Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 2. **(2005).** pp. 36 – 45

Steiner P-algebras

Sucheta Chakrabarti

Communicated by V.A. Artamonov

Abstract. General algebraic systems are able to formalize problems of different branches of mathematics from the algebraic point of view by establishing the connectivity between them. It has lots of applications in theoretical computer science, secure communications etc. Combinatorial designs play significant role in these areas. Steiner Triple Systems (STS) which are particular case of Balanced Incomplete Block Designs (BIBD) from combinatorics can be regarded as algebraic systems. Steiner quasigroups (Squags) and Steiner loops (Sloops) are two well known algebraic systems which are connected to STS. There is a one-to-one correspondence between STS and finite Squags and finite Sloops. A new algebraic system w.r.to a ternary operation P based on a Steiner Triple System introduced in [3].

In this paper the abstraction and the generalization of the properties of the ternary operation defined in [3] has been made. A new class of algebraic systems *Steiner P-algebras* has been introduced. The one-to-one correspondence between STS on a linearly ordered set and finite Steiner P-algebras has been established. Some identities have been proved.

Author is grateful to Dr.P.K.Saxena; Director, SAG, DRDO for his permission and constant encouragement for the research. My heartiest thanks to Dr.R.K.Khanna; Scientist 'E', SAG for valuable discussions and constant inspiration throughout this research work. Author also expressed her heartiest gratitude to Prof V.A.Artamonov for his valuable comments for the improvements of the paper.

2000 Mathematics Subject Classification: *08A62.*

Key words and phrases: *Balanced incomplete block designs, Steiner triple systems, General algebraic structures, Steiner quasigroup, Steiner loop, Steiner P-algebra.*

1. Preliminaries

In this section the basic definitions, properties and some results of combinatorial designs [4] and algebraic systems [1, 2, 3] have been given.

Definition 1.1. Let v, k, and λ be positive integers such that $v > k \geq 2$. A $(v,\ k,\ \lambda)$ - *balanced incomplete block design (BIBD)* is a pair $(X, \mathcal{A})$ such that the following properties are satisfied:

1. X is a set of v elements called *points*,
2. $\mathcal{A}$ is a collection of subsets of X called *blocks*,
3. each block contains exactly k *points*, and
4. every pair of distinct points is contained in exactly λ *blocks*

Two basic properties of BIBD are as follows:

Theorem 1.2 ([4]). *In a (v, k, λ)-BIBD, every point occurs in exactly $r = \frac{\lambda(v-1)}{k-1}$ blocks.*

Theorem 1.3 ([4]). *The number of blocks in a (v, k, λ)-BIBD is exactly $b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k}$.*

Definition 1.4. A *Steiner Triple System* (STS) of order v is a $(v,\ 3,\ 1)$ -*BIBD*. In other words a STS on the set X is a collection $\mathcal{S}_X$ of three elements subsets of X called blocks such that, any pair of distinct elements of X is contained in a unique block of $\mathcal{S}_X$.

Theorem 1.5 ([2]). *If $\mathcal{S}_X$ is a STS on a finite set X of order v, then*

1. $v \equiv 1,\ 3 \mod 6$
2. $|\mathcal{S}_X| \equiv \frac{v(v-1)}{6}$.

Definition 1.6. A *groupoid* satisfying the following identities :

1. $x \cdot x = x$
2. $x \cdot y = y \cdot x$
3. $x \cdot (x \cdot y) = y$

is called a *Steiner quasigroup* (Squag).

Theorem 1.7 ([2]). *If $< \mathcal{A},\ \cdot >$ is a Squag define $\mathcal{S}_A$ to be the set of three element subsets $\{x, y, z\}$ of $\mathcal{A}$ such that the product of any two elements gives the third. Then $\mathcal{S}_A$ is a STS on $\mathcal{A}$.*

Definition 1.8 ([2])**.** A *groupoid* with a distinguished element $< \mathcal{A},\ \cdot,\ 1 >$ is called a *Steiner loop* (Sloop) if the following identities hold :

1. $x \cdot x = 1$
2. $x \cdot y = y \cdot x$
3. $x \cdot (x \cdot y) = y$

Theorem 1.9 ([2])**.** *If* $< \mathcal{A},\ \cdot,\ 1 >$ *is a* Sloop *and* $|\mathcal{A}| \geq 2$, *define* $\mathcal{S}_A$ *to be the three element subsets of* $\mathcal{A} \setminus \{1\}$ *such that the product of any two distinct elements gives the third. Then* $\mathcal{S}_A$ *is a STS on* $\mathcal{A} \setminus \{1\}$.

