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Some properties of primitive matrices over Bezout B-domain

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Abstract. The properties of primitive matrices (matrices for which the greatest common divisor of the minors of maximal order is equal to 1) over Bezout B - domain, i.e. commutative domain finitely generated principal ideal in which for all a, b, c with $(a, b, c) = 1, c \neq 0$, there exists element $r \in R$, such that $(a+rb, c) = 1$ is investigated. The results obtained enable to describe invariants transforming matrices, i.e. matrices which reduce the given matrix to its canonical diagonal form.

The notation of elementary divisor ring, as rings over which every matrix admits diagonal reduction were introduced by I. Kaplansky in 1949 [1]. Such concept has appeared rather effective at the decision of many tasks in different areas of modern algebra. The whole direction in the theory of rings was formed in which the properties of rings of elementary divisor are studied, new classes of rings were described which possess the property of a diagonal reduction [2-5]. With current of time the interest to such rings has not died away – a lot of publications regularly occur in mathematical journals [6-9]. One of examples of a class elementary divisor ring are the Bezout $B-$rings, i.e. commutative domain of finitely generated principal ideal in which for all a, b, c with $(a, b, c) = 1$, $c \neq 0$, there exists an element $r \in R$, such that $(a + rb, c) = 1$ [10]. This paper is devoted to study of Bezout $B-$domain from the point of view of research of properties of their elements, and also to describe the invariants of transformable matrices, i.e. invertible matrices which the given matrix reduces to its canonical diagonal form.

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Let R be Bezout $B-$domain. A matrix is called *primitive* if the greatest common divisor of minors of the maximal order is equal to 1. In the first part of this paper the properties of primitive rows and columns are studied.

Property 1. *If* $(a_1, \ldots, a_n) = 1, a_n \neq 0, n \geq 3$*, then there are elements* $u_2, \ldots, u_{n-1}$*, such that*

$$(a_1 + u_2 a_2 + \ldots + u_{n-1} a_{n-1}, a_n) = 1.$$

Proof. Let $(a_2, \ldots, a_{n-1}) = \gamma$. Then there are elements $v_2, \ldots, v_{n-1}$, such that

$$v_2 a_2 + \ldots + v_{n-1} a_{n-1} = \gamma.$$

Since $(a_1, \gamma, a_n) = 1$, there is an element $r \in R$, for which $(a_1 + r\gamma, a_n) = 1$. Thus

$$(a_1 + (rv_2)a_2 + \ldots + (rv_{n-1})a_{n-1}, a_n) = 1.$$

□

Property 2. *If* $(a_1, \ldots, a_n) = 1, a_n \neq 0, n \geq 3$*, then there are invertible matrices of the form*

$$\left\| \begin{array}{cccccc} a_1 & v_1 & v_2 & \ldots & v_{n-2} & v_{n-1} \\ a_2 & 1 & 0 & \ldots & 0 & 0 \\ a_3 & 0 & 1 & & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & 0 & 0 & & 1 & 0 \\ a_n & 0 & 0 & \ldots & 0 & v_n \end{array} \right\| = V,$$

$$\left\| \begin{array}{cccccc} u_n & 0 & \ldots & 0 & 0 & u_{n-1} \\ 0 & 1 & & 0 & 0 & u_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & & 1 & 0 & u_2 \\ 0 & 0 & \ldots & 0 & 1 & u_1 \\ a_n & a_{n-1} & \ldots & a_3 & a_2 & a_1 \end{array} \right\| = U.$$

Proof. First we shall show, that the elements $v_1, \ldots, v_n$ can be chosen such that the matrix V will be invertible. By property 1 there are elements $v_1, \ldots, v_{n-2}$, such that

$$(a_1 - v_1 a_2 - \ldots - v_{n-2} a_{n-1}, a_n) = 1.$$

Since $\det V = v_n \gamma_{n-1} - v_{n-1} a_n$, where $\gamma_{n-1} = a_1 - v_1 a_2 - \ldots - v_{n-2} a_{n-1}$, and taking into consideration $(\gamma_{n-1}, a_1) = 1$ we can choose elements v_{n-1}, v_n so, that $\det V = 1$. It is similarly shown, that there are $u_1, \ldots, u_n$ for which $\det U = 1$.

