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On bounded m-reducibilities

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Abstract. Conditions for classes $\mathfrak{F}^1, \mathfrak{F}^0$ of non-decreasing total one-place arithmetic functions to define reducibility $\leq_m [{}^{\mathfrak{R}^1}_{\mathfrak{R}^0}] \leftrightharpoons \{(A,B) | A, B \subseteq \mathbb{N} \ \& \ (\exists \text{ r.f. } h)(\exists f_1 \in \mathfrak{F}^1)(\exists f_0 \in \mathfrak{F}^0) [A \leq_m^h B \ \& \ f_0 \trianglelefteq h \trianglelefteq f_1]\}$ where $k \trianglelefteq l$ means that function l majors function k almost everywhere are studied. It is proved that the system of these reducibilities is highly ramified, and examples are constructed which differ drastically $\leq_m [{}^{\mathfrak{R}^1}_{\mathfrak{R}^0}]$ from the standard m-reducibility with respect to systems of degrees. Indecomposable and recursive degrees are considered.

1. Introduction

In this paper we consider the consequences for m-reducibility enforced by the restrictions on the volume of oracle access. We mean using of reducibilities with imposing not time or zonal but information restrictions on the access to oracles. Analogous limitations (only upper restrictions) for Turing reducibilities were considered in [4].

We adopt the standard notation [6]. Additionally, C_A is the characteristic function of set A. $\mathcal{I}$ is the identity function on $\mathbb{N}$, i.e., $\mathcal{I}(x) = x$, $x \in \mathbb{N}$. As a shorthand, *r.f.* is often used instead *total recursive function*. W_e and φ_e are respectively the recursively enumerable (r.e.) set and the partial recursive function with the index e.

We write $f \trianglelefteq g$, when $\exists x_0 \forall y [y > x_0 \Rightarrow f(y) \leqslant g(y)]$. We call g as majorant almost everywhere for function f, and f – minorant almost everywhere for g.

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$\mathfrak{T}$ ($\mathfrak{T}^+$) is the class of all total (non-decreasing) functions $f : \mathbb{N} \to \mathbb{N}$. $\mathfrak{T}^{+,\infty}$ is the class of all functions from $\mathfrak{T}^+$ with an infinite set of values. Let $\mathfrak{R}^i \subseteq \mathfrak{T}^+,\ i = 0,1$. For any total function f we define two functions:

$$e^1_f(x) = \max\{f(y) | y \leqslant x\},\ e^0_f(x) = \min\{f(y) | x \leqslant y\}.$$

Lemma 1. *If f is a r.f. then e^1_f is non-decreasing r.f. and e^0_f is non-decreasing limit r.f.*

Proof. The first statement is trivial.
$e^0_f(x) = \lim\limits_{t\to\infty} h(x,t)$, where $h(x,t) = \min\{f(x') | x \leqslant x' \leqslant t\}$. □

Let us introduce the main definition: $\leq_m [^{\mathfrak{R}^1}_{\mathfrak{R}^0}] \rightleftharpoons \{(A,B) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} :$

$$\exists \text{ r.f. } f\ \exists h^1 \in \mathfrak{R}^1\ \exists h^0 \in \mathfrak{R}^0 [A \leq^f_m B\ \&\ h^0 \trianglelefteq e^0_f\ \&\ e^1_f \trianglelefteq h^1]\}.$$

It is obvious that $\leq_m [^{\mathfrak{R}^1}_{\mathfrak{R}^0}] \subseteq \leq_m$.

We will call the relation $\leq_m [^{\mathfrak{R}^1}_{\mathfrak{R}^0}]$ as a *bounded m-reducibility with upper and low restrictions* if this relation is reflexive and transitive. Here $\mathfrak{R}^1$, $\mathfrak{R}^0$ are the classes of upper and low restrictions respectively. As a shorthand, we refer to them as *upper* and *low classes.*

The introducing of low restrictions allows to estimate the complexity of reduced set by the complexity of the oracle and transfers (in some sense) the structure of reduced set on the oracle. For example, it is impossible to reduce an infinite recursive set to an immune set when the low bound of reduction belongs to $\mathfrak{T}^{+,\infty}$.

The rest of the paper is organized as follows. Section 2 introduces additional concepts and results about structural properties of introduced reducibilities. The reflexivity and transitivity of bounded reducibilities are considered in section 3. In sections 4, 6 and 7 we consider some types of introduced reducibilities with respect of systems of unsolvability degrees. In section 5 we consider some algebraic features of the system of bounded reducibilities.

2. Classes of upper and low restrictions

Let $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{T}^+$. We write $\mathfrak{A} \trianglelefteq^1 \mathfrak{B}$ if $(\forall f \in \mathfrak{A})(\exists g \in \mathfrak{B})[f \trianglelefteq g]$ and $\mathfrak{A} \trianglelefteq^0 \mathfrak{B}$ if $(\forall g \in \mathfrak{B})(\exists f \in \mathfrak{A})[f \trianglelefteq g]$.

Lemma 2.

(1) *For $\mathfrak{B}^i \subseteq \mathfrak{T}^+$ and $\mathfrak{A}^i \subseteq \mathfrak{T}^+$ $(i = 0,1)$ such that $\mathfrak{B}^1 \trianglelefteq^1 \mathfrak{A}^1\ \&\ \mathfrak{A}^0 \trianglelefteq^0 \mathfrak{B}^0$ the inclusion $\leq_m [^{\mathfrak{B}^1}_{\mathfrak{B}^0}] \subseteq \leq_m [^{\mathfrak{A}^1}_{\mathfrak{A}^0}]$ holds.*

(2) *Every relation* $\leq_m [\substack{\mathfrak{R}^1\\\mathfrak{R}^0}]$ *can be formed by countable subclasses* $\mathfrak{R}^1$, $\mathfrak{R}^0$ *of* $\mathfrak{T}^+$.

Proof. Indeed, let $C \leq_m [\substack{\mathfrak{B}^1\\\mathfrak{B}^0}]\ D$. Therefore by the main definition $(\exists h^1 \in \mathfrak{B}^1)(\exists h^0 \in \mathfrak{B}^0)[C \leq_m^f D\ \&\ h^0 \trianglelefteq e_f^0\ \&\ h^1 \trianglerighteq e_f^1]$. By the lemma's conditions $(\exists g^0 \in \mathfrak{A}^0)(\exists g^1 \in \mathfrak{A}^1)[g^0 \trianglelefteq h^0\ \&\ g^1 \trianglerighteq h^1]$ true. Hence $[C \leq_m^f D\ \&\ (g^0 \trianglelefteq e_f^0)\ \&\ (e_f^1 \trianglelefteq g^1)]$ holds due to transitivity of $\trianglelefteq$, i.e. $C \leq_m [\substack{\mathfrak{A}^1\\\mathfrak{A}^0}]D$.

At last, the second statement holds because $\{e_f^i | f \text{ r.f.},\ i = 0, 1\}$ are countable sets. $\square$

Further we consider classes $\mathfrak{R}^0, \mathfrak{R}^1$ defining the relations $\leq_m [\substack{\mathfrak{R}^1\\\mathfrak{R}^0}]$ as countable unless otherwise is specified.

Corollary 1. *There exists* $h^\star, l^\star \in \mathfrak{T}^+$ *such that* $\leq_m = \leq_m [\substack{h^\star\\l^\star}]$.

Proof. As it is easy to see, $l^\star$ can be defined in the following way: $\forall x\ l^\star(x) = 0$. We can use an arbitrary non-decreasing total majorant of the class of all r.f. Let

$$h(x, t) = \max\{\varphi_i(y) | i \leqslant x,\ y \leqslant x,\ \varphi_i(y) \downarrow \text{ for } t \text{ steps}\}.$$

It is clear that $h^\star(x) = \lim_{t\to\infty} h(x, t)$ is total and majors almost everywhere all the r.f. $\square$

3. Reflexivity and transitivity

As it was defined in the beginning, only reflexive and transitive relations could be bounded reducibilities. We begin this section with the following lemma.

Lemma 3. *Let* $\mathfrak{R} \subseteq \mathfrak{T}^+$.

