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Ternary Hopf algebras

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Abstract. In this paper we introduce the notion of a ternary Hopf algebra and prove that it can be embedded into a universal enveloping Hopf algebra.

Introduction

The investigations of Hopf algebras are important since it is one of important classes of algebras, because these algebras are connected with physics, noncommutative geometry, algebraic topology, the theory of algebraic groups, the theory of quantum groups. Hopf algebras arise in investigations of the cohomology of the Lie groups.

In this paper, the notation of Hopf algebra is generalized from binary to ternary case. The ternary systems play important role in Jordan and Lie algebras. In the paper it is proved that all obtained results do not hold for arbitrary n, $n > 3$. All additional necessary notations and definitions can be found in the paper, listed in References.

1. Preliminaries

In papers [1] and [2], notions of $(n, 2)$-bialgebras and $(2, n)$-bialgebras are introduced. In particular, for $n = 3$, we obtain notions of ternary bialgebras, of the type $(3, 2)$ and $(2, 3)$. Now, we introduce the notion of a ternary $(3, 3)$-bialgebra.

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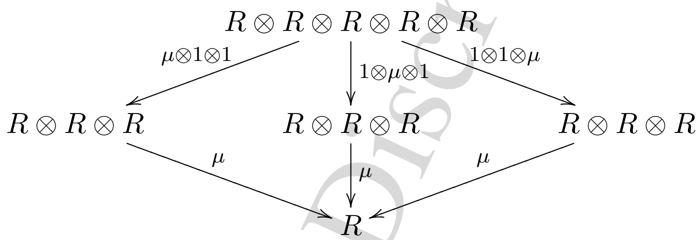
Definition 1.1. Let k be a commutative associative ring with a unit, R a module over k and

$$\mu : R \otimes R \otimes R \to R \quad \eta : k \to R$$

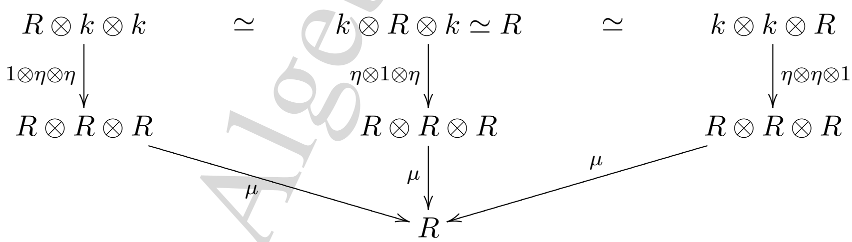
k-module morphisms, notation $\mu(a \otimes b \otimes c) = abc$, $\mu(1) = 1$, such that the following diagrams are commutative:

(1) associativity of the ternary multiplication μ:



or equivalently $(abc)de = a(bcd)e = ab(cde)$;

(2) the property of a unit element



equivalently $x11 = 1x1 = 11x = x$.

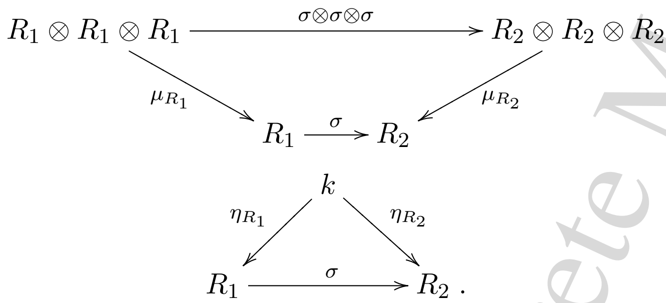
A triple (R, μ, η) is called a *ternary k-algebra*. The map μ is a *multiplication* and η a *unit* of the ternary k-algebra R.