□

Since R is finitely generated principal ideal domain then for all finitely set of relatively prime elements $a_1, \dots, a_n, n \geq 2$, there are elements $u_1, \dots, u_n$, such that

$$u_1 a_1 + \ldots + u_n a_n = 1. \tag{1}$$

Write the elements $u_1, \dots, u_n$ as $\|u_1 \quad \dots \quad u_n\|$. We shall say that elements of the row $\|u_1 \quad \dots \quad u_n\|$ satisfies equation (1). The following statement suggests a method of finding of such all rows with elements which satisfy the equation (1).

Property 3. *Let $(a_1, \dots, a_n) = 1, n \geq 2$, and A be any invertible matrix for which $\|a_1 \quad \dots \quad a_n\|^T$ is its first column. The set*

$$\mathbf{U} = \{\|1 \quad x_2 \quad \dots \quad x_n\| \, A^{-1} | x_i \in R, i = 2, \dots, n\}$$

consist of all rows with elements which satisfy equation (1).

Proof. Let $\|v_1 \quad \dots \quad v_n\| \in \mathbf{U}$, i.e.

$$\|v_1 \quad \dots \quad v_n\| = \|1 \quad x_2 \quad \dots \quad x_n\| \, A^{-1},$$

where $x_i \in R, i = 2, \dots, n$. Then

$$\|v_1 \dots v_n\| \, \|a_1 \quad \dots \quad a_n\|^T = \|1 \quad x_2 \quad \dots \quad x_n\| \, A^{-1} \, \|a_1 \quad \dots \quad a_n\|^T =$$

$$= \|1 \quad x_2 \quad \dots \quad x_n\| \, \|1 \quad 0 \quad \dots \quad 0\|^T = 1.$$

This means that elements of all rows from $\mathbf{U}$ satisfy equation (1).

Let elements of the row $\|u_1 \quad \dots \quad u_n\|$ satisfy equation (1) and $A^{-1} = \|b_{ij}\|_1^n$. Consider the matrix

$$\begin{Vmatrix} u_1 & u_2 & \dots & u_n \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{Vmatrix} = U.$$

Then

$$UA = \begin{Vmatrix} 1 & x_2 & \dots & x_n \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{Vmatrix}.$$

It follows from this that $\|u_1 \quad \dots \quad u_n\| \in \mathbf{U}$. This concludes the proof of our statement.

□

Property 4. *If* $(a_1,\dots,a_n)=1, n\ge 2$, *and* $\varepsilon_1|\varepsilon_2|\dots|\varepsilon_k, \varepsilon_i\ne 0, i=1,\dots,k, 1\le k<n$, *then*

$$(a_1\varepsilon_k, a_2\varepsilon_{k-1},\dots,a_k\varepsilon_1,a_{k+1},\dots,a_n)=(\varepsilon_k, a_2\varepsilon_{k-1},\dots,a_k\varepsilon_1,a_{k+1},\dots,a_n).$$

Proof. Denote $(a_1\varepsilon_k, a_2\varepsilon_{k-1},\dots,a_k\varepsilon_1,a_{k+1},\dots,a_n)=\delta_k$. In order to prove this statement it suffices to show that $\delta_k|\varepsilon_k$. In the case where $k=1$ we have

$$\delta_1=(a_1\varepsilon_1,a_2,\dots,a_n)=(a_1\varepsilon_1,(a_2,\dots,a_n))=(\varepsilon_1,(a_2,\dots,a_n)).$$

