to (W, H)$, is the pair (α, β) of two homomorphisms where $\alpha : V \to W$ is the homomorphism of K-modules and $\beta : G \to H$ is the homomorphism of groups subject to condition $(v \circ g)^\alpha = v^\alpha \circ g^\beta$, for every $v \in V$ and $g \in G$. If (V, G) is a representation, $V_0 \leq V$ is a K-submodule of V, $G_0 \leq G$ is a subgroup of G and V_0 is a KG_0-submodule, then we say that (V_0, G_0) is a subrepresentation of (V, G) (denoted by $(V_0, G_0) \le (V, G)$).

If $(\alpha, \beta) : (V, G) \to (W, H)$ is a homomorphism of representations, $V_0 = \ker \alpha$, $G_0 = \ker \beta$, then we denote $\ker(\alpha, \beta) = (V_0, G_0)$. We have $\ker(\alpha, \beta) \le (V, G)$, $G_0 \trianglelefteq G$, V_0 is a KG-module and G_0 acts trivially on the V/V_0. On the other hand, if $(V_0, G_0) \le (V, G)$ is a subrepresentation, which satisfies the conditions 1) $G_0 \trianglelefteq G$, 2) V_0 is a KG-module and 3) G_0 acts trivially on the V/V_0, then one can define the action of G/G_0 on the V/V_0 by the rule: $(v + V_0) \circ (gG_0) = v \circ g + V_0$ ($v \in V$, $g \in G$). Then $(V/V_0, G/G_0)$ is the representation and the pair of natural homomorphisms $\alpha : V \to V/V_0$, $\beta : G \to G/G_0$ is the homomorphism of representations $(\alpha, \beta) : (V, G) \to (V/V_0, G/G_0)$. Subrepresentation $(V_0, G_0) \le (V, G)$, which satisfies the conditions 1), 2) and 3) is called a normal subrepresentation (denoted by $(V_0, G_0) \trianglelefteq (V, G)$). We denote $(V/V_0, G/G_0) = (V, G) / (V_0, G_0)$.

Free objects in the $Rep - K$ are representations $(XKF(Y), F(Y))$, where $F(Y)$ is the free group with the set of free generators Y, $KF(Y)$ the group ring over this group, and $XKF(Y) = \bigoplus\limits_{x \in X} xKF(Y)$ is the free $KF(Y)$-module with the basis X. We denote the representation $(XKF(Y), F(Y))$ as $W(X, Y)$. Below in this paper we suppose (if we do not say anything else specifically) that X and Y are finite subsets of the countable sets X_0 and Y_0 respectively.

In the variety $Rep - K$ we can consider subvarieties Θ defined simultaneously by a set of identities in acting groups $\{f = 1 \mid f \in F(Y_0)\} = H$ and by a set of identities of the form $\{w = 0 \mid w \in X_0KF(Y_0)\} = A$. Elements of A are identities which describe action of groups on modules. These identities are called action type identities. In other words Θ is the set of representations $(V, G) \in Rep - K$ such that $\forall (\alpha, \beta) \in \mathrm{Hom}((X_0KF(Y_0), F(Y_0)), (V, G))$ holds $(f^\beta = 1) \wedge (w^\alpha = 0)$ $\forall f \in H$, $\forall w \in A$. Subvarieties of this kind are called in [PV] "bivarieties". For every $(V, G) \in \Theta$ we can consider the set of group identities satisfied by (V, G)

$$Id_{gr}(V, G) =$$

$$= \left\{ f \in F(Y_0) \mid \forall (\alpha, \beta) \in \mathrm{Hom}((X_0KF(Y_0), F(Y_0)), (V, G)) \left(f^\beta = 1 \right) \right\},$$

and the set of action-type identities satisfied by (V, G)

$$Id_{a.t.}(V, G) =$$

$$= \{ w \in X_0KF(Y_0) \mid \forall (\alpha, \beta) \in \mathrm{Hom}((X_0KF(Y_0), F(Y_0)), (V, G)) (w^\alpha = 0) \}.$$

$Id_{gr}(V,G) \supseteq H$ and $Id_{a.t.}(V,G) \supseteq A$ hold. Clearly, $(Id_{a.t.}(V,G), Id_{gr}(V,G))$ is the normal subrepresentation of the representation $(X_0KF(Y_0), F(Y_0))$. Denote

$$Id_{gr}\Theta = \bigcap_{(V,G)\in\Theta} Id_{gr}(V,G), Id_{a.t.}\Theta = \bigcap_{(V,G)\in\Theta} Id_{a.t.}(V,G).$$

$(Id_{a.t.}\Theta, Id_{gr}\Theta)$ is also the normal subrepresentation of the $(X_0KF(Y_0), F(Y_0))$ and $Id_{gr}\Theta \supseteq H$, $Id_{a.t.}\Theta \supseteq A$. By [PV], action type identities (elements of $Id_{a.t.}\Theta$) can be reduced to the identities in the cyclic module $\{x\}KF(Y_0) \cong KF(Y_0)$, i.e., to the identities of the form $x \circ u(y_1,\ldots,y_n) = 0$, where $y_1,\ldots,y_n$ are some generators of $F(Y_0)$, and $u \in KF(Y_0)$.

Example 0.1. The identity

$$x \circ (y_1 - 1)(y_2 - 1)\ldots(y_n - 1)$$

defines the n-stable variety of group representations. This variety we denote by $\mathfrak{S}^n$.

Denote

$$Id_{a.t.}\Theta \cap XKF(Y) = Id_{a.t.}(\Theta, X, Y), Id_{gr}\Theta \cap F(Y) = Id_{gr}(\Theta, Y),$$

$$Id(\Theta, X, Y) = (Id_{a.t.}(\Theta, X, Y), Id_{gr}(\Theta, Y))$$

($Id(\Theta, X, Y)$ is a normal subrepresentation of $W(X,Y)$)

$$XKF(Y)/Id_{a.t.}(\Theta, X, Y) = E_\Theta(X,Y),\ F(Y)/Id_{gr}(\Theta, Y) = F_\Theta(Y).$$

Then

$$W_\Theta(X,Y) = W(X,Y)/Id(\Theta,X,Y) = (E_\Theta(X,Y), F_\Theta(Y))$$

is the free representation in the variety of representations Θ (relatively free representation). Below in this paper we suppose (if we do not say anything else specifically) that Θ is a subvariety of $Rep - K$.

Let (V,G) be a fixed representation, such that $(V,G) \in \Theta$. We consider *affine spaces* of finite rank over the (V,G) in the variety Θ. These are the sets $\mathrm{Hom}(W_\Theta(X,Y), (V,G))$.

We have two kinds of equations in the algebraic geometry over representations: equations in the acting group of the form $f = 1$, where $f \in F_\Theta(Y)$, and the action type equations of the form $w = 0$, where $w \in E_\Theta(X,Y)$. Action type equations describe action of the group on a module.

In action type algebraic geometry of representations we consider only action type equations. If $T \subset E_{\Theta}(X,Y)$ is a set of these equations, it defines the algebraic set

$$T_{(V,G)}^{\nabla} = \{(\alpha,\beta) \in \mathrm{Hom}\left(W_{\Theta}\left(X,Y\right),(V,G)\right) \mid \ker\alpha \supset T\}.$$

over a representation (V,G). If $A \subset \mathrm{Hom}\left(W_{\Theta}\left(X,Y\right),(V,G)\right)$ is a set of points of the affine space, then we have the "ideal" of action type equations (in fact the $KF_{\Theta}\left(Y\right)$-submodule in $E_{\Theta}(X,Y)$), defined by the set A:

$$A_{(V,G)}^{\nabla} = \bigcap_{(\alpha,\beta)\in A} \ker\alpha.$$

Now we can consider the action type (V,G)-closure of a set of action type equations $T \subset E_{\Theta}(X,Y)$:

$$T_{(V,G)}^{\nabla\nabla} = \bigcap_{(\alpha,\beta)\in T_{(V,G)}^{\nabla}} \ker\alpha$$

and the action type (V,G)-closure of a set of points $A \subset \mathrm{Hom}(W_{\Theta}(X,Y),$ $(V,G))$:

$$A_{(V,G)}^{\nabla\nabla} = \left\{(\alpha,\beta) \in \mathrm{Hom}\left(W_{\Theta}\left(X,Y\right),(V,G)\right) \mid \ker\alpha \supset A_{(V,G)}^{\nabla}\right\}.$$

Definition 0.1. *Representations $(V_1,G_1),(V_2,G_2) \in \Theta$ are called* **action type geometrically equivalent** *(*denoted $(V_1,G_1) \sim_{a.t.} (V_2,G_2)$*) if $T_{(V_1,G_1)}^{\nabla\nabla} = T_{(V_2,G_2)}^{\nabla\nabla}$ for every X and Y and for every set $T \subset E_{\Theta}(X,Y)$.*

By Proposition 4.3 this definition is correct, i.e., action type geometric equivalence of representations does not depend on subvariety Θ.

Definition 0.2. *The universal logic formula of the form*

$$\left(\bigwedge_{i=1}^{n}(w_i = 0)\right) \Rightarrow (w_0 = 0), \tag{0.1}$$

where $\{w_0,w_1,\ldots,w_n\} \subset XKF\left(Y\right)$, is called an **action type quasi-identity**.

We say that a representation (V,G) is fulfilled on the formula (0.1) and denote:

$$(V,G) \vDash \left(\left(\bigwedge_{i=1}^{n}(w_i = 0)\right) \Rightarrow (w_0 = 0)\right)$$

if $w_0^{\alpha} = 0$ for every $(\alpha,\beta) \in \mathrm{Hom}\left(W\left(X,Y\right),(V,G)\right)$, such that $w_i^{\alpha} = 0$ for every $i \in \{1,\ldots,n\}$.

Also we can consider the **infinite action type "quasi-identity"**:

$$\left(\bigwedge_{i\in I}(w_i = 0)\right) \Rightarrow (w_0 = 0)\,, \tag{0.1'}$$

where $\{w_0\} \cup \{w_i \mid i \in I\} \subset XKF(Y)$, $|I| \geq \aleph_0$. This is not a logic formula in the usual sense, but we can interpret its meaning by the rule: a representation (V, G) satisfies (0.1') if $w_0^\alpha = 0$ for every $(\alpha, \beta) \in \mathrm{Hom}\,(W\,(X, Y)\,, (V, G))$, such that $w_i^\alpha = 0$ for every $i \in I$.

Definitions 0.3. *A representation $(V, G) \in \Theta$ is called* **action type geometrically Noetherian** *if for every sets X and Y and every set $T \subset E_\Theta(X, Y)$, there is a finite set $T_0 \subset T$, such that $(T_0)^\nabla_{(V,G)} = T^\nabla_{(V,G)}$.*

A representation $(V, G) \in \Theta$ is called **action type logically Noetherian** *if for every sets X and Y, every set $T \subset E_\Theta(X, Y)$ and every $w \in T^{\nabla\nabla}_{(V,G)}$, there is a finite set $T_0 \subset T$, such that $w \in (T_0)^{\nabla\nabla}_{(V,G)}$.*

Also by Proposition 4.3, action type geometric Noetherianity and action type logic Noetherianity of representation does not depend on subvariety Θ.

The paper is organized as follows. We start with two auxiliary sections. For the sake of completeness we recall in Section 1 some basic definitions and constructions for modules and representations of groups which will be needed later. In Section 2 we consider operators on classes of algebras. Some of these operators act specifically on classes of representations of groups and can be found in [PV]. In this paper we continue to study the properties of these operators.

In Section 3 we study the basic notions related to algebraic geometry in representations of groups. We distinguish two kinds of equations: equations in the acting group and action type equations. The main concepts of the action type algebraic geometry of representations, i.e. the geometry determined by action type equations, are defined in Section 4. Section 5 deals with the notion of action type geometrical equivalence of representations. In Corollary 2 from Theorem 5.1 an approximation-like criterion for two representations of groups to be action type geometrically equivalent is presented.

