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RESEARCH ARTICLE

Extended G-vertex colored partition algebras as centralizer algebras of symmetric groups

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Abstract. The Partition algebras $P_k(x)$ have been defined in [M1] and [Jo]. We introduce a new class of algebras for every group G called "Extended G-Vertex Colored Partition Algebras," denoted by $\widehat{P}_k(x, G)$, which contain partition algebras $P_k(x)$, as subalgebras. We generalized Jones result by showing that for a finite group G, the algebra $\widehat{P}_k(n, G)$ is the centralizer algebra of an action of the symmetric group S_n on tensor space $W^{\otimes k}$, where $W = \mathbb{C}^{n|G|}$. Further we show that these algebras $\widehat{P}_k(x, G)$ contain as subalgebras the "G-Vertex Colored Partition Algebras $P_k(x, G)$," introduced in [PK1].

1. Introduction

In 1937, Brauer [Br] analyzed Schur-Weyl duality for the orthogonal groups O_n and gave a combinatorial description of their centralizer algebras

$$\mathrm{End}_{O_n}(V^{\otimes k}) = \{\alpha \in \mathrm{End}(V^{\otimes k}) \mid \alpha\beta = \beta\alpha \text{ for } \beta \in O_n\}, \tag{1.1}$$

on tensor space, in relation to the decomposition of $V^{\otimes k}$ into irreducible representations of O_n, where $V = \mathbb{C}^n$. He defined the Brauer algebra $B_k(n)$ and showed that $\mathrm{End}_{O_n}(V^{\otimes k})$ is always a quotient of $B_k(n)$ (see [Br]). The signed Brauer algebra, which is a coloring of the Brauer algebra, was introduced in [PK] and has been realized as the centralizer

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algebra of direct product of two orthogonal groups (see [PS]). The study of such algebras is interesting, for the span of signed Brauer diagrams having only vertical edges is isomorphic to the group algebra of the hyperoctahedral group whereas in the case of Brauer algebra, the span of Brauer diagrams having only vertical edges is isomorphic to the group algebra of the symmetric group.

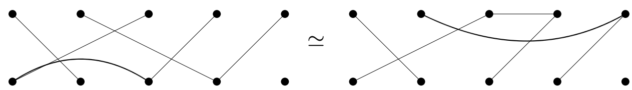
The partition algebras $P_k(x)$ have been studied independently by Martin and Jones as a generalization of the Temperley-Lieb algebras and the Potts model in statistical mechanics. The algebras appear implicitly in [M1; M2, Chap.3] and explicitly in [M3]. In 1993, Jones considered $P_k(n)$, as the centralizer algebra of the symmetric group S_n on $V^{\otimes k}$ (see [Jo]).

The G-edge colored partition algebra $\overline{P_k(x,G)}$, introduced in [Bl] by Bloss, has a basis consisting of partition diagrams with oriented edges, where edges are labelled by the group elements of G, whereas in the case of G-vertex colored partition algebra $P_k(x,G)$, introduced in [PK1], has a basis consisting of partition diagrams where vertices are labelled by the group elements of G. This basis in $P_k(x,G)$, gives a natural embedding of the algebra $P_k(n,G)$ into the centralizer algebra $\text{End}_{G\times S_n}(W^{\otimes k})$, where $W=\mathbb{C}^{n|G|}$. Moreover, it is easier to work in the G-vertex colored partition algebra $P_k(x,G)$ than in the G-edge colored partition algebra $\overline{P_k(x,G)}$. We are interested in a generalization of Jones's result. That is, we are interested in finding a suitable colored partition algebra which can be realized as the centralizer of suitable symmetric group acting on a vector space. In this paper we introduce a new class of algebras $\widehat{P}_k(x,G)$ which are called "Extended G-Vertex Colored Partition Algebras," and which contain as subalgebras the "G-Vertex Colored Partition Algebras $P_k(x,G)$" and "G-Edge Colored Partition Algebras $\overline{P_k(x,G)}$." We show that the algebra $\widehat{P}_k(n,G)$ is the centralizer algebra of the symmetric group S_n on $W^{\otimes k}$.

2. Preliminaries

2.1. The structure of $P_k(x)$

A k-partition diagram is a simple graph on two rows of k-vertices, one above the other. The connected components of such a graph partition the $2k$ vertices into l disjoint subsets with $1\le l\le 2k$. We say that two k-partition diagrams are equivalent if they give rise to the same partition of the $2k$ vertices. For example, the following are equivalent 5-diagrams.

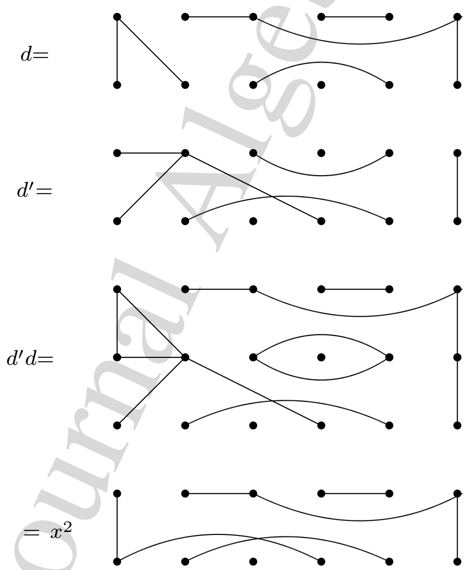


When we speak of diagrams, we are really talking about the equivalence classes of k-partition diagrams. Number the vertices of a k-diagram $1, 2, \ldots, k$ from left to right in the top row, and $k+1, k+2, \ldots, 2k$ from left to right in the bottom row.

Let x be an indeterminate. The multiplication of two k-partition diagrams d and d' is defined as follows:

- Place d on the top and d' at the bottom.
- Identify the $(k+j)^{\text{th}}$ vertex of d with the j^{th} vertex of d'. The partition diagram now has a top row, a bottom row, and a middle row of vertices.
- Let d'' be the resulting diagram obtained by using only the top and bottom row, replacing each "component" which is contained in the middle row by the variable x. That is, $d'd = x^{\lambda} d''$, where λ is the number of components in the middle row.

For example,

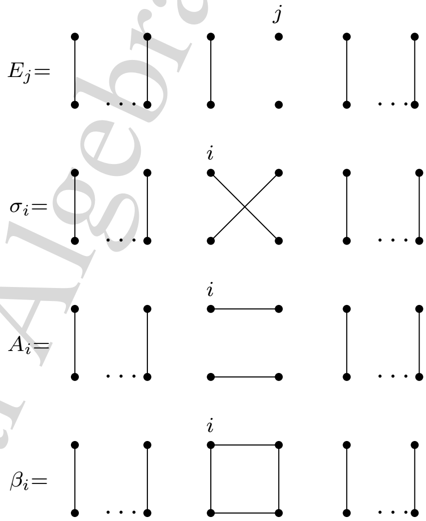


This product is associative and is independent of the graph that we choose to represent the k-partition diagram.

Let $\mathbb{C}(x)$ be the field of rational functions in the variable x with complex coefficients. The partition algebra $P_k(x)$ is defined to be the $\mathbb{C}(x)$-span of the k-partition diagrams, which is an associative algebra with identity. The identity is given by the partition diagram having each vertex in the top row connected to the vertex below it in the bottom row. The dimension of $P_k(x)$ is the Bell number $B(2k)$, where

$$B(2k) = \sum_{l=1}^{2k} S(2k, l), \tag{2.1}$$

and where $S(2k, l)$ is a Stirling number (see [St]). For a set with $2k$ elements $S(2k, l)$ is the number of equivalence relations with exactly l parts. By convention, $P_0(x) = \mathbb{C}(x)$. The span of the partition diagrams in which each component has exactly two vertices is the Brauer algebra $B_k(x)$ (see [Br]). The span of the partition diagrams in which each component has exactly two vertices, one in each row, is the group algebra $\mathbb{C}(x)[S_k]$ of the symmetric group S_k. For the remainder of this section, and for section 2.2, we will be closely following the exposition in [Ha]. For $1 \leq i \leq k-1$ and $1 \leq j \leq k$, define



Clearly (a) $A_i^2 = xA_i$, (b) $E_j^2 = xE_j$, (c) $A_i = \beta_i E_i E_{i+1} \beta_i$. Define $a_i = \frac{1}{x} A_i$ and $e_j = \frac{1}{x} E_j$. Then a_i and e_i are idempotent. The elements $\{\sigma_i\}$ generate the group algebra $\mathbb{C}(x)[S_k]$, the elements $\{\sigma_i, A_i\}$ generate the Brauer algebra $B_k(x)$ and the elements $\{\sigma_i, \beta_i, E_j\}$ generate the partition algebra $P_k(x)$ (see [Jo] and [Ha]). Replacing the variable x above with a complex number ξ, we obtain a $\mathbb{C}$-algebra $P_k(\xi)$.