Definition 1.2. Let $(R_1, \mu_{R_1}, \eta_{R_1})$ and $(R_2, \mu_{R_2}, \eta_{R_2})$ be two ternary k-algebras. A k-module morphism $\sigma : R_1 \to R_2$ is called a *k-algebra morphism* if the the following equivalent conditions are satisfied:

(i)

$$\sigma \circ \mu_{R_1} = \mu_{R_2} \circ (\sigma \otimes \sigma \otimes \sigma) \quad \text{and} \quad \sigma \circ \eta_{R_1} = \eta_{R_2},$$

(ii) the following diagrams are commutative:



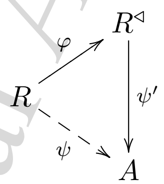
In [Z1], [Z2] a more general situation of the n-ary algebra was considered. In [Z1] it was proved that every n-ary algebra R can be embedded into a universal enveloping associative k-algebra $R^{\triangleleft}$. The algebra $R^{\triangleleft}$ as a k-module has a direct decomposition

$$R^{\triangleleft} = R \oplus Rx \oplus Rx \oplus \cdots \oplus Rx^{n-2},$$

where the multiplication in $R^{\triangleleft}$ is defined as follows:

$$(rx^i)(r'x^j) = (rr' \underbrace{1 \cdots 1}_{n-2})x^m \in Rx^m,$$

where $m \equiv (i + j) \mod (n - 1)$ and 1 is the identity element of R. An embedding $\varphi : R \to R^{\triangleleft}$ is defined by the rule $\varphi(r) = rx \in Rx$. It is assume that $x^n = 1$ [ZA]. The universal property means that if A is any associative k-algebra and $\psi : R \to A$ a k-module map such that $\psi(r_1 \cdots a_n) = \phi(r_1) \cdots \phi(r_n)$ then there exists a unique k-algebra homomorphism $\psi' : R^{\triangleleft} toA$ such that the following diagram is commutative



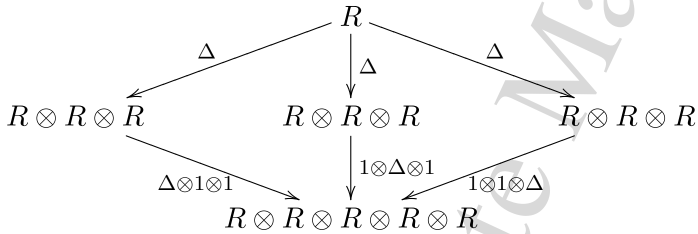
For any k-modules R_1 and R_2 there is a k-module map $\tau : R_1 \otimes R_2 \to R_2 \otimes R_1$ called a *twist* such that $\tau(x \otimes y) = y \otimes x$, for $x \in R_1,\ y \in R_2$. [Z1].

Definition 1.3. A triple (R, Δ, ε) is a *ternary k-coalgebra* if there exist k-module morphisms

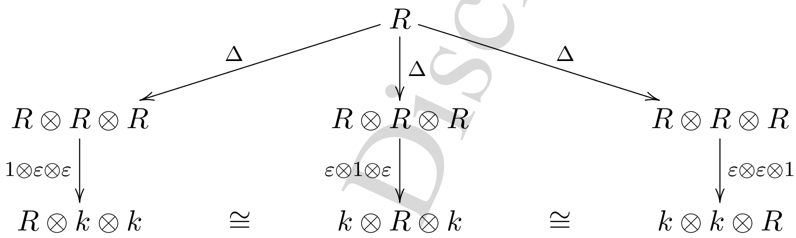
$$\Delta : R \to R \otimes R \otimes R \text{ and } \varepsilon : R \to k,$$

such that the following diagrams are commutative:

(i) the diagram of coassociativity of the ternary comultiplication Δ:



(ii) and the counity property



Following [Ab] we shall use the Σ-notation

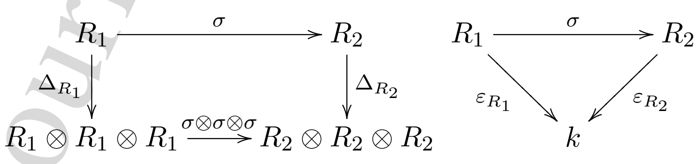
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}, \quad x_{(i)} \in R.$$

The maps Δ and ε are called a *comultiplication* and a *counit*, respectively, of the ternary k-coalgebra R.