The notions of Noetherian variety of algebras and geometrically (logically) Noetherian algebra play an important role in the theory (see, [Pl3–Pl4]). The corresponding notions of action type Noetherian variety of representations and action type geometrically (logically) Noetherian representation are discussed in Section 6. Theorems 6.1 and 6.2 establish relations between geometrical and logical properties of logically Noetherian representations. Two examples are presented: the n-stable variety of representation over the Noetherian ring K is the action type Noetherian variety for every $n \in \mathbb{N}$ (Proposition 6.2), and every finite dimension representation over the field K is action type geometrically Noetherian (Corollary from the Proposition 6.3).

R.Gobel and S. Shelah ([GSh]) proved that there is a non logically Noetherian group. A.Myasnikov and V.Remeslennikov [MR] proved that for every non logically Noetherian group there exists an ultrapower of this group, which, of course, has the same quasi-identities as the original group, but is not geometrically equivalent to the original group. Our target in the three final sections is to prove a similar result in the action type algebraic geometry of representations. In Section 7 we consider action type quasi-varieties of representations, i.e. quasi-varieties of representations, defined by action type quasi-identities. We give a description of the action type quasi-variety generated by a class of representations in terms of operators on classes of representations.

In Section 8 we prove Theorem 8.1: *There exists a continuum of non isomorphic simple modules over KF_2, where F_2 is a free group with 2 generators* (K is a countable field). This theorem is similar to the result of R.Camm [Ca]: there is a continuum of non isomorphic simple 2-generated groups. The latter theorem has been used by R.Gobel and S. Shelah in the construction of a non logically Noetherian group. We use Theorem 8.1 in the Section 9 (Theorem 9.1) for the construction of a non action type logically Noetherian representation. Then we show that there is an ultrapower of this representation which has the same action type quasi-identities as the original representation, but is not action type geometrically equivalent to it.

1. Some basics on modules and representations of groups

For the sake of completeness we will present in this section some well-known basic notions and facts about the representations of groups and modules which we will use later.

A representation (V, G) is finitely generated if G is a finitely generated group and V is a finitely generated KG-module.

The Cartesian product of the family of representations $\{(V_i, G_i) \mid i \in I\}$ ($(V_i, G_i) \in Rep-K$ for every $i \in I$) is the representation $\left(\prod\limits_{i\in I} V_i, \prod\limits_{i\in I} G_i\right)$, with componentwise action.

If $\{A_i \mid i \in I\}$ is a family of sets and $\mathfrak{F}$ is a filter over the set of indices I, then consider equivalence $\sim_{\mathfrak{F}}$ on the set $A = \prod\limits_{i\in I} A_i$: $a_1 \sim_{\mathfrak{F}} a_2$ if $\{i \in I \mid a_1^{\pi_i} = a_2^{\pi_i}\} \in \mathfrak{F}$, where $a_1, a_2 \in A$, $\pi_i : A \to A_i$ are projections. We denote by $[a]_{\sim_{\mathfrak{F}}}$ the equivalence class $\sim_{\mathfrak{F}}$, generated by the element

$a \in A$. The filtered product of the family of sets $\{A_i \mid i \in I\}$ by the filter $\mathfrak{F}$ is the factor set $A/\sim_{\mathfrak{F}} = \left(\prod_{i\in I} A_i\right)/\sim_{\mathfrak{F}}$. If $\{(V_i, G_i) \mid i \in I\}$ is a family of representations and $\mathfrak{F}$ is a filter over the set of indices I, then the filtered product of this family of representations by the filter $\mathfrak{F}$ is the representation

$$\left(\prod_{i\in I}(V_i, G_i)\right)/\sim_{\mathfrak{F}} = \left(\left(\prod_{i\in I} V_i\right)/\sim_{\mathfrak{F}}, \left(\prod_{i\in I} G_i\right)/\sim_{\mathfrak{F}}\right),$$

where action of the group $\left(\prod_{i\in I} G_i\right)/\sim_{\mathfrak{F}}$ on the module $\left(\prod_{i\in I} V_i\right)/\sim_{\mathfrak{F}}$ is defined by $[v]_{\sim_{\mathfrak{F}}} \circ [g]_{\sim_{\mathfrak{F}}} = [v \circ g]_{\sim_{\mathfrak{F}}}$ ($v \in \prod_{i\in I} V_i$, $g \in \prod_{i\in I} G_i$).

The regular representation (KG, G) is defined by: $v \circ g = vg$, where $v \in KG$, $g \in G$. If $U \leq KG_{KG}$ (in this way we denote both a right ideal in a ring and a right submodule in a module) then KG/U is the KG-module and we have the representation $(KG/U, G)$: $v^{\nu} \circ g = (vg)^{\nu}$, where $v \in KG$, $g \in G$, $\nu : KG \to KG/U$ is the natural homomorphism.

Let $\rho : G \to \mathrm{Aut}_K(V)$ be the group homomorphism, which defines a representation (V, G). We denote

$$\ker(V, G) = \ker\rho = \{g \in G \mid \forall v \in V\,(v \circ g = v)\}.$$

The $\ker(V, G)$ is the normal subgroup of G. If $\ker(V, G) = \{1\}$, then the representation is called faithful. Let (V, G) be an arbitrary representation, $\widetilde{G} = G/\ker(V, G)$, $\nu : G \to G/\ker(V, G)$ is the natural homomorphism. We can define the action of $\widetilde{G}$ on module V by $v * g^{\nu} = v \circ g$ ($v \in V$, $g \in G$). The representation $\left(V, \widetilde{G}\right)$ is faithful and called the faithful image of (V, G).

Let R be a ring with unit, M is a R-module, $X \subseteq M$. We can consider the annihilator of the set X: $\mathrm{ann}_R X = \{r \in R \mid \forall x \in X\,(xr = 0)\}$, and the stabilizer of the set X: $\mathrm{stab}_R X = \{r \in R \mid \forall x \in X\,(xr = x)\}$. It is well-known that $\mathrm{ann}_R X$ is the right ideal of R and if X is a submodule of M, then the $\mathrm{ann}_R X$ is a two-sided ideal of R and the $\mathrm{stab}_R X$ is a semigroup of R. It is clear that $\mathrm{stab}_R X = 1 + \mathrm{ann}_R X$. If (V, G) is a representation, $X \subseteq V$, we denote $(\mathrm{stab}_{KG} X) \cap G = \mathrm{stab} X$; by this notation we have $\mathrm{stab} V = \ker(V, G)$.

Proposition 1.1. *If R is a ring with unit, $U \leq R_R$ right ideal of the ring R, then $\mathrm{ann}_R(R/U)$ is the maximal two-sided ideal of the R, contained in U.* ([Pi, 2.1])

Proposition 1.2. *Let K be a commutative ring with unit, G be a group, $U \leq KG_{KG}$, then $\ker(KG/U, G)$ is the maximal normal subgroup of* *the G contained in the group $(1+U)\cap G=\{g\in G\mid g-1\in U\}$.* This proposition is similar to Proposition 1.1.

Corollary. *If $U\leq_{KG} KG_{KG}$ is two-sided ideal of the KG, then*

$$(1+U)\cap G=\{g\in G\mid g-1\in U\}=\ker\left(KG/U,G\right).$$

If $\varphi:S\to R$ is a homomorphism of rings, then over every R-module V_R we can define the structure of an S-module: $v\circ s=vs^{\varphi}$ $(v\in V,\ s\in S)$. We say in this case that S-module V is defined by the homomorphism φ and sometimes it is denoted by $(V)_{\varphi}$.

Proposition 1.3. *If $\varphi:S\to R$ is an epimorphism of rings and $U\leq R_R$ is a right ideal of the ring R, then $R/U\cong S/U^{\varphi^{-1}}$ as S-modules. If $U\leq_R R_R$ a two-sided ideal, then $R/U\cong S/U^{\varphi^{-1}}$ as rings.*

Corollary 1. *Let $\varphi:S\to R$ is an epimorphism of rings, $U\leq R_R$ a right ideal of the ring R. Then*

$$S/\mathrm{ann}_S\left(R/U\right)\cong R/\left(\mathrm{ann}_S\left(R/U\right)\right)^{\varphi}=R/\mathrm{ann}_R\left(R/U\right)$$

as rings.

The epimorphism of groups $\varphi:F\to G$ can be extended to the epimorphism of associative algebras $\varphi:KF\to KG$. So, we have

Corollary 2. *If $F=F\left(X\right)$ is the free n-generated group, G is another n-generated group, $U\leq KG_{KG}$ is a right ideal of the ring KG, then*

$$KF/\mathrm{ann}_{KF}\left(KG/U\right)\cong KG/\mathrm{ann}_{KG}\left(KG/U\right)$$

as associative algebras.

Proposition 1.4. *Let $V_R\cong W_R$ as R-modules, then $\mathrm{ann}_R\left(V_R\right)=\mathrm{ann}_R\left(W_R\right)$.* ([Pi, 2.1])

2. Operators on classes of representations of groups

Let $\mathfrak{X}$ be a class of algebras in some variety Θ (many sorted in general). We consider the following operators on classes of algebras:

$\mathcal{S}$: algebra $H\in\mathcal{S}\mathfrak{X}$ if and only if H is a subalgebra of some algebra $G\in\mathfrak{X}$;

$\mathcal{C}$: algebra $H\in\mathcal{C}\mathfrak{X}$ if and only if H is a Cartesian product of a family of algebras from the class $\mathfrak{X}$;

$\mathcal{F}$: algebra $H\in\mathcal{F}\mathfrak{X}$ if and only if H is a filtered product of a family of algebras $\{G_i\mid i\in I\}$ from the class $\mathfrak{X}$ by an arbitrary filter over the set I;

$\mathcal{C}_{up}$: algebra $H\in\mathcal{C}_{up}\mathfrak{X}$ if and only if H is a filtered product of a family of algebras $\{G_i\mid i\in I\}$ from the class $\mathfrak{X}$ by an arbitrary ultrafilter over the set I;

$\mathcal{L}$: algebra $H \in \mathcal{L}\mathfrak{X}$ if and only if $H_0 \in \mathfrak{X}$ for every finitely generated subalgebra $H_0 \leq H$.

Definitions 2.1. *Let $\mathfrak{X}$ be a class of algebras in some variety Θ.*

If $\mathcal{U}$ is an operator on the classes of algebras, we say that class of algebras $\mathfrak{X}$ is **closed under the operator** $\mathcal{U}$ *if $\mathcal{U}\mathfrak{X} = \mathfrak{X}$.*

An operator $\mathcal{U}$ on the classes of algebras is called **closed operator** *if $\mathcal{U}\mathcal{U}\mathfrak{X} = \mathcal{U}\mathfrak{X}$ for every class of algebras $\mathfrak{X}$.*

An operator $\mathcal{U}$ on the classes of algebras is called **monotone** *if $\mathcal{U}\mathfrak{X}_1 \subset \mathcal{U}\mathfrak{X}_2$ holds when $\mathfrak{X}_1 \subset \mathfrak{X}_2$ ($\mathfrak{X}_1$, $\mathfrak{X}_2$ - classes of algebras of the variety Θ).*

An operator $\mathcal{U}$ on the classes of algebras is called the **operator of extension on** *the fixed* **class** $\mathfrak{X}$ *if $\mathcal{U}\mathfrak{X} \supset \mathfrak{X}$, an operator $\mathcal{U}$ on the classes of algebras is called the* **operator of extension** *if it is an operator of extension on all class of algebras.*

If $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are operators on the classes of algebras and $\mathfrak{X}$ is a class of algebras, we denote by $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}\mathfrak{X}$ the minimal class of algebras which contain the class $\mathfrak{X}$ and closed under all operators $\mathcal{U}_1, \ldots, \mathcal{U}_n$. Of course, $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ will also be the operator on the classes of algebras.