So $\delta_1|\varepsilon_1$. Hence the results holds for $k=1$. Let $k\ge 2$ and suppose that the result is established for $m<k$. Then

$$\delta_k=(a_1\varepsilon_k, a_2\varepsilon_{k-1},\dots,a_k\varepsilon_1,a_{k+1},\dots,a_n)=$$
$$=\left(\varepsilon_1\left(a_1\frac{\varepsilon_k}{\varepsilon_1},\dots,a_{k-1}\frac{\varepsilon_2}{\varepsilon_1},a_k\right),a_{k+1},\dots,a_n\right).$$

Since $\frac{\varepsilon_2}{\varepsilon_1}|\frac{\varepsilon_3}{\varepsilon_1}|\dots|\frac{\varepsilon_k}{\varepsilon_1}$ we have by the induction hypothesis

$$\left(a_1\frac{\varepsilon_k}{\varepsilon_1},\dots,a_{k-1}\frac{\varepsilon_2}{\varepsilon_1},a_k,a_{k+1},\dots,a_n\right)=d_1|\frac{\varepsilon_k}{\varepsilon_1}.$$

Therefore

$$\delta_k=d_1\left(\varepsilon_1\frac{\left(a_1\frac{\varepsilon_k}{\varepsilon_1},\dots,a_{k-1}\frac{\varepsilon_2}{\varepsilon_1},a_k\right)}{d_1},\left(\frac{a_{k+1}}{d_1},\dots,\frac{a_n}{d_1}\right)\right)=$$

$$=d_1\left(\varepsilon_1,\left(\frac{a_{k+1}}{d_1},\dots,\frac{a_n}{d_1}\right)\right)=d_1d_2,$$

where $d_2|\varepsilon_1$. Thus $\delta_k=d_1d_2|\frac{\varepsilon_k}{\varepsilon_1}\varepsilon_1=\varepsilon_k$. □

Property 5. *Let* $(a,b,\varphi)=(a_1,b_1,\varphi)=1, aba_1b_1\varphi\ne 0$. *If*

$$ab_1\equiv a_1b(mod\varphi),$$

then

$$(ax+b,\varphi)=(a_1x+b_1,\varphi)$$

for all $x\in R$.

Proof. Set $(a,\varphi)=\alpha$. Then $(a,b)=1$. As $ab_1-a_1b=\varphi t$, we have $\alpha|a_1b$. By the property 4 $\alpha|a_1$. Hence, $\alpha|(a_1,\varphi)=\alpha_1$. From similar reasons $\alpha_1|(a,\varphi)=\alpha$. Hence $\alpha=\alpha_1$. As

$$(a_1(ax+b),\varphi)=(a_1ax+a_1b,\varphi)=(a_1ax+(a_1b+\varphi t),\varphi)=$$

$$=(a_1ax+a_1b,\varphi)=(a(a_1x+b_1),\varphi),$$

so

$$\left(\frac{a_1}{\alpha}(ax+b),\frac{\varphi}{\alpha}\right)=\left(\frac{a}{\alpha}(a_1x+b_1),\frac{\varphi}{\alpha}\right).$$

Therefore

$$\left(ax+b,\frac{\varphi}{\alpha}\right)=\left(a_1x+b_1,\frac{\varphi}{\alpha}\right)=\delta.$$

Since $\alpha|a$ and $\alpha|a_1$, and also $(\alpha,b)=(\alpha,b_1)=1$ then for all elements $x\in R$ the equality

$$(ax+b,\alpha)=(a_1x+b_1,\alpha)=1$$

holds. Hence

$$\delta=\left(ax+b,\frac{\varphi}{\alpha}\right)=\left(ax+b,\frac{\varphi}{\alpha}\alpha\right)=(ax+b,\varphi).$$

Similarly $\delta=(a_1x+b_1,\varphi)$. □

Property 6. *Let $(a_1,\dots,a_n)=1, n\geq 2$, and $\psi\in R$ be any fixed nonzero element, which is not unit. Then there are elements $u_1,\dots,u_n$, which satisfy the following conditions simultaneously:*

a) $u_1a_1+\ldots+u_na_n=1$;

b) $(u_1,\dots,u_i)=1$, *for any fixed* $2\leq i\leq n$;

c) $(u_i,\psi)=1$, *for any fixed* $2\leq i\leq n$.

