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RESEARCH ARTICLE

Presentations and word problem for strong semilattices of semigroups

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Abstract. Let I be a semilattice, and S_i $(i \in I)$ be a family of disjoint semigroups. Then we prove that the strong semilattice $S = \mathcal{S}[I, S_i, \phi_{j,i}]$ of semigroups S_i with homomorphisms $\phi_{j,i} : S_j \to S_i$ $(j \geq i)$ is finitely presented if and only if I is finite and each S_i $(i \in I)$ is finitely presented. Moreover, for a finite semilattice I, S has a soluble word problem if and only if each S_i $(i \in I)$ has a soluble word problem. Finally, we give an example of non-automatic semigroup which has a soluble word problem.

Introduction

Finite presentability of semigroup constructions has been widely studied in recent years (see, for example, [1, 2, 8, 9, 10]). One of semigroup structures is a strong semilattice of semigroups which is an important structure for completely regular semigroups (see [7]).

Let I be a semilattice, and let S_i $(i \in I)$ be a family of disjoint semigroups. Suppose that, for any two elements $i, j \in I$, with $i \leq j$ ($i \leq j$ if and only if $ij = i$), there is a homomorphism $\phi_{j,i} : S_j \longrightarrow S_i$, and that these homomorphisms satisfy the following conditions:

1. for each $i \in I$, $\phi_{i,i}$ is the identity map 1_{S_i};

2. for all $i, j, k \in I$, such that $i \leq j \leq k$, $\phi_{k,j}\phi_{j,i} = \phi_{k,i}$.

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Define a multiplication on $S = \cup_{i \in I} S_i$ in terms of the multiplications in the components S_i and the homomorphisms $\phi_{j,i}$, by the rule that, for each $x \in S_i$ and $y \in S_j$,

$$xy = (x\phi_{i,ij})(y\phi_{j,ij}).$$

Respect to this multiplication, S is a semigroup, called *a strong semilattice of semigroups*, and denoted by $\mathcal{S}[I; S_i, \phi_{j,i}]$. Finite presentability of strong semilattice of semigroups was investigated in [1]. The authors of [1] first proved that a band of semigroups S_i $(i \in I)$ is finitely presented if I is finite and each S_i is finitely presented (see [1, Corollary 4.5]). Moreover, they proved that a band of monoids S_i $(i \in I)$ is finitely presented if and only if I is finite and each S_i is finitely presented (see [1, Corollary 5.6]). Then, from Theorem 1.3 in [11] they concluded that a strong semilattice of semigroups $\mathcal{S}[I; S_i, \phi_{j,i}]$ is finitely presented if each S_i $(i \in I)$ is finitely presented and if I is finite. In this paper we construct more specific presentations for strong semilattice of semigroups from given presentations for S_i $(i \in I)$ and a presentation for each S_i from a given presentation for $\mathcal{S}[I; S_i, \phi_{j,i}]$. Moreover, if $\mathcal{S}[I; S_i, \phi_{j,i}]$ is finitely generated then we prove that $\mathcal{S}[I; S_i, \phi_{j,i}]$ has a soluble word problem if and only if each S_i has a soluble word problem.

1. The presentations

For a given alphabet A, let A^+ be the free semigroup on A (i.e. the set of all non-empty words over A) and let A^* be the free monoid on A (i.e. A^+ together with the empty word, denoted by ε). A *semigroup presentation* is a pair $\langle A \mid R\rangle$, with $R \subseteq A^+ \times A^+$. A semigroup S is *defined by the presentation* $\langle A \mid R\rangle$ if S is isomorphic to the semigroup A^+/ρ, where ρ is the congruence on A^+ generated by R. For any two words $w_1, w_2 \in A^+$ we write $w_1 \equiv w_2$ if they are identical words and write $w_1 = w_2$ if $w_1\rho = w_2\rho$ (i.e. if they represent the same element in S). It is well known that $w_1 = w_2$ if and only if this relation is a consequence of R, that is there is a finite sequence $w_1 \equiv \eta_1, \eta_2, ..., \eta_k \equiv w_2$ of words from A^+, in which every term η_i $(1 < i \leq k)$ is obtained from the previous one by applying one relation from R. Let $\mathcal{P} = \langle A \mid R\rangle$ be a presentation for S and let $W \subseteq A^+$. If W contains exactly one element representing each element of S then W is called a *canonical form* for S respect to $\mathcal{P}$. The semigroup S is *finitely presented* if S has a presentation $\langle A \mid R\rangle$ such that both A and R are finite. A *(finite) monoid presentation* is defined similarly with respect to the free monoid A^*. Notice that a monoid M is finitely presented as a monoid if and only if it is finitely presented as a semigroup. Furthermore, if $\langle A \mid R\rangle$ is a semigroup presentation for a monoid M then $\langle A \mid R, w = 1\rangle$ is a monoid presentation for M, where $w \in A^+$ is a word representing the identity of M.