It is clear that operators $\mathcal{L}$, $\mathcal{S}$, $\mathcal{C}$, $\mathcal{F}$, $\mathcal{C}_{up}$ are monotone. Operators $\mathcal{S}$, $\mathcal{C}$ are closed and operators of extension. $\mathcal{F}$ is also an operator of extension, because over the set $\{1\}$ the family of sets $\{\{1\}\}$ is a filter. And $\mathcal{F}$ is a closed operator (see [Ma]).

It is well-known that for every class of algebras $\mathfrak{X}$ fulfills

$$\mathcal{C}\mathcal{S}\mathfrak{X} \subset \mathcal{S}\mathcal{C}\mathfrak{X} \tag{2.1}$$

and

$$\mathcal{F}\mathcal{S}\mathfrak{X} \subset \mathcal{S}\mathcal{F}\mathfrak{X}. \tag{2.2}$$

About operator $\mathcal{L}$ in [PPT, Theorem 3] it was proved that if $\mathfrak{X}$ is a class of algebras, then

$$\mathcal{S}\mathcal{L}\mathfrak{X} = \mathcal{L}\mathfrak{X}; \tag{2.3}$$

$$\mathcal{C}\mathcal{L}\mathfrak{X} \subset \mathcal{L}\mathcal{S}\mathcal{C}\mathfrak{X}; \tag{2.4}$$

$$\text{if } \mathcal{S}\mathfrak{X} = \mathfrak{X}, \text{ then } \mathfrak{X} \subset \mathcal{L}\mathfrak{X} \tag{2.5}$$

and

$$\mathcal{L}\mathcal{L}\mathfrak{X} = \mathcal{L}\mathfrak{X}, \tag{2.6}$$

and it was induced from this that $\{\mathcal{L}, \mathcal{S}, \mathcal{C}\} = \mathcal{L}\mathcal{S}\mathcal{C}$.

Let $\mathfrak{X}$ be a class of representations. On classes of representations we can consider some special operators:

$\mathcal{Q}^r$: a representation $(V, G) \in \mathcal{Q}^r\mathfrak{X}$ if and only if there exists a representation $(V, D) \in \mathfrak{X}$, such that $(id_V, \varphi) : (V, D) \to (V, G)$ is a homomorphism of representations and $\varphi : D \to G$ is an epimorphism ([PV, 1.3]);

$\mathcal{Q}^0$: a representation $(V, G) \in \mathcal{Q}^0\mathfrak{X}$ if and only if there exists a representation $(V, D) \in \mathfrak{X}$, such that $(id_V, \varphi) : (V, G) \to (V, D)$ is a homomorphism of representations and $\varphi : G \to D$ is an epimorphism ([PV, 1.3]);

$\mathcal{S}_r$: a representation $(V, G) \in \mathcal{S}_r\mathfrak{X}$ if and only if $G \leq H$ and $(V, H) \in \mathfrak{X}$.

It is clear that operators $\mathcal{Q}^0$, $\mathcal{Q}^r$, $\mathcal{S}_r$ are monotone, closed and operators of extension.

Now we prove

Lemma 2.1.

$$\mathcal{Q}^r\mathcal{L}\mathfrak{X} \subset \mathcal{L}\mathcal{Q}^r\mathfrak{X}, \tag{2.7}$$

$$\mathcal{Q}^0\mathcal{L}\mathfrak{X} \subset \mathcal{L}\mathcal{Q}^0\mathfrak{X}, \tag{2.8}$$

$$\mathcal{F}\mathcal{Q}^0\mathfrak{X} \subset \mathcal{Q}^0\mathcal{F}\mathfrak{X}, \tag{2.9}$$

$$\mathcal{F}\mathcal{Q}^r\mathfrak{X} \subset \mathcal{Q}^r\mathcal{F}\mathfrak{X}, \tag{2.10}$$

for every class of representations $\mathfrak{X}$.

Proof:

Let $\mathfrak{X}$ be a class of representations.

Let $(V, G) \in \mathcal{Q}^r\mathcal{L}\mathfrak{X}$. Then there exists representation $(V, D) \in \mathcal{L}\mathfrak{X}$ and epimorphism $\varphi : D \to G$, such that (id_V, φ) is the homomorphism of representations. Let (V_0, G_0) be a finitely generated subrepresentation of (V, G) and $G_0 = \langle g_1, \ldots, g_n \rangle$. Let $d_i^\varphi = g_i$ ($d_i \in D$, $1 \leq i \leq n$). Denote $D_0 = \langle d_1, \ldots, d_n \rangle$. $v \circ d = v \circ d^\varphi$ for every $v \in V$, $d \in D$. So $(V_0, D_0) \leq (V, D)$ and (V_0, D_0) is a finitely generated representation, because V_0 is finitely generated KG_0-module. $(V, D) \in \mathcal{L}\mathfrak{X}$, so $(V_0, D_0) \in \mathfrak{X}$. Therefore $(V_0, G_0) \in \mathcal{Q}^r\mathfrak{X}$ and $(V, G) \in \mathcal{L}\mathcal{Q}^r\mathfrak{X}$. (2.7) is proved.

Similarly we can prove (2.8).

Let $(V, G) \in \mathcal{F}\mathcal{Q}^0\mathfrak{X}$. Then

$$(V, G) = \left(\left(\prod_{i \in I} V_i\right) / \sim_{\mathfrak{F}}, \left(\prod_{i \in I} G_i\right) / \sim_{\mathfrak{F}}\right),$$

where $\{(V_i, G_i) \mid i \in I\} \subset \mathcal{Q}^0\mathfrak{X}$. So, there is an epimorphism $\varphi_i : G_i \to D_i$ and a representation (V_i, D_i), such that $(V_i, D_i) \in \mathfrak{X}$ and $(id_{V_i}, \varphi_i) : (V_i, G_i) \to (V_i, D_i)$ is a homomorphism of representations, exist for every $i \in I$. Hence,

$$\left(id_{\prod\limits_{i \in I} V_i}, \varphi\right) : \left(\prod_{i \in I} V_i, \prod_{i \in I} G_i\right) \to \left(\prod_{i \in I} V_i, \prod_{i \in I} D_i\right),$$

where $\varphi = \prod_{i \in I} \varphi_i$, is a homomorphism of representations. Let $\pi_i : \prod_{i \in I} G_i \to G_i$ and $\rho_i : \prod_{i \in I} D_i \to D_i$ be projections. Let $g_1, g_2 \in \prod_{i \in I} G_i$ and $g_1 \sim_{\mathfrak{F}} g_2$, i.e. $\{i \in I \mid g_1^{\pi_i} = g_2^{\pi_i}\} \in \mathfrak{F}$. Then

$$\mathfrak{F} \ni \{i \in I \mid g_1^{\varphi\rho_i} = g_2^{\varphi\rho_i}\} \supset \{i \in I \mid g_1^{\pi_i} = g_2^{\pi_i}\},$$

so $g_1^{\varphi} \sim_{\mathfrak{F}} g_2^{\varphi}$. Then we can define

$$\widetilde{\varphi} : \left(\prod_{i \in I} G_i\right) / \sim_{\mathfrak{F}} \ni [g]_{\sim_{\mathfrak{F}}} \to [g^{\varphi}]_{\sim_{\mathfrak{F}}} \in \left(\prod_{i \in I} D_i\right) / \sim_{\mathfrak{F}} .$$

$\widetilde{\varphi}$ is an epimorphism and $\left(id_{\left(\prod_{i \in I} V_i\right)/\sim_{\mathfrak{F}}}, \widetilde{\varphi} \right)$ is a homomorphism of representations.

$$\left(\left(\prod_{i \in I} V_i\right) / \sim_{\mathfrak{F}}, \left(\prod_{i \in I} D_i\right) / \sim_{\mathfrak{F}} \right) \in \mathcal{F}\mathfrak{X},$$

so

$$(V, G) = \left(\left(\prod_{i \in I} V_i\right) / \sim_{\mathfrak{F}}, \left(\prod_{i \in I} G_i\right) / \sim_{\mathfrak{F}} \right) \in \mathcal{Q}^0\mathcal{F}\mathfrak{X}.$$

(2.9) is proved.

Similarly we can prove (2.10).

Corollary.

$$\left\{\mathcal{Q}^r, \mathcal{Q}^0, \mathcal{L}, \mathcal{S}, \mathcal{C}\right\} = \mathcal{L}\mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{C} \tag{2.11}$$

and

$$\left\{\mathcal{Q}^0, \mathcal{Q}^r, \mathcal{S}, \mathcal{F}\right\} = \mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}. \tag{2.12}$$

Proof:

By the results of [PV, 1.3.2] and by (2.2), (2.9) and (2.10) we have immediately (2.12).

To prove (2.11) we must use (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), and methods of [PPT, Theorem 3].

3. Basic notions of algebraic geometry of representations

Let $(V, G) \in \Theta$ be a fixed representation. Let $\mathrm{Hom}\left(W_{\Theta}(X, Y), (V, G)\right)$ be the affine space of finite rank over the (V, G).

Consider in this affine space the algebraic set

$$A = (T_1, T_2)'_{(V,G)} = \\ = \{(\alpha, \beta) \in \mathrm{Hom}\,(W_\Theta\,(X, Y)\,, (V, G)) \mid \ker \alpha \supset T_1, \ker \beta \supset T_2\}\,,$$

defined by an arbitrary pair (T_1, T_2) of sets of equations ($T_1 \subset E_\Theta(X, Y)$, $T_2 \subset F_\Theta\,(Y)$).

On the other hand, for an arbitrary set of points $A \subset \mathrm{Hom}(W_\Theta\,(X, Y)\,, (V, G))$ in the affine space we have the "ideal of equations" (in our case - normal subrepresentation of the $W_\Theta\,(X, Y)$), defined by this set:

$$T = A'_{(V,G)} = \left(\bigcap_{(\alpha,\beta)\in A} \ker \alpha, \bigcap_{(\alpha,\beta)\in A} \ker \beta \right).$$

Then, the (V, G)-closure of the pair (T_1, T_2) of sets of equations is:

$$(T_1, T_2)''_{(V,G)} = \left(\bigcap_{(\alpha,\beta)\in(T_1,T_2)'_{(V,G)}} \ker \alpha, \bigcap_{(\alpha,\beta)\in(T_1,T_2)'_{(V,G)}} \ker \beta \right).$$

The (V, G)-closure of the set of points $A \subset \mathrm{Hom}\,(W_\Theta\,(X, Y)\,, (V, G))$ is:

$$A''_{(V,G)} = \{(\mu, \nu) \in \mathrm{Hom}(W_\Theta(X, Y), (V, G)) \mid \ker \mu \supset T_1, \ker \nu \supset T_2\}$$

where $(T_1, T_2) = A'_{(V,G)}$.

We say that a pair of sets of equations (S_1, S_2) is contained in a pair of sets of equations (T_1, T_2) ($S_1, T_1 \subset E_\Theta(X, Y)$, $S_2, T_2 \subset F_\Theta\,(Y)$) and denote

$$(S_1, S_2) \subset (T_1, T_2) \tag{3.1}$$

if $S_1 \subset T_1$ and $S_2 \subset T_2$. The correspondence $'$ is the Galois correspondence between sets of points and pairs of sets of equations, that is:

1) $((S_1, S_2) \subset (T_1, T_2) \subset W_\Theta\,(X, Y)) \Rightarrow \left((S_1, S_2)'_{(V,G)} \supset (T_1, T_2)'_{(V,G)}\right)$,

2) $(A \subset B \subset \mathrm{Hom}\,(W_\Theta\,(X, Y)\,, (V, G))) \Rightarrow \left(B'_{(V,G)} \supset A'_{(V,G)}\right)$,

3) $((T_1, T_2) \subset W_\Theta\,(X, Y)) \Rightarrow \left((T_1, T_2) \subset (T_1, T_2)''_{(V,G)}\right)$,

4) $(A \subset \mathrm{Hom}\,(W_\Theta\,(X, Y)\,, (V, G))) \Rightarrow \left(A \subset A''_{(V,G)}\right)$ -

for every X and Y and every representation (V, G).