Theorem 2.1.1 ([MS]). *For each integer $k \geq 0$, the partition algebra*

$P_k(x)$ is semisimple over $\mathbb{C}(x)$. The algebra $P_k(\xi)$ is semisimple over $\mathbb{C}$ whenever ξ is not an integer in the range $[0, 2k-1]$.

2.2. Schur-Weyl duality

We follow the notations given in [Ha]. Let $V = \mathbb{C}^n$ be the permutation module for the symmetric group S_n with standard basis $v_1, v_2, \dots, v_n$. Then $\pi(v_i) = v_{\pi(i)}$, for $\pi \in S_n$ and $1 \le i \le n$. For each positive integer k, the tensor product space $V^{\otimes k}$ is a module for the group S_n with a standard basis given by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, where $1 \le i_j \le n$. The action of $\pi \in S_n$ on a basis vector is given by

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\pi(i_1)} \otimes v_{\pi(i_2)} \otimes \cdots \otimes v_{\pi(i_k)}. \quad (2.2)$$

For each k-partition diagram d and each integer sequence $i_1, i_2, \dots, i_{2k}$ with $1 \le i_s \le n$, define

$$\psi(d)^{i_1, i_2, \dots, i_k}_{i_{k+1}, \dots, i_{2k}} = \\ = \begin{cases} 1 & \text{if } i_r = i_s \text{ whenever vertex } r \text{ is connected to vertex } s \text{ in } d, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

Define the action of a partition diagram $d \in P_k(n)$ on $V^{\otimes k}$ by defining it on the standard basis by

$$d(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \\ = \sum_{1 \le i_{k+1}, \dots, i_{2k} \le n} \psi(d)^{i_1, i_2, \dots, i_k}_{i_{k+1}, \dots, i_{2k}} v_{i_{k+1}} \otimes v_{i_{k+2}} \otimes \cdots \otimes v_{i_{2k}}. \quad (2.4)$$

Theorem 2.2.1 (Jones [Jo])**.** *The algebras $\mathbb{C}[S_n]$ and $P_k(n)$ generate full centralizers of each other in* $\mathrm{End}(V^{\otimes k})$*. In particular, for $n \ge 2k$,*
(a) $P_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$.
(b) S_n *generates* $\mathrm{End}_{P_k(n)}(V^{\otimes k})$.

2.3. The G-vertex colored partition algebras $P_k(x, G)$

Let G be a group. We denote $[m]$ for the set $\{1, 2, \dots, m\}$. Let $G^{2k} = \{f \mid f : [2k] \longrightarrow G\}$. We say that each $f \in G^{2k}$ is a coloring of $[2k]$ by G. We define a multiplication on G^{2k} by $ff'(p) = f(p)f'(p)$, for $f, f' \in G^{2k}$ and $p \in [2k]$. Note that under this multiplication on G^{2k} is a group, called the coloring group of $[2k]$ by G. Let d be a partition diagram in the partition algebra $P_k(x)$. Let $G_d = \{f_d \in G^{2k} \mid f_d(p) = f_d(q)$, whenever $p \sim q$ in $d\}$. We say that each $f_d \in G_d$ is a class coloring of $[2k]$ with respect to d by G. Clearly, G_d is a subgroup of G^{2k}, for every partition d of the set $[2k]$. For each $g \in G$, define $\overline{g} : [2k] \longrightarrow G$ by $\overline{g}(p) = g$, for all $p \in [2k]$. Under this identification $G \cong \overline{G} := \{\overline{g} \mid g \in G\}$ is a subgroup of G_d, for every partition d of $[2k]$. Let $f \in G^{2k}$. We can write $f = (f_1, f_2)$, where $f_1, f_2 \in G^k$ are defined on $[k]$ by $f_1(p) = f(p)$, $f_2(p) = f(k+p)$, for all $p \in [k]$. We say that f_1 and f_2 are the first and the second component of f respectively.

A (G, k)-partition diagram (or simply G-diagram) is a k-partition diagram, where each vertex is labelled by an element of the group G. We can identify each (G, k)-diagram as a pair (d, f), where d is the underlying k-partition diagram and $f \in G^{2k}$ such that $f(i)$ is the label of the ith vertex. We say that $(f(1), f(2), \dots, f(k))$ and $(f(k+1), f(k+2), \dots, f(2k))$ are the top label sequence and the bottom label sequence of (d, f) respectively.

Let (d, f) and (d', f') be two (G, k)-diagrams, where d, d' are any two k-partition diagrams and $f = (f_1, f_2)$, $f' = (f'_1, f'_2) \in G^{2k}$. In [PK1], we defined an equivalence relation $\sim$ on (G, k)-diagrams and a multiplication on (G, k)-diagrams, which is associative and well-defined up to equivalence of such diagrams, as follows:

- $(d, f) \sim (d', f') \Leftrightarrow d \sim d'$ and $f = \overline{g} f'$ for some (unique) $\overline{g} \in \overline{G}$

  $\Leftrightarrow d \sim d'$ and $f \in \overline{G} f'$.
- $(d', f')(d, f) = \begin{cases} x^{\lambda}(d'', f'') & \text{if } f_2 = (\overline{g} f')_1 \text{ for some (unique) } \overline{g} \in \overline{G} \\ 0 & \text{otherwise,} \end{cases}$

  where $d'd = x^{\lambda} d''$ and $f'' = (f_1,\ (\overline{g} f')_2)$.

When we speak of a G-diagram, we are really speaking of its equivalence class. The $\mathbb{C}(x)$-span of all $\sim$-classes of (G, k)-diagrams is denoted as $P_k(x, G)$, called G-vertex colored partition algebra, which is an associative algebra with identity. For each $\sim$-class, we can choose a (G, k)-diagram (d, f) such that $f(1) = e$. Now we may consider the set $\{(d, f) \mid f(1) = e\}$ as a basis for the algebra $P_k(x, G)$. The identity in $P_k(x, G)$ is

$$\sum_{\substack{f \in G^{2k} \\ f(1)=e,\quad f_1 = f_2}} (d, f),$$

where d is the identity partition diagram. The dimension of the algebra $P_k(x, G)$ is $|G|^{2k-1} B(2k)$, if G is finite. Note that the algebra $P_k(x, H)$ is a subalgebra of $P_k(x, G)$ if H is a subgroup of G. In particular, if $H = \{e\}$ then $P_k(x, H) \simeq P_k(x)$. If G is an infinite group, the algebra $P_k(x, G)$ is an infinite dimensional associative algebra. When $x = \xi \in \mathbb{C}$, we obtain the $\mathbb{C}$-algebra $P_k(\xi, G)$. Let G be a finite group. The wreath product of G with S_n, denoted $G \wr S_n$, is the group of order $|G|^n n!$ with elements of the form $\pi_{(g_1, g_2 \ldots, g_n)}$ where $\pi \in S_n$ and $g_1, g_2, \ldots, g_n \in G$. The multiplication in $G \wr S_n$ is given by

$$\pi_{(g_1,\, g_2, \ldots,\, g_n)} \pi'_{(g'_1,\, g'_2, \ldots,\, g'_n)} = (\pi\pi')_{(g_{\pi'(1)} g'_1,\, g_{\pi'(2)} g'_2, \ldots,\, g_{\pi'(n)} g'_n)}. \tag{2.5}$$

Let $W = \mathrm{Span}_{\mathbb{C}}\{v_{(i,g)} \mid 1 \leq i \leq n \text{ and } g \in G\}$. In [Bl], Bloss defined an action of $G \wr S_n$ on $W^{\otimes k}$ as follows:

$$\pi_{(g_1,\, g_2, \ldots,\, g_n)}(v_{(i_1,h_1)} \otimes \cdots \otimes v_{(i_k,h_k)}) = \\ = v_{(\pi(i_1), g_{i_1} h_1)} \otimes \cdots \otimes v_{(\pi(i_k), g_{i_k} h_k)} \tag{2.6}$$

for all $\pi_{(g_1,\, g_2, \cdots,\, g_n)} \in G \wr S_n$ where $\pi \in S_n$ and $g_j, h_r \in G$ $(1 \leq j, i_r \leq n)$. Note that $\{\pi_{(g,g,\ldots,g)} \in G \wr S_n \mid \pi \in S_n \text{ and } g \in G\}$ is a subgroup of $G \wr S_n$, which is isomorphic to $S_n \times G$. Using the action of $G \wr S_n$ defined in (2.6), we identified $G \wr S_n$ as a subgroup of the set of all permutation matrices $S_{n|G|}$ in $GL_{n|G|}$ (see [PK1]). Hence we have $S_n \times G \subseteq G \wr S_n \subseteq S_{n|G|} \subseteq GL_{n|G|}$. In [PK1], we defined a map $\phi : P_k(n, G) \longrightarrow \mathrm{End}(W^{\otimes k})$ by defining it on a basis element (d, f) such that $f(1) = e$, as follows:

$$\phi(d,f) = \left(\phi(d,f)^{(i_1,h_1),(i_2,h_2),\ldots,(i_k,h_k)}_{(i_{k+1},h_{k+1}),(i_{k+2},h_{k+2}),\ldots,(i_{2k},h_{2k})}\right) \tag{2.7}$$

$$= \left(\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} \delta^{h_1(f(1),f(2),\ldots,f(2k))}_{h_1,h_2,\ldots,h_{2k}}\right)$$

$$= \sum_{\substack{g \in G \\ p \sim q \text{ in } d \Rightarrow j_p = j_q}} E^{(j_1, gf(1)),(j_2, gf(2)),\ldots,(j_k, gf(k))}_{(j_{k+1}, gf(k+1)),(j_{k+2}, gf(k+2)),\ldots,(j_{2k}, gf(2k))} \tag{2.8}$$

where $\psi(d)^{i_1,i_2,\cdots,i_k}_{i_{k+1},i_{k+2},\cdots,i_{2k}}$ is defined as in equation (2.3). Then ϕ is a algebra homomorphism and we have an action of the algebra $P_k(n, G)$ on $W^{\otimes k}$ with respect to ϕ, defined by

$$(d,f).(v_{(i_1,h_1)} \otimes v_{(i_2,h_2)} \otimes \cdots \otimes v_{(i_k,h_k)}) = \delta^{h_1(e,f(2),f(3),\ldots,f(2k))}_{(h_1,h_2,\ldots h_{2k})}. \\ \cdot \sum_{1 \leq i_{k+1},i_{k+2},\ldots,i_{2k} \leq n} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} v_{(i_{k+1},h_{k+1})} \otimes \cdots \otimes v_{(i_{2k},h_{2k})}.$$

Note that when G is the group with one element, W specializes to V, the permutation representation of S_n. The following theorem is the Schur-Weyl duality for the restricted action of $S_n \times G$ with $P_k(n, G)$.

Theorem 2.3.1 ([PK1]). *The algebras $\mathbb{C}[S_n \times G]$ and $P_k(n, G)$ generate full centralizers of each other in $\mathrm{End}(W^{\otimes k})$. In particular, for $n \geq 2k$*
(a) $P_k(n, G) \cong \mathrm{End}_{S_n \times G}(W^{\otimes k})$
(b) $S_n \times G$ generates $\mathrm{End}_{P_k(n,G)}(W^{\otimes k})$.

Proof. This is a condensed sketch of the proof that appears in [PK1]. For each $2k$-sequence $((i_1, g_1), (i_2, g_2), \ldots, (i_{2k}, g_{2k}))$, where $1 \leq i_r \leq n$ and $g_r \in G$ $(1 \leq r \leq 2k)$, define the matrix

$$L_{((i_1,g_1),(i_2,g_2),\ldots,(i_{2k},g_{2k}))} = \sum_{\substack{(1\leq j_1,j_2\ldots,j_{2k}\leq n)\\ [i_p=i_q \Rightarrow j_p=j_q], \quad g\in G}} E^{(j_1,gg_1),(j_2,gg_2),\ldots,(j_k,gg_k)}_{(j_{k+1},gg_{k+1}),\ldots,(j_{2k},gg_{2k})}, \tag{2.9}$$

where $E^{(j_1,gg_1),(j_2,gg_2),\ldots,(j_k,gg_k)}_{(j_{k+1},gg_{k+1}),\ldots,(j_{2k},gg_{2k})}$ is the matrix unit. The set of all such matrices is a basis for $\mathrm{End}_{S_n\times G}(W^{\otimes k})$ with dimension $|G|^{2k-1}\sum_{l=1}^{l=n} S(2k,l)$. When $n \geq 2k$, $\dim\ \mathrm{End}_{S_n\times G}(W^{\otimes k}) = |G|^{2k-1}\ B(2k)$. Clearly $\phi(d, f)$ is an element in the above basis of $\mathrm{End}_{S_n\times G}(W^{\otimes k})$. Since $n \geq 2k$, $\dim P_k(n, G) = \dim\ \mathrm{End}_{P_k(n,G)}(W^{\otimes k})$ so ϕ is an isomorphism from $P_k(n, G)$ onto $\mathrm{End}_{S_n\times G}(W^{\otimes k})$. Hence $P_k(n, G) \cong \mathrm{End}_{S_n\times G}(W^{\otimes k})$. Proof of (b) follows from (a) and the double centralizer Theorem. □

Also, we defined another equivalence relation ρ and a corresponding class multiplication $(\star)$ as follows:

- $(d, f)\rho(d', f') \Leftrightarrow d \sim d'$ and $f = f_{d'}f'$ for some (unique) $f_{d'} \in G_d$

  $$\Leftrightarrow d \sim d' \text{ and } f \in G_{d'}f'.$$

- $(d', f') \star (d, f) =$

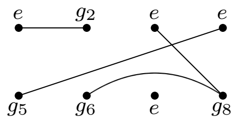
  $$\begin{cases} (x|G|)^{\lambda}(d'', f'') & \text{if } (f_d f)_2 = (f_{d'}f')_1 \text{ for some } f_d \in G_d \text{ and } f_{d'} \in G_{d'} \\ 0 & \text{otherwise,} \end{cases}$$

  where $d'd = x^{\lambda}d''$ and $f'' = ((f_d f)_1,\ (f_{d'}f')_2)$.

This multiplication $(\star)$ is well defined up to the equivalence relation ρ of G-diagrams. The $\mathbb{C}(x)$-algebra with basis consisting of (G, k)-diagrams with respect to the equivalence relation ρ and the multiplication operation $(\star)$ is an associative algebra with identity and it is denoted by $\overline{P}_k(x, G)$. In [Bl], Bloss introduced an edge coloring of partition algebra, denoted by $\overline{P_k(x, G)}$. We have proved in [PK1] that the G-edge colored partition algebras $\overline{P_k(x, G)}$ are isomorphic to the algebras $\overline{P}_k(x, G)$.

Definition 2.3.2. *Let d be a partition diagram with vertex set $[2k]$. For each class of d, we can choose the vertex with the smallest label called a minimal vertex of d. A k-partition diagram, with each vertex is labelled by an element of the group G such that all its minimal vertices are labelled by the identity (e) of G is called a minimal G-diagram.*

For example,



is a minimal G-diagram, where $g_s \in G$ $(s = 2, 5, 6, 8)$. For each ρ-class, we can choose a minimal G-diagram. Now, we may consider the set of all minimal G-diagrams as a basis for the algebra $\overline{P}_k(x, G)$, hence the dim $\overline{P}_k(x, G) = \sum_{l=1}^{l=2k} |G|^{2k-l} S(2k, l)$. If $G = \{e\}$ then the algebra $\overline{P}_k(x, G)$ is isomorphic to the partition algebra $P_k(x)$. The identity in the algebra $\overline{P}_k(x, G)$ is the G-diagram, whose underlying partition diagram is $1 \in P_k(x)$ and with all vertex labels e. Note that if H is a subgroup of G then the span of the diagrams in the algebra $\overline{P}_k(x, G)$ which are labelled by the elements of H is a subalgebra of the algebra $\overline{P}_k(x, G)$, denoted by $\overline{P}_k(x, G_H)$, which is isomorphic to the algebra $\overline{P}_k(\frac{|G|}{|H|}x, H)$. In particular, if $H = \{e\}$ then the algebra $\overline{P}_k(x, G_H)$ is isomorphic to the partition algebra $P_k(|G|x)$. When $x = \xi \in \mathbb{C}$, we obtain the $\mathbb{C}$-algebra $\overline{P}_k(\xi, G)$. In $P_k(x, G)$, for each minimal G-diagram (d, f), we define the sum

$$\overline{(d, f)} = \sum_{f_d \in [G_d]} (d, f_d f)$$

where $[G_d] = \{f_d \in G_d \mid f_d(1) = e\}$.

Theorem 2.3.3 ([PK1]). *The algebra $\overline{P}_k(x, G)$ is a subalgebra of the G-vertex colored partition algebra $P_k(x, G)$. Moreover, the mapping $(d, f) \longrightarrow \overline{(d, f)}$ is an isomorphism from $\overline{P}_k(x, G)$ into $P_k(x, G)$.*

The following theorem is the Schur-Weyl duality for $G \wr S_n$ with $\overline{P}_k(n, G)$ in [Bl] (see also [PK1]).

Theorem 2.3.4 ([Bl]). *The algebras $\mathbb{C}[G \wr S_n]$ and $\overline{P}_k(n, G)$ generate full centralizers of each other in $\text{End}(W^{\otimes k})$. In particular, for $n \geq 2k$*
(a) $\overline{P}_k(n, G) \cong \text{End}_{G \wr S_n}(W^{\otimes k})$
(b) $G \wr S_n$ generates $\text{End}_{\overline{P}_k(n,G)}(W^{\otimes k})$.