Now we construct a presentation for $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ from given presentations $\langle A_i \mid R_i\rangle$ for S_i $(i \in I)$. Since S is a disjoint union of semigroups S_i $(i \in I)$, it is clear that $A = \cup_{i\in I} A_i$ generates S. For simplicity of notation we assume that $A_i \subseteq S_i$.

Consider the following presentation

$$\mathcal{P} = \langle A \mid R,\ a_i a_j = (a_i\phi_{i,ij})(a_j\phi_{j,ij})\ (a_i \in A_i, a_j \in A_j)\rangle$$

where $A = \cup_{i\in I} A_i$ and $R = \cup_{i\in I} R_i$.

Proposition 1. *With above notation if $a \in A_i$ and $u \in A_j^+$ with $i \neq j$, then there exists $v \in A_{ij}^+$ such that the relation $ua = v$ is a consequence of the relations $a_j a = (a_j\phi_{j,ij})(a\phi_{i,ij})$ $(a \in A_i,\ a_j \in A_j)$.*

Proof. We use induction on the length of u. If $\mid u \mid= 1$ then take $v \equiv (u\phi_{j,ij})(a\phi_{i,ij})$. Next we suppose that $u \equiv a_1 \cdots a_n$ $(a_1, \ldots, a_n \in A_j)$ with $n \geq 2$ and that the result is true for any word of length $n - 1$. By the inductive hypothesis, there exists bv_1 with $b \in A_{ij}$ and $v_1 \in A_{ij}^*$ such that the relation $(a_2 \cdots a_n)a = bv_1$ is a consequences of the relations $a_j a = (a_j\phi_{j,ij})(a\phi_{i,ij})$ $(a \in A_i,\ a_j \in A_j)$. Since $jij = ij$ in I and $\phi_{ij,ij}$ is the identity map on S_{ij}, it follows that

$$ua \equiv a_1((a_2 \cdots a_n)a) = (a_1 b)v_1 = ((a_1\phi_{j,jij})(b\phi_{ij,jij})v_1 \equiv ((a_1\phi_{j,ij})b)v_1.$$

Since $(a_1\phi_{j,ij}) \in A_{ij}^+$, taking $v \equiv (a_1\phi_{j,ij})bv_1$ completes the proof. $\square$

Proposition 2. *If $u \equiv a_{i_1} \cdots a_{i_m}$ $(a_{i_1}, \ldots, a_{i_m} \in A$ $)$ and $k = i_1 \cdots i_m$, then there exists $\overline{u} \in A_k^+$ such that the relation $u = \overline{u}$ is a consequence of the relations $a_i a_j = (a_i\phi_{i,ij})(a_j\phi_{j,ij})$ $(a_i \in A_i, a_j \in A_j)$.*

Proof. We use induction on m. If $m = 1$ then we take $u \equiv \overline{u} \equiv a_{i_1}$. Now we suppose that $m \geq 2$ and that the result is true for $m - 1$. Let $k_1 = i_1 \cdots i_{m-1}$. By inductive hypothesis there exists $\overline{u_1} \in A_{k_1}^+$ such that the relation $u_1 \equiv a_{i_1} \cdots a_{i_{m-1}} = \overline{u_1}$ is a consequence of the relations $a_i a_j = (a_i\phi_{i,ij})(a_j\phi_{j,ij})$ $(a_i \in A_i, a_j \in A_j)$. By the previous proposition there exists $\overline{u} \in A_{k_1 i_m}^+$ such that $\overline{u_1} a_{i_m} = \overline{u}$. Since $k_1 i_m = k$, for $\overline{u} \in A_k^+$ we have $u \equiv (a_{i_1} \cdots a_{i_{m-1}})a_{i_m} = \overline{u_1} a_{i_m} = \overline{u}$, as required. $\square$

Theorem 1. *Let $\mathcal{S}[I; S_i, \phi_{j,i}]$ be a strong semilattice of semigroups and let $\mathcal{P}_i = \langle\ A_i \mid R_i\ \rangle$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, be a semigroup presentation for S_i $(i \in I)$. If we take $A = \cup_{i\in I} A_i$ and $R = \cup_{i\in I} R_i$ then*

$$\mathcal{P} = \langle A \mid R,\ a_i a_j = (a_i\phi_{i,ij})(a_j\phi_{j,ij})\ (a_i \in A_i,\ a_j \in A_j \text{ with } i \neq j)\rangle$$

is a presentation for $\mathcal{S}[I; S_i, \phi_{j,i}]$.