(1) *If* $\{f\} \ntrianglelefteq^1 \mathfrak{R}$ *holds for a r.f.* f *with an infinite set of values then there exist* $A, B \subseteq \mathbb{N}$ *such that* $A \leq_m^f B\ \&\ A \nleq_m [\substack{\mathfrak{R}\\\mathfrak{T}^+}]B$.

(2) *If* $e_f^0 \in \mathfrak{T}^{+,\infty}, \mathfrak{R} \ntrianglelefteq^0 \{e_f^0\}$ *holds for a r.f.* f, *then there exist* $A, B \subseteq \mathbb{N}$ *such that* $A \leq_m^f B\ \&\ A \nleq_m [\substack{\mathfrak{T}^+\\\mathfrak{R}}]B$.

Proof. (1) Let $f_0, f_1, \dots$ be enumeration of all r.f. with infinite repetitions such that $e_{f_0}^1, e_{f_1}^1, \dots$ have a minorants in $\mathfrak{R}^1$. Let $g_n \trianglerighteq e_{f_n}^1, n \in \mathbb{N}$ holds for $g_n \in \mathfrak{R}^1$. Since $\{f\} \ntrianglelefteq^1 \mathfrak{R}^1$, we have $\forall n \in \mathbb{N}[f \ntrianglelefteq g_n]$. For every n there exists an infinite sequence of numbers $x_0^n < x_1^n < x_2^n < \dots$ such that:

$$e_{f_n}^1(x_k^n) \leqslant g_n(x_k^n) < f(x_k^n) < f(x_{k+1}^n),\ k \in \mathbb{N};$$

(the last inequality is possible for f has an infinite set of values) and

$$\forall y[e^1_{f_n}(y+x^n_0) \leqslant g_n(y+x^n_0)].$$

Let us choose $\sigma : \mathbb{N} \to \mathbb{N}$ in such a way that

$$\forall n[f(x^n_{\sigma(n)}) > g_n(x^n_{\sigma(n)}) \ \&\ x^n_{\sigma(n)} < x^{n+1}_{\sigma(n+1)}].$$

Define $B \rightleftharpoons f(A)$, $A \subset \{x^n_{\sigma(n)} |\ n \in \mathbb{N}\}$. Next, let us suppose C_A is defined on the initial segment $[0, x^{n-1}_{\sigma(n-1)}]$. Now we define

$$C_A(x^n_{\sigma(n)}) = 1 \Leftrightarrow f_n(x^n_{\sigma(n)}) \notin B.$$

It is possible, since $f(x^n_{\sigma(n)}) > e^1_{f_n}(x^n_{\sigma(n)})$ and the definition of $C_A(x^n_{\sigma(n)})$ does not influence inclusion $f_n(x^n_{\sigma(n)})$ into B.

It is clear that $A \leq^f_m B$ but $A \not\leq^{f_n}_m B$, since otherwise $x^n_{\sigma(n)}$ brings a contradiction.

(2) Let $(r_i)_{i\in\mathbb{N}}$ be a list of r.f. such that $\mathfrak{R} \trianglelefteq^0 \{e^0_{r_i} | i \in \mathbb{N}\}$. If this list is empty then the lemma is proven. By the condition $\forall i[e^0_{r_i} \ntrianglelefteq e^0_f]$ holds. Hence $\forall i[r_i \ntrianglelefteq f]$. That is, there exist infinitely many numbers $z^i_k, k = 0, 1, \ldots$, such that $\forall k[r_i(z^i_k) > f(z^i_k)]$ for every $i \in \mathbb{N}$.

Now choose a subsequence $(y^i_s)_{s\in\mathbb{N}}$ from a sequence $(z^i_k)_{k\in\mathbb{N}}$ in such a way that
$\forall s[r_i(y^i_s) > f(y^i_s)]$ & $\forall s[r_i(y^i_s) < f(y^i_{s+1})]$, i.e., the following systems of inequalities holds

$$\cdots < f(y^i_s) < r_i(y^i_s) < f(y^i_{s+1}) < r_i(y^i_{s+1}) < \ldots$$

There exists a function $\sigma : \mathbb{N} \to \mathbb{N}$ and a diagonal sequence $y^i_{\sigma(i)}$ such that the following systems of inequalities hold for any i

$$f(y^i_{\sigma(i)}) < r_i(y^i_{\sigma(i)}) < f(y^{i+1}_{\sigma(i+1)}) < r_{i+1}(y^{i+1}_{\sigma(i+1)}).$$

Let us construct the set B by choosing exactly one number from the pair $\{f(y^i_{\sigma(i)}), r_i(y^i_{\sigma(i)})\}$ for all $i \in \mathbb{N}$. Next, define the set A as $A \rightleftharpoons f^{-1}(B)$. Thus $A \leq^f_m B$. However, as it is easy to see, $\forall i[A \not\leq^{r_i}_m B]$. Therefore, $A \leq^f_m B$ & $A \not\leq_m [{}^{\mathfrak{T}^+}_{\mathfrak{R}}]\ B$. □

Theorem 1. *For any $\mathfrak{R}^1, \mathfrak{R}^0 \in \mathfrak{T}^+$ a relation $\leq_m [{}^{\mathfrak{R}^1}_{\mathfrak{R}^0}]$ is reflexive iff $\mathfrak{R}^0 \trianglelefteq^0 \{\mathcal{I}\} \trianglelefteq^1 \mathfrak{R}^1$.*

Proof. Sufficiency is obvious, for $A \leq_m [{}^{\mathfrak{R}^1}_{\mathfrak{R}^0}]A$ with the reducing function $\mathcal{I}$.

Necessity. If $\mathcal{I}$ has no minorant in $\mathfrak{R}^0$ then by Lemma 3 there exist $A, B \subseteq \mathbb{N}$ such that $A \leq^{\mathcal{I}}_m B$, i.e. $A = B$, but $A \not\leq_m [{}^{\mathfrak{T}^+}_{\mathfrak{R}^0}]A$.

The same situation occurs when there is no majorant in $\mathfrak{R}^1$ for $\mathcal{I}$. □

In paper [2] the following theorem was proved.

Theorem 2. $\leq_m [^{\mathfrak{R}^1}_{\mathfrak{R}^0}]$*is transitive iff* $(\forall$ *r.f.* $f,g)[\{e^1_f, e^1_g\} \trianglelefteq^1 \mathfrak{R}^1$ & $\mathfrak{R}^0 \trianglelefteq^0 \{e^0_f, e^0_g\} \Rightarrow \{e^1_f, e^1_g \in \mathfrak{T}^{+,\infty} \Rightarrow (\exists H^1 \in \mathfrak{R}^1)[e^1_{f\circ g} \trianglelefteq H^1]\}$ & $\{e^0_f, e^0_g \in \mathfrak{T}^{+,\infty} \Rightarrow (\exists H^0 \in \mathfrak{R}^0)[H^0 \trianglelefteq e^0_{f\circ g}]\}]$ & $\{(\exists d \in \mathbb{N})[\mathfrak{R}^0 \trianglelefteq^0 \{d\} \trianglelefteq^1 \mathfrak{R}^1 \Rightarrow (\forall d \in \mathbb{N})[\mathfrak{R}^0 \trianglelefteq^0 \{d\} \trianglelefteq^1 \mathfrak{R}^1]]\}$.

Remark 1. In according with the above results it is possible to choose

(1) countable systems of non-decreasing total recursive functions that majorize $\mathcal{I}$ almost everywhere as *upper classes* of restrictions and

(2) countable systems of a e^0_f-type (where f is a r.f.) limit recursive functions from $\mathfrak{T}^+$ that minor $\mathcal{I}$ almost everywhere as *low classes* of restrictions.

Let us refer to the such systems as *reduced classes.* We say reduced classes $\mathfrak{R}^0$ and $\mathfrak{R}^1$ are *transitive* when they define a reducibility, i.e., $\leq_m [^{\mathfrak{R}^1}_{\mathfrak{R}^0}]$ is a reflexive and transitive relation.

Corollary 2.

(1) *Upper reduced class* $\mathfrak{R}$ *is a transitive class iff* $\forall f,g \in \mathfrak{R}\ \exists h \in \mathfrak{R}$ *s.t.*$f \circ g \trianglelefteq h$.