Definitions 3.1. *We say that a pair of sets of equations* (T_1, T_2) *is* (V, G)**-closed** *if* $(T_1, T_2) = (T_1, T_2)''_{(V,G)}$ *and that a set of points* A *is* (V, G)**-closed** *if* $A = A''_{(V,G)}$.

If the pair of sets of equations (T_1, T_2) is (V, G)-closed, then (T_1, T_2) is the normal subrepresentation of the $W_\Theta(X, Y)$.

As usual:

Proposition 3.1. The (V, G)-closure of a pair of sets (T_1, T_2) is equal to the smallest (V, G)-closed pair containing the pair (T_1, T_2).

By [Pl2, Proposition 3] we have

Proposition 3.2. *Let Θ_1, Θ_2 are a subvariety of $Rep - K$, $(V, G) \in \Theta_1 \subset \Theta_2$. There is a one-to-one order preserving correspondence between lattices of (V, G)-closed subrepresentations in $W_{\Theta_2}(X, Y)$ and in $W_{\Theta_1}(X, Y)$.*

By this proposition we can consider the lattices of (V, G)-closed subrepresentations in the biggest variety of representations: in $Rep - K$.

Quasi-identity in $Rep - K$ can have the forms:

$$\left(\left(\bigwedge_{i=1}^{n_1}(w_i = 0)\right) \wedge \left(\bigwedge_{i=1}^{n_2}(f_i = 1)\right)\right) \Rightarrow (w_0 = 0) \tag{3.2.1}$$

and

$$\left(\left(\bigwedge_{i=1}^{n_1}(w_i = 0)\right) \wedge \left(\bigwedge_{i=1}^{n_2}(f_i = 1)\right)\right) \Rightarrow (f_0 = 1), \tag{3.2.2}$$

where $w_i \in XKF(Y)$ $(0 \leq i \leq n_1)$, $f_i \in F(Y)$ $(0 \leq i \leq n_2)$. We say that a representation (V, G) satisfies (3.2.1) and denote:

$$(V, G) \vDash \left(\left(\left(\bigwedge_{i=1}^{n_1}(w_i = 0)\right) \wedge \left(\bigwedge_{i=1}^{n_2}(f_i = 1)\right)\right) \Rightarrow (w_0 = 0)\right)$$

if $w_0^\alpha = 0$, for every homomorphism $(\alpha, \beta) \in \mathrm{Hom}(W(X, Y), (V, G))$ which satisfies $w_i^\alpha = 0$ for every $i \in \{1, \dots, n_1\}$ and $f_i^\beta = 1$ for every $i \in \{1, \dots, n_2\}$. Similarly, a representation (V, G) satisfies (3.2.2) if $f_0^\beta = 1$, for every $(\alpha, \beta) \in \mathrm{Hom}(W(X, Y), (V, G))$ such that $w_i^\alpha = 0$ for every $i \in \{1, \dots, n_1\}$ and $f_i^\beta = 1$ for every $i \in \{1, \dots, n_2\}$.

Also we can consider the infinite "quasi-identities":

$$\left(\left(\bigwedge_{i \in I_1}(w_i = 0)\right) \wedge \left(\bigwedge_{i \in I_2}(f_i = 1)\right)\right) \Rightarrow (w_0 = 0) \tag{3.2.1'}$$

and

$$\left(\left(\bigwedge_{i \in I_1}(w_i = 0)\right) \wedge \left(\bigwedge_{i \in I_2}(f_i = 1)\right)\right) \Rightarrow (f_0 = 1), \tag{3.2.2'}$$

where $w_i \in XKF(Y)$ $(i \in I_1 \cup \{0\})$, $f_i \in F(Y)$ $(i \in I_2 \cup \{0\})$, I_1, I_2 is not necessary finite. We say that a representation (V, G) satisfies (3.2.1') if $w_0^\alpha = 0$, for every $(\alpha, \beta) \in \mathrm{Hom}(W(X,Y), (V,G))$ such that $w_i^\alpha = 0$ for every $i \in I_1$ and $f_i^\beta = 1$ for every $i \in I_2$. Similarly, a representation (V, G) satisfies (3.2.2') if $f_0^\beta = 1$, for every $(\alpha, \beta) \in \mathrm{Hom}(W(X,Y), (V,G))$, such that $w_i^\alpha = 0$ for every $i \in I_1$ and $f_i^\beta = 1$ for every $i \in I_2$.

Let $\mathfrak{X}$ be a class of representations. Denote by $qId\mathfrak{X}$ the set of quasi-identities satisfied by all representations of this class. Let now $\mathfrak{Q}$ be a set of quasi-identities of representations. Denote by $qVar\mathfrak{Q}$ the class of all representations, which satisfy all quasi-identities from the $\mathfrak{Q}$; this class is called a quasi-variety of representations. The quasi-variety generated by the class $\mathfrak{X}$, i.e. $qVarqId\mathfrak{X}$, is denoted by $qVar\mathfrak{X}$. If $\mathfrak{X} = \{(V,G)\}$ then we denote $qId\mathfrak{X} = qId(V,G)$, $qVar\mathfrak{X} = qVar(V,G)$.

It is easy to see that

Proposition 3.3. *If* $\{w_i \mid i \in I_1\} \cup \{w_0\} \subset XKF(Y)$, $\{f_i \mid i \in I_2\} \cup \{f_0\} \subset F(Y)$, $(T_1, T_2) = (\{w_i \mid i \in I_1\}, \{f_i \mid i \in I_2\})''_{(V,G)}$ *then*

$$(V,G) \vDash \left(\left(\left(\bigwedge_{i \in I_1} (w_i = 0)\right) \wedge \left(\bigwedge_{i \in I_2} (f_i = 1)\right)\right) \Rightarrow (w_0 = 0)\right)$$

if and only if $w_0 \in T_1$ *and*

$$(V,G) \vDash \left(\left(\left(\bigwedge_{i \in I_1} (w_i = 0)\right) \wedge \left(\bigwedge_{i \in I_2} (f_i = 1)\right)\right) \Rightarrow (f_0 = 1)\right)$$

if and only if $f_0 \in T_2$.

Proposition 3.4. $(T_1, T_2) \subset W_\Theta(X,Y)$ *is a* (V,G)*-closed subrepresentation (*$(V,G) \in \Theta$*) if and only if* $W_\Theta(X,Y)/(T_1,T_2) \in SC(V,G)$.

Proof:

We apply the Remak theorem for representations to the representation $W_\Theta(X,Y)/(T_1,T_2)$.

Corollary. *Let* $(V,G), (W,H) \in \Theta$. *Every* (W,H)*-closed representation is a* (V,G)*-closed representation if and only if* $(W,H) \in \mathcal{LSC}(V,G)$.

This corollary we can prove by the method of [Pl2, Proposition 14].

Definition 3.2. *Representations* $(V_1,G_1), (V_2,G_2) \in \Theta$ *are called* **geometrically equivalent** *(*denoted $(V_1,G_1) \sim (V_2,G_2)$*) if* $(T_1,T_2)''_{(V_1,G_1)} = (T_1,T_2)''_{(V_2,G_2)}$ *for every* X *and* Y *and for every pair of sets* $(T_1,T_2) \subset W_\Theta(X,Y)$.

Corollary from Proposition 3.1. $(V_1, G_1) \sim (V_2, G_2)$ *if and only if every* (V_1, G_1)*-closed representation is a* (V_2, G_2)*-closed representation and vice versa.*

This corollary and Proposition 3.2 imply that the definition of geometrical equivalence is correct: if we can state the geometrical equivalence of two representations (V_1, G_1) and (V_2, G_2) in a variety Θ, containing both of them, then these representations are geometrically equivalent in every variety with the same property, in particular, in the biggest variety of representations $Rep - K$. Below in this paper we calculate the geometrical equivalence of representations only in the variety $Rep - K$.

Corollary 1 from Proposition 3.3. *If* $(V_1, G_1) \sim (V_2, G_2)$ *then* $qId(V_1, G_1) = qId(V_2, G_2)$.

Corollary 2 from Proposition 3.3. $(V_1, G_1) \sim (V_2, G_2)$ *if and only if representations* (V_1, G_1) *and* (V_2, G_2) *have same infinite "quasi-identities".*

By [PPT, Theorem 3], we have:

Proposition 3.5. *Let* (V_1, G_1) *and* (V_2, G_2) *be representations. Then* $(V_1, G_1) \sim (V_2, G_2)$ *if and only if* $\mathcal{LSC}(V_1, G_1) = \mathcal{LSC}(V_2, G_2)$.

Corollary. *Let* (V_1, G_1) *and* (V_2, G_2) *be representations. If* $(V_1, G_1) \sim (V_2, G_2)$ *then* $G_1 \sim G_2$ *as groups.*

Proof: Let $G_1^0 \leq G_1$ be a finitely generated subgroup. There is a finitely generated subrepresentation $\left(V_1^0, G_1^0\right) \subset (V_1, G_1)$ and there is an embedding of representations $\left(V_1^0, G_1^0\right) \hookrightarrow (V_2, G_2)^I = \left(V_2^I, G_2^I\right)$ (I - some set of indices). So, there is the embedding of groups: $G_1^0 \hookrightarrow G_2^I$. Therefore $G_1 \in \mathcal{LSC}(G_2)$. By symmetry, $G_2 \in \mathcal{LSC}(G_1)$. By [PPT, Theorem 3], the proof is complete.

A subrepresentation $(V_0, G_0) \leq (V, G)$ is finitely generated as a normal subrepresentation if this is a normal subrepresentation, the group G_0 is finitely generated as a normal subgroup and V_0 is a finitely generated KG_0-module.

Define now the notions of Noetherian variety of representations (subvariety of $Rep-K$), geometrically Noetherian representation and logically Noetherian representation:

Definitions 3.3. *We call a variety* $\Theta \subset Rep - K$ **Noetherian** *if for every* X *and* Y *every normal subrepresentation of* $W_\Theta(X, Y)$ *is finitely generated as a normal subrepresentation.*

A representation $(V, G) \in \Theta$ *is called* **geometrically Noetherian** *if for every sets* X *and* Y *and every pair of sets* $(T_1, T_2) \subset W_\Theta(X, Y)$*, there is a pair of finite subsets* $(R_1, R_2) \subset (T_1, T_2)$*, such that* $(T_1, T_2)'_{(V,G)} = (R_1, R_2)'_{(V,G)}$.

A representation $(V, G) \in \Theta$ *is called* **logically Noetherian** *if for every sets* X *and* Y*, every pair of sets* $(T_1, T_2) \subset W_\Theta(X, Y)$ *and every*

$w \in E_\Theta(X,Y)$ ($f \in F_\Theta(Y)$) belongs to the first (second) component of the pair $(T_1,T_2)''_{(V,G)}$ exists a pair of finite subsets $(R_1,R_2) \subset (T_1,T_2)$, such that w (f) belongs to the first (second) component of the pair $(R_1,R_2)''_{(V,G)}$.

It is clear that Noetherianity of the variety Θ of representations is equivalent to the ascending chain condition for normal subrepresentations in every finitely generated relatively free representation $W_\Theta(X,Y)$ and geometrical Noetherianity of the representation $(V,G) \in \Theta$ is equivalent to the ascending chain condition for (V,G)-closed normal subrepresentations in every $W_\Theta(X,Y)$.