It is evident that any STS, $\mathcal{S}_A$ on a set $\mathcal{A}$ enables to construct Squag on the set $\mathcal{A}$ and Sloop on the set $\mathcal{A} \cup \{1\}$ w.r.to the following binary operation

1. $x \cdot y = \begin{cases} z & \text{if } \{x, y, z\} \in \mathcal{S}_A, \\ x & \text{if } x = y. \end{cases}$
2. $x \cdot y = \begin{cases} z & \text{if } \{x, y, z\} \in \mathcal{S}_A, \\ 1 & \text{if } x = y \end{cases}$

for all $x,\ y \in \mathcal{A}$, respectively. So, there is a one-to-one correspondence between STS and finite Squags and Sloops.

The ternary operation P for a given STS on a finite linear order (l.o) set $\mathcal{A}$ (see [3]) is given below.

Definition 1.10. Let $\mathcal{A}$ be a finite linear ordered set and $\mathcal{S}_A$ be the STS on $\mathcal{A}$. Define the ternary operation P on $\mathcal{A}$ as follows:

1. $P(x, y, z) = \min\{x, y, z\}$ if $\{x, y, z\} \in \mathcal{S}_A$
2. $P(x, y, z) = a$ where $\{x, z, b\}$, $\{y, b, a\} \in \mathcal{S}_A$ for some elements $b,\ a \in \mathcal{A}$ if $\{x, y, z\} \notin \mathcal{S}_A$ and $x \neq z$,
3. $P(x, y, z) = a$ where $\{x, y, a\} \in \mathcal{S}_A$ if $x = z \neq y$
4. $P(x, y, z) = x$ if $x = y = z$.

Theorem 1.11 ([3])**.** *Let "$\cdot$"' be the binary operation and "P" be the ternary operation as defined in definition 1.6 and Definition 1.10 respectively on a finite l.o set* $\mathcal{A}$ *for a given STS* $\mathcal{S}_A$ *on* $\mathcal{A}$. *Then*

$$P(x, y, z) = \begin{cases} \min\{x, y, z\} & \text{if } \{x, y, z\} \in \mathcal{S}_A \\ y \cdot (x \cdot z) & \text{if } \{x, y, z\} \notin \mathcal{S}_A. \end{cases}$$

Theorem 1.12 ([3]). *Let $\mathcal{A}$ be a finite l.o set, $\mathcal{S}_A$ be a STS on the given set and P be the ternary operation w.r.to $\mathcal{S}_A$ as defined in definition 1.10.The set $\mathcal{S}$ consists of all three element subsets $\{x, y, z\}$ of $\mathcal{A}$ such that*

$$P(x, y, z) \;=\; \min\;\{x, y, z\}.$$

Then $\mathcal{S} \;=\; \mathcal{S}_A$.

2. Steiner P - algebra and Some Identities

In this section a new algebraic structure on a linear ordered set has been defined. The connectivity between the STS on a l.o set and variety of finite algebraic structures has been established. Some important identities have been proved.

Definition 2.1. A Steiner P - algebra $< \mathcal{A},\ P >$ is an algebra on a l.o set $\mathcal{A}$ with one ternary operation P satisfying the following properties for all $x, y, z \in \mathcal{A}$:

1. For any two distinct elements x, y there exists an element $z \neq x \neq y$ such that $P(x, y, z) = \min\{x, y, z\}$ irrespective of positional order of x, y, z.

2. For $x \neq z$, $P(x, y, z) = a$ for some elements $b, a \in \mathcal{A}$ such that $P(x, z, b) = \min\{x, z, b\}$ and $P(y, b, a) = \min\{y, b, a\}$ irrespective of positional order of x, z, b, y, a.