(2) *Let* $\mathfrak{R}$ *be a low reduced class. If* $\forall f,g \in \mathfrak{R}\ \exists h \in \mathfrak{R}$ *s.t.* $h \trianglelefteq f \circ g$ *then* $\mathfrak{R}$ *is a transitive class.*

Proof. The first statement is obvious by criterion of transitivity.

For any r.f. t the inequality $t \geqslant e^0_t$ holds. In additional, if $t \in \mathfrak{T}^+$ then $t = e^0_t$. Hence $e^0_f \circ e^0_g \leqslant f \circ g$. Next, if $t \leqslant h$ then $e^0_t \leqslant e^0_h$. Therefore $e^0_f \circ e^0_g \leqslant e^0_{f\circ g}$ for any total f,g. Consequently, if $e^0_f \circ e^0_g$ has a minorant then $e^0_{f\circ g}$ has too. □

This corollary is useful in constructing of concrete bounded reducibilities that are considered in the following sections.

4. Singular reducibilities

In this section we consider the one-element classes which form reducibilities. We refer to them as *singular classes.* And as a shorthand, we refer to such reducibilities as *singular reducibilities.* They will play an important role further.

Let $f, g \in \mathfrak{T}^+,\ f \geqslant g$. Let us define $\widehat{fg}(0) = f(0)$ and

$$\widehat{fg}(x+1) = \begin{cases} \widehat{fg}(x), \widehat{fg}(x) \geqslant g(x+1), \\ f(x+1), \widehat{fg}(x) < g(x+1). \end{cases}$$

We say function g is a *h-hybrid* if it is equal to $\widehat{hg}$ or $\widehat{gh}$ almost everywhere.

We say x is a *minimal point* of function $t : \mathbb{N} \to \mathbb{N}$ iff $(\forall y > x)[t(y) > t(x)]$. Denote by M_t the set of all minimal points of t.

Lemma 4. *Let $\widehat{fg}$ is defined.*

(1) $\widehat{fg} \in \mathfrak{T}^+$.

(2) $g \leq \widehat{fg} \leq f$.

(3) *If $g \trianglelefteq \mathcal{I} \trianglelefteq f$ then $\widehat{fg} \circ \widehat{fg} = \widehat{fg}$ almost everywhere.*

(4) $(\forall h \in \mathfrak{T}^+)[h = \widehat{hh}]$.

(5) *For all $h, g \in \mathfrak{T}^+$ if $h \trianglelefteq g$ and $g \circ g \trianglelefteq h \circ g$ or $g \trianglelefteq h$ and $h \circ g \trianglelefteq g \circ g$ then there exists h-hybrid function f such that $h \trianglelefteq g \trianglelefteq f$ or $f \trianglelefteq g \trianglelefteq h$ respectively. In additional, if h is equal to $\mathcal{I}$ almost everywhere then g is $\mathcal{I}$-hybrid.*

Proof. (1),(2) Obvious.

(3) By the definition $\widehat{fg}$ is a stair-like non-decreasing function. Let us consider an equivalence relation $x \equiv_{\widehat{fg}} y \Leftrightarrow \widehat{fg}(x) = \widehat{fg}(y)$. Starting with some x' every $\equiv_{\widehat{fg}}$-equivalence class is a segment of $\mathbb{N}$ which includes (since $g \trianglelefteq \mathcal{I} \trianglelefteq f$) the value of $\widehat{fg}$ at its elements. That is $\widehat{fg} \circ \widehat{fg} = \widehat{fg}$ almost everywhere.

(4) Obvious.

(5) Let us consider the case $g \trianglelefteq h$ & $h \circ g \trianglelefteq g \circ g$. By the non-decreasing of g, h the equality $h \circ g = g \circ g$ holds almost everywhere. Thus h and g are equal on M_g almost everywhere. We assume $|M_g| = \infty$ (the case $|M_g| < \infty$ is obvious.)

Let $\{x_k | k \in \mathbb{N}\}$ be an enumeration of M_g in increasing order. Then the system of segments $\mathbb{N}_k = \{y | x_k \leqslant y < x_{k+1}\}, k \in \mathbb{N}$ is a partition of $\mathbb{N}$. Next, define $f(x) = g(x_k), x \in \mathbb{N}_k, k \in \mathbb{N}$. Now the statement is obvious.

At last, let $h = \mathcal{I}$ almost everywhere. Then $g = g \circ g$ almost everywhere. Let us consider a sufficiently large $x \in M_g$ such that $g(x) = g \circ g(x)$. Then $x + 1 \geqslant g(x+1) > g(x)$. Next, if $x + 1 > g(x+1)$ then $g(x+1) \leqslant x$ and, therefore, $g(x+1) > g(g(x+1))$. Obtained contradiction holds the equality $g = \mathcal{I}$ for all sufficiently large $x + 1, x \in M_g$. Hence, g is $\mathcal{I}$-hybrid. $\square$

Theorem 3. *For every transitive upper and low classes $\mathfrak{R}^1, \mathfrak{R}^0 \subseteq \mathfrak{T}^+$ and every $f, g \in \mathfrak{T}^+$ such that $g \trianglelefteq \mathcal{I} \trianglelefteq f$ the following relations are reducibilities:*

$$\leq_m [\substack{\widehat{f\mathcal{I}} \\ \mathfrak{R}^0}], \quad \leq_m [\substack{\mathfrak{R}^1 \\ \widehat{\mathcal{I}g}}].$$

Proof. Since $f, g \in \mathfrak{T}^+$ then $\widehat{f\mathcal{I}}, \widehat{\mathcal{I}g} \in \mathfrak{T}^+, \widehat{\mathcal{I}g} \trianglelefteq \mathcal{I} \trianglelefteq \widehat{f\mathcal{I}}$ by the Lemma 4. And in according with the third point the classes $\{\widehat{f\mathcal{I}}\}$ and $\{\widehat{\mathcal{I}g}\}$ are closed with respect to superpositions. Hence, the result holds by Theorem 1 and 2. $\square$

Theorem 4.

(1) *Upper reduced singular class is transitive iff it consist of hybrid function $\widehat{h\mathcal{I}}$ for some r.f. h.*

(2) *If low reduced singular transitive class consists of function e_h^0, where h is a r.f. then $e_{tot}^0 = e_h^0 = e_t^0$, where $t = \min(h, \mathcal{I})$. If e_h^0 is $\mathcal{I}$-hybrid, where h is a such r.f. that $h \trianglelefteq \mathcal{I}$, then $\{e_h^0\}$ is a singular transitive low class.*

Proof. (1) At first, let us consider an upper class $\{f\}$. The function f is a non-decreasing r.f. and by Corollary 2 $\mathcal{I} \trianglelefteq f$, $f \circ f \trianglelefteq f$ holds. Hence, f is a $\mathcal{I}$-hybrid by Lemma 4(5).

(2) Let us suppose $\{e_h^0\}$ is given singular reduced transitive class. By the criterion of reflexivity $e_h^0 \trianglelefteq \mathcal{I}$ must hold. Next, if $t = \min(h, \mathcal{I})$ then from $e_h^0 \trianglelefteq \mathcal{I}$ the equality $e_t^0 = e_h^0$ is true almost everywhere. Indeed, the values of functions h, e_h^0 are equal on all sufficiently large $x \in M_h$ and not exceed x. But t and h take on the same values at such x. We assume $e_t^0 \in \mathfrak{T}^{+,\infty}$ (otherwise e_t^0 is hybrid obviously.)

Next, we have $e_t^0 \trianglelefteq e_{tot}^0$ by the condition of transitivity. Let x be a large enough in M_h again. From $t(x) = h(x) = e_h^0(x) = e_t^0(x)$ the inequality $t(t(x)) \leqslant e_t^0(x)$ follows. If the last inequality is a strict one for infinitely many elements of M_h then we have the contradiction with $e_t^0 \trianglelefteq e_{tot}^0$. That is e_t^0 and e_{tot}^0 must be equal almost everywhere on M_h $(= M_t)$. Hence, $e_t^0 = e_{tot}^0$ almost everywhere.

It is well known that the set of minimal points of some r.f. is hyperimmune or recursive. In the latter case e_h^0 is recursive too. Then, since e_h^0 is non-decreasing and by the condition of transitivity the relation $e_h^0 \trianglelefteq e_h^0 \circ e_h^0$ must hold. Thus, e_h^0 is $\mathcal{I}$-hybrid by Lemma 4(5).