3. The extended G-vertex colored partition algebras $\widehat{P}_k(x, G)$

In this section, we introduce extended G-vertex colored partition algebras and study their structure.

3.1. The structure of $\widehat{P}_k(x, G)$

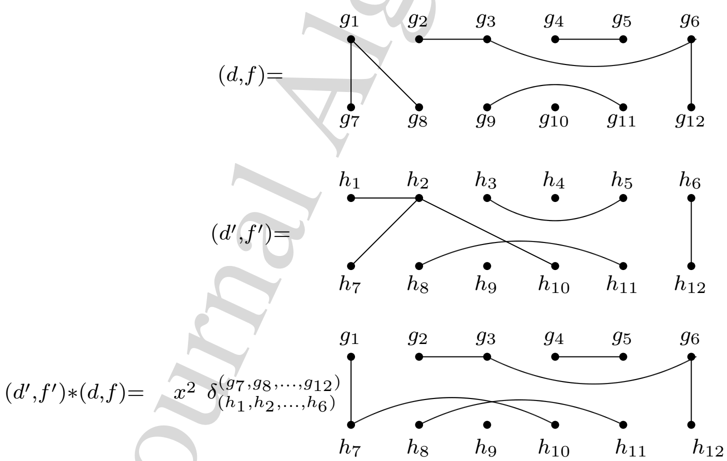
In this section, we define another multiplication $(*)$ on (G, k)-diagrams without defining any equivalence relation, as follows:
Let (d, f) and (d', f') be two (G, k)-diagrams, where d, d' are any two partitions and $f = (f_1, f_2),\ f' = (f'_1, f'_2) \in G^{2k}$.

$$(d', f') * (d, f) = \begin{cases} x^{\lambda}(d'', (f_1, f'_2)) & \text{if } f_2 = f'_1 \\ 0 & \text{otherwise,} \end{cases}$$

where $d'd = x^{\lambda} d''$. The multiplication $*$ of two G-diagrams (d, f) and (d', f') defined above can be equivalently stated in other words as follows:

- Multiply the underlying partition diagrams d and d'. This will give the underlying partition diagram of the G-diagram $(d', f') * (d, f)$.
- If the bottom label sequence of (d, f) is equal to the top label sequence of (d', f') then the top label sequence and the bottom label sequence of $(d', f') * (d, f)$ are the top label sequence of (d, f) and the bottom label sequence of (d', f') respectively.
- If the bottom label sequence of (d, f) is not equal to the top label sequence of (d', f'), then $(d', f') * (d, f) = 0$.
- For each connected component entirely in the middle row, a factor of x appears in the product.

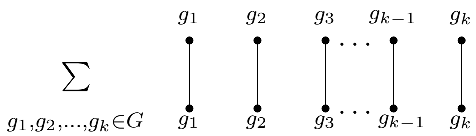
For example, let $g_r, h_s \in G$ $(1 \le r, s \le 12)$.



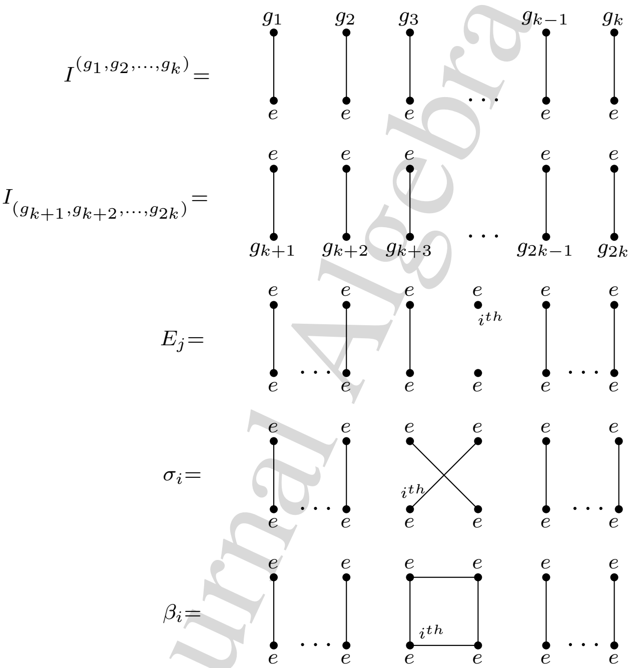
Note that $\delta^{(g_7, g_8, \ldots, g_{12})}_{(h_1, h_2, \ldots, h_6)}$ is the Kronecker delta, that is

$$\delta^{(g_7, g_8, \ldots, g_{12})}_{(h_1, h_2, \ldots, h_6)} = \begin{cases} 1 & \text{if } (g_7, g_8, \ldots, g_{12}) = (h_1, h_2, \ldots, h_6) \\ 0 & \text{if } (g_7, g_8, \ldots, g_{12}) \neq (h_1, h_2, \ldots, h_6). \end{cases}$$

The multiplication $*$ is associative on (G,k)-diagrams. The $\mathbb{C}(x)$-span of all (G,k)-diagrams under the above multiplication is denoted as $\widehat{P}_k(x,G)$, called extended G-vertex colored partition algebra, which is an associative algebra with identity. The identity in $\widehat{P}_k(x,G)$ is $\sum_{\substack{f\in G^{2k}\\ f_1=f_2}}(d,f)$ where d is the identity partition diagram, that is



The dimension of the algebra $\widehat{P}_k(x,G)$ is the number of (G,k)-diagrams, so that if G is finite, dim $\widehat{P}_k(x,G)=|G|^{2k}B(2k)$, where $B(2k)$ is the Bell number of $2k$, the number of equivalence relations of $2k$ vertices. Note that the algebra $\widehat{P}_k(x,H)$ is a subalgebra of $\widehat{P}_k(x,G)$ if H is a subgroup of G. In particular, if $H=\{e\}$ then $\widehat{P}_k(x,H)\simeq P_k(x)$. If G is an infinite group, the algebra $\widehat{P}_k(x,G)$ is an infinite dimensional associative algebra. When $x=\xi\in\mathbb{C}$, we obtain the $\mathbb{C}$-algebra $\widehat{P}_k(\xi,G)$. Define elements in $\widehat{P}_k(x,G)$ by



where $g_l\in G$ $(1\le l\le 2k)$, e is the identity of G, $(1\le i\le k-1)$ and $(1\le j\le k)$.

Lemma 3.1.1. *The algebra $\widehat{P}_k(x,G)$ is generated by the above elements $I^{(g_1,g_2,\ldots,g_k)}$, $I_{(g_{k+1},g_{k+2},\ldots,g_{2k})}$, E_j, σ_i and β_i.*

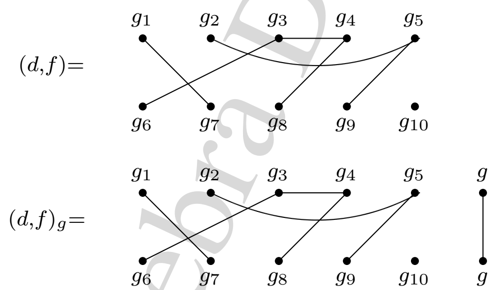
Proof. We have from § 2.1, the elements E_j, σ_i and β_i generate all partition diagrams with labels e in all its vertices. Let (d, f) be a (G, k)-partition diagram, where $f = (g_1, g_2, ..., g_{2k})$. Let (d, e) be the (G, k)-partition diagram with underlying partition diagram d and labels e in all its vertices. Then, $(d, f) = I_{(g_{k+1}, g_{k+2}, \dots, g_{2k})} (d, e) I^{(g_1, g_2, \dots, g_k)}$. □

Lemma 3.1.2. *The algebra $\widehat{P}_{k-1}(x, G)$ is a subalgebra of $\widehat{P}_k(x, G)$, for an indeterminate x and for all $x = \xi \in \mathbb{C}$.*

Proof. Define $\widehat{\pi} : \widehat{P}_{k-1}(x, G) \longrightarrow \widehat{P}_k(x, G)$ by defining on a basis element (d, f),

$$\widehat{\pi}(d, f) = \sum_{g \in G} (d, f)_g \tag{3.1}$$

where $(d, f)_g$ is the (G, k)-partition diagram obtained by adding an isolated vertical edge with d in the rightmost place with vertex labels g. For example,



Then $\widehat{\pi}$ is an isomorphism. □

Lemma 3.1.3. *For an indeterminate x and for all $x = \xi \in \mathbb{C}$, the algebra $\widehat{P}_{k-1}(x, G)$ is a subalgebra of the G-vertex colored partition algebra $P_k(x, G)$.*

Proof. Define the mapping

$$\Upsilon : \widehat{P}_{k-1}(x, G) \longrightarrow P_k(x, G)$$

by defining it on the basis element

$$(d, f) \longrightarrow (d, f)_e,$$

where $(d, f)_e$ is defined in Lemma 3.1.2. Then Υ is an isomorphism. □

Note that if $G = \{e\}$, then Lemma 3.1.2 and 3.1.3 are identical with the usual partition algebra inclusion $P_{k-1}(x) \subseteq P_k(x)$. (i.e., The partition algebra $P_{k-1}(n)$ is a subalgebra of $P_k(n)$, since we can identify each $k-1$-partition diagram as a k-partition diagram by adding an isolated vertical edge in the rightmost place.)

