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Two-generated graded algebras

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To my parents

Abstract. The paper is devoted to classification of two-generated graded algebras. We show that under some general assumptions there exist two classes of these algebras, namely quantum polynomials and Jordanian plane. We study prime spectrum, the semigroup of endomorphisms and the Lie algebra of derivations of Jordanian plane.

Introduction

Let $A_0 = \mathbb{K}$ be a field and $A = \bigoplus_{n=0}^{\infty} A_n$ the associative graded algebra generated over $\mathbb{K}$ by elements $X, Y \in A_1$. Suppose that $\dim A_2 = 3$. In the paper we find a criterion for A to be a domain when $\dim A_{n+1} = n+1$ (see Corollary 5.5). We also show that if $\mathbb{K}$ has no quadratic extensions, A is a domain and either $\dim A_{n+1} = n+1$ or A is a central algebra then A is either the algebra of quantum polynomials in two variables

$$\Lambda_1(\mathbb{K}, \lambda) = \mathbb{K}\langle X, Y\rangle / (YX - \lambda XY), \quad \lambda \in \mathbb{K}^*,$$

or Jordanian plane

$$\Lambda_2(\mathbb{K}) = \mathbb{K}\langle X, Y\rangle / (YX - XY - Y^2)$$

(see Theorems 5.3, 5.4). The other sections of this paper are devoted to Jordanian plane. We describe its center (see Theorem 2.2), derivations (see Theorem 4.2) and the Lie algebra of outer derivations for an

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arbitrary field $\mathbb{K}$ (see Theorems 4.6, 4.10, 4.16, 4.20, 4.23). In the case $char\mathbb{K} = 0$ we describe prime spectrum (see Theorem 2.4), the group of automorphisms (see Theorem 3.1), the endomorphisms with non-trivial kernels (see Proposition 3.2). Similar problems for quantum polynomials have been considered by V. A. Artamonov [1]. Some properties of quantum polynomials are also considered in details in [4]. Note that a study of non-commutative graded algebras is motivated by non-commutative algebraic geometry [6].

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1. Definitions

Definition 1.1. The single parameter **algebra of quantum polynomials in two variables** over $\mathbb{K}$ is the $\mathbb{K}-$algebra $\Lambda_1(\mathbb{K}, \lambda)$, $\lambda \in \mathbb{K}^*$, given by generators X and Y and defining relation $YX = \lambda XY$, i.e. $\Lambda_1(\mathbb{K}, \lambda) = {}^{\mathbb{K}\langle X, Y\rangle}/_{(YX - \lambda XY)}$. **Jordanian plane** over $\mathbb{K}$ is the $\mathbb{K}-$algebra $\Lambda_2(\mathbb{K})$ given by generators X and Y and defining relation $YX = XY + Y^2$, i.e. $\Lambda_2(\mathbb{K}) = {}^{\mathbb{K}\langle X, Y\rangle}/_{(YX - XY - Y^2)}$.

Proposition 1.2. *The basis of* $\Lambda_2(\mathbb{K})$ *is* $\{X^iY^j | i, j \in \mathbb{N}_0\}$. *In particular,*

$$Y^mX^n = \sum_{l=0}^{n} \binom{n}{l} \frac{(m+n-l-1)!}{(m-1)!} X^lY^{m+n-l}, \qquad m, n \in \mathbb{N},$$

and $\Lambda_2(\mathbb{K})$ *is a domain.*

Proof. 1. We claim that the monomials $X^\bullet Y^\bullet$ are linear independent. Let us consider the linear space $L = \langle U^iV^j | i, j \in \mathbb{N}_0\rangle$ with basis $U^\bullet V^\bullet$. Denote by $\rho : \Lambda_2(\mathbb{K}) \to \mathcal{L}(L)$ the linear map such that

$$\rho(X)(U^mV^n) = U^{m+1}V^n, \quad \rho(Y)(V^n) = V^{n+1},$$

$$\rho(Y)\left(U^{m+1}V^n\right) = \rho(Y)\rho(Y)(U^mV^n) + \rho(X)\rho(Y)(U^mV^n),$$