The last statement of the theorem was actually proved in Theorem 3. $\square$

Some functions can form singular classes not being hybrid. For example, the total majorant of the class of all r.f. forms the upper transitive singular class. Analogously, the total minorant $h \in \mathfrak{T}^{+,\infty}$ of all $e^0_f \in \mathfrak{T}^{+,\infty}$ (f is a r.f.) forms the low transitive singular class.

5. The hierarchy of reducibilities

We saw that bounded m-reducibilities are partially ordered by the relation of inclusion. In this section we study the poset. We denote by $\mathfrak{M}$ the set of bounded reducibilities and by $\subseteq$ the considered relation on $\mathfrak{M}$. By Remark 1 for every bounded m-reducibility there is an equal $\leq_m [{}^{\mathfrak{F}^1}_{\mathfrak{F}^0}]$-reducibility, where $\mathfrak{F}^1, \mathfrak{F}^0$ are reduced classes.

Let us consider arbitrary reducibilities $\leq_\alpha = \leq_m [{}^{\mathfrak{A}^1}_{\mathfrak{A}^0}]$ and $\leq_\beta = \leq_m [{}^{\mathfrak{B}^1}_{\mathfrak{B}^0}]$, where $\mathfrak{A}^i, \mathfrak{B}^i$ are reduced classes ($i = 0, 1$).

By Lemma 2 and 3 it is possible to show, that

1) $\leq_\alpha \subseteq \leq_\beta$ if and only if $\mathfrak{A}^1 \trianglelefteq^1 \mathfrak{B}^1,\ \mathfrak{B}^0 \trianglelefteq^0 \mathfrak{A}^0$, so that,

2) $\leq_\alpha = \leq_\beta$ if and only if $\mathfrak{B}^1 \trianglelefteq^1 \mathfrak{A}^1,\ \mathfrak{A}^0 \trianglelefteq^0 \mathfrak{B}^0\ \&\ \mathfrak{A}^1 \trianglelefteq^1 \mathfrak{B}^1,\ \mathfrak{B}^0 \trianglelefteq^0 \mathfrak{A}^0$

Define $\bigvee$ and $\bigwedge$ on $\mathfrak{M}$ as follows: $\leq_\alpha \bigwedge \leq_\beta \rightleftharpoons \leq_m [{}^{\widetilde{\mathfrak{C}^1}}_{\widetilde{\mathfrak{C}^0}}]$, where $\mathfrak{C}^1,\ \mathfrak{C}^0$ are subclasses of $\mathfrak{T}^+$

$$\mathfrak{C}^1 \rightleftharpoons \{f | f -\ \text{r.f.}, \mathcal{I} \trianglelefteq f\ \exists g \in \mathfrak{A}^1\ \exists h \in \mathfrak{B}^1 :\ f \trianglelefteq g\ \&\ f \trianglelefteq h\},$$

$$\mathfrak{C}^0 \rightleftharpoons \{f | f -\ \text{limit r.f.}, \mathcal{I} \trianglerighteq f\ \exists g \in \mathfrak{A}^0\ \exists h \in \mathfrak{B}^0 :\ f \trianglerighteq g\ \&\ f \trianglerighteq h\},$$

and $\widetilde{\mathfrak{F}}$ is the closure of $\mathfrak{F}$ under composition. Define similarly $\bigvee$:
$\leq_\alpha \bigvee \leq_\beta \rightleftharpoons \leq_m [{}^{\widetilde{\mathfrak{D}^1}}_{\widetilde{\mathfrak{D}^0}}]$, where $\mathfrak{D}^1,\ \mathfrak{D}^0$ are subclasses of $\mathfrak{T}^+$

$$\mathfrak{D}^1 \rightleftharpoons \{f | f -\ \text{r.f.}, \mathcal{I} \trianglelefteq f [\exists g \in \mathfrak{A}^1,\ f \trianglelefteq g] \vee [\exists h \in \mathfrak{B}^1 :\ f \trianglelefteq h]\},$$

$$\mathfrak{D}^0 \rightleftharpoons \{f | f -\ \text{limit r.f.}, \mathcal{I} \trianglerighteq f\ [\exists g \in \mathfrak{A}^0,\ f \trianglerighteq g] \vee [\exists h \in \mathfrak{B}^0 :\ f \trianglerighteq h]\}.$$

Observe that $\mathcal{I} \in \mathfrak{C}^i$ and $\mathcal{I} \in \mathfrak{D}^i$ ($i = 0, 1$), so classes in definition are not empty. The proof of the following lemma is obvious:

Lemma 5. *For every* $\leq_\alpha, \leq_\beta \in \mathfrak{M}$

1) $\leq_\alpha \bigvee \leq_\beta \in \mathfrak{M},\ \leq_\alpha \bigwedge \leq_\beta \in \mathfrak{M}$;

2) $\leq_\alpha \subseteq \leq_\alpha \bigvee \leq_\beta,\ \leq_\alpha \bigwedge \leq_\beta \subseteq \leq_\alpha$.

Example. Let R denote the set of all even integers. Then functions $x + C_R(x)$ and $x + C_{\overline{R}}(x)$ are $\mathcal{I}$-hybrid and relations $\leq_\alpha \rightleftharpoons \leq_m [{}^{\{\mathcal{I}+C_R\}}_{\{\mathcal{I}\}}]$ and $\leq_\alpha \rightleftharpoons \leq_m [{}^{\{\mathcal{I}+C_{\overline{R}}\}}_{\{\mathcal{I}\}}]$ are reducibilities by Theorem 3. It is easy to see that

$$\leq_\alpha \bigwedge \leq_\beta = \leq_m [{}^{\{\mathcal{I}\}}_{\{\mathcal{I}\}}],\ \leq_\alpha \bigvee \leq_\beta = \leq_m [{}^{\mathfrak{I}^+}_{\{\mathcal{I}\}}],$$

where $\mathfrak{I}^+ \rightleftharpoons \{\mathcal{I} + c | c \in \mathbb{N}\}$. The first reducibility is minimal element of $\mathfrak{M}$.

Let $\leq_\gamma \rightleftharpoons \leq_m [{}^{\mathfrak{C}^1}_{\mathfrak{C}^0}]$, $\mathfrak{C}^i$ are reduced classes ($i = 0, 1$) and $\leq_\gamma$ is some upper bound for $\leq_\alpha$ and $\leq_\beta$. This means that $\mathfrak{A}^1 \trianglelefteq^1 \mathfrak{C}^1$, $\mathfrak{B}^1 \trianglelefteq^1 \mathfrak{C}^1$, $\mathfrak{C}^0 \trianglelefteq^0 \mathfrak{A}^0$, $\mathfrak{C}^0 \trianglelefteq^0 \mathfrak{B}^0$. Now, let $\leq_\lambda = \leq_m [{}^{\mathfrak{D}^1}_{\mathfrak{D}^0}]$, where $\leq_\lambda \rightleftharpoons \leq_\alpha \bigvee \leq_\beta$. Let us show that $\mathfrak{D}^1 \trianglelefteq^1 \mathfrak{C}^1$, $\mathfrak{C}^0 \trianglelefteq^0 \mathfrak{D}^0$. Really, in the opposite case there is a function f from $\mathfrak{D}^1$ such that $f \ntrianglelefteq \mathfrak{C}^1$. But $f = f_1 \cdot f_2 \ldots f_{n-1} \cdot f_n$ and $\forall i = \overline{1,n}\ \exists h_i :\ [h_i \in \mathfrak{A}^1 \vee h_i \in \mathfrak{B}^1]\ \&\ f_i \trianglelefteq h_i$ holds by the definition of $\mathfrak{D}^1$. On the other hand $\forall i = \overline{1,n}\ \exists c_i :\ c_i \in \mathfrak{C}^1 : c_i \trianglerighteq h_i$. Thus

$$f \trianglelefteq h_1 \cdot h_2 \ldots h_{n-1} \cdot h_n \trianglelefteq c_1 \cdot c_2 \ldots c_{n-1} \cdot c_n$$

Since $\mathfrak{C}^1$ must contain the majorant for the composition of arbitrary own elements, we have

$$\exists c^* \in \mathfrak{C}^1 : c_1 \cdot c_2 \ldots c_{n-1} \cdot c_n \trianglelefteq c^*,$$

therefore $f \trianglelefteq c^* \trianglelefteq \mathfrak{C}^1$. Analogously $\mathfrak{C}^0 \trianglelefteq^0 \mathfrak{D}^0$. Therefore, $\leq_\lambda \subseteq \leq_\gamma$ and $\leq_\lambda$ is the least upper bound for $\leq_\alpha$ and $\leq_\beta$.