Proof. Consider the invertible matrix A with first column $\|a_1 \ \ldots \ a_n\|^T$. Let's show, that matrix A can be chosen in such a way that the elements of the matrix $A^{-1}=\|b_{ij}\|_1^n$ satisfy $b_{3i}=\ldots=b_{ni}=0$. Indeed, let A_1 be any invertible matrix with first column $\|a_1 \ \ldots \ a_n\|^T, A^{-1}=\|\bar{b}_{ij}\|_1^n$ and among elements $\bar{b}_{2i},\dots,\bar{b}_{ni}$ there is at least one not zero. Then there is such a matrix $D\in GL_{n-1}(R)$, that

$$D\left\|b_{2i} \ \ldots \ b_{ni}\right\|^T=\left\|\gamma \ 0 \ \ldots \ 0\right\|^T.$$

Thus, the matrix

$$\left((1\oplus D)A_1^{-1}\right)^{-1}=A_1(1\oplus D^{-1})=A$$

will be found.

Let the matrix consisting of the first i columns of the matrix A^{-1} has the form

$$\left\| \begin{matrix} b_{11} & \dots & b_{1,i-1} & b_{1i} \\ b_{21} & \dots & b_{2,i-1} & \gamma \\ b_{31} & \dots & b_{3,i-1} & 0 \\ \dots & \dots & \dots & \dots \\ b_{i1} & \dots & b_{i,i-1} & 0 \\ \hline b_{i+1,1} & \dots & b_{i+1,i-1} & 0 \\ \dots & \dots & \dots & \dots \\ b_{n1} & \dots & b_{n,i-1} & 0 \end{matrix} \right\| = \left\| \begin{matrix} M \\ N \end{matrix} \right\|.$$

By property 3 every set of elements $u_1, \dots, u_n$ satisfying condition a) can be presented as follows:

$$\left\| u_1 \quad \dots \quad u_n \right\| = \left\| 1 \quad x_2 \quad \dots \quad x_n \right\| A^{-1},$$

where $x_i \in R, i = 2, \dots, n$. In order that the our statement be valid it is sufficient, that there are elements $x_2 \dots x_n$, such that

$$\left\| 1 \quad x_2 \quad \dots \quad x_n \right\| \left\| \begin{matrix} M \\ N \end{matrix} \right\| = \left\| q_1 \quad \dots \quad q_i \right\|,$$

where $(q_1, \dots, q_i) = (q_i, \psi) = 1$.

Let $\gamma = 0$. Since the matrix $\left\| \begin{matrix} M \\ N \end{matrix} \right\|$ is primitive, we conclude that $b_{1i} \in U(R)$. Therefore $b_{11}, \dots, b_{1i}$ will be found elements.

Let $\gamma \neq 0$ and $b_{tj} \neq 0, i+1 \leq t \leq n, 1 \leq j \leq i-1$. As $(b_{1i}, \gamma) = 1$ then $(b_{1i}, \gamma, \psi b_{tj}) = 1$. Therefore there is element l, such that $(b_{1i} + \gamma l, \psi b_{tj}) = 1$. This equality implies

i) $(d_{1i}, \psi) = 1$; (2)

ii) $(d_{1i}, b_{tj}) = 1$,

where $d_{1i} = b_{1i} + \gamma l \neq 0$. Then $(d_{1j}, b_{tj}, d_{1i}) = 1$, where $d_{1j} = b_{1j} + b_{2j} l$. Therefore, there is m, such that $(d_{1j} + b_{tj} m, d_{1i}) = 1$. Taking into account equality (2), we are convinced, that elements of the first row of the matrix

$$\left(\left\| \begin{matrix} 1 & l & 0 & \dots & 0 & m \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{matrix} \right\| \oplus E_{n-t} \right) \left\| \begin{matrix} M \\ N \end{matrix} \right\|,$$

where E_{n-t} is identity $(n-t)\times(n-t)$ matrix, will satisfy to all the requirements of our statement.