3. For $x \neq y \neq z$, $P(x, y, z) = \min\{x, y, z\}$ where for any two distinct elements x, y the third element z satisfies the condition 1.

4. For $x \neq y$, $P(x, y, x) = a$ where $P(x, y, a) = \min\{x, y, a\}$ irrespective of positional order of x, y, a.

5. $P(x, x, x) = x$.

Proposition 2.2. *The class of Steiner P -algebras is axiomatic.*

Proof. Let K be the class of Steiner P-algebras. Let $< \mathcal{A},\ P > \in K$ and $\phi(x, y, z) = (P(x, y, z) = \min\{x, y, z\})$. Then from the definition of Steiner P-algebra it follows that the following formulas satisfy in $\mathcal{A}$:

1. $(x \neq y) \rightarrow (\exists z(z \neq x \neq y)(\phi(x, y, z) \wedge \phi(x, z, y) \wedge \phi(y, x, z) \wedge \phi(y, z, x)$ $\wedge \phi(z, x, y) \wedge \phi(z, y, x)))$.

2. $((x \neq z) \rightarrow (P(x,y,z) = a)) \leftrightarrow ((x \neq z) \rightarrow (\exists b(b \neq y \neq x \neq z)(\phi(x,z,b) \wedge \phi(x,b,z) \wedge \phi(z,x,b) \wedge \phi(z,b,x) \wedge \phi(b,x,z) \wedge \phi(b,z,x))) \wedge ((y \neq b) \rightarrow (\exists a(a \neq y \neq b)(\phi(y,b,a) \wedge \phi(y,a,b) \wedge \phi(b,y,a) \wedge \phi(b,a,y) \wedge \phi(a,y,b) \wedge \phi(a,b,y))))$.

3. $((x \neq y \neq z) \rightarrow \phi(x,y,z)) \leftrightarrow ((x \neq z) \rightarrow (\exists y(y \neq x \neq z)(\phi(x,z,y) \wedge \phi(x,y,z) \wedge \phi(z,x,y) \wedge \phi(z,y,x) \wedge \phi(y,x,z) \wedge \phi(y,z,x)))$.

4. $((x \neq y) \rightarrow (P(x,y,x) = a)) \leftrightarrow ((x \neq y) \rightarrow (\exists a(a \neq x \neq y)(\phi(x,y,a) \wedge \phi(x,a,y) \wedge \phi(y,x,a) \wedge \phi(y,a,x) \wedge \phi(a,x,y) \wedge \phi(a,y,x)))$.

5. $P(x,x,x) = x$.

So K is said to be axiomatized by the set of above mentioned axioms for K. Hence the class of Steiner P-algebras is axiomatic. □

Example 2.3. Let $\mathcal{A} = \{1,2,3,4,5,6,7\}$, the STS on $\mathcal{A}$ is $\mathcal{S}_A = \{\{1,2,4\}\ \{2,3,5\}\ \{3,4,6\}\ \{4,5,7\}\ \{1,5,6\}\ \{2,6,7\}\ \{1,3,7\}\}$ and the ternary operation P is as defined in Definition 1.10. Then it is evident that $< \mathcal{A},\ P >$ is a Steiner P-algebra.

Proposition 2.4. *For any two distinct elements x, y of a Steiner P-algebra $< \mathcal{A},\ P >$ there exists a unique third element $z \neq x \neq y$ such that $P(x,y,z) = \min\{x,y,z\}$.*

Proof. Let $x \neq y$ and suppose there exist $z, z' \neq x \neq y$ such that $\phi(x,y,z) = \min\{x,y,z\}$ and $\phi(x,y,z') = \min\{x,y,z'\}$ hold. By the axiom (3) of Proposition 2.2 it follows that

$$\phi(x,y,z) \wedge \phi(x,z,y) \wedge \phi(y,x,z) \wedge \phi(y,z,x) \wedge \phi(z,x,y) \wedge \phi(z,y,x)$$

and $\phi(x,y,z') \wedge \phi(x,z',y) \wedge \phi(y,x,z') \wedge \phi(y,z',x) \wedge \phi(z',x,y)$ holds. This implies that $z = P(x,y,x) = z'$ by (4) of Proposition 2.2. □