Analogously, let $\leq_\gamma \rightleftharpoons \leq_m [{}^{\mathfrak{C}^1}_{\mathfrak{C}^0}]$, $\mathfrak{C}^i$ are reduced classes ($i = 0, 1$) and $\leq_\gamma$ is some lower bound for reducibilities $\leq_\alpha$ and $\leq_\beta$. This means that $\mathfrak{A}^1 \trianglerighteq^1 \mathfrak{C}^1$, $\mathfrak{B}^1 \trianglerighteq^1 \mathfrak{C}^1$, $\mathfrak{C}^0 \trianglerighteq^0 \mathfrak{A}^0$, $\mathfrak{C}^0 \trianglerighteq^0 \mathfrak{B}^0$. Now, let $\leq_\lambda = \leq_m [{}^{\mathfrak{D}^1}_{\mathfrak{D}^0}]$, where $\leq_\lambda \rightleftharpoons \leq_\alpha \bigwedge \leq_\beta$. Let us show that $\mathfrak{D}^1 \trianglerighteq^1 \mathfrak{C}^1$, $\mathfrak{C}^0 \trianglerighteq^0 \mathfrak{D}^0$. Really, let $c \in \mathfrak{C}^1$. By condition there are $a \in \mathfrak{A}^1$, $b \in \mathfrak{B}^1$ such that $c \trianglelefteq a$ and $c \trianglelefteq b$. Then $f = \min(a, b)$ belongs to $\mathfrak{D}^1$ and $c \trianglelefteq f$. Similarly, it can be proved $\mathfrak{C}^0 \trianglerighteq^0 \mathfrak{D}^0$. Therefore $\leq_\gamma \subseteq \leq_\lambda$ and $\leq_\lambda$ is the greatest lower bound for $\leq_\alpha$ and $\leq_\beta$.

Thus the following proposition has been proved:

Proposition 1. *$\mathfrak{M}$ is lattice under $\bigvee$ and $\bigwedge$.*

Denote this lattice as $\mathfrak{L}$. We saw that $\mathfrak{L}$ has the greatest element – the unbounded m-reducibility and the least element – the reducibility $\leq_m [{}^{\{\mathcal{I}\}}_{\{\mathcal{I}\}}]$. Let us consider other features of $\mathfrak{L}$. It is easy to see the relation $\leq_m [{}^{\mathfrak{I}^+}_{\mathfrak{I}^+,\infty}]$ is reducibility. It plays an important role, so we introduce special notation $\leq_{m^*} \rightleftharpoons \leq_m [{}^{\mathfrak{I}^+}_{\mathfrak{I}^+,\infty}]$ and will call it as m^*-reducibility.

Theorem 5. *$\mathfrak{L}$ has unique co-atom. $\mathfrak{L}$ has no atoms.*

Proof. In opposite to the latter statement assume that $\leq_m [{}^{\mathfrak{A}^1}_{\mathfrak{A}^0}]$ is an atom in $\mathfrak{L}$. Then at least in one of the classes $\mathfrak{A}^1$ or $\mathfrak{A}^0$ there is a function $\mathfrak{a}$ and infinite recursive set R such that one from the following take place:

$$\forall x \in R\ \mathfrak{a}(x) > x \text{ if } \mathfrak{a} \in \mathfrak{A}^1$$
$$\forall x \in R\ \mathfrak{a}(x) < x \text{ if } \mathfrak{a} \in \mathfrak{A}^0$$

Assume $\mathfrak{a} \in \mathfrak{A}^0$. Define $R^1 \rightleftharpoons \{r(2i) | i \in \mathbb{N}\}$, where $r(i)$ runs through R. Now, define $\mathfrak{a}' = \mathcal{I} - C_{R^1}$, $\mathfrak{B}^0 \rightleftharpoons \{\mathfrak{a}'\}$. Then $\mathfrak{a}'$ is $\mathcal{I}$-hybrid and $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{B}^0}$ is a reducibility. In additional

$$\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{B}^0} \subseteq \leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{A}^0} \text{ and } \leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{B}^0} \neq \leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{A}^0}.$$

The latter holds by Lemma 3. Obvious, $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{B}^0} \neq \leq_m \genfrac{[}{]}{0pt}{}{\{\mathcal{I}\}}{\{\mathcal{I}\}}$.

In order to prove the first statement we note that reducibility $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{F}^1}{\mathfrak{F}^0}$ can include m^*-reducibility only if $\mathfrak{F}^0 \trianglelefteq^0 \mathfrak{T}^{+,\infty}$. But we can essentially expand the class of lower bounds $\mathfrak{T}^{+,\infty}$ by including constants only. But then the function 0 must be in $\mathfrak{F}^0$ due to transitivity criterion.

Therefore, a reducibility $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{F}^1}{\mathfrak{F}^0}$ must be equal to the unbounded m-reducibility. So m^*-reducibility is a co-atom in $\mathfrak{L}$.

Assume $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{A}^0}$ is another co-atom in $\mathfrak{L}$. Since $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{A}^1}{\mathfrak{A}^0}$ and $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{T}^+}{\mathfrak{T}^{+,\infty}}$ are incomparable with respect to $\subseteq$ relations $\mathfrak{A}^1 \trianglelefteq^1 \mathfrak{T}^+$ & $\mathfrak{T}^{+,\infty} \ntrianglelefteq^0 \mathfrak{A}^0$ or $\mathfrak{A}^1 \ntrianglelefteq^1 \mathfrak{T}^+$ & $\mathfrak{T}^{+,\infty} \trianglelefteq^0 \mathfrak{A}^0$ must take place. The second case is impossible, obvious. In the first case $\mathfrak{T}^{+,\infty} \ntrianglelefteq^0 \mathfrak{A}^0$ means that $\mathfrak{A}^0$ must contain the function 0 (see above.)

To finish the proof we need the following auxiliary notions (see [7]): recursive function $\Phi(m,x)$ is *universal for class* S if $\forall \varphi \in S\ \exists m:\ \varphi(\cdot) = \Phi(m,\cdot)$ and $\forall m \in \mathbb{N}\ \Phi(m,x) \in S$.

Let S and S_1 are classes of r.f. If for every $f_j{}' \in S_1$ there is $f_i \in S$ and infinite set A such that:$(\forall x \in A)[f'{}_j(x) \leqslant f_i(x)]$, then S *weakly majors* S_1.

In paper [7] the following proposition was proved:

Proposition 2.

(1) *If S has universal r.f. then there exists r.f. h such that $S \trianglelefteq^1 \{h\}$*

(2) *If S weakly majors the class of all r.f. then S has no universal r.f.*

Let us return to the theorem proof. It follows from above that $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{F}^1}{\mathfrak{F}^0}$ cannot be a co-atom in $\mathfrak{L}$ if $\mathfrak{F}^1$ is majorized by some class S with universal r.f.

As has been stated above, if the reducibility $\leq_m \genfrac{[}{]}{0pt}{}{\mathfrak{F}^1}{\mathfrak{F}^0}$ is another co-atom then $\mathfrak{F}^0$ must contain function 0. On the other hand it must exist r.f. h such that $\{h\} \ntrianglelefteq^1 \mathfrak{F}^1$ and $\widetilde{\mathfrak{F}^1 \cup \{h\}} \trianglerighteq^1 \mathfrak{T}^+$. But this means that class $\{h\}$ weakly majors class $\mathfrak{F}^1$. By the assumption $\widetilde{\mathfrak{F}^1 \cup \{h\}}$ majors the class of all r.f. Now, define $\mathfrak{H} \rightleftharpoons \widetilde{\{h\}}$. It's obvious, $\mathfrak{H}$ has universal r.f. In additional $\widetilde{\mathfrak{F}^1 \cup \{h\}}$ is weakly majored by $\mathfrak{H}$. It follows from this that $\mathfrak{H}$ weakly majors the class of all r.f. This contradicts the assumption that $\mathfrak{L}$ has co-atoms not equal to $\leq_{m^*}$. This completes the proof of the theorem. $\square$

The reducibility hierarchy is not trivial. In paper [2] the following theorem was proved:

Theorem 6. *Every countable partial ordering can be isomorphically embedded into* $\mathfrak{L}$.