By (3.1) the order on a family of pairs of sets $\{(R,T) \mid (R,T) \subset W(X,Y)\}$ is defined. We can consider directed systems by this order. Also we can consider the union of two pairs of sets: $(R_1,T_1) \cup (R_2,T_2) = (R_1 \cup R_2, T_1 \cup T_2)$. According to [Pl4, Proposition 7]:

Proposition 3.6. *A representation $(V,G) \in \Theta$ is logically Noetherian if and only if the union of any directed system of (V,G)-closed subrepresentations in the $W_\Theta(X,Y)$ for every X and Y is also a (V,G)-closed subrepresentation.*

So, by Proposition 3.2, geometric Noetherianity and logic Noetherianity of representation is not depend in what subvariety Θ we consider those. Also every representation (V,G) from the Noetherian variety Θ is geometrically Noetherian. And every geometrically Noetherian representation is logically Noetherian.

By Proposition 3.3, if a representation (V,G) is logically Noetherian then for every infinite quasi-identity of the form (3.2.1') ((3.2.2')) which is fulfilled in the (V,G) there is a finite quasi-identity with the minor premise and the same conclusion which is also fulfilled in the (V,G). A representation (V,G) is geometrically Noetherian if and only if in this reduction choosing of a premise is not dependent on the conclusion, but only on the infinite premise.

Proposition 3.7. *If a representation (V,G) is geometrically (logically) Noetherian, then the group G is geometrically (logically) Noetherian too.*

Proof:

Let (V,G) be a logically Noetherian representation. Let the group G satisfies the infinite group quasi-identity

$$\left(\bigwedge_{i\in I}(f_i = 1)\right) \Rightarrow (f_0 = 1),$$

where $\{f_i \mid i \in I\} \cup \{f_0\} \subset F(Y)$. This quasi-identity can be considered as a special case of (3.2.2') and can be reduced in (V,G) to the finite quasi-identity

$$\left(\bigwedge_{i\in I_0}(f_i=1)\right)\Rightarrow(f_0=1),$$

where $I_0\subset I$, $|I_0|<\aleph_0$. Every group homomorphism $\beta\in\mathrm{Hom}\left(F\left(Y\right),G\right)$ can be realized as the second component in a homomorphism of representations $(\alpha,\beta)\in\mathrm{Hom}\left(W\left(X,Y\right),(V,G)\right)$, for example, as the second component in the pair $(0,\beta)\in\mathrm{Hom}\left(W\left(X,Y\right),(V,G)\right)$ (for every X). Therefore, G fulfill the quasi-identity

$$\left(\bigwedge_{i\in I_0}(f_i=1)\right)\Rightarrow(f_0=1).$$

Hence, the group G is logically Noetherian.

Analogously, we can prove that if representation (V,G) is geometrically Noetherian, then the group G is geometrically Noetherian too. The proof is complete.

This proposition and the Corollary from Proposition 3.5 show that algebraic geometry over representations of groups in the regular sense, i.e. the algebraic geometry which deals with equations on acting groups and action-type equations, is very closely connected with the algebraic geometry over groups. For example, if a group G is non geometrically (logically) Noetherian, then every representation of this group is non geometrically (logically) Noetherian and this fact does not depend on the action of this group on a module. So, in order to study the geometry which enjoys the peculiarities of the action one has to consider not the "two-sided" geometry above, but the one-sided action-type geometry.

4. Basic notions of action type algebraic geometry of representations

In action type algebraic geometry of representations, we consider algebraic sets in the affine space $\mathrm{Hom}\left(W_\Theta\left(X,Y\right),(V,G)\right)$ defined only by action type equations: $w=0$ - where $w\in E_\Theta(X,Y)$.

We have, as above, the Galois correspondence between sets of "points" and sets of action type equations:

1) $(S\subset T\subset E_\Theta(X,Y))\Rightarrow\left(S^\nabla_{(V,G)}\supset T^\nabla_{(V,G)}\right)$,

2) $(A\subset B\subset\mathrm{Hom}\left(W_\Theta\left(X,Y\right),(V,G)\right))\Rightarrow\left(B^\nabla_{(V,G)}\supset A^\nabla_{(V,G)}\right)$,

3) $(T\subset E_\Theta(X,Y))\Rightarrow\left(T\subset T^{\nabla\nabla}_{(V,G)}\right)$,

4) $(A\subset\mathrm{Hom}\left(W_\Theta\left(X,Y\right),(V,G)\right))\Rightarrow\left(A\subset A^{\nabla\nabla}_{(V,G)}\right)$ -

for every X and Y and every arbitrary representation $(V,G) \in \Theta$.

Definition 4.1. *We say that a set of action type equations T is* **action type** (V,G)**-closed** *if* $T = T^{\nabla\nabla}_{(V,G)}$.

If the set of equations T is action type (V,G)-closed, then T is a $KF_\Theta(Y)$-submodule of the $E_\Theta(X,Y)$.

The Galois correspondence implies

Proposition 4.1. *The action type (V,G)-closure of the set T is equal to the smallest action type (V,G)-closed submodule, containing the set T.*

Proposition 4.2. *Let $(V,G) \in \Theta$. A $KF_\Theta(Y)$-submodule $T \leq E_\Theta(X,Y)$ is an action type (V,G)-closed if and only if there exists a normal subgroup $H \trianglelefteq F_\Theta(Y)$ such that $(T,H) \subset W_\Theta(X,Y)$ is the (V,G)-closed subrepresentation.*

Proof:

It is clear that $(T,H)'_{(V,G)} \subset T^{\nabla}_{(V,G)}$ for every $H \subset F_\Theta(Y)$. So, if $(T,H) = (T,H)''_{(V,G)}$ then $T \supset T^{\nabla\nabla}_{(V,G)} \supset T$. Therefore T is an action type (V,G)-closed submodule.

Let $T = T^{\nabla\nabla}_{(V,G)}$. Denote $\bigcap\limits_{\beta \in \mathrm{Hom}(F_\Theta(Y),G)} \ker\beta = Id_\Theta(G,Y)$. It is clear that $(T, Id_\Theta(G,Y))'_{(V,G)} = T^{\nabla}_{(V,G)}$. Thus, $(T, Id_\Theta(G,Y))''_{(V,G)} = (T, Id_\Theta(G,Y))$ since $(0,\beta) \in T^{\nabla}_{(V,G)}$ for every group homomorphism $\beta : F_\Theta(Y) \to G$. The proof is complete.

Corollary 1. *$T \leq XKF(Y)$ is an action type (V,G)-closed submodule if and only if there exists a normal subgroup $H \trianglelefteq F(Y)$, such that $(XKF(Y)/T, F(Y)/H) \in \mathcal{SC}(V,G)$.*

Proof: By Proposition 3.4.

Remark 4.1. We can see from the proof, that in Proposition 4.2 and its Corollary 1 one can always take $H = Id_\Theta(G,Y)$.

From Proposition 4.2 and Proposition 3.2 we can easy conclude

Proposition 4.3. *Let Θ_1, Θ_2 are a subvariety of $Rep-K$, $(V,G) \in \Theta_1 \subset \Theta_2$. There is a one-to-one order preserving correspondence between lattices of action type (V,G)-closed submodules in $E_{\Theta_2}(X,Y)$ and in $E_{\Theta_1}(X,Y)$.*

By this proposition we can consider the lattices of action type (V,G)-closed submodules in the biggest variety of representations: in $Rep-K$.

We have immediately

Proposition 4.4.

$$(V,G) \vDash \left(\left(\bigwedge_{i \in I} (w_i = 0) \right) \Rightarrow (w_0 = 0) \right),$$

where $\{w_0\} \cup \{w_i \mid i \in I\} \subset XKF(Y)$, if and only if $w_0 \in \{w_i \mid i \in I\}^{\nabla\nabla}_{(V,G)}$.

Definition 4.2. *We say that a quasi-variety of representations* $\mathfrak{X}$ *is an* **action type quasi-variety** *if it can be defined by a set of action type quasi-identities.*

It means that $\mathfrak{X}$ is an action type quasi-variety of representations if and only if there exists a set of action type quasi-identities $\mathfrak{Q}$ such that $\mathfrak{X} = qVar\mathfrak{Q}$.

The set of all action type quasi-identities satisfied by representation (V,G) is denoted by $qId_{a.t.}(V,G)$. Let $\mathfrak{X} \subset Rep{-}K$ be a class of representations. Denote by $qId_{a.t.}\mathfrak{X}$ the set of action type quasi-identities satisfied by all representations from $\mathfrak{X}$. Clearly, $qId_{a.t.}\mathfrak{X} = \bigcap\limits_{(V,G)\in\mathfrak{X}} qId_{a.t.}(V,G)$.

We denote $qVar\left(qId_{a.t.}\mathfrak{X}\right) = qVar_{a.t.}\mathfrak{X}$.

Definition 4.3. *The action type quasi-variety* $qVar_{a.t.}\mathfrak{X}$ *we call* **action type quasi-variety, generated by the class** $\mathfrak{X}$.

5. Action type geometrical equivalence of representations

Corollary from Proposition 4.1. *Let* $(V_1,G_1),(V_2,G_2) \in \Theta$. *Then* $(V_1,G_1) \sim_{a.t.} (V_2,G_2)$ *if and only if for every finite* X *and* Y *every action type* (V_1,G_1)*-closed submodule of* $E(X,Y)$ *is the action type* (V_2,G_2)*-closed submodule and vice versa.*

By this Corollary and by Proposition 4.3, action type geometrical equivalence of representations (V_1,G_1) and (V_2,G_2) can be recognized in all subvariety $\Theta \subseteq Rep - K$, such that $(V_1,G_1),(V_2,G_2) \in \Theta$. Below we use for this purpose the biggest variety of representations: $Rep - K$.

Corollary 1 from Proposition 4.4. *If* $(V_1,G_1) \sim_{a.t.} (V_2,G_2)$ *then* $qId_{a.t.}(V_1,G_1) = qId_{a.t.}(V_2,G_2)$.

Corollary 2 from Proposition 4.4. $(V_1,G_1) \sim_{a.t.} (V_2,G_2)$ *if and only if representations* (V_1,G_1) *and* (V_2,G_2) *have the same infinite action type quasi-identities.*

Also we have

Corollary 2 from Proposition 4.2. *If two representations* (V_1,G_1) *and* (V_2,G_2) *are geometrically equivalent then they are action type geometrically equivalent.*

Remark 5.1. In spite of this Corollary and Corollary from Proposition 3.5, if two representations (V_1,G_1) and (V_2,G_2) are action type geometrically equivalent and groups G_1 and G_2 are geometrically equivalent, the representations (V_1,G_1) and (V_2,G_2) are not necessarily geometrically equivalent.

Definitions 5.1. *Two representations* (V_1,G_1) *and* (V_2,G_2) *are called* **(algebraically) equivalent** *if the corresponding faithful representations are isomorphic.*

A class of representations $\mathfrak{X}$ is called **saturated** *if with a representation $(V,G) \in \mathfrak{X}$ it contains all representations which are algebraically equivalent to the representation (V,G).*

By [PV, 1.3], two representations (V_1,G_1) and (V_2,G_2) are algebraically equivalent if and only if $\mathcal{Q}^0\mathcal{Q}^r(V_1,G_1) = \mathcal{Q}^0\mathcal{Q}^r(V_2,G_2)$, so a class of representations $\mathfrak{X}$ is saturated if and only if $\{\mathcal{Q}^0,\mathcal{Q}^r\}\mathfrak{X} = \mathcal{Q}^0\mathcal{Q}^r\mathfrak{X} = \mathfrak{X}$.

Theorem 5.1. *Let $(Z,H),(V,G) \in Rep-K$. Every action type (Z,H)-closed submodule $T \leq (XKF(Y))_{KF(Y)}$* is an action type *(V,G)-closed submodule if and only if $(Z,H) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G)$.*

Proof:

Let $(Z,H) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G)$. Let $T \leq (XKF(Y))_{KF(Y)}$ be an action type (Z,H)-closed submodule. By Proposition 4.2, there exists a normal subgroup $P \trianglelefteq F(Y)$ such that $(T,P) \subset W(X,Y)$ is the (Z,H)-closed subrepresentation. By Proposition 3.4 and by (2.11) $(XKF(Y)/T,F(Y)/P) \in \mathcal{SC}(Z,H) \subset \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G)$. $(XKF(Y)/T,F(Y)/P)$ is finitely generated, so, by projectivity of the free groups, there exists $S \trianglelefteq F(Y)$ such that $(XKF(Y)/T,F(Y)/S) \in \mathcal{SC}(V,G)$. So, by Proposition 3.4 and Proposition 4.2, T is an action type (V,G)-closed submodule.