If $N = \mathbf{0}$, or $i = n$ (in this case matrix N is empty) it follows from invertibility of a matrix A_1, that $M \in GL_i(R)$. Therefore $(b_{1i}, \gamma) = 1$, so $(b_{1i}, \gamma, \psi) = 1$. As well as in the previous cases there is r, that $(b_{1i} + \gamma r, \psi) = 1$. Hence, the elements of the first row of the matrix

$$\left(\begin{Vmatrix} 1 & r \\ 0 & 1 \end{Vmatrix} \oplus E_{i-2}\right) M$$

also will be found elements. The statement is proved. □

By the theorem 5.2 of [1] R is elementary divisor domain. Therefore for every nonsingular $n \times n$ matrix A over R exist invertible matrices P and Q (further we shall call them as transformable matrices), such that

$$PAQ = \mathrm{diag}(\varphi_1, \dots, \varphi_n) = \Phi, \varphi_i|\varphi_{i+1}, i = 1, \dots, n-1. \qquad (3)$$

Denote $\mathbf{P}_A$ the set of invertible matrices P, which satisfy equality (3). In a final part of this paper the properties of set of transformable matrices $\mathbf{P}_A$ will be studied. It was shown in papers [11-13], that $\mathbf{P}_A = \mathbf{G}_\Phi P$, where P be any fixed matrix from equality (3), and $\mathbf{G}_\Phi$ is multiplicative group, which consists of all invertible matrices of the form

$$\begin{Vmatrix} h_{11} & h_{12} & \dots & h_{1,n-1} & h_{1n} \\ \frac{\varphi_2}{\varphi_1} h_{21} & h_{22} & \dots & h_{2,n-1} & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_n}{\varphi_1} h_{n1} & \frac{\varphi_n}{\varphi_2} h_{22} & \dots & \frac{\varphi_n}{\varphi_{n-1}} h_{n,n-1} & h_{nn} \end{Vmatrix}.$$

In [12,14] it is proved that the group $\mathbf{G}_\Phi$ and the set of left transformable matrices $\mathbf{P}_A$ play the main role in the description of the associated matrices, which have the given canonical diagonal form Φ.

Proposition ([12, 14]). *Let $A = P_A^{-1}\Phi Q_A^{-1}$, $B = P_B^{-1}\Phi Q_B^{-1}$. The following are equivalent:*

a) A and B are right associates ($B = AU$, $U \in GL_n(R)$);

b) $P_B = HP_A$, where $H \in \mathbf{G}_\Phi$;

c) $\mathbf{P}_B = \mathbf{P}_A$.

We apply the obtained results to describe the properties of transformable matrices. Denote

$$\Phi_1 = E_n, \Phi_i = \operatorname{diag}\left(\frac{\varphi_i}{\varphi_1}, \dots, \frac{\varphi_i}{\varphi_{i-1}}, \underbrace{1, \dots, 1}_{n-i+1}\right), i = 2, \dots, n.$$

Definition 1. *Let* $\left\|a_1 \quad \dots \quad a_n\right\|^T$ *be primitive column and*

$$\Phi_i \left\|a_1 \quad \dots \quad a_n\right\|^T \sim \left\|\delta_i \quad 0 \quad \dots \quad 0\right\|^T,$$

$i = 1, \dots, n$. *The column* $\left\|\delta_1 \quad \dots \quad \delta_n\right\|^T$ *is called* $\Phi-$*rod of the column* $\left\|a_1 \quad \dots \quad a_n\right\|^T$.