In Fig.1 the structure of $\mathfrak{L}$ is shown schematically. Three reducibilities of "special" type are picked out here. The other reducibilities may be named as reducibilities of "general" type. In the area between $\leq_m \left[{\{\mathcal{I}\} \atop \{\mathcal{I}\}}\right]$ and $\leq_{m^*}$ the reducibilities with low restrictions from $\mathfrak{T}^{+,\infty}$ are placed.

In the following sections we will focus on the special subclass of the latter reducibilities. They are reducibilities with enumerable classes of restrictions.

6. The structure of recursive degrees

In this section we consider the structure of degrees of recursive sets for bounded m-reducibilities. For arbitrary bounded reducibility we are interested in the quantity of degrees, the density of their partial order and the possibility to embed other posets into the order.

In fact, we almost don't use the information of the oracles in reductions here. For the classical m-reducibility there are three recursive degrees only: $\{\varnothing\}$, $\{\mathbb{N}\}$ and all other recursive sets (since $\leq_m$ does not suppose any restriction on the reducing function apart from its recursiveness.)

Clearly, the same is true when we set upper restrictions only, i.e. when we deal with reducibility $\leq_m \left[{\mathfrak{F} \atop \mathfrak{T}^+}\right]$ for some $\mathfrak{F}$.

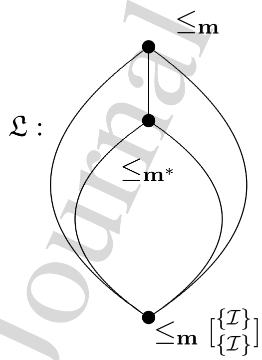

Figure 1: The structure of bounded m-reducibilities

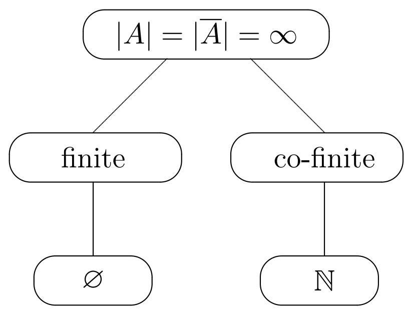

Figure 2: The structure of recursive $\leq_{m^*}$-degrees

The quantity of degrees can increase when we introduce non-trivial low restrictions on reducing function. Actually, if there are no upper bounds and low restrictions are represented by the subclass of $\mathfrak{T}^{+,\infty}$ then we have five degrees (see Fig.2): $\{\varnothing\}$, $\{\mathbb{N}\}$, all finite sets, all co-finite sets, other recursive sets (i.e. infinite recursive sets with infinite complement.)Yet, if in addition we put upper restrictions, the fifth class can fall to other degrees.

We call $\mathfrak{R}$ as *enumerable* class if there is a r.f. s such that

$$\mathfrak{R} = \{\varphi_{s(i)} | i \in \mathbb{N}\}.$$

Proposition 3. *Let* $\leq_m [^{\mathfrak{F}^1}_{\mathfrak{F}^0}]$ *a reducibility such that* $\mathfrak{F}^0 \subseteq \mathfrak{T}^{+,\infty}$ *and there are enumerable classes* Φ^0, Φ^1 *of recursive functions from* $\mathfrak{T}^{+,\infty}$ *which minors* $\mathfrak{F}^0$ *and majors* $\mathfrak{F}^1$ *respectively. Then there are recursive sets* A *and* B *such that* $|A| = |\overline{A}| = |B| = |\overline{B}| = \infty$ *and* $A \not\leq_m [^{\mathfrak{F}^1}_{\mathfrak{F}^0}] B$.

Proof. The case when for every r.f. f such that $\mathfrak{F}^0 \trianglelefteq f \trianglelefteq \mathfrak{F}^1 \Rightarrow f = \mathcal{I}$ almost everywhere is obvious (we can take then two disjoint infinite recursive sets.) Let γ be some recursive function of large oscillation and r.f. e_0 and e_1 enumerate Φ^0 and Φ^1 respectively. We construct recursive A and B in such a way that C_A and C_B are defined at step n on initial segment σ_n :

S t e p 0: Define $C_A(0) = C_B(0),\ \sigma_0 = \{0\}$;

S t e p $n+1$: Define $C_B(\|\sigma_n\|) = 1,\ C_A(\|\sigma_n\|) = 0$, where $\|\sigma_n\|$ is the length of σ_n. Next, let $\gamma(n+1) = \langle l, r\rangle$. Let us find first $x :\ x > \|\sigma_n\|$ such that $\varphi_{e_0(l)}(x) > \|\sigma_n\|$ (it is possible because $\Phi^0, \Phi^1 \subseteq \mathfrak{T}^{+,\infty}$) and define

$$C_A(x) = 1,\ C_B(x) = 0,\ \sigma_{n+1} = [0, \varphi_{e_1(r)}(x)].$$

At last, let $C_A(t) = C_B(t) = 0$ for every t from $\sigma_{n+1} \setminus ([0, \|\sigma_n\|] \cup \{x\})$.

Since A and B are enumerating in increasing order, they are recursive.Let assume for some r.f. t $A \leq^t_m [^{\mathfrak{F}^1}_{\mathfrak{F}^0}] B$ holds. This means that there are $f_0 \in \mathfrak{F}^0,\ f_1 \in \mathfrak{F}^1$ and numbers l^*, r^* such that

$$\varphi_{e_0(l^*)} \trianglelefteq f_0 \trianglelefteq t \trianglelefteq f_1 \trianglelefteq \varphi_{e_1(r^*)}$$

On the other hand for l^* and r^* there is an infinite computable sequence $\{n_i\} : \forall i\ \gamma(n_i) = \langle l^*, r^*\rangle$. Let us consider an infinite recursive set

$$A^* \rightleftharpoons A \cap (\cup_{i\in\mathbb{N}}(\sigma_{n_i} \setminus \sigma_{n_i-1}))$$

By construction for every x from A^* there are no elements from B in the segment $[\varphi_{e_0(l^*)}(x), \varphi_{e_1(r^*)}(x)]$. The latter is true and for the segment $[f_0(x), f_1(x)]$. We come to a contradiction with the reduction. $\square$

Below by a *non-trivial interval* for reducibility $\leq_\alpha$ we mean a pair (A,B) such that:

1) $A <_\alpha B$, i.e. $A \leq_\alpha B$ and $B \not\leq_\alpha A$;

2) $A \notin \{\varnothing, \mathbb{N}\}$.

Theorem 7. *Let* $\leq_\alpha \rightleftharpoons \leq_m \left[\begin{smallmatrix}\mathfrak{F}^1\\ \mathfrak{F}^0\end{smallmatrix}\right]$ *a reducibility such that* $\mathfrak{F}^0 \subseteq \mathfrak{T}^{+,\infty}$ *and there are enumerable classes* Φ^0, Φ^1 *of recursive functions from* $\mathfrak{T}^{+,\infty}$ *which minors* $\mathfrak{F}^0$ *and majors* $\mathfrak{F}^1$ *respectively. Then there is an isomorphical embedding of an arbitrary finite poset into any non-trivial interval of the structure of recursive* $\leq_\alpha$*-degrees.*

Proof. Let $A <_\alpha^f B$ and r.f. e_0 and e_1 enumerates Φ^0 and Φ^1 respectively. Since $B \not\leq_\alpha A$ then for every pair (m, n) there is an infinite set $R_{m,n}$ such that

$$\forall x \in R_{m,n}\ C_B(x) = \sigma\ \&\ \forall y:\ y \in [\varphi_{e_0(m)}(x), \varphi_{e_1(n)}(x)]\ C_A(y) = 1 - \sigma,$$

where $\sigma = 0, 1$ (in the opposite case the functions $\varphi_{e_0(m)}$ and $\varphi_{e_1(n)}$ are bounds of reduction $B \leq_m \left[\begin{smallmatrix}\mathfrak{F}^1\\ \mathfrak{F}^0\end{smallmatrix}\right] A$ and the reducing function can be constructed by the choosing of appropriate numbers from segment $[\varphi_{e_0(m)}(x), \varphi_{e_1(n)}(x)]$ for every x sufficiently large.) It is easy to see all $R_{m,n}$ are recursive.