Definition 1.4. Let $(R_1, \Delta_{R_1}, \varepsilon_{R_1})$ and $(R_2, \Delta_{R_2}, \varepsilon_{R_2})$ be two ternary k-coalgebras. A k-module morphism $\sigma : R_1 \to R_2$ is a *k-coalgebra morphism* if

$$\Delta_{R_2} \circ \sigma = (\sigma \otimes \sigma \otimes \sigma) \circ \Delta_{R_1} \text{ and } \varepsilon_{R_2} \circ \sigma = \varepsilon_{R_1}.$$

It means that the following diagrams are commutative:



Theorem 1.1. *Let a triple (R, μ, η) be a ternary k-algebra and a triple (R, Δ, ε) a ternary k-coalgebra. Then, the following conditions are equivalent:*

(i) *μ, η are k-coalgebra morphisms;*

(ii) *Δ, ε are k-algebra morphisms;*

(iii) $\Delta(fgh) = \sum_{(f),(g),(h)} f_1 g_1 h_1 \otimes f_2 g_2 h_2 \otimes f_3 g_3 h_3, \quad \Delta(1) = 1$ *and*

$$\varepsilon(fgh) = \varepsilon(f)\varepsilon(g)\varepsilon(h), \quad \varepsilon(1) = 1.$$

Proof. The conditions, under which Δ is a ternary k-algebra morphism, are as follows:

1) $\Delta \circ \mu = (\mu \otimes \mu \otimes \mu) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes \Delta \otimes \Delta)$

2) $\Delta \circ \eta = \eta \otimes \eta \otimes \eta$, where k is identified with $k \otimes k \otimes k$.

The conditions, under which ε is a ternary k-algebra morphism, are as follows:

a) $\varepsilon \circ \mu = \varepsilon \otimes \varepsilon \otimes \varepsilon$,

b) $\varepsilon \circ \eta = 1_k$, where k is identified with $k \otimes k \otimes k$.

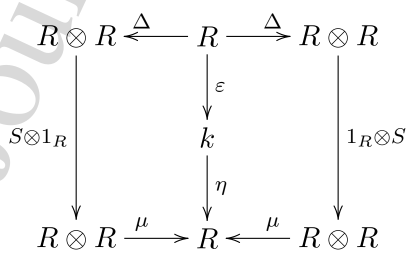
On the other hand, μ is a ternary k-coalgebra morphism if it satisfies conditions 1), a); and η is a ternary k-coalgebra morphism if it satisfies conditions 2) and b).

This fact allows us to conclude that i)⇔ii). Equivalently ii)⇔iii) follows from the definition. □

Definition 1.5. The system $(R, \mu, \eta, \Delta, \varepsilon)$ or simply R is called a *ternary k-bialgebra* or (3,3)-*k-bialgebra,* if the k-module R together with k-module maps $\mu, \eta, \Delta, \varepsilon$ satisfies one of the equivalent conditions of Theorem 1.1

2. Basic constructions

In the binary case [A], an *antipode* S in a k-bialgebra $(R, \mu, \eta, \Delta, \varepsilon)$ is defined as a morphism $S : R \to R$, such that the following diagram is commutative:



In other words,

$$\varepsilon(x) = \sum_{(x)} x_{(1)} S(x_{(2)}) = \sum_{(x)} S(x_{(1)}) x_{(2)}, \quad \forall x \in R.$$

Antipode S is anti-bialgebra morphism, i.e.

$$S(xy) = S(y)S(x), \qquad \Delta S(x) = \sum_{(x)} S(x_{(2)}) \otimes S(x_{(1)}),$$
$$S(1) = 1, \qquad \varepsilon S(x) = \varepsilon(x).$$

The k-bialgebra R with an antipode S is called a *Hopf algebra.*

We consider the following question: in what way that definition can be generalized to n-ary case? In the papers [Z1], [Z2] the notion of a binary k-bialgebras are generalized to n-ary case. In [Z1] it was proved that each $(2, n)$-ring R can be embedded into a universal enveloping ring $R^{\triangleleft}$ [ZA]. The algebra homomorphism

$$(\varphi \otimes \varphi) \circ \Delta_R : R \to R^{\triangleleft} \otimes R^{\triangleleft},$$

by the university of $R^{\triangleleft}$, induces algebra-homomorphism

$$\Delta_{R^{\triangleleft}} : R^{\triangleleft} \to R^{\triangleleft} \otimes R^{\triangleleft},$$

under which the following diagram is commutative

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R^{\triangleleft} \\ {\scriptstyle \Delta_R}\downarrow & & \downarrow{\scriptstyle \Delta_{R^{\triangleleft}}} \\ R \otimes R & \xrightarrow{\varphi\otimes\varphi} & R^{\triangleleft} \otimes R^{\triangleleft} \end{array}$$

where

$$\Delta_{R^{\triangleleft}}(rx^j) = \Delta_R(r)(x^j \otimes x^j), \quad j = 0, 1, \ldots, n-2$$
$$1 = \Delta_{R^{\triangleleft}}(x^{n-1}) = x^{n-1} \otimes x^{n-1} = 1 \otimes 1.$$

It is proved that $(2, n)$-ring R is n-ary k-bialgebra if and only if the universal enveloping ring $R^{\triangleleft}$ is k-bialgebra.