Theorem 1. *If* $H \in \mathbf{G}_\Phi$, *then* Φ*-rods of columns* $\left\|a_1 \quad \dots \quad a_n\right\|^T$, $H\left\|a_1 \quad \dots \quad a_n\right\|^T$ *coincide.*

Proof. Since j column of the matrix H has the form

$$\left\|h_{1j} \quad \dots \quad h_{jj} \quad \tfrac{\varphi_{j+1}}{\varphi_j} h_{j+1,j} \quad \dots \quad \tfrac{\varphi_n}{\varphi_j} h_{nj}\right\|^T, 1 \le j \le n-1,$$

then $\Phi_i h_j =$

$$= \left\|\tfrac{\varphi_i}{\varphi_1} h_{1j} \dots \tfrac{\varphi_i}{\varphi_{j-1}} h_{j-1,j} \quad \tfrac{\varphi_i}{\varphi_j} h_{jj} \quad \tfrac{\varphi_i}{\varphi_j} h_{j+1,j} \dots \tfrac{\varphi_i}{\varphi_j} h_{ij} \quad \tfrac{\varphi_{i+1}}{\varphi_j} h_{i+1,j} \dots \tfrac{\varphi_n}{\varphi_j} h_{nj}\right\|^T$$

$$= \frac{\varphi_i}{\varphi_j} \left\|\tfrac{\varphi_j}{\varphi_1} h_{1j} \quad \dots \quad \tfrac{\varphi_j}{\varphi_{j-1}} h_{j-1,j} \quad h_{jj} \quad \dots \quad h_{ij} \quad \tfrac{\varphi_{i+1}}{\varphi_i} h_{i+1,j} \quad \dots \quad \tfrac{\varphi_n}{\varphi_i} h_{nj}\right\|^T.$$

Hence, $\dfrac{\varphi_i}{\varphi_j} | \Phi_i h_j, i = 2, \dots, n, j = 1, \dots, n-1, i > j$. It means that the equalities

$$\Phi_i H = K_i \Phi_i, \tag{4}$$

$i = 2, \dots, n$, holds. As all the matrices Φ_i are nonsingular, and the matrix H is invertible, then from equality (4) follows $K_i \in GL_n(R)$. Therefore

$$\Phi_i H \left\|a_1 \quad \dots \quad a_n\right\|^T = K_i \Phi_i \left\|a_1 \quad \dots \quad a_n\right\|^T \sim$$
$$\sim \Phi_i \left\|a_1 \quad \dots \quad a_n\right\|^T \sim \left\|\delta_i \quad 0 \quad \dots \quad 0\right\|^T,$$

$i = 2, \dots, n$. It remains to note that $\delta_1 = 1$ which concludes the proof of the theorem. ∎

Theorem 2. *Let $\|\delta_1 \ \ \dots \ \ \delta_n\|^T$ be Φ-rod of the primitive column $\|a_1 \ \ \dots \ \ a_n\|^T$. Then there is a matrix $H \in \mathbf{G}_\Phi$ for which*

$$H\,\|a_1 \ \ \dots \ \ a_n\|^T = \|b \ \ \delta_2 \ \ \dots \ \ \delta_n\|^T.$$

Proof. Property 6 implies that there are elements $u_1, \dots, u_n$, such that

$$\frac{\varphi_n}{\varphi_1}u_1a_1 + \dots + \frac{\varphi_n}{\varphi_{n-1}}u_{n-1}a_{n-1} + u_na_n = \delta_n,$$

where $\left(u_n, \frac{\varphi_n}{\varphi_1}\right) = 1$. Since $\frac{\varphi_n}{\varphi_{n-1}} | \frac{\varphi_n}{\varphi_{n-2}} | \dots | \frac{\varphi_n}{\varphi_1}$, taking in account the property 4

$$\left(\frac{\varphi_n}{\varphi_1}u_1, \dots, \frac{\varphi_n}{\varphi_{n-1}}u_{n-1}, u_n\right) = \left(\frac{\varphi_n}{\varphi_1}, \frac{\varphi_n}{\varphi_2}u_2, \dots, \frac{\varphi_n}{\varphi_{n-1}}u_{n-1}, u_n\right) =$$

$$= \left(\frac{\varphi_n}{\varphi_2}u_2, \dots, \frac{\varphi_n}{\varphi_{n-1}}u_{n-1}, \left(u_n, \frac{\varphi_n}{\varphi_1}\right)\right) = 1.$$