$m, n \in \mathbb{N}_0$. We shall check that the map ρ is well defined. It is enough to prove that $\rho(Y)(U^mV^n) \subseteq \left\langle U^iV^j \mid i \le m\right\rangle$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. We shall proceed by induction on m. If $m = 1$, then

$$\rho(Y)(UV^n) = V^{n+2} + UV^{n+1} \subseteq \left\langle U^iV^j \mid i \le 1\right\rangle.$$

We now assume that $\rho(Y)(U^mV^n) \subseteq \langle U^iV^j \mid i \le m \rangle$ for all $m = 0, ..., l$. If $m = l+1$, then by the inductive assumption we have

$$\begin{aligned} &\rho(Y)\left(U^{l+1}V^n\right) = \rho(Y)\rho(Y)\left(U^lV^n\right) + \rho(X)\rho(Y)\left(U^lV^n\right) \\ &= \rho(Y)\big(\sum_{i\le l,j}\alpha_{ij}U^iV^j\big) + \rho(X)\big(\sum_{i\le l,j}\alpha_{ij}U^iV^j\big) \\ &= \sum_{i\le l,j}\alpha_{ij}\sum_{i'\le i,j'}\delta_{iji'j'}U^{i'}V^{j'} + \sum_{i\le l,j}\alpha_{ij}U^{i+1}V^j \subseteq \left\langle U^iV^j \,\middle|\, i \le l+1\right\rangle. \end{aligned}$$

Thus, the linear map ρ is well defined and

$$\rho(Y)\rho(X) = \rho(X)\rho(Y) + \rho(Y)\rho(Y).$$

In fact, for basic elements we have

$$\begin{aligned} &\rho(Y)\rho(X)(U^mV^n) = \rho(Y)\left(U^{m+1}V^n\right) \\ &= \rho(X)\rho(Y)(U^mV^n) + \rho(Y)\rho(Y)(U^mV^n). \end{aligned}$$

Thus, ρ is an algebra homomorphism. Note that

$$\rho\left(X^mY^d\right)(1) = U^mV^d.$$

Now assume that monomials $X^\bullet Y^\bullet$ are linear dependent, i.e.

$$\sum_{i,j}\alpha_{ij}X^iY^j = 0$$

for some coefficients $\alpha_{ij} \in \mathbb{K}$. Then

$$\sum_{i,j}\alpha_{ij}U^iV^j = \sum_{i,j}\alpha_{ij}\rho\left(X^i\right)\rho\left(Y^j\right)(1) = 0,$$

which is impossible since the monomials $U^\bullet V^\bullet$ are linear independent. So, the monomials $X^\bullet Y^\bullet$ are linear independent too.

2. We claim that the monomials $X^\bullet Y^\bullet$ span $\Lambda_2(\mathbb{K})$. It is enough to check that $Y^mX^n = \sum_{l=0}^{n}\binom{n}{l}\frac{(m+n-l-1)!}{(m-1)!}X^lY^{m+n-l}$ for all $m, n \in \mathbb{N}$. We shall proceed by induction on n. If $n = 1$, then an easy induction on m shows that $Y^mX = XY^m + mY^{m+1}$, $m \in \mathrm{N}$. Assume that for $n = l$ our statement is holds. If $n = l+1$, then using the inductive assumption and

the equality $Y^m X = XY^m + mY^{m+1}$, $m \in \mathbb{N}$, we obtain

$$Y^m X^{l+1} = (Y^m X^l)X = \sum_{k=0}^{l} \binom{l}{k} \frac{(m+l-k-1)!}{(m-1)!} X^k Y^{m+l-k} X$$