Let every action type (Z,H)-closed submodule be an action type (V,G)-closed submodule. Let $(Z_0,H_0) \leq (Z,H)$ be a finitely generated subrepresentation of the (Z,H). $(Z_0,H_0) \cong W(X,Y)/(T,L)$, where (T,L) is a normal subrepresentation of $W(X,Y)$). By Proposition 3.4 and Proposition 4.2, T is an action type (Z,H)-closed submodule and an action type (V,G)-closed submodule. Hence, by Proposition 4.2 and by Proposition 3.4, there exists a normal subrepresentation $(T,D) \leq W(X,Y)$ such that $W(X,Y)/(T,D) \in \mathcal{SC}(V,G)$. Therefore $(Z_0,H_0) \in \mathcal{Q}^0\mathcal{Q}^r\mathcal{SC}(V,G)$ and $(Z,H) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G)$. The proof is complete.

We shall denote $(Z,H) \prec (V,G)$ if and only if $(Z,H) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G)$. By consideration of action type closed submodules we have

Proposition 5.1. *The relation "$\prec$" is the preorder in the $Rep-K$.*

By Corollary from the Proposition 4.1 we have

Corollary 1 from the Theorem 5.1. *Let $(V_1,G_1),(V_2,G_2) \in Rep-K$. $(V_1,G_1) \sim_{a.t.} (V_2,G_2)$ if and only if $(V_1,G_1) \prec (V_2,G_2)$ and $(V_2,G_2) \prec (V_1,G_1)$ i.e., if and only if $(V_1,G_1) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V_2,G_2)$ and $(V_2,G_2) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V_1,G_1)$.*

By (2.11), (2.5) and monotony of operators: $\mathcal{Q}^r$, $\mathcal{Q}^0$, $\mathcal{L}$, $\mathcal{S}$, $\mathcal{C}$ we have

Corollary 2 from the Theorem 5.1. *Let $(V_1,G_1),(V_2,G_2) \in Rep -$*

K. $(V_1, G_1) \sim_{a.t.} (V_2, G_2)$ if and only if $\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V_1, G_1) = \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V_2, G_2)$.

Corollary 3 from the Theorem 5.1. *Let $(V, G) \in Rep - K$. Then*

$$\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, G) \subset qVar_{a.t.}(V, G).$$

Proof:

Let $(Z, H) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, G)$. If

$$(V, G) \vDash \left(\left(\bigwedge_{i \in I} w_i = 0 \right) \Rightarrow (w_0 = 0) \right),$$

then, by Proposition 4.4, $w_0 \in \{w_i \mid i \in I\}^{\nabla\nabla}_{(V,G)}$. Every action type (Z, H)-closed submodule is also an action type (V, G)-closed submodule, therefore $\{w_i \mid i \in I\}^{\nabla\nabla}_{(Z,H)} \supset \{w_i \mid i \in I\}^{\nabla\nabla}_{(V,G)} \ni w_0$. Thus,

$$(Z, H) \vDash \left(\left(\bigwedge_{i \in I} w_i = 0 \right) \Rightarrow (w_0 = 0) \right).$$

The proof is complete.

Corollary 4 from the Theorem 5.1. *If two representations (V_1, G_1), $(V_2, G_2) \in Rep - K$ are equivalent then they are action type geometrically equivalent. In particular, every representation $(V, G) \in Rep - K$ is action type geometrically equivalent to its faithful image $\left(V, \widetilde{G}\right)$.*

Proof:

Let representations (V_1, G_1) and (V_2, G_2) be equivalent; then

$$\mathcal{Q}^0\mathcal{Q}^r(V_1, G_1) = \mathcal{Q}^0\mathcal{Q}^r(V_2, G_2).$$

So

$$\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V_1, G_1) \supset \mathcal{Q}^0\mathcal{Q}^r(V_2, G_2) \ni (V_2, G_2),$$

i.e., $(V_2, G_2) \prec (V_1, G_1)$. By symmetry the proof is complete.

6. Action type Noetherianity of representations

Definitions 6.1. *We call a variety $\Theta \subset Rep - K$ action type Noetherian if for every finite X and Y every $KF_\Theta(Y)$-submodule of $E_\Theta(X, Y)$ is finitely generated.*

It is clear that action type Noetherianity of the variety Θ is equivalent to the ascending chain condition for $KF_\Theta(Y)$-submodules of $E_\Theta(X, Y)$

for every X and Y. Action type geometrical Noetherianity of the representation (V,G) is equivalent to the ascending chain condition for action type (V,G)-closed submodules of $E_\Theta(X,Y)$. So, by Proposition 4.3, every representation (V,G) from the action type Noetherian variety Θ is action type geometrically Noetherian.

Similarly to [Pl4, Proposition 7] one can prove that:

Proposition 6.1. *A representation $(V,G)\in\Theta$ is action type logically Noetherian if and only if the union of any directed system of action type (V,G)-closed submodules of $E_\Theta(X,Y)$ for every X and Y is also an action type (V,G)-closed submodule.*

Hence every action type geometrically Noetherian representation is also action type logically Noetherian.

If (V,G) is an action type logically Noetherian representation, then, by Proposition 4.4, every infinite action type quasi-identity (0.1') can be reduced to the finite action type quasi-identity (0.1).

We shall give some examples of these notions.

Proposition 6.2. *The variety $\mathfrak{S}^n$ of representation over the Noetherian ring K is the action type Noetherian variety for every $n\in\mathbb{N}$.*

Proof:

We denote $\bigoplus\limits_{x\in X} x\left(KF\left(Y\right)/\Delta^n\right) = X\left(KF\left(Y\right)/\Delta^n\right)$, where Δ is the augmentation ideal of the $KF\left(Y\right)$.

$$W_{\mathfrak{S}^n}\left(X,Y\right)=\left(XKF\left(Y\right)/\bigoplus_{x\in X}x\Delta^n,F\left(Y\right)\right)\cong\left(X\left(KF\left(Y\right)/\Delta^n\right),F\left(Y\right)\right).$$

If $|Y|=m$, then, by the Taylor formula for Fox derivation ([Vvs]),

$$w=w^{\varepsilon}+\sum_{i_1=1}^{m}\left(\partial_{i_1}w\right)^{\varepsilon}\left(y_{i_1}-1\right)+\sum_{i_1,i_2=1}^{m}\left(\partial_{i_1i_2}w\right)^{\varepsilon}\left(y_{i_1}-1\right)\left(y_{i_2}-1\right)+\dots$$

$$\dots+\sum_{i_1,\dots,i_{k-1}=1}^{m}\left(\partial_{i_1,\dots,i_{k-1}}w\right)^{\varepsilon}\left(y_{i_1}-1\right)\dots\left(y_{i_{k-1}}-1\right)+$$

$$\sum_{i_1,\dots,i_k=1}^{m}\left(\partial_{i_1,\dots,i_k}w\right)\left(y_{i_1}-1\right)\dots\left(y_{i_k}-1\right)$$

for every $w\in KF\left(Y\right)$ and for every $k\in\mathbb{N}$, where $\varepsilon: KF\left(Y\right)\to K$ is the augmentation homomorphism, $\partial_{i_1,\dots,i_s}: KF\left(Y\right)\to KF\left(Y\right)$ is the s-th Fox derivation by the variables $y_{i_1},\dots,y_{i_s}$ $(1\leq s\leq k)$. So, $KF\left(Y\right)/\Delta^n$ is the finitely generated K-module for every Y. Hence, $X\left(KF\left(Y\right)/\Delta^n\right)$ is the finitely generated K-module for every X and Y.

K is the Noetherian ring, so, every K-submodule and every $KF(Y)$-submodule of $X(KF(Y)/\Delta^n)$ is finitely generated. The proof is complete.

Proposition 6.3. *Every faithful finitely dimension representation (V,G) over the field K is action type geometrically Noetherian.*

This proposition we can prove by using ideas from [BMR, Theorem B1].

Corollary. *Every finite dimension representation (V,G) over the field K is action type geometrically Noetherian.*

Proof: By Corollary 4 from the Theorem 5.1 and Corollary from the Proposition 4.1.

Theorem 6.1. *Let (V_1,G_1) and (V_2,G_2) be action type logically Noetherian representations. Then $(V_1,G_1) \sim_{a.t.} (V_2,G_2)$ if and only if $qId_{a.t.}(V_1,G_1) = qId_{a.t.}(V_2,G_2)$.*

Proof: By Proposition 4.4, Corollary 1 from Proposition 4.4 and Corollary from Proposition 4.1.

Corollary. *In an action type Noetherian variety of representations Θ there is bijection between classes of action type geometrical equivalent representations and action type quasi-varieties generated by one representation.*

Proof:

Let $(V,G) \in \Theta$. We denote by $[(V,G)]$ the class of all representations in Θ which are action type geometrically equivalent to the representation (V,G). It is easy to check that the correspondence $[(V,G)]^{\varphi} = qVar_{a.t.}(V,G)$ is well defined, and bijection.

Proposition 6.4. *Let (V,G) be an action type logically Noetherian representation. Then $qVar_{a.t.}(V,G) \subset \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G)$.*

Proof:

Let $(Z,H) \in qVar_{a.t.}(V,G)$ and $T \subset XKF(Y)$ is the action type (Z,H)-closed submodule, but not action type (V,G)-closed submodule. Let $w \in T^{\nabla\nabla}_{(V,G)} \setminus T$. There is $T_0 = \{w_1, \ldots, w_n\} \subset T$, such that $w \in (T_0)^{\nabla\nabla}_{(V,G)}$. Therefore,

$$(V,G) \vDash \left(\left(\bigwedge_{i=1}^{n} (w_i = 0) \right) \Rightarrow (w = 0) \right)$$

and

$$(Z,H) \vDash \left(\left(\bigwedge_{i=1}^{n} (w_i = 0) \right) \Rightarrow (w = 0) \right).$$

So $w \in (T_0)^{\nabla\nabla}_{(Z,H)} \subset T^{\nabla\nabla}_{(Z,H)}$, but $w \notin T$. By this contradiction, T is the action type (V,G)-closed submodule. By Theorem 5.1, the proof is complete.

Theorem 6.2. $\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G) = qVar_{a.t.}(V,G)$ *if and only if* (V,G) *is an action type logically Noetherian representation.*

Proof:

By Corollary 3 from the Theorem 5.1, we always have

$$\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G) \subset qVar_{a.t.}(V,G).$$

If (V,G) is an action type logically Noetherian representation, then, by Proposition 6.4,

$$\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G) = qVar_{a.t.}(V,G).$$

Let

$$\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G) = qVar_{a.t.}(V,G).$$

Let $\{T_i \mid i \in I\}$ be a direct system of action type (V,G)-closed submodules of $XKF(Y)$ and $T = \bigcup_{i\in I} T_i$.

Let $(V,G) \vDash \mathfrak{q}$ where $\mathfrak{q}$ is an action type quasi-identity. By Proposition 4.2 and Proposition 3.4

$$(XKF(Y)/T_i, F(Y)) \in \mathcal{Q}^0\mathcal{SC}(V,G) \subset qVar_{a.t.}(V,G)$$

for every $i \in I$, so, using the method of [Pl4, Theorem 1], we can prove that $(XKF(Y)/T, F(Y)) \vDash \mathfrak{q}$. Hence

$$(XKF(Y)/T, F(Y)) \in qVar_{a.t.}(V,G) = \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V,G).$$

Consequently, there exists $H \trianglelefteq F(Y)$, such that

$$(XKF(Y)/T, F(Y)/H) \in \mathcal{SC}(V,G).$$

Therefore, by Proposition 3.4 and Proposition 4.2 T is an action type (V,G)-closed submodule. The proof is complete.

