Algebra and Discrete Mathematics
Number 2. **(2005).** pp. 80 – 89
© Journal "Algebra and Discrete Mathematics"
RESEARCH ARTICLE

On strongly graded Gorestein orders

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Communicated by D. Simson

Abstract. Let G be a finite group and let $\Lambda = \oplus_{g\in G}\Lambda_g$ be a strongly G-graded R-algebra, where R is a commutative ring with unity. We prove that if R is a Dedekind domain with quotient field K, Λ is an R-order in a separable K-algebra such that the algebra Λ_1 is a Gorenstein R-order, then Λ is also a Gorenstein R-order. Moreover, we prove that the induction functor $ind: \ Mod\Lambda_H \rightarrow Mod\Lambda$ defined in Section 3, for a subgroup H of G, commutes with the standard duality functor.

1. Introduction

Throughout this paper,R is a Dedekind domain with quotient field K,G is a finite group and A a finite dimensional separable K-algebra. An R-order Λ in A is a subring of A such that i) the center of Λ contains R, ii) Λ is finitely generated R-module and iii) $K\Lambda = A$. A Λ-lattice is a left Λ-module which is a finitely generated and projective R-module. Let us denote by ${}_\Lambda M$ (resp. M_Λ) a left (resp. right) Λ-module M. An R-order Λ in A is Gorenstein if $({}_\Lambda\Lambda)^* = Hom_R(\Lambda, R)$ is projective as a right Λ-lattice (see [3], p.778).

The importance of Gorenstein orders in integral representation theory appears in the next fact, taking into account that the property of being a Gorenstein order is a local property: Let Λ be a Gorenstein R-order in a separable K-algebra, where R is a complete discrete valuation ring. Then

2000 Mathematics Subject Classification: *16H05, 16G30, 16S35, 16G10, 16W50.*

Key words and phrases: *strongly graded rings, Gorenstein orders, symmetric algebras.*

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every nonprojective indecomposable left Λ-lattice M is a lattice relative to some strictly larger R-order in A ([3], 37.13).

We recall from [8] that, for a group G, a ring Λ is a strongly G-graded ring if $\Lambda = \bigoplus_{g\in G} \Lambda_g$, where each Λ_g is an additive subgroup, and $\Lambda_g\Lambda_h = \Lambda_{gh}$, for every $g, h \in G$. In particular, if Λ_g has a unit for every $g \in G$, then Λ is said to be a crossed product $\Lambda_1 * G$ of Λ_1 by G.

In the Section 2 of this paper we prove that if Λ is a strongly graded R-order in a finite dimensional separable K-algebra A, for a finite group G, and Λ_1 is a Gorenstein order, then Λ is also a Gorenstein order.

In the special case of the group ring, $\Lambda = RG$ is a Gorenstein order in KG ([3], 10.29). In case Λ is the classical crossed product $S * G$, where S is the integral closure of R in a Galois extension L/K with Galois group G, it was proved in [10] that Λ is not only a Gorenstein order, but a symmetric order, that is, $({}_\Lambda\Lambda)^* \cong \Lambda$ as a two-sided Λ-module. This extends the corresponding result for group algebras over a field given in [10], Theorem 1.

The proof of the symmetricity of a classical crossed product order was one of the motivations of this paper. The question that appears is the following: When a strongly G-graded R-algebra Λ, for a finite group G, R a commutative artinian ring and an artinian R-algebra Λ_1, is symmetric? It seems that the question is extremely complicated even in the case of algebras over a field. M.E. Harris has proved in [5] that if Λ_1 is a finite dimensional semisimple algebra over a field K and G is a finite group, then any crossed product $\Lambda_1 * G$ is a symmetric K-algebra. On the other hand E.C. Dade has proved in [4] (see also [7], p.62) that there exists a finite dimensional algebra A over a field K such that A is a crossed product $A_1 * G$ of a K-algebra A_1 by a finite group G, and A is symmetric K-algebra, while A_1 is not symmetric. Continuing this direction, we prove that if Λ is a twisted group ring of Λ_1 by a finite group G, R is a commutative ring and Λ_1 is a symmetric artinian R-algebra, then Λ is a symmetric R-algebra. Moreover we prove that if A is a strongly G-graded K-algebra for a finite group such that A_1 is a Frobenius K-algebra, then A is also Frobenius. This extends the relative result for crossed product algebras given in [7], Ch2, Theorem 2.4.