$$= \sum_{k=0}^{l} \binom{l}{k} \tfrac{(m+l-k-1)!}{(m-1)!} X^k \left(XY^{m+l-k} + (m+l-k)Y^{m+l+1-k} \right)$$

$$= \sum_{k=1}^{l+1} \binom{l}{k-1} \tfrac{(m+l-k)!}{(m-1)!} X^k Y^{m+l+1-k}$$

$$+ \sum_{k=0}^{l} \binom{l}{k} \frac{(m+l-k)!}{(m-1)!} X^k Y^{m+l+1-k}$$

$$= \sum_{k=1}^{l} \left(\binom{l}{k-1} + \binom{l}{k} \right) \tfrac{(m+l-k)!}{(m-1)!} X^k Y^{m+l+1-k} + X^{l+1} Y^m$$

$$+ \frac{(m+l)!}{(m-1)!} Y^{m+l+1} = \sum_{k=0}^{l+1} \binom{l+1}{k} \tfrac{(m+l-k)!}{(m-1)!} X^k Y^{m+l+1-k}.$$

□

In the same way one can prove

Proposition 1.3. *The basis of* $\Lambda_1(\mathbb{K}, \lambda)$ *is* $\left\{ X^i Y^j | i, j \in \mathbb{N}_0 \right\}$. *In particular,* $Y^m X^n = \lambda^{mn} X^n Y^m$ *for all* $m, n \in \mathbb{N}$ *and* $\Lambda_1(\mathbb{K}, \lambda)$ *is a domain.*

In what follows we assume that elements of $\Lambda_1(\mathbb{K}, \lambda)$ or $\Lambda_2(\mathbb{K})$ are presented in the canonical form $\sum \alpha_{ij} X^i Y^j$. The $X-$degree of an element $w = \sum_{i=0}^{n} X^i \varphi_i(Y)$ of $\Lambda_1(\mathbb{K}, \lambda)$ or of $\Lambda_2(\mathbb{K})$ is equal to n, provided $\varphi_n \neq 0$. We shall write $\deg_X w = n$. It is clear that the family of linear spans of monomials of degree n, $n \in \mathbb{N}_0$, in X and Y induces the gradings of $\Lambda_1(\mathbb{K}, \lambda)$ and of $\Lambda_2(\mathbb{K})$. So, these algebras are graded. Note that the algebras $\Lambda_1(\mathbb{K}, \lambda)$ and $\Lambda_2(\mathbb{K})$ have the structure of iterated skew polynomial rings [4], [7].

Theorem 1.4. $\Lambda_1(\mathbb{K}, \lambda) \not\cong \Lambda_2(\mathbb{K})$, $\lambda \in \mathbb{K}^*$.

Proof. Put

$$A = {}^{\mathbb{K}\langle X_1, Y_1 \rangle} / _{(Y_1 X_1 - \lambda X_1 Y_1)},$$

$$B = {}^{\mathbb{K}\langle X_2, Y_2 \rangle} / _{\left(Y_2 X_2 - X_2 Y_2 - Y_2^2\right)}.$$

Assume that there exists an isomorphism $\psi : B \to A$. Let $\psi(X_2) = \sum\limits_{i,j} \alpha_{ij} X_1^i Y_1^j$, $\psi(Y_2) = \sum\limits_{i,j} \beta_{ij} X_1^i Y_1^j$. Since $Y_2 X_2 = X_2 Y_2 + Y_2^2$, we have

$$(\sum_{i,j} \beta_{ij} X_1^i Y_1^j)(\sum_{i,j} \alpha_{ij} X_1^i Y_1^j) = (\sum_{i,j} \alpha_{ij} X_1^i Y_1^j)(\sum_{i,j} \beta_{ij} X_1^i Y_1^j) + (\sum_{i,j} \beta_{ij} X_1^i Y_1^j)^2.$$