Further, we construct recursive set X. Denote t-th element of X as $X(t)$ and define it at step t:
S t e p 0: $X(0) = R_{0,0}(0)$;
S t e p t: if $t = p^k$ for some prime p, $p > 2$ then

$$X(t) = \min y\{y \in R_{m,n}\ \&\ y > X(t-1)\},$$

where $k = \langle m, n, l\rangle$; else $X(t) = X(t-1) + 1$.

Therefore, for every prime p, $p > 2$ we have

$$\forall m, n\ |\{X(p^t) | t \in \mathbb{N}\} \cap R_{m,n}| = \infty, \tag{$*$}$$

that is the set $\{X(p^t) | t \in \mathbb{N}\}$ is "bad" for all r.f. which have a minorants in $\mathfrak{F}^0$ and a majorants in $\mathfrak{F}^1$.

It is well known that every partial ordering $(M, \leq_l)$ can be isomorphically embedded into partial ordering $(\{\mathfrak{S}_i | i \in M\}, \subseteq)$, where $\mathfrak{S}_i = \{m \in M | m \leq_l i\}$. Let ψ is one-to-one map from given finite poset M into set $\{p | p$ - prime , $p > 2\}$. Define $\mathfrak{S}_i^\psi = \bigcup\limits_{p \in \psi(\mathfrak{S}_i)} \{p^t | t \geqslant 1\}$ and recursive $X^i = X(\mathfrak{S}_i^\psi)$ (we write $X(S)$ as a shortening of $\{X(k) | k \in S\}$.) It is clear that $(M, \leq_l)$ can be isomorphically embedded into $(\{X^i | i \in M\}, \subset)$. At last, define the family of sets $A^i,\ i \in M$:

$$C_{A^i}(x) = (C_A(x) + C_{X^i}(x)) \ \mathbf{mod}\ 2.$$

Let $i, j \in M$ and $i <_l j$. Then $X^i \subset X^j$ and $|X^j \setminus X^i| = \infty$. It follows from this that $A^i \leq_\alpha^{f_{ij}} A^j$, where

$$f_{ij}(x) = \begin{cases} f(x) & \text{if } x \in X^j \setminus X^i \\ x & \text{otherwise} \end{cases}$$

On the other hand $A^j \not\leq_\alpha A^i$ since construction of X^i and X^j. Indeed, in the opposite case $A^j \leq_\alpha^h A^i$ and there are $m^\star,\ n^\star$ such that $\varphi_{e_0(m^\star)}$ and $\varphi_{e_1(n^\star)}$ are low and upper bounds for h respectively. Since $X^i \subset X^j$ there is prime $p_\star$ such that $\{X(p_\star^t) | t \in \mathbb{N}\} \subseteq X^j \setminus X^i$. On the other hand $|\{X(p_\star^t) | t \in \mathbb{N}\} \cap R_{m^\star, n^\star}| = \infty$ since condition (*). Thus, h fails on infinitely many numbers.

If $i ||_l j$ in $(M, \leq_l)$ then $|X^i \setminus X^j| = |X^j \setminus X^i| = \infty$. Thus $A^j \not\leq_\alpha A^i$ and $A^i \not\leq_\alpha A^j$ as above.

At last, for all $i \in M$

1) $A \leq_\alpha^{f_i} A^i$, where $f_i(x) = \begin{cases} f(x) & \text{if } x \in X^i; \\ x & \text{otherwise.} \end{cases}$

2) $A^i \not\leq_\alpha A$ because of property of X^i;

3) $A^i \leq_\alpha^{g_i} B$, where $g_i(x) = \begin{cases} x & \text{if } x \in X^i; \\ f(x) & \text{otherwise.} \end{cases}$

4) $B \not\leq_\alpha A^i$, because of property of $X \setminus X^i$.

That completes the proof. □

Corollary 3. *For the reducibility obeying the conditions of the theorem the structure of degrees for simultaneously infinite and co-infinite recursive sets is dense.*

A more detailed analysis shows that it is possible to replace finite poset by countable poset of some "constructive" type.

7. m^*-reducibility

I this section we compare $\leq_{m^*}$ with $\leq_1$ and study consequences for m-reducibility enforced by the essential low bounds using. First trivial fact concerns to isolated sets [6]: $A \leq_1 B$ & B is isolated $\Rightarrow$ A is isolated. Clearly, the same is true for m^*-reducibility.

Lemma 6. $\leq_1 \subsetneq \leq_{m^*} \subsetneq \leq_m$

Proof. Let $A \leq_1^f B$. It is obvious that $e_f^0 \in \mathfrak{T}^{+,\infty}$. That is $A \leq_{m^*} B$. Let C, D be finite sets and $|C| < |D|$. Then $C <_1 D$ and $C \equiv_{m^*} D$. That is $\leq_1 \neq \leq_{m^*}$. Thus, the first relation is proven.

Let A be an infinite recursive set and B be an immune set, $b_1 \in B$, $b_2 \in \bar{B}$. Then $A \leq_m^f B$ where $f(x) = b_1$ for $x \in A$ and $f(x) = b_2$ for $x \in \bar{A}$. But the existence of an infinite enumerable subset of B holds by reduction $A \leq_{m^*} B$. That is $\leq_{m^*} \neq \leq_m$. Thus, the second relation is proven. $\square$

The following theorem is an example of low bounds information using. Let us remind [6] that set A is *T-complete* iff $B \leq_T A$ for any r.e. B. It is well known that in contrast to m-reducibility T-complete set can be simple. It is convenient to describe classes of T-complete simple sets by notion of "weak effectivisation" [1]: recursively enumerable set A is *weakly effectively simple* if and only if there exists a total f such that $f \leq_T A$ and $\forall z[W_z \subset \overline{A} \Rightarrow |W_z| \leqslant f(x)]$. Recursively enumerable set A is *weakly 2-strictly recursively simple* if and only if there exists a total f such that $f \leq_T A$ and $\forall z[W_z \subset \overline{A} \Rightarrow \max\{W_z\} \leqslant f(x)]$.

Lachlan showed that A is T-complete and simple $\iff$ A is weakly effectively simple. Arslanov in [1] showed that A is weakly effectively simple $\iff$ A is weakly 2-strictly recursively simple.

Let us now introduce the following notations:
$f^{-\downarrow}(x) = \min\{y : \; f(y) \geqslant x\}$. It is clear that $f^{-\downarrow}$ is computable on f.

Theorem 8. *Let B be an r.e. set. If*

1) *$A \leq_{m^*} B$ and $h \in \mathfrak{T}^{+,\infty}$ is low bound in reduction,*

2) *B is weakly 2-strictly recursively simple with function f,*

3) *$h^{-\downarrow} \circ f \leq_T A$*

then A is T-complete.

Proof. Let t be reducing function in 1). Every r.e. subset of $\overline{B}$ is finite since condition 2). Obviously, the same is for $\overline{A}$. We need to estimate maximal elements of r.e. subsets of $\overline{A}$. That is we need to estimate partial function

$$m(z) \rightleftharpoons \max y\{y \in W_z, \; W_z \subseteq \overline{A}\}.$$

There exists recursive function g such that $W_{g(z)} = t(W_z)$. Since condition 1) the inequality $h(m(z)) \leqslant t(m(z))$ holds. Next, observe that $t(m(z)) \leqslant$

$\max\{W_{g(z)}\} \leqslant f(g(z))$. That is $h(m(z)) \leqslant f(g(z))$ and inequality $m(z) \leqslant h^{-\downarrow} f(g(z))$ take place. Hence

$$\forall z[W_z \subseteq \overline{A} \Rightarrow \max\{W_z\} \leqslant h^{-\downarrow} f(g(z))].$$

But $h^{-\downarrow} \circ f \circ g \leq_T A$ since condition 3), thus A is weakly 2-strictly recursively simple with function $h^{-\downarrow} \circ f \circ g$. Thus, A is T-complete. $\square$

We note that analogous statements are true for h-simple and pseudo-simple sets. The first condition ($h \in \mathfrak{T}^{+,\infty}$) is essential in above theorem. Indeed, let B be Post's simple set [6]. It is well known B is effectively simple (see [1]) with r.f. $f(x) = 2x + 1$, that is B is T-complete. Define $A = \mathbb{N} \oplus \varnothing$. It is easy to see other conditions are true. But A is not complete, obviously.