In the Section 3 of this paper we prove a result on induced modules of strongly graded modules relative to the Hom functor, extending the well-known results for group rings given in [9]. As a consequence, we get another proof of Theorem 2.1 by applying Corollary 3.3.

The reader is referred to [3] for a back-ground material on the representation theory and Gorenstein orders, to [1] and [2] for basic facts on the theory of artin algebras, to [7] for the properties of symmetric algebras, to [6] for basic results on Gorenstein orders and to [9], [11] for

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induced representations.

2. Strongly graded Gorenstein orders

Let R be an artinian commutative ring. If Λ is an artinian R-algebra, let $mod\Lambda$ be the category of finitely generated left Λ-modules and Λ^{op} the opposite ring of Λ. We denote by J the injective envelope of the direct sum of the non-isomorphic simple R-modules. For an object X in $mod\Lambda$, the R-module $Hom_R(X, J)$ becomes a right Λ-module by the rule $(f\lambda)(x) = f(\lambda x)$, for $f \in Hom_R(X, J)$ and $\lambda, x \in \Lambda$. Hence $Hom_R(X, J)$ is a left Λ^{op}-module. Similary, if X is a right Λ-module, then $Hom_R(X, J)$ becomes a left Λ-module. The contravariant R-functor:

$$D : mod\Lambda \to mod\Lambda^{op},\ X \mapsto Hom_R(X, J)$$

is a duality called the standard duality, ([2], II, Theorem 3.3). The R-algebra Λ is said to be symmetric if there exists an isomorphism $\Lambda \cong D(\Lambda)$ of Λ-Λ-bimodules.

We suppose now that

$$\Lambda = \bigoplus_{g \in G} \Lambda_g$$

is a strongly G-graded ring, for a finite group G, and Λ_1 an artinian R-algebra. Then Λ is also an artinian R-algebra. Since Λ is a strongly G-graded ring, from the relation $\Lambda_g\Lambda_{g^{-1}} = \Lambda_1$, for $g \in G$, it follows that are element $a_g^{(i)} \in \Lambda_g$ and $b_{g^{-1}}^{(i)} \in \Lambda_{g^{-1}}$ such that

$$\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)} = 1 \qquad (2.1)$$

for some positive integer n_g depending on $g \in G$. Using the above notation we prove the following result.

Theorem 2.1. *Assume that $\Lambda = \bigoplus_{g \in G} \Lambda_g$ is a strongly G-graded artinian R-algebra such that there exists an isomorphism $\Lambda_1 \cong D(\Lambda_1)$ of left (resp. right) $\Lambda_1 -$modules. Then there exists an isomorphism $\Lambda \cong D(\Lambda)$ of left (resp. right) $\Lambda -$modules.*

Proof. We prove the left part of the proposition because the "right" one is dual. Let $\phi : \Lambda_1 \to D(\Lambda_1)$ be an isomorphism of left Λ_1-modules. We define a map $\psi : \Lambda \to D(\Lambda)$ by the rule

$$\psi(\lambda)(x) = \phi(1)[(\lambda x)_1], \text{ for } \lambda, x \in \Lambda.$$