3.2. Two bases for $\mathrm{End}_{S_n}(W^{\otimes k})$

We have $\{\pi_{(e,e,\ldots,e)} \in G \wr S_n \mid \pi \in S_n$ and e is the identity of $G\}$ is a subgroup of $G \wr S_n$, which is isomorphic to S_n. The restricted action of S_n on W is given by $\pi(v_{(i,g)}) = v_{(\pi(i),g)}$. In this section, we give two bases for $\mathrm{End}_{S_n}(W^{\otimes k})$, where $W = \mathbb{C}^m$ and the action of S_n on $W^{\otimes k}$ is diagonal action defined as follows :

Let $\mathbb{S} = [n] \times G$ and let $I = ((i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k))$, $J = ((i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k}))$ are in $\mathbb{S}^k$. The action of S_n on $\mathbb{S}$ is defined by

$$\pi(i,g) = (\pi(i),g) \tag{3.2}$$

can be extended to an action on $\mathbb{S}^{2k}$ by $\pi(I,J) = (\pi(I),\pi(J))$, as component wise. Diagonally extend the action of S_n on W to an action of S_n on $W^{\otimes k}$:

$$\pi(v_{(i_1,g_1)} \otimes \cdots \otimes v_{(i_k,g_k)}) = v_{(\pi(i_1),g_1)} \otimes \cdots \otimes v_{(\pi(i_k),g_k)} \tag{3.3}$$

where $\pi \in S_n$. We will write the action above as $\pi(v_I) = v_{\pi(I)}$. Let $A \in \mathrm{End}(W^{\otimes k})$. Define $A(v_J) = \sum_I A_I^J(v_I)$, where $A_I^J \in \mathbb{C}$ is the $(I,J)^{th}$ entry of A $(I,J \in \mathbb{S}^k)$ and v_I is a basis element of $W^{\otimes k}$. We have

$$\mathrm{End}_{S_{n|G|}}(W^{\otimes k}) \subseteq \mathrm{End}_{G\wr S_n}(W^{\otimes k}) \subseteq \mathrm{End}_{S_n\times G}(W^{\otimes k}) \subseteq \mathrm{End}_{S_n}(W^{\otimes k}).$$

The following is our analogue of Jones result (see, for example, [Bl]).

Lemma 3.2.1. *$A \in \mathrm{End}_{S_n}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{\pi(I)}^{\pi(J)}$, for all $\pi \in S_n$.*

Proof. We have $A \in \mathrm{End}_{S_n}(W^{\otimes k}) \Leftrightarrow \pi A = A\pi$, for all $\pi \in S_n$.
$\Leftrightarrow \pi A(v_J) = A\pi(v_J)$, for all v_J. $\Leftrightarrow \pi \sum_I A_I^J(v_I) = A(v_{\pi(J)})$
$\Leftrightarrow \sum_I A_I^J \pi(v_I) = \sum_I A_I^{\pi(J)}(v_I)$
$\Leftrightarrow \sum_I A_I^J(v_{\pi(I)}) = \sum_I A_{\pi(I)}^{\pi(J)}(v_{\pi(I)})$, since the action of S_n is permutation representation. The result follows from linearly independence and equating the scalars. □

Lemma 3.2.2.

$$dim \ \ \mathrm{End}_{S_n}(W^{\otimes k}) = |G|^{2k} \sum_{l=1}^{l=n} S(2k,l).$$

When $n \geq 2k$,

$$dim \ \ \mathrm{End}_{S_n}(W^{\otimes k}) \ = \ |G|^{2k} \ B(2k).$$

Proof. Lemma 3.2.1 tells as that A commutes with the S_n-action on $W^{\otimes k}$ if and only if the matrix entries of A are equal on S_n-orbits. Thus dim $\operatorname{End}_{S_n}(W^{\otimes k})$ is the number of S_n-orbits on $\mathbb{S}^{2k}$. Fix a tuple of indices $(I,J) = ((i_1,g_1),(i_2,g_2)\dots,(i_{2k},g_{2k})) \in \mathbb{S}^{2k}$. This tuple determines a partition $d_{[(I,J)]} := d(i_1,i_2,\dots,i_{2k})$ of $[2k]$ (into at most n subsets) according to those that have an equal value. Let $[(I,J)]$ be the orbit of $(I,J) \in \mathbb{S}^{2k}$. Then $(I',J') \in [(I,J)] \Leftrightarrow (I',J') = \pi(I,J)$ for some $\pi \in S_n$ $\Leftrightarrow$ $(j_r,h_r) = \pi(i_r,g_r)$ for every r such that $1 \le r \le 2k$, where (j_r,h_r) and

(i_r,g_r) are the r^{th} component of (I',J') and (I,J) respectively
$\Leftrightarrow (j_r,h_r) = (\pi(i_r),g_r)$
$\Leftrightarrow j_r = \pi(i_r)$ and $h_r = g_r$
$\Leftrightarrow [j_p = j_q \text{ iff } i_p = i_q]$ $(1 \le p,q \le 2k)$ and $h_r = g_r$ $(1 \le r \le 2k)$ (3.4)
$\Leftrightarrow$ $d(j_1,j_2,\cdots,j_{2k}) = d(i_1,i_2,\cdots,i_{2k})$ and $h_r = g_r$, for all r, $(1 \le r \le 2k)$. Thus, for every S_n-orbit determines a partition of $[2k]$ and a $2k$-tuple $(g_1,g_2,\dots,g_{2k})$ and vice versa. Hence the result is proved. □

We define for each S_n-orbit $[(I,J)]$, a matrix $T^I_J \in \operatorname{End}(W^{\otimes k})$ by

$$T^I_J = \sum_{(I',J')\in[(I,J)]} E^{I'}_{J'},$$

where $E^{I'}_{J'}$ is the matrix unit, which has non-zero entry 1 in $(I',J')^{\text{th}}$ position. In fact, $T^I_J \in \operatorname{End}(W^{\otimes k})$, since such a matrix satisfies the condition in Lemma 3.2.1: the matrix entries are equal on S_n-orbits. Using equation (3.4), we have

$$T^{(i_1,g_1),(i_2,g_2),\cdots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\cdots,(i_{2k},g_{2k})} = \sum E^{(j_1,g_1),(j_2,g_2),\dots,(j_k,g_k)}_{(j_{k+1},g_{k+1}),(j_{k+2},g_{k+2}),\dots,(j_{2k},g_{2k})}, \tag{3.5}$$

where the sum is over $i_p = i_q \Leftrightarrow j_p = j_q$ (i.e., the sum is over $p \sim q$ in $d(i_1,i_2,\dots,i_{2k}) \Leftrightarrow j_p = j_q,\ 1 \le p,q \le 2k$). Since each matrix T^I_J is the sum of different matrix units, the set $\{T^I_J \mid [(I,J)] \text{ is a } S_n\text{-orbit}\}$ is a linearly independent set. For $A \in \operatorname{End}_{S_n}(W^{\otimes k})$, we use the Lemma 3.2.1 to obtain: $A = \sum_{[(I,J)]} A^I_J T^I_J$. Thus the matrix T^I_J span $\operatorname{End}_{S_n}(W^{\otimes k})$, and so is a basis for $\operatorname{End}_{S_n}(W^{\otimes k})$.