Proposition 2.5. *In a Steiner P-algebra $< \mathcal{A}, P >$ the following identity holds: $P(x,x,y) = y = P(y,x,x)$.*

Proof. If $x = y$ then the identity holds by (5) of the Definition 2.1. Now suppose $x \neq y$ and $P(x,x,y) = a$ From (2) of Proposition 2.2 and Proposition 2.4 it follows that
$((x \neq y) \rightarrow (\exists! b(b \neq x \neq y)(\phi(x,y,b) \wedge \phi(x,b,y) \wedge \phi(y,x,b) \wedge \phi(y,b,x) \wedge \phi(b,x,y) \wedge \phi(b,y,x)))) \wedge ((x \neq b) \rightarrow (\exists! a(a \neq x \neq b)(\phi(x,b,a) \wedge \phi(x,a,b) \wedge \phi(b,x,a) \wedge \phi(b,a,x) \wedge \phi(a,x,b) \wedge \phi(a,b,x))))$
Now let denote A= $((x \neq y) \rightarrow (\exists! b(b \neq x \neq y)(\phi(x,y,b) \wedge \phi(x,b,y) \wedge \phi(y,x,b) \wedge \phi(y,b,x) \wedge \phi(b,x,y) \wedge \phi(b,y,x))))$
and B= $((b \neq x) \rightarrow (\exists y(y \neq b \neq x)(\phi(x,y,b) \wedge \phi(x,b,y) \wedge \phi(y,x,b) \wedge$

$\phi(y, b, x) \wedge \phi(b, x, y) \wedge \phi(b, y, x)))).$
So A and $A \to B$ hold. It follows B holds. So $a = y$ (by the uniqueness property) Hence $P(x, x, y) = y$. Similarly $P(y, x, x) = y$. □

Proposition 2.6. *In a Steiner P-algebra $< \mathcal{A},\ P >$ the identity $P(x, y, x) = P(y, x, y)$ holds for all $x, y \in \mathcal{A}$.*

Proof. If $x = y$ it follows from (5) of the Definition 2.1.
Now suppose $x \neq y$ and $P(x, y, x) = a$. From (4) of the Proposition 2.2 and the Proposition 2.4 it follows that

$$A = ((x \neq y) \to (\exists! a(a \neq x \neq y)(\phi(x, y, a) \wedge \phi(x, a, y) \wedge \\ \wedge \phi(y, x, a) \wedge \phi(y, a, x) \wedge \phi(a, x, y) \wedge \phi(a, y, x))))$$

holds. This is equivalent to

$$B = ((y \neq x) \to (\exists! a(a \neq y \neq x)(\phi(y, x, a) \wedge \phi(y, a, x) \wedge \\ \wedge \phi(x, y, a) \wedge \phi(x, a, y) \wedge \phi(a, y, x) \wedge \phi(a, x, y)))).$$

So B holds in $< \mathcal{A},\ P >$ which implies that $P(y, x, y) = a$.
Hence $P(x, y, x) = P(y, x, y)$. □

Proposition 2.7. *In a Steiner P-algebra $< \mathcal{A},\ P >$ the identity $P(x, y, z) = P(z, y, x)\ \ \forall\ x, y, z \in \mathcal{A}$ holds.*

Proof. To prove this identity we have to consider the following two cases

(i) $P(x, y, z) = \min\{x, y, z\}\ for\ x \neq y \neq z$,

(ii) $P(x, y, z) = a\ for\ x \neq y \neq z$; where $a \in \mathcal{A}$ and $a \notin \{x, y, z\}$

Let us first consider case (i). From the axiom (3) of Proposition 2.2 it follows that

$$A = ((x \neq z) \to (\exists y(y \neq x \neq z)(\phi(x, z, y) \wedge \phi(x, y, z) \wedge \\ \wedge \phi(z, x, y) \wedge \phi(z, y, x) \wedge \phi(y, x, z) \wedge \phi(y, z, x))))$$

holds. Let

$$B = ((z \neq x) \to (\exists y(y \neq z \neq x)(\phi(z, x, y) \wedge \phi(z, y, x) \wedge \\ \wedge \phi(x, z, y) \wedge \phi(x, y, z) \wedge \phi(y, z, x) \wedge \phi(y, x, z))))$$