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Let $\lambda = \sum_{g\in G}\lambda_g$ and $x = \sum_{g\in G} x_g$. Then $(\lambda x)_1 = \sum_{g\in G}\lambda_g x_{g^{-1}}$. It is clear that $\psi(\lambda)$ is an R-homomorphism. We prove that $\psi(\lambda)$ is a Λ-homomorphism of left Λ-modules. Let $\lambda, \mu, x \in \Lambda$, then

$$[\psi(\lambda)\mu](x) = \psi(\lambda)(\mu x) = \phi(1)\left([\lambda(\mu x)]_1\right) = \phi(1)\left([(\lambda\mu)x]_1\right) = \psi(\lambda\mu)(x).$$

First we show that ψ is a monomorphism. For, let $\lambda \neq 0$ be an element of Λ. Then $\lambda_t \neq 0$, for some $t \in G$. We consider the relation on (2.1) for $g = t^{-1}$,

$$\sum_{i\in I} a^{(i)}_{t^{-1}} b^{(i)}_t = 1$$

for a finite index set I depending on t. Since $\lambda_t \neq 0$, it follows that $\lambda_t a^{(j)}_{t^{-1}} \neq 0$ for some $j \in I$. Now since ϕ is an isomorphism we get that $\phi(\lambda_t a^{(j)}_{t^{-1}}) \neq 0$. Hence there exists an element $x_1 \in \Lambda_1$ such that $\phi(\lambda_t a^{(j)}_{t^{-1}})(x_1) \neq 0$. Therefore

$$\psi(\lambda)(a^{(j)}_{t^{-1}} x_1) = \phi(1)\left[(\lambda a^{(j)}_{t^{-1}} x_1)_1\right] = \phi(1)\left(\lambda_t a^{(j)}_{t^{-1}} x_1\right) =$$

$$= \left(\phi(1)\lambda_t a^{(j)}_{t^{-1}}\right)(x_1) = \phi\left(\lambda_t a^{(j)}_{t^{-1}}\right)(x_1) \neq 0.$$

Hence $\psi(\lambda) \neq 0$ and ψ is a monomorphism. Next we prove that ψ is an epimorphism. Since Λ is a finitely generated R-module, it follows that ([2] II 3.1), $l(\Lambda) = l(Hom_R(\Lambda, J))$, where $l(X)$ denotes the length of the Λ-module X. We consider the exact sequence

$$0 \to \Lambda \xrightarrow{\psi} Hom_R(\Lambda, J) \to Coker\psi \to 0.$$

It follows that

$$l(\Lambda) + l(Cocker\psi) = l(D(\Lambda))$$

and hence $l(Cocker(\psi)) = 0$, so $Cocker(\psi) = 0$ and hence ψ is surjective. This completes the proof. □

If R in the above theorem is a field K, then A is a strongly G-graded K-algebra and from the above proposition $A \cong Hom_K(A, K)$ as left A-modules. So extending the relevant result for crossed product algebras ([7], Ch.2, Theorem 2.4) we get the following useful fact.

Corollary 2.2. *Let K be a field, G a finite group and $A = \sum_{g\in G} A_g$ a strongly G-graded K-algebra. If A_1 is a Frobenius K-algebra, then A is also a Frobenius K-algebra.*

Below, we apply again the notation introduced above.

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Proposition 2.3. *Let* $\Lambda = \bigoplus_{g\in G}\Lambda_g$ *be a strongly* G*-graded* R*-algebra for a finite group* G*. Let* Λ_1 *be a symmetric artinian* R*-algebra, with a* Λ_1-Λ_1*-bimodule isomorphism* $\phi : \Lambda_1 \to D(\Lambda_1)$*. The following conditions are equivalent:*
i) $\phi(1)\left[(\mu\lambda)_1\right] = \phi(1)\left[(\lambda\mu)_1\right]$*, for* $\lambda, \mu \in \Lambda$*.*
ii) $\phi(1)(\lambda_g\mu_{g^{-1}}) = \phi(1)(\mu_{g^{-1}}\lambda_g)$*, for* $\lambda_g \in \Lambda_g$*,* $\mu_{g^{-1}} \in \Lambda_{g^{-1}}$*,* $g \in G$*.*
iii) $\phi(1)\left(\sum_{i=1}^{n_g} b_{g^{-1}}^{(i)}\lambda_1 a_g^{(i)}\right) = \phi(1)(\lambda_1)$*, for* $\lambda_1 \in \Lambda_1$ *and* $a_g^{(i)}$*,* $b_{g^{-1}}^{(i)}$*,* n_g *as in the relation (2.1).*