Definition 3.2.3. *Let d and d' be partitions of the $[2k]$ into subsets. We say that d' is coarser than d if any subset in d is contained in some subset in d'. In this case we write $d' \le d$.*

We now define another basis of $\operatorname{End}_{S_n}(W^{\otimes k})$ as follows: Define $L^I_J = \sum T^{I'}_{J'}$, the sum is over $[(I',J')]$ such that $d_{[(I',J')]} \le d_{[(I,J)]}$. By Möbius inversion (see [St]) the T^I_J can be expressed in terms of the L^I_J, so they also span $\operatorname{End}_{S_n}(W^{\otimes k})$. Using equation (3.5), we get

$$L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} = \sum E^{(j_1,g_1),(j_2,g_2),\cdots,(j_k,g_k)}_{(j_{k+1},g_{k+1}),(j_{k+2},g_{k+2}),\ldots,(j_{2k},g_{2k})}, \tag{3.6}$$

where the sum is over $i_p = i_q \Rightarrow j_p = j_q$ (i.e., the sum over $p \sim q$ in $d(i_1, i_2, \cdots, i_{2k}) \Rightarrow j_p = j_q,\ 1 \leq p, q \leq 2k$). Now the multiplication of the matrices L^I_J in the basis of $\mathrm{End}_{S_n}(W^{\otimes k})$ has a nice form as follows:

Lemma 3.2.4.

$$\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right) = 0$$
$$\Leftrightarrow (g_1, g_2, \ldots, g_k) \neq (h_{k+1}, h_{k+2}, \ldots, h_{2k}).$$

Proof.

$$\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},h_{k+1}),(j_{k+2},h_{k+2}),\ldots,(j_{2k},h_{2k})}\right)$$

$$= \left(\sum_{i_p=i_q \Rightarrow i'_p=i'_q} E^{(i'_1,g_1),(i'_2,g_2),\ldots,(i'_k,g_k)}_{(i'_{k+1},g_{k+1}),\ldots,(i'_{2k},g_{2k})}\right)\left(\sum_{j_p=j_q \Rightarrow j'_p=j'_q} E^{(j'_1,h_1),(j'_2,h_2),\ldots,(j'_k,h_k)}_{(j'_{k+1},h_{k+1}),\ldots,(j_{2k},h_{2k})}\right)$$

$$= \sum_{\substack{i_p=i_q \Rightarrow i'_p=i'_q \\ j_p=j_q \Rightarrow j'_p=j'_q}} E^{(i'_1,g_1),(i'_2,g_2),\ldots,(i'_k,g_k)}_{(i'_{k+1},g_{k+1}),(i'_{k+2},g_{k+2}),\ldots,(i'_{2k},g_{2k})} E^{(j'_1,h_1),(j'_2,h_2),\ldots,(j'_k,h_k)}_{(j'_{k+1},h_{k+1}),(j'_{k+2},h_{k+2}),\ldots,(j'_{2k},h_{2k})}$$

$$= \sum_{\substack{i_p=i_q \Rightarrow i'_p=i'_q \\ j_p=j_q \Rightarrow j'_p=j'_q}} \delta^{(i'_1,g_1),(i'_2,g_2),\ldots,(i'_k,g_k)}_{(j'_{k+1},h_{k+1}),(j'_{k+2},h_{k+2}),\ldots,(j'_{2k},h_{2k})} E^{(j'_1,h_1),(j'_2,h_2),\ldots,(j'_k,h_k)}_{(i'_{k+1},g_{k+1}),(i'_{k+2},g_{k+2}),\ldots,(i'_{2k},g_{2k})},$$

since $E^q_p E^s_r = \delta_{qr} E^s_p$, where δ_{qr} is the Kronecker delta.
$= 0$ iff $(g_1, g_2, \ldots, g_k) \neq (h_{k+1}, h_{k+2}, \ldots, h_{2k})$. □

We denote the product $[d(i_1, i_2, \ldots, i_{2k})]\ [d(j_1, j_2, \ldots, j_{2k})]$ for the multiplication of the corresponding partition diagrams.

Lemma 3.2.5.

$$\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},g_1),(j_{k+2},g_2),\ldots,(j_{2k},g_k)}\right)$$
$$= (n)^{\lambda}\quad L^{(s_1,h_1),(s_2,h_2),\ldots,(s_k,h_k)}_{(s_{k+1},g_{k+1}),(s_{k+2},g_{k+2}),\ldots,(s_{2k},g_{2k})}, \tag{3.7}$$

where $1 \leq s_1, s_2, \ldots, s_{2k} \leq n$ *such that* $[d(i_1, i_2, \ldots, i_{2k})]\ [d(j_1, j_2, \ldots, j_{2k})] = (n)^{\lambda}\ [d(s_1, s_2, \ldots, s_{2k})]$ *in* $P_k(n)$.

Proof.

$$\left(L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}\right)\left(L^{(j_1,h_1),(j_2,h_2),\ldots,(j_k,h_k)}_{(j_{k+1},g_1),(j_{k+2},g_2),\ldots,(j_{2k},g_k)}\right)$$

$$= \left(\sum_{i_p=i_q\Rightarrow i'_p=i'_q} E^{(i'_1,g_1),(i'_2,g_2),\ldots,(i'_k,g_k)}_{(i'_{k+1},g_{k+1}),\ldots,(i'_{2k},g_{2k})}\right)\left(\sum_{j_p=j_q\Rightarrow j'_p=j'_q} E^{(j'_1,h_1),(j'_2,h_2),\ldots,(j'_k,h_k)}_{(j'_{k+1},g_1),\ldots,(j'_{2k},g_k)}\right)$$

$$= \sum_{\substack{i_p=i_q\Rightarrow i'_p=i'_q \\ j_p=j_q\Rightarrow j'_p=j'_q}} E^{(i'_1,g_1),(i'_2,g_2),\ldots,(i'_k,g_k)}_{(i'_{k+1},g_{k+1}),(i'_{k+2},g_{k+2}),\ldots,(i'_{2k},g_{2k})} E^{(j'_1,h_1),(j'_2,h_2),\ldots,(j'_k,h_k)}_{(j'_{k+1},g_1),(j'_{k+2},g_2),\ldots,(j'_{2k},g_k)}$$

$$= \sum_{\substack{i_p=i_q\Rightarrow i'_p=i'_q \\ j_p=j_q\Rightarrow j'_p=j'_q}} \delta^{(j'_{k+1},j'_{k+2},\ldots,j'_{2k})}_{(i'_1,i'_2,\ldots,i'_k)} E^{(j'_1,h_1),(j'_2,h_2),\ldots,(j'_k,h_k)}_{(i'_{k+1},g_{k+1}),(i'_{k+2},g_{k+2}),\ldots,(i'_{2k},g_{2k})}. \tag{3.8}$$

where δ_{qr} is the Kronecker delta.

$$= (n)^{\lambda} \quad L^{(s_1,h_1),(s_2,h_2),\ldots,(s_k,h_k)}_{(s_{k+1},g_{k+1}),(s_{k+2},g_{k+2}),\ldots,(s_{2k},g_{2k})}, \tag{3.9}$$

where $1 \leq s_1, s_2, \ldots, s_{2k} \leq n$ such that $[d(i_1, i_2, \ldots, i_{2k})]\, [d(j_1, j_2, \ldots, j_{2k})] = (n)^{\lambda}\ [d(s_1, s_2, \ldots, s_{2k})]$ in $P_k(n)$. □

3.3. Schur-Weyl duality

We have an action of $\widehat{P}_k(n, G)$ on $W^{\otimes k}$, defined as follows: Number the vertices of a (G, k)-diagram $1, 2, \ldots k$ from left to right in the top row, and $k + 1, k + 2, \ldots, 2k$ from left to right in the bottom row. Define a map $\widehat{\phi} : \widehat{P}_k(n, G) \longrightarrow \mathrm{End}(W^{\otimes k})$ by defining it on a G-diagram (d, f), as follows:

$$\begin{aligned}\widehat{\phi}(d, f) &= \left(\widehat{\phi}(d, f)^{(i_1,h_1),(i_2,h_2),\ldots,(i_k,h_k)}_{(i_{k+1},h_{k+1}),(i_{k+2},h_{k+2}),\ldots,(i_{2k},h_{2k})}\right) \\ &= \left(\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} \quad \delta^{(g_1,g_2,\ldots,g_{2k})}_{(h_1,h_2,\ldots,h_{2k})}\right),\end{aligned}$$

where $f = (g_1, g_2, \ldots, g_{2k})$ is the label sequence of d, and where $\psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}}$ is defined as in equation 2.3, and $\delta^{(g_1,g_2,\ldots,g_{2k})}_{(h_1,h_2,\ldots,h_{2k})}$ is the Kronecker delta. Alternatively, in terms of matrix units we have

$$\widehat{\phi}(d, f) = \sum_{\substack{p\sim q \ \ in \ \ d\Rightarrow i_p=i_q \\ 1\leq i_1,i_2,\ldots,i_{2k}\leq n}} E^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}.$$

Then we have an action of $\widehat{P}_k(n,G)$ on $W^{\otimes k}$ defined by

$$(d,f)(v_J) = \widehat{\phi}(d,f)(v_J), \text{ for all } J \in \mathbb{S}^k.$$

When G is a group with one element, this action restricts to the action of the partition algebra defined by Jones in [Jo] on tensors.