Since the algebra A is graded, we can conclude that $\alpha_{00}\beta_{00} = \beta_{00}^2 + \alpha_{00}\beta_{00}$ and

$$\begin{aligned}&(\beta_{10}X_1 + \beta_{01}Y_1)(\alpha_{10}X_1 + \alpha_{01}Y_1) + \left(\beta_{20}X_1^2 + \beta_{11}X_1Y_1 + \beta_{02}Y_1^2\right)\alpha_{00}\\ &= (\alpha_{10}X_1 + \alpha_{01}Y_1)(\beta_{10}X_1 + \beta_{01}Y_1) + \alpha_{00}\left(\beta_{20}X_1^2 + \beta_{11}X_1Y_1 + \beta_{02}Y_1^2\right)\\ &+ (\beta_{10}X_1 + \beta_{01}Y_1)^2.\end{aligned}$$

Therefore, $\beta_{00} = 0$ and $\beta_{10}X_1^2 + \xi X_1 Y_1 + \beta_{01} Y_1^2 = 0$ for some $\xi \in \mathbb{K}$. Since the monomials X_1^2, X_1Y_1 and Y_1^2 are linear independent, we can conclude that $\beta_{10} = \beta_{01} = 0$. Consider the inverse isomorphism $\varphi = \psi^{-1} : A \to B$. Let $\varphi(X_1) = \sum\limits_{i,j} c_{ij} X_2^i Y_2^j$, $\varphi(Y_1) = \sum\limits_{i,j} d_{ij} X_2^i Y_2^j$. Since $Y_1 X_1 = \lambda X_1 Y_1$, we have

$$(\sum_{i,j} d_{ij} X_2^i Y_2^j)(\sum_{i,j} c_{ij} X_2^i Y_2^j) = \lambda(\sum_{i,j} c_{ij} X_2^i Y_2^j)(\sum_{i,j} d_{ij} X_2^i Y_2^j).$$

Since the algebra B is graded, we have $c_{00}d_{00} = \lambda c_{00} d_{00}$. The algebra B is non-commutative, so $\lambda \neq 1$. Hence $c_{00}d_{00} = 0$. Suppose that $c_{00} = 0$ and $d_{00} \neq 0$. Since $\varphi(X_1) \neq 0$, there exists a positive integer n such that $c_{ij} = 0$ provided that $i + j < n$ and $c_{i'j'} \neq 0$ for some $i', j' \in \mathbb{N}$, $i' + j' = n$. Since the algebra B is graded, we have $d_{00}(\sum\limits_{i,j:i+j=n} c_{ij} X_2^i Y_2^j) = \lambda d_{00}(\sum\limits_{i,j:i+j=n} c_{ij} X_2^i Y_2^j)$. The monomials $X_2^\bullet Y_2^\bullet$ are linear independent and $\lambda \neq 1$, thus $c_{i'j'} = 0$, a contradiction. Similarly, the case $c_{00} \neq 0$ and $d_{00} = 0$ is impossible too. Thus, $c_{00} = d_{00} = 0$. Finally, we obtain $\psi(Y_2) = \sum\limits_{i,j:i+j\geq 2} \beta_{ij} X_1^i Y_1^j$, and

$$\begin{aligned}Y_2 &= \varphi(\psi(Y_2))\\ &= \sum_{i,j:i+j\geq 2} \beta_{ij} \left(\sum_{i',j':i'+j'\geq 1} c_{ij} X_2^{i'} Y_2^{j'}\right)^i \left(\sum_{i',j':i'+j'\geq 1} d_{ij} X_2^{i'} Y_2^{j'}\right)^j.\end{aligned}$$

But the polynomial $\varphi(\psi(Y_2))$ either vanishes or the degree of each monomial of $\varphi(\psi(Y_2))$ is at least 2. This contradiction proves Theorem 1.4. $\square$