7. Action type quasi-varieties of representations

Definition 7.1. *We say that a class of representations $\mathfrak{X}$ is right hereditary if $\mathcal{S}_r\mathfrak{X} = \mathfrak{X}$.*

In [Ma] it was proved that a class of algebras which contains the unit algebra is a quasi-variety if and only if this class is closed under the operators $\mathcal{S}$ and $\mathcal{F}$. Later on in [Gv] this result was established for the case of many sorted algebras. We use this fact in order to describe the action type quasi-varieties of representations. It is clear that every non empty class of representations which is closed under the operators $\mathcal{S}$ contains the unit representation $(\{0\}, \{1\})$, so the non empty class of representations is a quasi-variety if and only if this class is closed under the operators $\mathcal{S}$ and $\mathcal{F}$.

Let $\mathfrak{X}$ be a class of representations. Denote by $\mathfrak{X}_G$ the class of all KG-modules V_{KG}, such that the corresponding representation (V, G) belongs to the class $\mathfrak{X}$.

Lemma 7.1. A class $\mathfrak{X}$ *is a saturated quasi-variety of representations if and only if* $\mathfrak{X}$ *is saturated, right hereditary and* $\mathfrak{X}_G$ *is a quasi-variety of* KG*-modules for every group* G.

Proof:

Let $\mathfrak{X}$ be a saturated quasi-variety. It is clear that $\mathfrak{X}$ is a right hereditary class.

If G is a group and M is a submodule of the KG-module V, then (M, G) is a subrepresentation of the (V, G). So $\mathcal{S}\mathfrak{X}_G \subset \mathfrak{X}_G$.

Let $\{(V_i)_{KG} \mid i \in I\} \subset \mathfrak{X}_G$, $\mathfrak{F}$ be a filter in the I. $(V_i, G) \in \mathfrak{X}$ for every $i \in I$. The filtered product of the family $\{(V_i)_{KG} \mid i \in I\}$ as KG-modules is $\left(\prod_{i\in I} (V_i)_{KG}\right) / \sim_{\mathfrak{F}}$. The filtered product of representations $\{(V_i, G) \mid i \in I\}$ is $\left(\left(\prod_{i\in I} V_i\right) / \sim_{\mathfrak{F}}, \left(G^I\right) / \sim_{\mathfrak{F}}\right)$. The representation $\left(\left(\prod_{i\in I} V_i\right) / \sim_{\mathfrak{F}}, G\right)$ is its subrepresentation, because the diagonal of $\left(G^I\right) / \sim_{\mathfrak{F}}$is isomorphic to G, so $\left(\prod_{i\in I} (V_i)_{KG}\right) / \sim_{\mathfrak{F}} \in \mathfrak{X}_G$. Therefore $\mathcal{F}\mathfrak{X}_G \subset \mathfrak{X}_G$ and, by [Ma], $\mathfrak{X}_G$ is a quasi-variety of KG-modules for every group G.

Let $\mathfrak{X}$ be saturated, right hereditary and $\mathfrak{X}_G$ be a quasi-variety of KG-modules for every group G. Let $(M, H) \leq (V, G)$, $(V, G) \in \mathfrak{X}$. Then $V_{KH} \in \mathfrak{X}_H$ and $M_{KH} \in \mathcal{S}\mathfrak{X}_H = \mathfrak{X}_H$. Therefore $\mathcal{S}\mathfrak{X} = \mathfrak{X}$.

Let $\{(V_i, G_i) \mid i \in I\} \subset \mathfrak{X}$, $\mathfrak{F}$ be a filter over the I. Denote $\prod_{i\in I} G_i = G$. The filtered product of the family of representations $\{(V_i, G_i) \mid i \in I\}$ is $\left(\left(\prod_{i\in I} V_i\right) / \sim_{\mathfrak{F}}, G/ \sim_{\mathfrak{F}}\right)$. Let $\pi_i : G \to G_i$ be projections. Epimorphism π_i defines representation (V_i, G) and (id_{V_i}, π_i) is a homomorphism of representations for every $i \in I$. Hence, $(V_i, G) \in \mathcal{Q}^0\mathfrak{X} \subset \mathfrak{X}$ and $(V_i)_{KG} \in \mathfrak{X}_G$ for every $i \in I$. So $\left(\prod_{i\in I} (V_i)_{KG}\right) / \sim_{\mathfrak{F}} \in \mathfrak{X}_G$ and

$\left(\left(\prod_{i\in I} V_i\right)/\sim_{\mathfrak{F}}, G\right) \in \mathfrak{X}$. So, $\left(\left(\prod_{i\in I} V_i\right)/\sim_{\mathfrak{F}}, G/\sim_{\mathfrak{F}}\right) \in \mathcal{Q}^r\mathfrak{X} \subset \mathfrak{X}$. Therefore $\mathcal{F}\mathfrak{X} \subset \mathfrak{X}$ and, by [Gv], $\mathfrak{X}$ is a quasi-variety. The proof is complete.

Theorem 7.1. *A quasi-variety of representations $\mathfrak{X}$ is an action type quasi-variety of representations if and only if $\mathfrak{X}$ is a saturated quasi-variety.*

Proof:

Let $\mathfrak{X}$ be an action type quasi-variety of representations. Let $(V,G) \in \mathfrak{X}$ and a representation (M,H) is equivalent to the representation (V,G). By Corollary 4 from the Theorem 5.1, $(V,G) \sim_{a.t.} (M,H)$ and, by Corollary 1 from Proposition 4.4, $qId_{a.t.}(V,G) = qId_{a.t.}(M,H)$, so $(M,H) \in \mathfrak{X}$. Therefore $\mathfrak{X}$ is a saturated class of representations.

Let $\mathfrak{X}$ be a saturated quasi-variety of representations. By Lemma 7.1, $\mathfrak{X}_{F(Y_0)}$ is a quasi-variety of $KF(Y_0)$-modules, i.e., $\mathfrak{X}_{F(Y_0)} = qVar\mathfrak{Q}$, where $\mathfrak{Q} = \{\mathfrak{q}_i \mid i \in I\}$,

$$\mathfrak{q}_i \equiv \left(\forall x_1 \ldots \forall x_{n_i}\left(\left(\bigwedge_{j=1}^{k_i}(w_{ij}=0)\right) \Rightarrow (w_{i0}=0)\right)\right), \qquad (7.1)$$

$w_{ij} = w_{ij}(x_1,\ldots,x_{n_i},y_1,\ldots,y_{m_i}) \in X_0KF(Y_0)$. In (7.1) we consider $x_1,x_2,\ldots$ as variables and $y_1,y_2,\ldots$ as constants. But we can consider $y_1,y_2,\ldots$ also as variables. By this point of view, $\widetilde{\mathfrak{q}_i} = \forall y_1\ldots\forall y_{m_i}\mathfrak{q}_i$ is an action type quasi-identity in $Rep-K$ and the set $\widetilde{\mathfrak{Q}} = \{\widetilde{\mathfrak{q}_i} \mid i \in I\}$ will be a set of action type quasi-identities in $Rep-K$. We shall prove that $\mathfrak{X} = qVar\widetilde{\mathfrak{Q}}$.

Let $(V,G) \vDash \widetilde{\mathfrak{Q}}$. Let $G_0 \leq G$ be a finitely generated subgroup of the group G. Also, $(V,G_0) \vDash \widetilde{\mathfrak{Q}}$. There is an epimorphism $\beta : F(Y_0) \to G_0$. Denote by $(V)_\beta$ the $KF(Y_0)$-module defined by the homomorphism β and by $(V,F(Y_0))$ the representation corresponding to this module. Let $\widetilde{\mathfrak{q}_i} \in \widetilde{\mathfrak{Q}}$. The mapping $\alpha : \{x_l \mid l \in X_0\} \to V$ can be extended to the homomorphism of $KF(Y_0)$-modules $\alpha : X_0KF(Y_0) \to (V)_\beta$. It is clear that in this situation the pair (α,β) will be a homomorphism of representations: $(\alpha,\beta) : (X_0KF(Y_0), F(Y_0)) \to (V,G_0)$. The result w_{ij}^α does not depend on point of view on α: as a homomorphism of $KF(Y_0)$-modules or as a left component of homomorphism of representations. So $(V,G_0) \vDash \widetilde{\mathfrak{q}_i}$ if and only if $(V)_\beta \vDash \mathfrak{q}_i$. Hence, $(V)_\beta \vDash \mathfrak{Q}$. Therefore, $(V)_\beta \in \mathfrak{X}_{F(Y_0)}$ and $(V,G_0) \in \mathcal{Q}^r\mathfrak{X} \subset \mathfrak{X}$. $\mathfrak{X}$ is a quasi-variety, so $(V,G) \in \mathfrak{X}$, because all quasi-identities which define $\mathfrak{X}$ are checked in finitely generated representations.

Let $(V,G) \in \mathfrak{X}$. Let $(\alpha,\beta) : (X_0KF(Y_0), F(Y_0)) \to (V,G)$ be a homomorphism of representation. Denote by $(V)_\beta$ the $KF(Y_0)$-module, defined by the homomorphism β, and $(V, F(Y_0))$ - the representation corresponding to this module. We have that $(V, \mathrm{im}\beta) \in \mathfrak{X}$, $(V, F(Y_0)) \in \mathcal{Q}^0\mathfrak{X} \subset \mathfrak{X}$ and $(V)_\beta \in \mathfrak{X}_{F(Y_0)}$. Therefore $(V)_\beta \vDash \mathfrak{Q}$. Because $\alpha : X_0 KF(Y_0) \to (V)_\beta$ is a homomorphism of $KF(Y_0)$-modules, as above, $(V, G) \vDash \widetilde{\mathfrak{Q}}$. The proof is complete.

Corollary 1. *$qVar_{a.t.}\mathfrak{X} = \mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X}$ for every class of representations $\mathfrak{X}$.*

Proof:

$qVar_{a.t.}\mathfrak{X} = qVar(qId_{a.t.}\mathfrak{X})$ is a quasi-variety, so $qVar_{a.t.}\mathfrak{X}$ is closed by $\mathcal{S}$ and $\mathcal{F}$. By Theorem 7.1, $qVar_{a.t.}\mathfrak{X}$ is a saturated class of representations. Thus it is closed under $\mathcal{Q}^0$ and $\mathcal{Q}^r$. $\mathfrak{X} \subset qVar_{a.t.}\mathfrak{X}$ thus, by (2.12),

$$\mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X} \subset qVar_{a.t.}\mathfrak{X}.$$

Also, by (2.12), $\mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X}$ is closed under $\mathcal{S}$, $\mathcal{F}$, $\mathcal{Q}^0$ and $\mathcal{Q}^r$. By Theorem 7.1, $\mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X}$ is an action type quasi-variety of representations, i.e.,

$$\mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X} = qVar\mathfrak{Q},$$

where $\mathfrak{Q}$ is the set of action type quasi-identities. $\mathcal{Q}^0$, $\mathcal{Q}^r$, $\mathcal{S}$, $\mathcal{F}$ are operators of extension, so

$$\mathfrak{X} \subset \mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X} = qVar\mathfrak{Q}.$$

Hence $\mathfrak{X} \vDash \mathfrak{Q}$ and $qId_{a.t.}\mathfrak{X} \supset \mathfrak{Q}$. Therefore

$$qVar_{a.t.}\mathfrak{X} \subset qVar\mathfrak{Q} = \mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{F}\mathfrak{X}.$$

The proof is complete.