By property 2 we shall complete a primitive row $\left\|\frac{\varphi_n}{\varphi_1}u_1 \ \ \dots \ \ \frac{\varphi_n}{\varphi_{n-1}}u_{n-1} \ \ u_n\right\|$ to an invertible matrix H_n in which this row will be last, and other elements of this matrix, which lies under the main diagonal will be zero. Then $H_n \in \mathbf{G}_\Phi$ and

$$H_n\,\|a_1 \ \ \dots \ \ a_n\|^T = \|b_1 \ \ \dots \ \ b_{n-1} \ \ \delta_n\|^T.$$

By theorem 1 this column will have again Φ-rod $\|\delta_1 \ \ \dots \ \ \delta_n\|^T$. Therefore

$$\Phi_{n-1}H_n\,\|a_1 \ \ \dots \ \ a_n\|^T \sim \|\delta_{n-1} \ \ 0 \ \ \dots \ \ 0\|^T.$$

Hence, there are elements $v_1, \dots, v_n$, such that

$$\frac{\varphi_{n-1}}{\varphi_1}v_1b_1 + \dots + \frac{\varphi_{n-1}}{\varphi_{n-2}}v_{n-2}b_{n-2} + v_{n-1}b_{n-1} + v_n\delta_n = \delta_{n-1}.$$

Moreover, as it follows from property 6, these elements can be chosen in such a manner that $(v_1, \dots, v_{n-1}) = 1$ and $\left(v_{n-1}, \frac{\varphi_{n-1}}{\varphi_1}\right) = 1$. Thus we have

$$\left(\frac{\varphi_{n-1}}{\varphi_1}v_1, \dots, \frac{\varphi_{n-1}}{\varphi_{n-2}}v_{n-2}, v_{n-1}\right) = 1.$$

It means, that in the group $\mathbf{G}_\Phi$ there is a matrix H_{n-1} with the following two last rows:

$$\left\|\begin{matrix} \frac{\varphi_{n-1}}{\varphi_1}v_1 & \dots & \frac{\varphi_{n-1}}{\varphi_{n-2}}v_{n-2} & v_{n-1} & v_n \\ 0 & \dots & 0 & 0 & 1 \end{matrix}\right\|.$$

Consequently,

$$H_{n-1}H_n \left\|a_1 \quad \dots \quad a_n\right\|^T = \left\|d_1 \quad \dots \quad d_{n-2} \quad \delta_{n-1} \quad \delta_n\right\|^T.$$

Continuing the described process, on (n-1) step we shall receive the matrix $H = H_2 \cdots H_n \in \mathbf{G}_\Phi$, such that $HA = \left\|b \quad \delta_2 \quad \dots \quad \delta_n\right\|^T$. The theorem is proved.

□

Denote

$$\Delta_i = \left(\frac{\varphi_i}{\varphi_{i-1}}, \frac{a_i}{\delta_{i-1}}, \dots, \frac{a_n}{\delta_{i-1}}\right), i = 2, \dots, n. \tag{5}$$

Theorem 3. *Let $\left\|\delta_1 \quad \dots \quad \delta_n\right\|^T$ be Φ-rod of the primitive column $\left\|a_1 \quad \dots \quad a_n\right\|^T$. Then the elements δ_i satisfy the following conditions:*

a) $\delta_i = \Delta_2 \cdots \Delta_i, i = 2, \dots, n;$

b) $\delta_i | \frac{\varphi_i}{\varphi_1}, i = 2, \dots, n.$

Proof. Since $\delta_1 = 1$, we obtain from property 4

$$\delta_2 = \left(\frac{\varphi_2}{\varphi_1}a_1, a_2, \dots, a_n\right) = \Delta_2,$$

$$\delta_3 = \left(\frac{\varphi_3}{\varphi_1}a_1, \frac{\varphi_3}{\varphi_2}a_2, a_3, \dots, a_n\right) = \left(\frac{\varphi_3}{\varphi_2}\left(\frac{\varphi_2}{\varphi_1}, a_2\right), a_3, \dots, a_n\right) =$$

$$= \delta_2\left(\frac{\varphi_3}{\varphi_2}\frac{\left(\frac{\varphi_2}{\varphi_1}, a_2\right)}{\delta_2}, \frac{a_3}{\delta_2}, \dots, \frac{a_n}{\delta_2}\right) = \Delta_2\left(\frac{\varphi_3}{\varphi_2}, \frac{a_3}{\delta_2}, \dots, \frac{a_n}{\delta_2}\right) = \Delta_2\Delta_3.$$