Proof. Let $\phi : \Lambda_1 \to D(\Lambda_1)$ be Λ_1-Λ_1-bimodule isomorphism and $\Lambda = \oplus_{g\in G}\Lambda_g$, a strongly G-graded R-algebra. We remark that, for λ_1, x_1 elements in Λ_1, we have

$$\phi(\lambda_1)(x_1) = \left[\lambda_1\phi(1)\right](x_1) = \phi(1)(x_1\lambda_1)$$

and

$$\phi(\lambda_1)(x_1) = \left[\phi(1)\lambda_1\right](x_1) = \phi(1)(\lambda_1 x_1).$$

Hence

$$\phi(1)(x_1\lambda_1) = \phi(1)(\lambda_1 x_1). \tag{2.2}$$

The implication i)⇒ii) is obvious.
ii)⇒i) Let $\lambda = \sum_{g\in G}\lambda_g$ and $\mu = \sum_{g\in G}\mu_g$ be elements of Λ. Then

$$\phi(1)\left[(\lambda\mu)_1\right] = \phi(1)\left(\sum_{g\in G}\lambda_g\mu_{g^{-1}}\right) = \sum_{g\in G}\phi(1)(\lambda_g\mu_{g^{-1}}) =$$
$$= \sum_{g\in G}\phi(1)(\mu_{g^{-1}}\lambda_g) = \phi(1)\left[(\mu\lambda)_1\right].$$

ii)⇒iii) Let $\lambda_1 \in \Lambda_1$. Then

$$\phi(1)\left(\sum_{i=1}^{n_g} b_{g^{-1}}^{(i)}\lambda_1 a_g^{(i)}\right) = \phi(1)\left(\sum_{i=1}^{n_g} a_g^{(i)} b_{g^{-1}}^{(i)}\lambda_1 =\right)\phi(1)(\lambda_1).$$

iii)⇒ii) Let $\lambda_g \in \Lambda_g$ and $\mu_{g^{-1}} \in \Lambda_{g^{-1}}$. Then using the equations (2.1) and (2.2) we get

$$\phi(1)(\mu_{g^{-1}}\lambda_g) = \phi(1)\left(\sum_{i=1}^{n_g}\mu_{g^{-1}} a_g^{(i)} b_{g^{-1}}^{(i)}\lambda_g\right) =$$
$$= \phi(1)\left(\sum_{i=1}^{n_g} b_{g^{-1}}^{(i)}\lambda_g\mu_{g^{-1}} a_g^{(i)}\right) = \phi(1)(\lambda_g\mu_{g^{-1}}).$$

□

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Remarks 2.4. *i) It is clear that if one of the relations of Proposition 2.3 holds, then the map ψ in the proof of Theorem 2.1 is an isomorphism of Λ-Λ-bimodules and hence Λ is symmetric.*
ii) Suppose that $\Lambda = \Lambda_1 * G$ is a crossed product of Λ_1 by G, that is $\Lambda = \oplus_{g\in G}\Lambda_1\overline{g}$ for some units $\overline{g}$ in Λ. It is clear that the condition iii) of Proposition 2.3 is equivalent to the following one

$$\phi(1)(x^g) = \phi(1)(x),\ x \in \Lambda_1,\ g \in G, \tag{2.3}$$

where $x^g = \overline{g}x\overline{g}^{\,-1}$.

We finish this section by a useful result for twisted group algebras.