Hence the action of a G-partition diagram $(d,f) \in \widehat{P}_k(n,G)$ on $W^{\otimes k}$ is given by defining it on the standard basis by

$$(d,f).(v_{(i_1,h_1)} \otimes v_{(i_2,h_2)} \otimes \cdots \otimes v_{(i_k,h_k)}) = \\ = \delta^{(g_1,g_2,\ldots,g_{2k})}_{(h_1,h_2,\ldots h_{2k})} \\ \sum_{i_{k+1},i_{k+2},\ldots,i_{2k}} \psi(d)^{i_1,i_2,\ldots,i_k}_{i_{k+1},i_{k+2},\ldots,i_{2k}} v_{(i_{k+1},h_{k+1})} \otimes v_{(i_{k+2},h_{k+2})} \otimes \cdots \otimes v_{(i_{2k},h_{2k})}$$

Lemma 3.3.1. *The map $\widehat{\phi} : \widehat{P}_k(n,G) \longrightarrow \mathrm{End}(W^{\otimes k})$ is an algebra homomorphism.*

Proof. It is enough to prove $\widehat{\phi}(d'_{f'} * d_f) = \phi(d'_{f'})\phi(d_f)$, where $d'_{f'}, d_f$ are (G,k)-diagrams. Let (d,f) be a (G,k)-diagram, where $f = (g_1, g_2, \ldots, g_{2k})$ is the label sequence. The connected components of d partition the $2k$ vertices into subsets. Having numbered the vertices from 1 to $2k$ as described above, we obtain a partition $\triangle_d$ to the set $[2k]$. This partition, together with the labels in d, naturally determine a matrix $L_{(d,f)} = L^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})} \in \mathrm{End}_{S_n}(W^{\otimes k})$. Here the indices $i_1, i_2, \ldots i_{2k}$ have equal values according to the partition $\triangle_d$, and if a vertex p is labelled by g_p, that is, the matrix

$$(\widehat{\phi}(d,f)^{(i_1,g_1),(i_2,g_2),\ldots,(i_k,g_k)}_{(i_{k+1},g_{k+1}),(i_{k+2},g_{k+2}),\ldots,(i_{2k},g_{2k})}),$$

is precisely $L_{(d,f)}$. So the result is an immediate consequence of Lemma 3.2.4 and Lemma 3.2.5. □

The following is our analogue of Theorem 2.2.1.

Theorem 3.3.2. *The algebras $\mathbb{C}[S_n]$ and $\widehat{P}_k(n,G)$ generate full centralizers of each other in $\mathrm{End}(W^{\otimes k})$. That is, for $n \geq 2k$, we have*
(a) $\widehat{P}_k(n,G) \cong \mathrm{End}_{S_n}(W^{\otimes k})$,
(b) S_n generates $\mathrm{End}_{\widehat{P}_k(n,G)}(W^{\otimes k})$.

Proof. Proof of (a). Since $n \geq 2k$, dim $\widehat{P}_k(n,G) =$ dim $\mathrm{End}_{S_n}(W \otimes k)$. In the proof of Lemma 3.3.1, we have $\widehat{\phi}(\widehat{P}_k(n,G)) \subseteq \mathrm{End}_{S_n}(W^{\otimes k})$. As (d,f) ranges over all G-diagrams, all $L_{(d,f)}$ are obtained. Thus the representation ϕ takes a basis of $\widehat{P}_k(n,G)$ to a basis of $\mathrm{End}_{S_n}(W^{\otimes k})$, so $\widehat{P}_k(n,G) \cong \mathrm{End}_{S_n}(W^{\otimes k})$. Proof of (b). This follows from (a) and the double centralizer Theorem. □

As the centralizer of the semisimple group algebra $\mathbb{C}(S_n)$, the $\mathbb{C}$-algebra $\widehat{P}_k(n,G)$ is semisimple for $n \geq 2k$.

3.4. Some interesting subalgebras of $\widehat{P}_k(x,G)$

In this section we study the subalgebras of $\widehat{P}_k(x,G)$.

Theorem 3.4.1. *For an indeterminate x and for all $x = \xi \in \mathbb{C}$, the G-vertex colored partition algebra $P_k(x,G)$ is a subalgebra of the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$.*

Proof. In the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$, for each G-diagram (d,f) such that $f(1) = e$, we define the sum

$$\widehat{(d,f)} = \sum_{\overline{g}\in\overline{G}} (d, \overline{g}f).$$

In other words, this sum is over all distinct G-diagrams in the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$, which are related to the G-diagram (d,f) with respect to the equivalence relation $\sim$ on G-diagrams. (i.e., $(d,f) \sim (d',f')$ if $d = d'$ and $f = \overline{g}f'$ for some $\overline{g} \in \overline{G}$). So, we say that this sum is the class sum of (d,f) under $\sim$ in the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$. Since any two class sums are the disjoint sums of G-diagrams in the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$, the set of all class sums is a linearly independent set in the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$. We are going to prove that $(d,f) \longrightarrow \widehat{(d,f)}$ is an algebra isomorphism from the G-vertex colored partition algebra$P_k(x,G)$ in to the extended G-vertex colored partition algebra $\widehat{P}_k(x,G)$. Clearly this map is well-defined and injective. □

Claim: $(d,f) \longrightarrow \widehat{(d,f)}$ is a ring homomorphism.
Step 1. $(d',f')(d,f) = 0$ in $P_k(x,G)$ $\Leftrightarrow$ $\widehat{(d',f')} * \widehat{(d,f)} = 0$ in $\widehat{P}_k(x,G)$.

Proof. We have, $(d',f')(d,f) = 0$ in $P_k(x,G)$

$\Leftrightarrow$ $f_2 \neq \overline{g'}f'_1$, for all $\overline{g'} \in \overline{G}$. $\Leftrightarrow$ $(d', \overline{g'}f') * (d, \overline{g}f) = 0$ in $\widehat{P}_k(x,G)$, for all $\overline{g}, \overline{g'} \in \overline{G}$. $\Leftrightarrow$ $\left(\sum_{\overline{g'}\in\overline{G}} (d', \overline{g'}f')\right) * \left(\sum_{\overline{g}\in\overline{G}} (d, \overline{g}f)\right) = 0$ in $\widehat{P}_k(x,G)$. $\Leftrightarrow$ $\widehat{(d',f')} * \widehat{(d,f)} = 0$ in $\widehat{P}_k(x,G)$. □

Suppose $(d', f')(d, f) \neq 0$. Let $(d', f')(d, f) = x^{\lambda}(d'', f'')$.

Claim. $\widehat{(d', f')} * \widehat{(d, f)} = (d', \widehat{f')(d}, f)$.

$$\text{i.e.,} \quad \Big(\sum_{\overline{g'} \in \overline{G}} (d', \overline{g'} f')\ \Big) * \Big(\sum_{\overline{g} \in \overline{G}} (d, \overline{g} f)\ \Big) = x^{\lambda} \widehat{(d'', f'')}$$

$$\text{i.e.,} \quad \sum_{\overline{g},\ \overline{g'} \in \overline{G}} (d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda} \sum_{\overline{g''} \in \overline{G}} (d'', \overline{g''} f'')$$

Step 2. If $(d', \overline{g'} f') * (d, \overline{g} f) \neq 0$ for some $\overline{g}$, $\overline{g'} \in \overline{G}$ then $(d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda}(d'', \overline{g''} f'')$ for some $\overline{g''} \in \overline{G}$.

Proof. If $(d', \overline{g'} f') * (d, \overline{g} f) \neq 0$ then $(\overline{g} f)_2 = (\overline{g'} f')_1$ and $(d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda}(d'', ((\overline{g} f)_1, (\overline{g'} f')_2))$. Since $(\overline{g} f)_2 = (\overline{g'} f')_1$, $(d', f')(d, f) = x^{\lambda}(d'', ((\overline{g} f)_1, (\overline{g'} f')_2))$. Since $(d', f')(d, f) = x^{\lambda}(d'', f'')$, $(d'', ((\overline{g} f)_1, (\overline{g'} f')_2))$ and (d'', f'') are equivalent G-diagram in $P_k(x, G)$. This implies $(d'', ((\overline{g} f)_1, (\overline{g'} f')_2)) = (d'', \overline{g''} f'')$ for some $\overline{g''} \in \overline{G}$. Hence $(d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda}(d'', \overline{g''} f'')$ for some $\overline{g''} \in \overline{G}$. □

Step 3. For every $\overline{g''} \in \overline{G}$ there exist a unique pair $\overline{g}$, $\overline{g'} \in \overline{G}$ such that $(d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda}(d'', \overline{g''} f'')$.