Having continued on analogy our reasons, we obtain $\delta_i = \Delta_2 \cdots \Delta_i, i = 2, \dots, n$.

By (5) $\Delta_i | \frac{\varphi_i}{\varphi_{i-1}}, i = 2, \dots, n$. Hence

$$\delta_i = \Delta_2\Delta_3 \cdots \Delta_{i-1}\Delta_i | \frac{\varphi_2}{\varphi_1}\frac{\varphi_3}{\varphi_2} \cdots \frac{\varphi_{i-1}}{\varphi_{i-2}}\frac{\varphi_i}{\varphi_{i-1}} = \frac{\varphi_i}{\varphi_1}, i = 2, \dots, n.$$

□

The following corollary follows from the theorems 2 and 3.

Corollary 1. *If* $\left\|\delta_1 \quad \dots \quad \delta_n\right\|^T$ *be* Φ*-rod of the primitive column* $\left\|a_1 \quad \dots \quad a_n\right\|^T$, *then there is a matrix* $H \in \mathbf{G}_\Phi$, *such that*

$$H\left\|a_1 \quad \dots \quad a_n\right\|^T = \left\|\begin{matrix} b \\ \Delta_2 \\ \Delta_2\Delta_3 \\ \dots \\ \Delta_2\Delta_3\cdots\Delta_n \end{matrix}\right\|,$$

where $\Delta_i|\frac{\varphi_i}{\varphi_{i-1}}, i = 2, \dots, n.$

Definition 2. *Let* $P \in GL_n(R)$ *and* $\bar{p}_1, \dots, \bar{p}_n$ *be its columns,* $\left\|\delta_{i1} \quad \dots \quad \delta_{in}\right\|^T$ *is* $\Phi - rod$ *of column* $\bar{p}_i, i = 1, \dots, n$. *The matrix* $\|\delta_{ij}\|_1^n$ *is called* $\Phi - rod$ *of the matrix* P.

Theorem 4. Φ*-rods of matrices from* $\mathbf{P}_A$ *coincide.*

Proof. Let P_1 be any matrix from $\mathbf{P}_A$. Since $\mathbf{P}_A = \mathbf{G}_\Phi P$ then there exists a matrix $H \in \mathbf{G}_\Phi$, such that $P_1 = HP$. According to the theorem 1 Φ-rods correspond columns of matrices P and P_1 coincide. Therefore will be Φ-rods of these matrices coincide. □

Since all matrices of the set $\mathbf{P}_A$ have identical Φ-rods, it is possible to speak about Φ-rod of set of transformable matrices $\mathbf{P}_A$, having identified it with Φ-rod of any matrix of this set. Using the theorem 2, we obtain.

Corollary 2. *Let* $\|\delta_{ij}\|_1^n$ *be* Φ*-rod of the set* $\mathbf{P}_A$. *Then there is matrix* $P_i \in \mathbf{P}_A$, *which have* i *column of the form* $\left\|* \quad \delta_{i1} \quad \dots \quad \delta_{in}\right\|^T, 1 \le i \le n.$

Corollary 3. *If the matrices* A *and* B *have the canonical diagonal form* Φ *and are right associates, then* Φ*-rods of sets* $\mathbf{P}_A$ *and* $\mathbf{P}_B$ *coincide.*

Proof. By proposition $\mathbf{P}_A = \mathbf{P}_B$, so that Φ-rods of these sets coincide. □

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