Theorem 2.5. *Let Λ be a twisted group ring of the artinian R-algebra Λ_1 by the finite group G. If Λ_1 is a symmetric R-algebra, then Λ is also symmetric.*

Proof. The fact that Λ is a twisted group ring is equivalent to the fact that the action of G on Λ_1 is inner, that is $\overline{g}x\overline{g}^{\,-1} = \varepsilon(g)x\varepsilon(g)^{-1}$, for $x \in \Lambda_1$ and $\varepsilon(g)$ is a unit of Λ_1 depending on g, for $g \in G$ (see [8] Ch.2, Proposition 14). Then the equalities (2.2), (2.3) yield

$$\phi(1)(x^g) = \phi(1)\left(\varepsilon(g)x\varepsilon(g)^{-1}\right) = \phi(1)\left(x\varepsilon(g)\varepsilon(g)^{-1}\right) = \phi(1)(x),$$

since $x\varepsilon(g)^{-1} \in \Lambda_1$. Hence the equality iii) of Proposition (2.3) holds and by Remarks (2.4) i) it follows that Λ is symmetric. □

Now we assume that R is a complete discrete valuation ring with quotient field K and A a finite dimensional separable K-algebra. Let Λ be an R-order in A which is a strongly G-graded ring by a finite group G, such that Λ_1 is an R-order in a separable K-algebra A_1. Then $D(\Lambda)$ (resp. $D(\Lambda_1)$) is $\Lambda^* = Hom_R(\Lambda, R)$ (resp. $\Lambda_1^* = Hom_R(\Lambda_1, R)$) (see [3]) and the proof of Theorem 2.1 also holds for Λ_1^* and Λ^*. Now the condition $\Lambda^* \cong \Lambda$ as right Λ-modules, becomes $\Lambda^* \cong \Lambda$ as right Λ-lattices, which means that Λ^* is projective as right Λ-lattice. In other words, Λ is a Gorenstein order and since the property of being a Gorenstein order is a local property we get the following

Theorem 2.6. *Let R be a Dedekind domain with quotient field K and A a finite dimensional separable K-algebra. Let $\Lambda = \oplus_{g\in G}\Lambda_g$ be a strongly G-graded R-algebra which is an R-order in A and let Λ_1 be an R-order in a separable K-algebra A_1. If Λ_1 is a Gorenstein R-order then Λ is also a Gorenstein R-order.*

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3. The induction functor commutes with the standard duality

Let G be a finite group and $\Lambda = \oplus_{g\in G}\Lambda_g$ a strongly G-graded ring. Let R be a commutative ring with unity such that

$$R \subseteq C_{\Lambda_1}(\Lambda) = \{\lambda \in \Lambda_1 : \ \lambda x = x\lambda, \text{ for every } x \in \Lambda\}.$$

Then Λ is an R-algebra. Let H be a subgroup of G, then $\Lambda_H = \oplus_{h\in H}\Lambda_h$ is a strongly H-graded R-algebra. If V is a left (resp. right) Λ_H-module by V^G we denote the left (resp. right) Λ-module $\Lambda \otimes_{\Lambda_H} V$ (resp. $V \otimes_{\Lambda_H} \Lambda$). We denote by

$$ind : Mod\Lambda_H \longrightarrow Mod\Lambda$$

the induction functor defined by $V \longmapsto V^G = \Lambda \otimes_{\Lambda_H} V$. The first result of this section generalizes a result on induced modules of group rings to strongly graded rings. We use the above notation.

Theorem 3.1. *Let G be a finite group, $H \subseteq G$ a subgroup and $\Lambda = \bigoplus_{g\in G}\Lambda_g$ a strongly G-graded R-algebra. Let V be a left Λ_H-module and B a left R-module.*
i) There is a functorial isomorphism $[Hom_R(V,B)]^G \cong Hom_R(V^G,B)$ of right Λ-modules.
ii) If Λ is a crossed product of Λ_1 over G, then there is a functorial isomorphism
$[Hom_R(B,V)]^G \cong Hom_R(B,V^G)$ of Λ-modules.