So B is equivalent to A. This implies that B holds. Hence $P(z, y, x) = \min\{x, y, z\}$ by (3) of Proposition 2.2.
Now let us consider case (ii). From (2) of Proposition 2.2 it follows that

$$((x \neq z) \to (\exists b(b \neq y \neq x \neq z)(\phi(x, z, b) \wedge \phi(x, b, z) \wedge \phi(z, x, b) \wedge \\ \wedge \phi(z, b, x) \wedge \phi(b, x, z) \wedge \phi(b, z, x))))$$

This implies $P(x,b,z) = \min\{x,b,z\}$ *for* $x \neq b \neq z$. So $P(x,b,z) = P(z,b,x)$ (by case (i)) -(I). Hence from the assumption and (I) we get the following

$$((z \neq b \neq x) \rightarrow \phi(z,b,x)) \wedge ((y \neq a \neq b) \rightarrow \phi(y,a,b))$$

This implies that $P(z,y,x) = a$ (by (2) and (3) of Proposition 2.2). Hence $P(x,y,z) = P(z,y,x) \ \forall\ x,y,z \in \mathcal{A}$. □

Theorem 2.8. *If* $< \mathcal{A},\ P >$ *is a Steiner* P*-algebra , define* $\mathcal{S}_A$ *to be the set of three element subsets* $\{x,y,z\}$ *of* $\mathcal{A}$ *such that* $P(x,y,z) = min\{x,y,z\}$ *irrespective of the positional order of* x,y,z *. Then* $\mathcal{S}_A$ *is a STS on* $\mathcal{A}$.

Proof. Let $x \neq y \neq z$ and $\{x,y,z\} \in \mathcal{S}_A$. This implies

$$\phi(x,y,z) \wedge \phi(x,z,y) \wedge \phi(y,x,z) \wedge \phi(y,z,x) \wedge \phi(z,x,y) \wedge \phi(z,y,x)$$

holds in A i.e. $P(x,y,z) = P(x,z,y) = P(y,x,z) = P(y,z,x) = P(z,x,y) = P(z,y,x) = \min\{x,y,z\}$. From (3) of Proposition 2.2 and Proposition 2.4 it follows that for any two distinct elements of three element subsets $\{x,y,z\}$ of $\mathcal{S}_A$ the third element is the required element so that the value under the operation P is $\min\{x,y,z\}$.

Now let $x = y \neq z$ and $\{x,y,z\} \in \mathcal{S}_A$. Hence

$$P(x,x,z) = P(x,z,x) = \min\{x,z\}. \tag{1}$$

It follows that $z = \min\{x,z\}$ (by Proposition 2.5). So $z < x$. By (1) and (4) of Proposition 2.2 and by the Proposition 2.4 it follows that $P(x,z,x) = a$ for unique

$$a \neq x \neq z. \tag{2}$$

So from (1) and (2) it follows that $z = a$ which is a contradiction. Hence $z = x$. So if any two are equal, all three are equal. Consequently for any two distinct elements of $\mathcal{A}$ there is a unique third element (distinct from the two) such that it gives the minimum of three irrespective of the positional order of the elements under the operation P. Thus $\mathcal{S}_A$ is indeed a STS on $\mathcal{A}$. □

It is trivial to prove that a finite l.o set $\mathcal{A}$ w. r. to the operation P as defined in Definition 1.10 for a given STS $\mathcal{S}_A$ on $\mathcal{A}$ is a Steiner P- algebra. From the theorem 2.8 it follows that the axiomatic class of Steiner P-algebras exactly captures the Steiner Triple Systems.