Proof. It is clear that A generates $\mathcal{S}[I; S_i, \phi_{j,i}]$ and that all the relations in $\mathcal{P}$ hold in $\mathcal{S}[I; S_i, \phi_{j,i}]$. Notice that since $\phi_{i,i}$ is the identity map on S_i and $i^2 = i$ $(i \in I)$, we can consider $\mathcal{P}$ as $\langle A \mid R,\ a_i a_j = (a_i\phi_{i,ij})(a_j\phi_{j,ij})\ (a_i \in A_i,\ a_j \in A_j)\rangle$ (i.e. without conditions $i \neq j$).

Let u and v be any two words in A^+ such that $u = v$ holds in $\mathcal{S}[I; S_i, \phi_{j,i}]$, that is they represent the same element of $\mathcal{S}[I; S_i, \phi_{j,i}]$. By Proposition 2, there exist $k, l \in I$, $\overline{u} \in A_k^+$ and $\overline{v} \in A_l^+$ such that $u = \overline{u}$ and $v = \overline{v}$ are consequences of the relations $(a_i\phi_{i,ij})(a_j\phi_{j,ij})$ $(a_i \in A_i, a_j \in A_j)$. Since the relation $\overline{u} = \overline{v}$ holds in both S_k and S_l, we have $k = l$ and $\overline{u} = \overline{v}$ as a consequence of R_k. Therefore $u = v$ is a consequence of relations in $\mathcal{P}$, as required. □

Notice that if $W_i \subseteq A_i^+$ $(i \in I)$ is a set of canonical forms for S_i $(i \in I)$ with respect to $\langle\ A_i \mid R_i\ \rangle$, then $W = \cup_{i\in I} W_i$ is a set of canonical forms for $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ with respect to $\mathcal{P}$.

Now we find a presentation for each S_i $(i \in I)$ from a presentation for $S = \mathcal{S}[I; S_i, \phi_{j,i}]$.

Proposition 3. *Let A be a generating set for $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ and $B_j = \{a \in A \mid a \in S_j\}$ $(j \in I)$. Then, for each $i \in I$,*

$$A_i = \bigcup_{j \geq i} \{a_{j,i} \in S_i \mid a_j \in B_j \text{ and } a_j\phi_{j,i} = a_{j,i}\}$$

generates S_i.

Proof. Since $A_i \subset S_i$, it is enough to show that $S_i \subseteq \langle A_i \rangle$.

For $s \in S_i$, there exist $a_{i(1)}, \ldots, a_{i(m)} \in A$ such that $s = a_{i(1)} \cdots a_{i(m)}$. By the multiplication defined on S, we must have $i(1), \ldots, i(m) \geq i$ and there exists at least one $k \in \{1, \ldots, m\}$ such that $i(k) = i$, and so $s = a_{i(1)} \cdots a_{i(m)} = (a_{i(1)}\phi_{i(1),i}) \cdots (a_{i(m)}\phi_{i(m),i}) \equiv a_{i(1),i} \cdots a_{i(m),i} \in \langle A_i \rangle$, as required. □

Notice that $a_i\phi_{i,i} = a_{i,i} \equiv a_i$. Moreover, for $a_j, a'_j \in B_j$, $a_j \neq a'_j$ it is possible to obtain $a_j\phi_{j,i} = a'_j\phi_{j,i}$. For every $a_{j,i} \in A_i$, we fix $a_j \in B_j$ such that $a_j\phi_{j,i} = a_{j,i}$ and denote this fixed a_j by $\overline{a_{j,i}}$.