Recall that an element $g \in R$ is a *group-like element* if $\Delta(g) = g \otimes g \otimes g$ and $\varepsilon(g) = 1$.

In order to define n-ary analogy of the antipode S, we need to prove that the following conditions are equivalent:

1) There is an antipode $S : R \to R$ with the properties:

a) $S(a_1 \dots a_n) = S(a_n) \dots S(a_1)$

b) $\varepsilon(a) = \sum a_{(1)} \dots S(a_{(j_1)}) \dots S(a_{(j_m)}) \dots a_{(n)}$ for some fixed m and all possible places $j_1 < \dots < j_m$.

2) There is an antipode $\bar{S} : R^{\triangleleft} \to R^{\triangleleft}$, such that $\bar{S}(ax^k) = x^{n-1-k}\bar{S}(a)$ (since $\bar{S}(ax^k) = \bar{S}(x)^k \bar{S}(a) = x^{n-1-k}\bar{S}(a)$, where $a, \bar{S}(a) \in R$, the group-like element maps to an inverse element. In the binary case [A] we have $S(x) = x^{-1}$). In particular

$$\bar{S}(ax) = x^{n-2}\bar{S}(a), \quad \bar{S}(a) = x\bar{S}(ax) \subseteq Rx^2,$$

since $\bar{S} : R \to R, \quad Rx \to Rx$. Therefore,

$$\bar{S} : R \to Rx^2 \to Rx^4 \to \dots$$

and thus $n - 2 = 1$, i.e. $n = 3$.

Hence, $R^{\triangleleft} = R \oplus Rx$, and we have to consider *ternary* k-bialgebras.

On the other hand, in the conditions 1), we had to ask about occurrences m of S in the definition of the morphism ε:

1^0 if n is an even number, then $m = \frac{n}{2}$ (if m would be less there is no n-ary factors)

2^0 if n is odd number, then $m = \frac{n-1}{2}$. Hence, the left side is equal to a. In the case $n = 3$ we have

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

i.e.

$$a = \sum_{(a)} S(a_{(1)})a_{(2)}a_{(3)} = \sum_{(a)} a_{(1)}S(a_{(2)})a_{(3)} = \sum_{(a)} a_{(1)}a_{(2)}S(a_{(3)}).$$

Therefore, if $S^2 = 1$, then

$$S(a) = \sum_{(a)} S(a_{(3)})S(a_{(2)})a_{(1)} = \sum_{(a)} S(a_{(3)})a_{(2)}S(a_{(1)}) = \sum_{(a)} a_{(3)})S(a_{(2)})S(a_{(1)}).$$

These are analogies of the inverse elements in semigroups. Further, we consider the following schema of the embeddings:

$$R \to R^{\triangleleft} = R \oplus Rx, \quad x^2 = 1,$$

i.e. $(3,3)$-bialgebra is embeded into a $(2,3)$-bialgebra. If R is a finitely generated projective k-module then $(R^{\triangleleft})^* \to \big((R^{\triangleleft})^*\big)^{\triangleleft}$, i.e. $(3,2)$-bialgebra is embeded into $(2,2)$-bialgebra. Here $R^* = \mathrm{Hom}_k(R,k)$ is the dual module of R over k.

3. Ternary Hopf algebras

Theorem 3.1. *Let R be a $(3,3)$-bialgebra. Then, R can be embeded into a universal enveloping ring $R^{\triangleleft}$, which is $(2,3)$-bialgebra, and the following conditions are equivalent:*

1) *there is an antipode $S : R \to R$ with the property*

$$S(a_1a_2a_3) = S(a_3)S(a_2)S(a_1), \quad \forall a_1,a_2,a_3 \in R$$

2) *there is an antipode $\bar{S} : R^{\triangleleft} \to R^{\triangleleft}$ with the property:*

$$\bar{S}(uv) = \bar{S}(v)\bar{S}(u), \quad \forall u,v \in R^{\triangleleft}$$

and $\bar(S)(R) = R$ (binary algebra-anti homomorphism).