Corollary 2. *$qVar_{a.t.}\mathfrak{X} = \mathcal{Q}^0\mathcal{Q}^r\mathcal{S}\mathcal{C}\mathcal{C}_{up}\mathfrak{X}$ for every class of representations $\mathfrak{X}$.*

Proof: By [GL], for every class of algebras $\mathfrak{X}$ we have $\mathcal{F}\mathfrak{X} = \mathcal{C}\mathcal{C}_{up}\mathfrak{X}$.

8. Existing of continuum non isomorphic simple F_2-modules

There is a continuum of non isomorphic simple 2-generated groups ([Ca]). Using this fact, R.Gobel and S. Shelah proved [GSh] that there is a non logically Noetherian group.

K in this and the next section is a countable field such that $charK \neq 2$. We shall prove that there is continuum of non isomorphic simple modules over KF_2, where F_2 is the free group with 2 generators. And we shall deduce from this fact that there is a non action type logically Noetherian representation.

Let Δ be the augmentation ideal of group algebra KG.

Proposition 8.1. *Let G be a non periodic group, i.e. $\exists\, g \in G$, such that $|g| = \infty$. Then the set $\Omega = \{U \lneqq KG_{KG} \mid U \nsubseteq \Delta\}$ is non empty and has a maximal element U_G which is a maximal right ideal in KG.*

Proof:

The element $g + 1$ is not invertible in KG, because $|g| = \infty$. So, $(g + 1)\, KG \lneqq KG_{KG}$. Also $(g + 1)\, KG \nsubseteq \Delta$. So, $\Omega \neq \varnothing$. By Zorn's lemma the set Ω has a maximal element U_G. It is easy to check that U_G is a maximal right ideal in KG.

Corollary. *KG/U_G is a simple KG-module in the situation of* Proposition 8.1.

The ideal U_G, of course, is not uniquely defined by the group G.

Proposition 8.2. *If Γ is a simple non periodic group, then the representation $(K\Gamma/\mathrm{ann}_{K\Gamma}\,(K\Gamma/U_\Gamma)\,, \Gamma)$ is faithful.*

Proof:

$\mathrm{ann}_{K\Gamma}\,(K\Gamma/U_\Gamma) = L_\Gamma \subset U_\Gamma$, by Proposition 1.1. We consider two representations: $(K\Gamma/L_\Gamma, \Gamma)$ and $(K\Gamma/U_\Gamma, \Gamma)$.

$$(1 + U_\Gamma) \cap \Gamma = \{g \in \Gamma \mid g - 1 \in U_\Gamma\} \supset (1 + L_\Gamma) \cap \Gamma.$$

$\{g \in \Gamma \mid g - 1 \in U_\Gamma\} \neq \Gamma$, otherwise $U_\Gamma = \Delta$. By Corollary from Proposition 1.2,

$$(1 + L_\Gamma) \cap \Gamma = \ker\,(K\Gamma/L_\Gamma, \Gamma) \lhd \Gamma$$

Γ is a simple group, so $\ker\,(K\Gamma/L_\Gamma, \Gamma) = \{1\}$. The proof is complete.

Corollary. *The group Γ is embedded into the associative algebra $K\Gamma/\mathrm{ann}_{K\Gamma}\,(K\Gamma/U_\Gamma)$ in the situation of Proposition 8.2.*

Proof:

$$\ker\,(K\Gamma/\mathrm{ann}_{K\Gamma}\,(K\Gamma/U_\Gamma)\,, \Gamma) = \{g \in \Gamma \mid g - 1 \in \mathrm{ann}_{K\Gamma}\,(K\Gamma/U_\Gamma)\} = \{1\}\,.$$

Theorem 8.1. *There exists a continuum of non isomorphic simple modules over KF_2, where F_2 is a free group with 2 generators.*

Proof:

Let $\Re$ be the set of all non isomorphic simple 2-generated groups, considered in [Ca] ($|\Re| = \aleph$). By constructions of [Ca], every $\Gamma \in \Re$ is a non periodic group. So, by Proposition 8.1, we can choose for every $\Gamma \in \Re$ the maximal right ideal in $K\Gamma$: $U_\Gamma < K\Gamma_{K\Gamma}$. It holds that $U_\Gamma \neq \Delta$. $K\Gamma/U_\Gamma$ is the simple $K\Gamma$-module and the simple KF_2-module defined by the natural homomorphism $F_2 \to \Gamma$. After the choosing of U_Γ for every $\Gamma \in \Re$, we define in $\Re$ the equivalence: $\Gamma_1 \approx \Gamma_2$ if $K\Gamma_1/U_{\Gamma_1} \cong K\Gamma_2/U_{\Gamma_2}$ as KF_2-modules ($\Gamma_1, \Gamma_2 \in \Re$).

If $K\Gamma_1/U_{\Gamma_1} \cong K\Gamma_2/U_{\Gamma_2}$ as KF_2-modules, then, by Proposition 1.4,

$$\mathrm{ann}_{KF_2}\left(K\Gamma_1/U_{\Gamma_1}\right) = \mathrm{ann}_{KF_2}\left(K\Gamma_2/U_{\Gamma_2}\right),$$

and by Corollary 2 from Proposition 1.3,

$$K\Gamma_1/\mathrm{ann}_{K\Gamma_1}\left(K\Gamma_1/U_{\Gamma_1}\right) \cong K\Gamma_2/\mathrm{ann}_{K\Gamma_2}\left(K\Gamma_2/U_{\Gamma_2}\right)$$

as associative algebras. By the Corollary from Proposition 8.2

$$\Gamma_1 \hookrightarrow K\Gamma_1/\mathrm{ann}_{K\Gamma_1}\left(K\Gamma_1/U_{\Gamma_1}\right) \cong K\Gamma_2/\mathrm{ann}_{K\Gamma_2}\left(K\Gamma_2/U_{\Gamma_2}\right),$$

so Γ_1 is isomorphic to one of the multiplicative subgroup of the associative algebra $K\Gamma_2/\mathrm{ann}_{K\Gamma_2}\left(K\Gamma_2/U_{\Gamma_2}\right)$. $|K\Gamma_2| = \aleph_0$, so $\left|K\Gamma_2/\mathrm{ann}_{K\Gamma_2}\left(K\Gamma_2/U_{\Gamma_2}\right)\right| \leq \aleph_0$ and there is a countable set of 2-generated subgroups of $K\Gamma_2/\mathrm{ann}_{K\Gamma_2}\left(K\Gamma_2/U_{\Gamma_2}\right)$. Therefore, the cardinality of classes by equivalence "$\approx$" is not bigger than $\aleph_0$. So, there are $\aleph$ classes by equivalence "$\approx$". The proof is complete.

9. Non action type logically Noetherian representation of the group F_2

In this section we shall prove that there is a non action type logically Noetherian representation. Let $\mathcal{P} \subseteq \Re$ be the set of all non isomorphic simple 2-generated groups such that simple KF_2-modules $\{K\Gamma/U_\Gamma \mid \Gamma \in \mathcal{P}\}$ are non isomorphic. By the Theorem 8.1, $|\mathcal{P}| = \aleph$.

If $\varphi_\Gamma : F_2 \to F_2/H = \Gamma \in \mathcal{P}$ is the natural homomorphism of groups, then, by Proposition 1.3, $K\Gamma/U_\Gamma \cong KF_2/U_\Gamma^{\varphi_\Gamma^{-1}}$. Denote $KF_2/U_\Gamma^{\varphi_\Gamma^{-1}} = V_\Gamma$. V_Γ is a simple KF_2-module.

Let $\{V_j \mid j \in J\}$ be the set of all finitely generated right ideals in KF_2. $V = \prod\limits_{j \in J}\left(KF_2/V_j\right)$ is the KF_2-module. So, we can consider the representation (V, F_2). $|KF_2/V_j| = \aleph_0$ for every $j \in J$, $|J| = \aleph_0$, so $|V| = \aleph_0$.

Theorem 9.1. *The representation (V, F_2) is non action type logically Noetherian.*

Proof:

We shall prove that there is $\Gamma \in \mathcal{P}$ such that $U_\Gamma^{\varphi_\Gamma^{-1}} < (KF_2)_{KF_2}$ is not action type (V, F_2)-closed. Let $\Gamma \in \mathcal{P}$ and $U_\Gamma^{\varphi_\Gamma^{-1}}$ be the action type (V, F_2)-closed. By Proposition 4.2 and Proposition 3.4, there exists $H \trianglelefteq F_2$ such that $\left(U_\Gamma^{\varphi_\Gamma^{-1}}, H\right) \trianglelefteq (KF_2, F_2)$ and $(V_\Gamma, F_2/H) \in \mathcal{SC}\,(V, F_2)$. So, there exists a homomorphism of representations $(\iota, \eta) : (V_\Gamma, F_2) \to$

$\left(V^{I}, F_2^{I}\right)$ (I is the set of indices), such that ι is a monomorphism. Since V_Γ is a simple KF_2-module, we can conclude, that there exists an embedding of KF_2-module $V_\Gamma \hookrightarrow (V)_{\widetilde{\eta}}$, where $\widetilde{\eta}$ is an endomorphism of F_2. $|V| = \aleph_0$, $|\mathrm{End}\,(F_2)| = \aleph_0$ (every endomorphism is defined by values on generators). In the module $(V)_{\widetilde{\eta}}$ there is a countable set of simple submodules (every simple submodule is a cyclic, so it is defined by a generator). So only the countable set of modules V_Γ can be embedded into the modules of the kind $(V)_{\widetilde{\eta}}$. Therefore, by Theorem 8.1, there is $\Gamma_0 \in \mathcal{P}$ such that the right side ideal $U_{\Gamma_0}^{\varphi_{\Gamma_0}^{-1}}$ is not action type (V, F_2)-closed.

On the other hand, by Proposition 4.2 and Proposition 3.4, V_j is action type (V, F_2)-closed for every $j \in J$. Therefore,

$$\left\{V_j \mid j \in J, V_j \subseteq U_{\Gamma_0}^{\varphi_{\Gamma_0}^{-1}}\right\} = \{V_j \mid j \in J_0\}$$

is the direct system of action type (V, F_2)-closed modules, which unit $\bigcup\limits_{j \in J_0} V_j = U_{\Gamma_0}^{\varphi_{\Gamma_0}^{-1}}$ is not a action type (V, F_2)-closed module. So, the representation (V, F_2) is non action type logically Noetherian. The proof is complete.

Corollary. *There exists* $\left(\widetilde{V}, \widetilde{F_2}\right)$ *an ultrapower of* (V, F_2) *which is not action type geometrically equivalent to the* (V, F_2).

Proof:

If $\mathcal{C}_{up}(V, F_2) \subset \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, F_2)$ then, by Corollary 2 from Theorem 7.1 and (2.11),

$$qVar_{a.t.}(V, F_2) = \mathcal{Q}^0\mathcal{Q}^r\mathcal{SC}\mathcal{C}_{up}(V, F_2) \subset \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, F_2).$$

So, by the Corollary 3 from Theorem 5.1,

$$qVar_{a.t.}(V, F_2) = \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, F_2)$$

and, by Theorem 6.2, (V, F_2) is action type logically Notherian. By this contradiction, there exists $\left(\widetilde{V}, \widetilde{F_2}\right)$ an ultrapower of (V, F_2), such that

$$\left(\widetilde{V}, \widetilde{F_2}\right) \notin \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, F_2).$$

On the other hand,

$$\left(\widetilde{V}, \widetilde{F_2}\right) \in \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}\left(\widetilde{V}, \widetilde{F_2}\right),$$

so

$$\mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}\left(\widetilde{V}, \widetilde{F_2}\right) \neq \mathcal{LQ}^0\mathcal{Q}^r\mathcal{SC}(V, F_2),$$

and, by Corollary 2 from Theorem 5.1, $\left(\widetilde{V}, \widetilde{F_2}\right) \nsim_{a.t.} (V, F_2)$. The proof is complete.

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