Proof. i) We remark that for rings S and R, X a left S-module and Y a left R-module, the group $Hom_R(X,Y)$ is a right S-module via the rule $fs(x) = f(sx)$, for $f \in Hom_R(X,Y)$, $s \in S$ and $x \in X$.

Let now V be a left Λ_H-module and B a left R-module, then $Hom_R(V,B)$ is a right Λ_H-module. We define the map

$$F : [Hom_R(V,B)]^G \to Hom_R(V^G,B)$$

by the formula $x = \sum_{t\in T} f_t \otimes \lambda_{t^{-1}} \mapsto F_x$, where T is a left transversal of H in G, $f_t \in Hom_R(V,R)$, $\lambda_g \in \Lambda_g$ for $g \in G$ and

$$F_x : V^G \to B$$

is defined by the formula $\sum_{t\in T} l_t \otimes v_t \mapsto \sum_{t\in T} f_t(\lambda_{t^{-1}}l_tv_t)$, for $l_t \in \Lambda_t$, $v_t \in V$ and $t \in T$. It is enough to define F on x and then to extent it on $\sum_{t\in T}\sum_{i\in I} f_t^{(i)} \otimes \lambda_{t^{-1}}^{(i)}$ for a finite index set I and similarly for the elements of V^G.

First we prove that F is independent of T. Let T' be another left transversal of H in G. Then for any $t' \in T'$ there exists a unique element h_t of H and a unique $t \in T$ such that $t' = th_t$. From the relations $\Lambda_{t'} = \Lambda_t \Lambda_{h_t}$ and $\Lambda_{t'^{-1}} = \Lambda_{h_t^{-1}} \Lambda_{t^{-1}}$ we get the next expressions of arbitrary elements $l_{t'} \in \Lambda_{t'}$ and $\lambda_{t'^{-1}} \in \Lambda_{t'^{-1}}$

$$l_{t'} = \sum_i l_t^{(i)} l_{h_t}^{(i)} \text{ and } \lambda_{t'^{-1}} = \sum_j \lambda_{h_t^{-1}}^{(j)} \lambda_{t^{-1}}^{(j)}$$

for $l_g^{(i)}$, $\lambda_g^{(j)} \in \Lambda_g$, $g \in G$ and i, j running over finite index sets. Then

$$F(\sum_{t' \in T'} f_{t'} \otimes \lambda_{t'^{-1}})(\sum_{t' \in T'} l_{t'} \otimes v_{t'}) =$$

$$= F(\sum_{t \in T} \sum_j f_{t'} \lambda_{h_t^{-1}}^{(j)} \otimes \lambda_{t^{-1}}^{(j)})(\sum_{t \in T} \sum_i l_t^{(i)} \otimes l_{h_t}^{(i)} v_{t'}) =$$

$$= \sum_{t \in T} \sum_i \sum_j (f_{t'} \lambda_{h_t^{-1}}^{(j)})(\lambda_{t^{-1}}^{(j)} l_t^{(i)} l_{h_t}^{(i)} v_{t'}) = \sum_{t \in T} f_{t'}(\lambda_{t'^{-1}} l_{t'} v_{t'}).$$

It is easy to see that F_x is an R-homomorphism, since $R \subseteq C_{\Lambda_1}(\Lambda)$. The map F is a monomorphism. Indeed; let $F(\sum_{t \in T} f_t \otimes \lambda_{t^{-1}}) = 0$, then for $a_g \in \Lambda_g$, $g \in G$ and $v \in V$ we conclude that the equality

$$F(\sum_{t \in T} f_t \otimes \lambda_{t^{-1}})(a_g \otimes v) = 0$$

implies the equalities $f_g \lambda_g a_g = 0$ for $g \in G$, $a_g \in \Lambda_g$. Hence we conclude, by taking $a_t^{(i)}$ as in the equality (2.1), the equalities

$$\sum_{t \in T} f_t \otimes \lambda_{t^{-1}} = \sum_{t \in T} \sum_i f_t \lambda_{t^{-1}} a_t^{(i)} \otimes b_{t^{-1}}^{(i)} = 0.$$