Proof. $(3,3)$-bialgebra R can be embeded into a universal enveloping ring $R^{\triangleleft}$:

$$R \to R^{\triangleleft} = R \oplus Rx, \quad x^2 = 1,$$

by the rule $r \mapsto rx$, $\forall r \in R$, and $R^{\triangleleft}$ is a (2,3)-bialgebra [Z1].

1)⇒ 2) Extend the map S to $R^{\triangleleft}$, $\bar{S} : Rx \to Rx$, $R \to R$, by setting

$$\bar{S}(u) = x\bar{S}(ux), \quad \bar{S}(x) = x, \quad \text{i.e.} \quad \bar{S}(ux) = x\bar{S}(u).$$

Take $u,v \in R$. Then there are the following cases

a) $\bar{S}(uv) = x\bar{S}(uvx) = x\bar{S}(vx)\bar{S}(u) = \bar{S}(v)\bar{S}(u)$, by 1b);

b) by the property 1b) we have

$$\begin{aligned}\bar{S}(uvx) &= \bar{S}(uxxvx) = \bar{S}(vx)\bar{S}(x)\bar{S}(ux)\\ &= \bar{S}(vx)x\bar{S}(ux) = \bar{S}(vx)\bar{S}(u);\end{aligned}$$

c) by the property 1) we have

$$\begin{aligned}\bar{S}(uxv) &= x\bar{S}(uxvx) = x\bar{S}(vx)\bar{S}(x)\bar{S}(u)\\ &= \bar{S}(v)x\bar{S}(u) = \bar{S}(v)\bar{S}(ux);\end{aligned}$$

d) $\bar{S}(uxvx) = x\bar{S}(uxv) = x\bar{S}(v)\bar{S}(ux) = \bar{S}(vx)\bar{S}(ux)$.

2)⇒1) An antipode $S : R \to R$ can be defined as a restriction $\bar{S} \mid_R$ of $\bar{S}$ to $R \subset R^{\triangleleft}$. □

Suppose now that k is a field and R is a finitely generated projective k-module. Denote by T the universal enveloping ring $R^{\triangleleft}$. Then T is again a finitely generated projective k-module. As in [Z2] it can be proved that T^* is a (3,2)-bialgebra, such that

$$(\Delta f)(r_1 \otimes r_2) = f(r_1 r_2)$$

since

$$\Delta f \in T^* \otimes T^* = (T \otimes T)^*, \quad \forall f \in T^*$$

and

$$(f \circ g \circ h)(r) = \sum_{(r)} f(r_1)g(r_2)h(r_3)$$

because

$$\Delta(r) = \sum_{(r)} r_1 \otimes r_2 \otimes r_3 \in T \otimes T \otimes T, \quad \forall r \in T.$$

Denote by V the dual module T^* of T.

Theorem 3.2. *Let V be a $(3,2)$-bialgebra. Then, V can be embedded into a universal enveloping ring $R^{\triangleleft}$, which is a $(2,2)$-bialgebra, and the following conditions are equivalent:*

(1) *there is an antipode $\bar{S}^* : V \to V$, with the properties:*

$$\begin{aligned}\bar{S}^*(a_1a_2a_3) &= \bar{S}^*(a_3)\bar{S}^*(a_2)\bar{S}^*(a_1), \quad \forall a_1, a_2, a_3 \in V\\ \bar{\varepsilon}^*(a) &= \sum_{(a)} a_{(1)}\bar{S}^*(a_{(2)}) = \sum_{(a)} \bar{S}^*(a_{(1)})a_{(2)}, \quad \forall a \in V\end{aligned}$$

(2) *there is an antipode $\widehat{S} : V^{\triangleleft} \to V^{\triangleleft}$, $\widehat{S}(V) = V$, with the properties:*

$$\widehat{S}(gh) = \widehat{S}(h)\,\widehat{S}(g), \quad \forall g, h \in V^{\triangleleft},$$

$$\widehat{\varepsilon}(g) = \sum_{(g)} g_1\,\widehat{S}(g_2) = \sum_{(g)} \widehat{S}(g_1)g_2, \quad \forall g \in V^{\triangleleft}.$$

Proof. Taking into account [Z1] and previous considerations we can deduce that $(3,2)$-bialgebra V can be embedded into a universal enveloping ring $V^{\triangleleft}$, i.e.

$$V \to V^{\triangleleft} = V \oplus Vy, \quad y^2 = 1,$$

by the rule $r \mapsto ry, \quad \forall r \in V$ and $V^{\triangleleft}$ is (2,2)-bialgebra.