The following theorem characterizes cylinders in terms of m^*:

Theorem 9. *A is a cylinder iff* $(\forall B)[B \leq_m A \Rightarrow B \leq_{m^*} A]$.

Proof. Necessity is obvious by Lemma 6 and the cylinder criterion [6] : A is a cylinder iff $(\forall B)[B \leq_m A \Rightarrow B \leq_1 A]$.
Sufficiency. We will use another cylinder criterion:

$$A \text{ is a cylinder iff } (\exists \text{ r.f. } g)(\forall x)[g(x) > x \ \& \ x \in A \Leftrightarrow g(x) \in A].$$

By the conditions of the theorem $A \times \mathbb{N} \leq^f_{m^*} A$ holds, where f is some r.f. Since f has a minorant in $\mathfrak{T}^{+,\infty}$ the set $I_x = \{i | f(\langle x, i\rangle) \leqslant x,\ i \in \mathbb{N}\}$ is finite. Define $g(x) = f(\langle x, m_x\rangle)$, where $m_x = \min\{i | i \notin I_x\}$. Hence, g is total r.f. and $g(x) > x$ for every x. By the theorem conditions and definition of $A \times \mathbb{N}$ we have:

$$(\forall i)[x \in A \iff \langle x, i\rangle \in A \times \mathbb{N} \iff f(\langle x, i\rangle) \in A].$$

Thus $x \in A$ iff $g(x) \in A$ and A is a cylinder. $\square$

Corollary 4. *A is a cylinder iff* $A \equiv_{m^*} A \times \mathbb{N}$

The degree structure created by a reducibility inside the degrees of another and weaker reducibility has been discussed in the literature. A.Degtev actually showed in [5] that the structure of 1-degrees inside the m^*-degree of any simple set neither is upper nor low semi-lattice.

An m-degree is *indecomposable* if it consists of a single 1-degree [5]. We call m-degree as *m^*-indecomposable* if it consists of a single m^*-degree.

Corollary 5. *m-degree of a set A is indecomposable iff it is m^*-indecomposable.*

Proof. We note only that for every B such that $B \equiv_m A$ the equivalence $B \equiv_{m^*} B \times \mathbb{N}$ holds. Thus A is a cylinder by Corollary 4. $\square$

Now it's naturally to ask about the structure of degrees of a stronger bounded reducibility inside the degrees of a weaker bounded reducibility.

Let us consider m^*-reducibility as a weaker and $\leq_m [\substack{\mathfrak{T}^+ \\ \mathfrak{R}}]$-reducibility as a stronger bounded reducibility, where $\mathfrak{R}$ is the subclass of $\mathfrak{T}^{+,\infty}$. We generalize the definition from [6] of the operator $\bigoplus$.

Let r.f. s enumerates indexes of some functions from $\mathfrak{T}^{+,\infty}$, i.e. $\{\varphi_{s(i)} | i \in \mathbb{N}\} \subseteq \mathfrak{T}^{+,\infty}$. Now, define r.f. t and a map $\psi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ in the following way: $t(0) = 0$,

$$t(x) = \mu k[k > t(x-1) \mathbin{\&}_{i \leqslant x} \varphi_{s(i)}(k) > x],$$

$$\psi(\{x\}) = [t(x), t(x+1))$$

It is clear $t \in \mathfrak{T}^{+,\infty}$. For simplicity we write $\psi(x)$ instead of $\psi(\{x\})$ when x is a number. Finally, define $\bigoplus\limits_s$ as follows:

$$\forall x, y \; y \in \psi(x) \; C_{\bigoplus\limits_s A}(y) = C_A(x),$$

therefore, $x \in A \iff \psi(x) \subseteq \bigoplus\limits_s A$.

Lemma 7. *If A is not a cylinder then $\bigoplus\limits_s A$ is not a cylinder too.*

Proof. In the opposite case there is a r.f. f such that

$$\forall y \; f(y) > y \mathbin{\&} y \in B \iff f(y) \in B$$

where $B \rightleftharpoons \bigoplus\limits_s A$. Now we define $g(x)$ for every x. Let $m(x)$ is the maximal element of $\psi(x)$. By construction of ψ we have:

$$x \in A \iff \psi(x) \subseteq B \iff m(x) \in B \iff f(m(x)) \in B.$$

We set $g(x)$ as $\psi^{-1}(f(m(x)))$. It is clear that $g(x) > x$ and g is a r.f. Therefore, A is a cylinder, contrary to the assumption. $\square$

Lemma 8. *Let A be not a cylinder and the class $M \rightleftharpoons \{\varphi_{s(i)} | i \in \mathbb{N}\}$ minors the class $\mathfrak{R}$ of low bounds of $\leq_m [\substack{\mathfrak{T}^+ \\ \mathfrak{R}}]$-reducibility.*
Then $\bigoplus\limits_s A \not\leq_m [\substack{\mathfrak{T}^+ \\ \mathfrak{R}}] A$.

Proof. Assume $\bigoplus_s A \leq_m^f [\frac{\mathfrak{T}}{\mathfrak{R}}]A$. By condition there is a number j such that $\varphi_{s(j)} \trianglelefteq f$. Now, according to the definition of the operator $\bigoplus_s$ we have $x \in A \iff \psi(x) \subseteq \bigoplus_s A$. But by the assumption

$$\psi(x) \subseteq \bigoplus_s A \iff f(\psi(x)) \subseteq A$$

The following inequalities are obvious:

$$\max\{f(\psi(x))\} \geqslant f(t(x)) \geqslant \varphi_{s(j)}(t(x)).$$

In additional, if $x \geqslant j$ then $\varphi_{s(j)}(t(x)) > x$ holds according to the definition of t. Therefore, $\forall x\ x \geqslant j\ \max\{f(\psi(x))\} > x$, and thus $g(x) = \max\{f(\psi(x))\}$ is a r.f. obeying $x \in A \iff g(x) \in A$. Yet, then A is a cylinder, that contradicts to the condition of the lemma. □

On the other hand $\bigoplus_s A \leq_{m^*}^h A$, where $h(x) = \max\{k | t(k) \leqslant x\}$. Therefore, $\bigoplus_s A \equiv_{m^*} A$.

Let $\mathfrak{R}$ is defined as in Lemma 8 above.

Theorem 10. *If m^*-degree includes more than one $\leq_m [\frac{\mathfrak{T}^+}{\mathfrak{R}}]$-degree then it includes infinite chain of $\leq_m [\frac{\mathfrak{T}^+}{\mathfrak{R}}]$-degrees.*

Proof. To finish the proof we note that a m^*-degree obeying the assumption must include non-cylinder A. Thus

$$A <_m [\tfrac{\mathfrak{T}^+}{\mathfrak{R}}] \bigoplus_s A <_m [\tfrac{\mathfrak{T}^+}{\mathfrak{R}}] \bigoplus_s \bigoplus_s A < \dots .$$

This is what the theorem states. □

8. Conclusion and future research

In the previous sections we discovered the system of reducibilities which arose from the classical many-to-one reducibility by imposing restrictions on reducing functions. The motivation is an attempt to refine (detail) some results of classical recursion theory.

In section 6 we saw that the restriction on oracle access separated the family of recursive sets and in such way it formed some classification of the latter. The construction and analysis of nontrivial models for such classifications is future research direction. The further investigation of the relationship between m^*-reducibility and 1-reducibility are interesting too. Here we note the problem of describing of m^*-degrees consisting of a single 1-degree. We analyze also the relationship between the restrictions and the completeness for corresponding reducibilities.

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