Now we prove that the map F is surjective. If $f \in Hom_R(V^G, B)$ we define $f_t^{(i)} : V \to B$ by $f_t^{(i)}(v) = f(a_t^{(i)} \otimes v)$ for $a_t^{(i)}$ as in (2.1). Since f is an R-homomorphism then for $r \in R$, $v \in V$ we get

$$f_t^{(i)}(rv) = f(a_t^{(i)} \otimes rv) = rf(a_t^{(i)} \otimes v) = rf_t^{(i)}(v).$$

Now we prove that $f = F(\sum_{t \in T} \sum_i f_t^{(i)} \otimes b_{t^{-1}}^{(i)})$. We remark that for $\sum_{t \in T} l_t \otimes v_t \in V^G$ we have

$$F(\sum_{t \in T} \sum_i f_t^{(i)} \otimes b_{t^{-1}}^{(i)})(\sum_{t \in T} l_t \otimes v_t) = \sum_{t \in T} \sum_i f_t^{(i)}(b_{t^{-1}}^{(i)} l_t v_t) =$$

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$$\sum_{t\in T}\sum_i f(a_t^{(i)}\otimes b_{t-1}^{(i)}l_tv_t)=\sum_{t\in T}\sum_i f(a_t^{(i)}b_{t-1}^{(i)}l_t\otimes v_t)=f(\sum_{t\in T}l_t\otimes v_t).$$

Finally we prove that the map F is a Λ-homomorphism. Let $g\in G$, $\lambda_g\in\Lambda_g$ and $K=\{g^{-1}t:\ t\in T\}$ another left transversal of H in G. If $k\in K$, $t\in T$, $l_k\in\Lambda_k$ and $v\in V$, it is enough to prove that

$$F\left[(f_t\otimes\lambda_{t^{-1}})\lambda_g\right](l_k\otimes v)=\left[F(f_t\otimes\lambda_{t^{-1}})\lambda_g\right](l_k\otimes v).$$

The left hand term of this equality is equal to zero, if $k\neq gt^{-1}$, and is equal to $f_t(\lambda_{t^{-1}}\lambda_g l_k v)$, if $k=gt^{-1}$. For the right hand term we have:

$$F(f_t\otimes\lambda_{t^{-1}})(\lambda_g l_k\otimes v)=\begin{cases}0, & \text{if}\quad gk\neq t\\ f_t(\lambda_{t^{-1}}\lambda_g l_k v), & \text{if}\quad gk=t.\end{cases}$$

It follows that F is a Λ-homomorphism and the statement i) is proved.

ii) Let $\Lambda=\sum_{t\in T}\bar{t}\Lambda_H$ be a crossed product. We consider the map

$$F:[Hom_R(B,V)]^G\to Hom_R(B,V^G)$$

defined by $\bar{t}\otimes\varphi_t\mapsto\varphi_t^*$, where $\varphi_t^*:B\to\Lambda\otimes_{\Lambda_H}V$, is defined by $b\mapsto\bar{t}\otimes\varphi_t(b)$, for $t\in T$ and $b\in B$. Similarly as in i), we prove that F has the required properties. □

Corollary 3.2. *Let G be an arbitrary group and H a subgroup of G of finite index, V a left Λ_H-module and W a right Λ-module. Then there exist isomorphism*

$$(D(V))^G\cong D(V^G)\ \ \textit{and}\ \ (V^*)^G\cong(V^G)^*$$

of right Λ-modules.

Corollary 3.3. *Let G be a finite group. If there exists an isomorphism $D(\Lambda_1)\cong\Lambda_1$ of right Λ_1-modules then there exists an isomorphism $D(\Lambda)\cong\Lambda$ of right Λ-modules.*

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Received by the editors: 28.05.2004
and final form in 06.07.2005.

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