Proof. Since $\overline{g''} \in \overline{G}$, $(d', f')(d, f) = x^{\lambda}(d'', \overline{g''} f'')$. This implies that there exist unique $\overline{h'} \in \overline{G}$ such that $(\overline{h'} f')_1 = f_2$ and $(d'', \overline{g''} f'')$ is $\sim$-equivalent to $(d'', (f_1, (\overline{h'} f')_2))$ in $P_k(x, G)$. This implies that there exist unique $\overline{g} \in \overline{G}$ such that $\overline{g''} f'' = \overline{g}(f_1, (\overline{h'} f')_2)$. Put $\overline{g'} = \overline{g}\overline{h'}$. Since $(\overline{g} f)_2 = (\overline{g'} f')_1$, $(d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda}(d'', ((\overline{g} f)_1, (\overline{g'} f')_2))$. Hence $(d', \overline{g'} f') * (d, \overline{g} f) = x^{\lambda}(d'', \overline{g''} f'')$. //

Thus $\widehat{(d', f')} * \widehat{(d, f)} = (d', \widehat{f')(d}, f)$. Hence $\text{Span}_{C(x)}\{\widehat{(d, f)} \mid (d, f)$ be a G-diagram such that $f(1) = e\}$ is a subalgebra and the map $(d, f) \longrightarrow \widehat{(d, f)}$ is an algebra isomorphism from $P_k(x, G)$ into $\widehat{P}_k(x, G)$. □

Theorem 3.4.2. *Let $\widehat{\phi}$ be the algebra isomorphism from $\widehat{P}_k(n, G) \longrightarrow \text{End}_{S_n}(W^{\otimes k})$ defined in §3.3. If $n \geq 2k$, then the restriction map of $\widehat{\phi}$ on $P_k(n, G)$ under the identification in Theorem 3.4.1 is equal to the algebra isomorphism ϕ from $P_k(n, G) \longrightarrow \text{End}_{S_n \times G}(W^{\otimes k})$, which is defined in §2.3.*

Proof. We have

$$\begin{aligned}\widehat{\widehat{\phi}(d,f)} &= \widehat{\phi}\left(\sum_{\overline{g}\in\overline{G}}(d,\overline{g}f)\right)\\ &= \sum_{\overline{g}\in\overline{G}}\widehat{\phi}(d,\overline{g}f)\\ &= \sum_{g\in G}\ \sum_{p\sim q\ in\ d\Rightarrow j_p=j_q} E^{(j_1,gf(1)),(j_2,gf(2)),\ldots,(j_k,gf(k))}_{(j_{k+1},gf(k+1)),(j_{k+2},gf(k+2)),\ldots,(j_{2k},gf(2k))}\\ &= \sum_{\substack{g\in G\\ p\sim q\ in\ d\ \Rightarrow j_p=j_q}} E^{(j_1,gf(1)),(j_2,gf(2)),\ldots,(j_k,gf(k))}_{(j_{k+1},gf(k+1)),(j_{k+2},gf(k+2)),\ldots,(j_{2k},gf(2k))}. \qquad (3.12)\end{aligned}$$

Thus the matrices $\widehat{\widehat{\phi}(d,f)} \in \mathrm{End}_{G\times S_n}(W^{\otimes k})$ (see (2.9)). By Theorem 3.3.2, the restriction of $\widehat{\phi}$ on $P_k(n,G)$, under the identification in Theorem 3.4.1 is the isomorphism ϕ from the algebra $P_k(n,G)$ into $\mathrm{End}_{G\times S_n}(W^{\otimes k})$, if $n \geq 2k$. □

In the algebra $\widehat{P}_k(x,G)$, for each minimal G-diagram (d,f), we define the sum

$$\overrightarrow{(d,f)} = \sum_{f_d\in G_d}(d,f_df).$$

In other words,

$$\overrightarrow{(d,f)} = \widehat{\widehat{(d,f)}} = \sum_{f_d\in[G_d]}\widehat{(d,f_df)}.$$

Corollary 3.4.3. *In the algebra* $\widehat{P}_k(x,G)$, $\mathrm{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ *ranges over all minimal* (G,k)*-diagrams*$\}$ *is isomorphic to the algebra* $\overline{P}_k(x,G)$. *Moreover the linear extension of the mapping defined on the basis minimal* (G,k)*-diagram by* $(d,f) \longrightarrow \overrightarrow{(d,f)}$ *is an algebra isomorphism from the algebra* $\overline{P}_k(x,G)$ *into* $\widehat{P}_k(x,G)$.

Proof. The Corollary follows from Theorems 2.3.3 and 3.4.1. □

Also, we conclude by exhibiting known algebras as subalgebras of extended G-vertex colored partition algebras $\widehat{P}_k(x,G)$.

1) $\mathrm{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ ranges over all minimal diagrams with label e in all its vertices$\}$ is isomorphic to the partition algebra $P_k(x|G|)$, which is the centralizer algebra of $S_{n|G|}$ on $W^{\otimes k}$ if $x = n$ $(n|G| \geq 2k)$.

2) $\text{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ ranges over all minimal diagrams with label e in all its vertices, whose underlying partition diagrams are Brauer diagrams$\}$ is isomorphic to the Brauer algebra $B_k(x|G|)$, which is the centralizer algebra of $O_{n|G|}$ on $W^{\otimes k}$ if $x = n$.

3) $\text{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ ranges over all minimal diagrams with label e in all its vertices, whose underlying partition diagrams are permutation diagrams$\}$ is isomorphic to the group algebra $\mathbb{C}(x)(S_k)$, which is the centralizer algebra of $GL_{n|G|}$ on $W^{\otimes k}$ if $x = n$.

4) $\text{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ ranges over all minimal diagrams with label e in all its top row vertices, whose underlying partition diagrams are permutation diagrams$\}$ is isomorphic to the group algebra $\mathbb{C}(x)(G \wr S_k)$.

5) $\text{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ ranges over all minimal diagrams with label e in all its top row vertices and g in all its bottom vertices, whose underlying partition diagrams are permutation diagrams$\}$ is isomorphic to the group algebra $\mathbb{C}(x)(G \times S_k)$.

6) $\text{Span}_{\mathbb{C}(x)}\{\overrightarrow{(d,f)} \mid (d,f)$ ranges over all minimal diagrams with label e in all its top row vertices and g in all its bottom vertices, whose underlying diagrams are identity partition diagram$\}$ is isomorphic to the group algebra $\mathbb{C}(x)(G)$.

Thus we have an algebra sequence

$$\widehat{P}_k(x,G) \supseteq P_k(x,G) \supseteq \overline{P}_k(x,G) \supseteq P_k(x|G|) \supseteq B_k(x|G|) \supseteq \mathbb{C}(x)(S_k).$$

If $x = n$ then the above algebra sequence is the centralizer of the following group sequence

$$S_n \subseteq S_n \times G \subseteq G \wr S_n \subseteq S_{n|G|} \subseteq O_{n|G|} \subseteq GL_{n|G|}$$

acting on $W^{\otimes k}$. Also we have an algebra sequence

$$\widehat{P}_k(x,G) \supseteq P_k(x,G) \supseteq \overline{P}_k(x,G) \supseteq \mathbb{C}(x)(G \wr S_k) \supseteq \mathbb{C}(x)(G \times S_k) \supseteq \mathbb{C}(x)(G).$$

References

[Bl] Matthew Bloss, *G-colored partition algebras as centralizer algebras of wreath products,* J. Algebra, **265**(2003), 690-710.

[Br] R. Brauer, *On algebras which are connected with the semisimple continuous groups,* Ann. of Math., **38**(1937), 854 -872.

[Ha] Tom Halverson, *Characters of the partition algebras,* J . Algebra, **238** (2001), 502 -533.

[Jo] V. F. R Jones, *The Potts model and the symmetric group,* in "Subfactors: Proceedings of the Taniguchi Symposium on Operater Algebra, Kyuzeso, 1993," pp. 259 -267, World Scientific, River edge, NJ, 1994.

[M1] P. Martin, *Representations of graph Temperley – Lieb algebras,* Pupl. Res. Inst. Math. Sci., **26** (1990), 485-503.

[M2] P. Martin, *Potts Models and Related Problems in statistical mechanics,* World Scientific, Singapore, 1991.

[M3] P. Martin, *Temperley – Lieb algebras for non planar statistical mechanics-The partition algebra construction,* J. Knot Theory Ramifications, **3** (1994), 51 -82.

[MS] P. Martin and H. Saleur, *On an algebraic approach to higher-dimensional statistical mechanics,* Comm. Math. Phys., **158** (1993), pp. 155-190.

[PK] M. Parvathi and M. Kamaraj, *Signed Brauer's algebras,* Comm. in Algebra, **26**(3) (1998), 839-855.

[PK1] M. Parvathi and A. Joseph Kennedy, *G-vertex colored partition algebras as centralizer algebras of direct products,* accepted for publication in Comm. in Algebra.

[PS] M. Parvathi and C. Selvaraj, *Signed Brauer's algebras as centralizer algebras,* Comm. in Algebra, **27**(12) (1999), 5985 -5998.

[Sc] I. Schur, *Uber eine Klass von Matrizen, die sich einer gegeben Matrix zuorden lassen,* Dissertation, 1901; reprinted in Gessamelte Abhandlungen, 1, pp. 1-72.

[St] R. Stanley, *Enumerative Combinatorics,* Vol **1**. Wadsworth and Books/Cole (1986).

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