Proposition 2.9. *In a finite Steiner P -algebra $< \mathcal{A},\ P >$ the number of disjoint blocks with each arbitrary but fixed block of the corresponding STS is $n = 6m^2 - 8m + 2$ and $6m^2 - 4m$ respectively for $|\mathcal{A}| = v = 1 + 6m$ and $3 + 6m$ where $m > 0$.*

Proof. Let $< \mathcal{A},\ P >$ be a finite Steiner P-algebra. Then by Theorem 2.8 the set $\mathcal{S}_A = \{\{x, y, z\}$ such that $P(x, y, z) = min\{x, y, z\}$irrespective of the positional order of $x, y, z\}$ be the corresponding STS on $\mathcal{A}$. So, either $|\mathcal{A}| = v = 1 + 6m$ -(i) or $v = 3 + 6m$ -(ii) where $m > 0$. Let n denote the number of disjoint blocks with each arbitrary but fixed block. By STS property $n = b - \{3(r - 1) + 1\} = b - 3r + 2$ where b denote the total number of blocks and r denote the number of blocks containing a fixed element.
Now for case (i)

$$r = \frac{v(v-1)3}{6v} = \frac{v-1}{2} = \frac{1+6m-1}{2} = 3m$$

and $b = \frac{v(v-1)}{6} = \frac{(1+6m)6m}{6} = 6m^2 + m$.
So $n = 6m^2 + m - 3(3m) + 2 = 6m^2 - 8m + 2$
and for case (ii)

$$r = \frac{3+6m-1}{2} = 1 + 3m$$

and $b = \frac{(3+6m)(2+6m)}{6} = (1 + 2m)(1 + 3m) = 6m^2 + 5m + 1$.
So $n = 6m^2 + 5m - 1 - 3(1 + 3m) + 3 = 6m^2 + 5m - 3 - 9m + 3 = 6m^2 - 4m$. □

Theorem 2.10. *In a Steiner P- algebra $< \mathcal{A},\ P >$ if the integer n from the Proposition 2.9 is equal to zero (i.e $n = 0$) then $P(x, y, z)$ is equal irrespective of the positional order of x, y, z and if $n = 2$ then $P(x, y, z) \neq P(x, z, y) \neq P(y, x, z)$ for all distinct $x, y, z \in \mathcal{A}$ such that $P(x, y, z) \neq min\{x, y, z\}$.*

Proof. Let $< \mathcal{A},\ P >$ be a finite Steiner P-algebra. Then by Theorem 2.8 it follows that the set $\mathcal{S}_A = \{\{x, y, z\} : x \neq y \neq z \text{ and } P(x, y, z) = \min\{x, y, z\}$ irrespective of the positional order of $x, y, z\}$ is the corresponding STS on $\mathcal{A}$. Now suppose $n = 0$. This implies that the intersection of any two elements of $\mathcal{S}_A$ (i.e. intersection of any two blocks of the corresponding STS of the Steiner P-algebra)is non empty. Let $x \neq y \neq z$ and $P(x, y, z) \neq \min\{x, y, z\}$. It implies that $\{x, y, z\} \notin \mathcal{S}_A$.

If possible let $P(x,y,z)=a$ and $P(x,z,y)=a'$.
So by (2) of Proposition 2.2 follows that

$$((x \neq z) \rightarrow (\exists b(b \neq y \neq x \neq z)(\phi(x,z,b) \wedge \phi(x,b,z) \wedge \\ \wedge \phi(z,x,b) \wedge \phi(z,b,x) \wedge \phi(b,x,z) \wedge \phi(b,z,x)))) \wedge \\ \wedge((y \neq b) \rightarrow (\exists a(a \neq y \neq b)(\phi(y,b,a) \wedge \phi(y,a,b) \wedge \\ \wedge \phi(b,y,a) \wedge \phi(b,a,y) \wedge \phi(a,y,b) \wedge \phi(a,b,y)))).$$

This implies that $\{x,z,b\}$ and $\{y,b,a\} \in \mathcal{S}_A$.

Similarly , $\exists b'(b' \neq z \neq x \neq y)$ such that $\{x,y,b'\}$ and $\{z,b',a'\} \in \mathcal{S}_A$. So $\{x,z,b\},\{x,y,b'\} \in \mathcal{S}_A$ and $\{x,z,b\} \cap \{x,y,b'\} \neq \emptyset$. Since $x \neq y \neq z$ and by the property of STS the intersection of any two blocks is atmost an one element set. So $b \neq b'$. By the assumption $\{y,b,a\} \cap \{z,b',a'\} \neq \emptyset$. Since $y \neq z \neq b' \neq b$, so $a=a'$. Hence $P(x,y,z)=P(x,z,y)$.