Let $\langle A \mid R \rangle$ be a finite presentation for $S = \mathcal{S}[I; S_i, \phi_{j,i}]$. Next we construct a presentation for S_i $(i \in I)$ in terms of A_i. Let Φ_i be the unique homomorphism from $\left(\bigcup_{j\geq i} B_j\right)^+$ to A_i^+ such that, for each $a \in B_j$, $j \geq i$, $a\Phi_i = a\phi_{j,i}$, and let $W_i = \left(\bigcup_{j\geq i} B_j\right)^* B_i \left(\bigcup_{j\geq i} B_j\right)^*$. Notice that, for $w \in A^+$, w represent an element of S_i if and only if $w \in W_i$.

Theorem 2. *Let $\mathcal{Q} = \langle A \mid R \rangle$ be a presentation for $S = \mathcal{S}[I; S_i, \phi_{j,i}]$. With above notation,*

$$\mathcal{Q}_i = \langle A_i \mid \{(r\Phi_i, s\Phi_i) \mid (r,s) \in R \cap (\left(\bigcup_{j\geq i} B_j\right)^* \times \left(\bigcup_{j\geq i} B_j\right)^*)\}\rangle$$

is a presentation for S_i $(i \in I)$.

Proof. By Proposition 3, A_i is a generating set for S_i. Let

$$R_i = \{(r\Phi_i, s\Phi_i) \mid (r,s) \in R \cap (\left(\bigcup_{j\geq i} B_j\right)^* \times \left(\bigcup_{j\geq i} B_j\right)^*)\}.$$

It is clear that all the relation in R_i hold in S_i. For

$$u \equiv a_{j(1),i} \cdots a_{j(m),i} \text{ and } v \equiv a_{\lambda(1),i} \cdots a_{\lambda(n),i}$$

where $a_{j(1),i}, \ldots, a_{j(m),i}, a_{\lambda(1),i}, \ldots, a_{\lambda(n),i} \in A_i$, let the relation $u = v$ holds in S_i. To complete the proof we have to show that $u = v$ is a consequence of R_i.

Let $\overline{u}$,$\overline{v} \in A^+$ denote the words A^+ obtained from u and v by replacing $a_{j,i}$ by $\overline{a_{j,i}}$, respectively. It is clear that the relation $\overline{u} = \overline{v}$ holds in S, and so there is a finite sequence $\overline{u} \equiv \alpha_1, \alpha_2, ..., \alpha_m \equiv \overline{v}$ of words from A^+, in which every term α_{k+1} $(1 \leq k < m)$ is obtained from α_k by applying one relation from R. If $\alpha_k \equiv \beta_k r_k \gamma_k$ and $\alpha_{k+1} \equiv \beta_k s_k \gamma_k$ where $\beta_k, \gamma_k \in A^*$ and $(r_k, s_k) \in R$ (or equivalently $(s_k, r_k) \in R$) then β_k, γ_k, r_k and s_k must be in $\left(\bigcup_{j\geq i} B_j\right)^*$. Thus $\alpha_k\Phi_i \equiv (\beta_k\Phi_i)(r_k\Phi_i)(\gamma_k\Phi_i)$ and $\alpha_{k+1}\Phi_i \equiv (\beta_k\Phi_i)(s_k\Phi_i)(\gamma_k\Phi_i)$ where $\beta_k\Phi_i, \gamma_k\Phi_i \in A_i^*$ and $(r_k\Phi_i, s_k\Phi_i) \in R_i$ (or equivalently $(s_k\Phi_i, r_k\Phi_i) \in R_i$), and so we have a finite sequence $u \equiv \alpha_1\Phi_i, \alpha_2\Phi_i, ..., \alpha_m\Phi_i \equiv v$ of words from A_i^+. Hence $u = v$ is a consequence of R_i, as required. □

With above notation notice that if $W_i = \left(\bigcup_{j\geq i} B_j\right)^* B_i \left(\bigcup_{j\geq i} B_j\right)^*$ then, for $w \in A^+$, w represent an element of S_i if and only if $w \in W_i$. Since there exist finitely many B_i, there exist finitely many W_i so that I is finite and every $\mathcal{Q}_i$ $(i \in I)$ in Theorem 2 is a finite presentation whenever $\mathcal{Q}$ is a finite presentation. Moreover, the presentation $\mathcal{P}$ in Theorem 1 is finite if I is finite and every $\mathcal{P}_i$ is finite. Therefore, we have the following corollary.