(1) $\Rightarrow$(2) Let us extend the map $\bar{S}^*$ to $V^{\triangleleft}$, $\widehat{S} : Vy \to Vy$, $V \to V$, such that

$$\widehat{S}(g) = y\,\widehat{S}(gy), \quad \widehat{S}(y) = y, \quad \text{i.e.} \quad \widehat{S}(gy) = y\,\widehat{S}(g).$$

There are the following cases:

a) $\widehat{S}(gh) = y\,\widehat{S}(ghy) = y\,\widehat{S}(hy)\,\widehat{S}(g) = \widehat{S}(h)\,\widehat{S}(g)$,

b) by 1) we have

$$\begin{aligned}\widehat{S}(ghy) &= \widehat{S}(gyyhy) = \widehat{S}(hy)\,\widehat{S}(y)\,\widehat{S}(gy)\\ &= \widehat{S}(hy)y\,\widehat{S}(gy) = \widehat{S}(hy)\,\widehat{S}(g);\end{aligned}$$

c) by 1) we have

$$\begin{aligned}\widehat{S}(gyh) &= y\,\widehat{S}(gyhy) = y\,\widehat{S}(hy)\,\widehat{S}(y)\,\widehat{S}(g)\\ &= \widehat{S}(h)y\,\widehat{S}(g) = \widehat{S}(h)\,\widehat{S}(gy);\end{aligned}$$

d) $\widehat{S}(gyhy) = y\,\widehat{S}(gyh) = y\,\widehat{S}(h)\,\widehat{S}(gy) = \widehat{S}(hy)\,\widehat{S}(gy)$.

If $g \in V$, then:

$$\begin{aligned}\bar{\varepsilon}^*(g) &= \sum_{(g)} g_{(1)}\bar{S}^*(g_{(2)}) = \sum_{(g)} \bar{S}^*(g_{(1)})g_{(2)}\\ &= \sum_{(g)} g_{(1)}\,\widehat{S}\left(g_{(2)}\right) = \sum_{(g)} \widehat{S}(g_{(1)})g_{(2)} = \widehat{\varepsilon}(g).\end{aligned}$$

If $g \in Vy$, then $g = hy, \quad h \in V$:

$$\begin{aligned}\bar{\varepsilon}^*(h) &= \sum_{(h)} h_{(1)}\bar{S}^*(h_{(2)}) = \sum_{(h)} h_{(1)}\,\widehat{S}\left(h_{(2)}\right)\\ &= \sum_{(h)} h_{(1)}y\,\widehat{S}\left(h_{(2)}y\right) = \sum_{(g)} g_{(1)}\,\widehat{S}\left(g_{(2)}\right) = \widehat{\varepsilon}(g).\end{aligned}$$

Note that antipode $\bar{S}^*: T^* \to T^*$ is defined by the rule

$$(\bar{S}^*)(r) = f(\bar{S}(r)), \quad \forall r \in T,$$

where

$$\bar{S}: T \to T$$

is an extension of the antipode $S: R \to R$. □

References

[Z1] Zekovitch B. On n-ary bialgebra (I). Tchebyshev sbornik, 4, N 3 (7), 2003, 65-73.

[Z2] Zekovitch B. On n-ary bialgebra (II). Tchebyshev sbornik, 4, N 3 (7), 2003, 73-88.

[A] Artamonov V. A. Structure of Hopf algebras. Itogi nauki i tech. Ser. Algebra. Topology. Geometry. 29, Moscow, 1991, 3-63.

[ZA] Zekovitch B., Artamonov V. A. n-group rings and their radicals, Abelian groups and modules (1992), Tomsk univeristy, 11, 3-7.

[Ab] Abe Eiichi, Hopf algebras, Cambridge University Press, Cambridge, 1980.

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