Similarly , $P(x,y,z)=P(y,x,z)$. Hence by Proposition 2.7 we get that for all distinct $x,y,z \in \mathcal{A}$ the value of $P(x,y,z)$ is same irrespective of the positional order of the elements.

Let us now consider the case $n=2$.

Let $P(x,y,z)=a$ and $P(x,z,y)=a'$, then as above we get $\{x,z,b\}$, $\{y,b,a\} \in \mathcal{S}_A$ and $\{x,y,b'\},\{z,b',a'\} \in \mathcal{S}_A$ for some $b \neq b' \in \mathcal{A}$. Now suppose $a=a'$. Since $n=2$, so $|\mathcal{A}|=v=9$ (by Proposition 2.9) and by the property of STS for any arbitrary but fixed block b_i there exist two blocks viz. b_1,b_2 such that $b_i \cap b_k=\emptyset$ for $k=1,2$. Now if possible let $b_1 \cap b_2 \neq \emptyset$. By the property of STS the intersection is a single element set. Let $w \in b_1 \cap b_2$. By Theorem 1.2 w belongs to $r=4$ blocks of $\mathcal{S}_A$. Now, w belongs to 3 blocks each of which contains also either one of the element of b_i. So w belongs to 5 blocks since $b_i \cap b_k=\emptyset$ for $k=1,2$. It leads to a contradiction to the fact that each element of $\mathcal{A}$ belongs to $r=4$ blocks of $\mathcal{S}_A$. Hence $b_1 \cap b_2=\emptyset$. It follows that $b_i \cup b_1 \cup b_2=\mathcal{A}$–(I). Let $b_i=\{x,y,b'\}$, then $z,a \in \mathcal{A} \setminus b_i$ and they can't belong to the same b_k, $k=1,2$. So let $z \in b_1$ and $a \in b_2$. It follows that $b \notin b_1$ and $b \notin b_2$ which contradicts (I). So $a \neq a'$. Hence $P(x,y,z) \neq P(x,z,y)$.

Similarly we can prove the other inequalities of the theorem. □

Conjecture 2.11. If the integer n from the Proposition 2.9 is positive then $P(x,y,z) \neq P(x,z,y) \neq P(y,x,z)$ for all distinct $x,y,z \in \mathcal{A}$ such that $P(x,y,z) \neq \min\{x,y,z\}$.

Theorem 2.12. *In a Steiner P- algebra $<\mathcal{A},\ P>$ for all distinct $x,y,z \in \mathcal{A}$ such that $P(x,y,z) \neq \min\{x,y,z\}$ the following identities hold:*

1. $P(x,P(x,y,z),z)=y$,

2. $P(P(x, y, z), x, y) = z$,

3. $P(P(x, y, z), z, y) = x$.

Proof. The identities follow from the Proposition 2.2 and Proposition 2.4 □

References

[1] Artamonov V. A, Universal Algebra in the book *General Algebra*, Editor -L. A. Skornyakov, Moscow Nauka,Vol.2, pp. 295 - 367, 1991.

[2] Burris S., Sankappanavar H. P, *A Course in Universal Algebra*, Springer - Verlag, 1978.

[3] Chakrabarti S, New Algebraic Structure of Steiner Triple Systems, *Fundamantalnaya I Prikladnaya Matematika (Fundamental and Applied Mathematics)* , Vol.8, No.1, pp.313-318 (Russian), 2002.

[4] Hall Marshall, Jr. *Combinatorial Theory*, Second Edition, A Wiley - Interscience Publication, 1986.

Contact information

S. Chakrabarti

Scientific Analysis Group
DRDO, Metcalfe House Complex
DELHI–110 054
E-Mail: suchetadrdo@hotmail.com

Received by the editors: 30.05.2005
and final form in 05.07.2005.