Corollary 1. *The strong semilattice $\mathcal{S}[I; S_i, \phi_{j,i}]$ of disjoint semigroups S_i $(i \in I)$ is finitely presented if and only if I is finite and every S_i is finitely presented.*

2. Word Problem

A finitely generated semigroup S is said to have a soluble word problem with respect to a finite generating set A if there exists an algorithm which, for any two words u, $v \in A^+$, decides whether or not the relation $u = v$ holds in S (in finite steps). It is easy to see that the solubility of the word problem does not depend on the choice of the finite generating set for a finitely generated semigroup ([11]).

Theorem 3. *Let I be a finite semilattice, and let S_i $(i \in I)$ be a family of disjoint finitely generated semigroups. The strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ has a soluble word problem if and only if, for each $i \in I$, S_i has a soluble word problem.*

Proof. ($\Rightarrow$) Suppose that $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ has a soluble word problem. Let A_i be a finite generating set for S_i. We know that $A = \bigcup_{i \in I} A_i$ is a finite generating set for S. Then S has a soluble word problem with respect to generating set A. Since, for each $i \in I$, if $u, v \in A_i^+$, then $u, v \in A^+$ and there exists an algorithm which decides whether or not $u = v$ holds in S and so in S_i. Thus S_i has a soluble word problem.

($\Leftarrow$) Suppose that, for each $i \in I$, S_i has a soluble word problem and B_i is a finite generating set for S_i. Let $A_i = \bigcup_{j \geq i} B_j \phi_{j,i}$ for each $i \in I$. It is clear that, for each $i \in I$, A_i is a finite generating set for S_i, and moreover, S_i has a soluble word problem with respect to the generating set A_i.

Then we show that $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ has a soluble word problem with respect to the generating set $A = \bigcup_{i \in I} A_i$. Let $u, v \in A^+$ be any two words. Then we have $u \equiv a_{i(1)} \cdots a_{i(m)}$ and $v \equiv a_{j(1)} \cdots a_{j(n)}$ where $a_{i(k)} \in A_{i(k)}$ and $a_{j(l)} \in A_{j(l)}$ $(1 \leq k \leq m,\ 1 \leq l \leq n)$. Take $i = i(1) \cdots i(n)$ and $j = j(1) \cdots j(n)$. Since u represents an element of S_i and v represents an element of S_j, if $i \neq j$, then by the multiplication defined on S, $u = v$ does not hold in S. Now suppose that $i = j$ and take $\overline{u} \equiv (a_{i(1)}\phi_{i(1),i}) \cdots (a_{i(m)}\phi_{i(m),i}) \in A_i^+$ and $\overline{v} \equiv (a_{j(1)}\phi_{j(1),j}) \cdots (a_{j(m)}\phi_{j(m),j}) \in A_i^+$. Since $u = \overline{u}$ and $v = \overline{v}$ holds in S, $u = v$ holds in S if and only if $\overline{u} = \overline{v}$ holds in S_i. Since, for each $i \in I$, S_i has a soluble word problem with respect to A_i, there exists an algorithm which decides whether $\overline{u} = \overline{v}$ holds in S_i in finite steps. Therefore S has a soluble word problem. □

Automatic semigroups were first introduced in [5] and they have been widely studied for semigroup structures, such as direct product of semigroups, Rees matrix semigroups, etc. (see [4, 6]). It is shown that if a semigroup S is automatic then S has a soluble word problem (see [5, Corollary 3.7]). In general the converse of Corollary 3.7 in [5] is not true. For this, consider the free group $G_1 = \langle a, b \mid \ \rangle$ of rank two and the free product $G_2 = \langle c, d \mid c^2 = 1\ d^2 = 1 \rangle$ of two cyclic groups of order two. It is a well-known fact that free groups of finite rank, and finite groups has soluble word problem. In addition, the free product of two groups which have soluble word problems has also soluble word problem. Therefore, G_1 and G_2 have soluble word problems. Moreover the groups G_1 and G_2 are automatic (see [3]).

Let $\phi_{1,2} : G_1 \to G_2$ be the homomorphism defined by $a\phi_{1,2} = c$ and $b\phi_{1,2} = d$, $\phi_{1,1}$ be the identity map of G_1, $\phi_{2,2}$ be the identity map of G_2. Then consider the strong semilattice $G = \mathcal{S}[I; G_i, \phi_{j,i}]$ of the groups G_1 and G_2 where $I = \{1, 2\}$ is the semilattice with the multiplication $i \cdot j = \max\{i, j\}$. It follows from Theorem 3 that the strong semilattice G of the groups G_1 and G_2 has a soluble word problem however it is shown that G is not automatic (see [6]).

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