2. Centre and Spectrum of $\Lambda_1(\mathbb{K},\lambda)$ and $\Lambda_2(\mathbb{K})$

As in Proposition 1.2 we have

Proposition 2.1. *Let* $w = \sum_{i\geq 0, j\geq 0} \alpha_{ij} X^i Y^j \in \Lambda_2(\mathbb{K})$. *Then*

$$Yw = \sum_{k\geq 0} w_X^{(k)} Y^{k+1},$$

where $w_X^{(k)} = \sum_{i\geq k, j\geq 0} \alpha_{ij} \frac{i!}{(i-k)!} X^{i-k} Y^j$ *is the formal partial derivative* w *by* X *and*

$$wX = Xw + w'_Y Y^2,$$

where $w'_Y = \sum_{i\geq 0, j\geq 1} j\alpha_{ij} X^i Y^{j-1}$ *is the formal partial derivative* w *by* Y.

The following Theorem 2.2 describes the centre $Z(\Lambda_2(\mathbb{K}))$ of $\Lambda_2(\mathbb{K})$ depending on the characteristic of $\mathbb{K}$ and the centre $Z(\Lambda_1(\mathbb{K},\lambda))$ of $\Lambda_1(\mathbb{K},\lambda)$ depending on the parameter λ.

Theorem 2.2. (i) *If* $char\mathbb{K} = 0$, *then* $Z(\Lambda_2(\mathbb{K})) = \mathbb{K}$; *if* $char\mathbb{K} = p > 0$, *then* $Z(\Lambda_2(\mathbb{K}))$ *is the subalgebra generated by* X^p, Y^p. (ii) *If* λ *is not a root of unity, then* $Z(\Lambda_1(\mathbb{K},\lambda)) = \mathbb{K}$; *if* λ *is a root of unity of the degree* m, $m \in \mathrm{N}$, *then* $Z(\Lambda_1(\mathbb{K},\lambda))$ *is the subalgebra generated by* X^m, Y^m.

Proof. (i) Let $f \in Z(\Lambda_2(\mathbb{K}))\setminus\{0\}$, $f = \sum_{i=0}^{n} X^i \psi_i(Y)$. Then $Xf = fX$ and $Yf = fY$. By Proposition 2.1 we have $fX = Xf + f'_Y Y^2$ and

$$Yf(X) = \sum_{k\geq 0} f^{(k)}(X) Y^{k+1}.$$

Therefore, $f'_Y Y^2 = 0$ and $\sum_{k\geq 1} f_X^{(k)}(X) Y^{k+1} = 0$. Since the algebra $\Lambda_2(\mathbb{K})$ is a domain, we conclude that $f'_Y = 0$. We shall consider two cases.

Let first $char\mathbb{K} = 0$. Since $f'_Y = 0$, we have $\psi'_i(Y) = 0$, i.e. $\psi_i(Y) = a_i \in \mathbb{K}$ for all $i = 0, ..., n$. Hence $f = \sum_{i=0}^{n} a_i X^i$, $a_n \neq 0$. If $n \geq 1$ then the coefficient in $X^{n-1}Y^2$ in $\sum_{k\geq 1} f_X^{(k)}(X) Y^{k+1} = 0$ is equal to $na_n \neq 0$, which is impossible. Hence $n = 0$ and $f = a_0$. Thus, $Z(\Lambda_2(\mathbb{K})) = \mathbb{K}$.

Suppose secondly that $char\mathbb{K} = p > 0$. By Proposition 2.1 elements X^p, Y^p are central in $\Lambda_2(\mathbb{K})$. Since $f'_Y = 0$, we have $\psi_i(Y) = \tilde{\psi}_i(Y^p)$ for all $i = 0, ..., n$ and $\tilde{\psi}_i \in \mathbb{K}[V]$. Set $f_1 = \sum_1 X^i \tilde{\psi}_i(Y^p)$, where the sum Σ_1

is taken over all $i = 0, ..., p$ such that $p \nmid i$. Since $f = \sum_2 X^i \tilde{\psi}_i (Y^p) + f_1$, where the sum Σ_2 is taken over all $i = 0, ..., p$ such that $p \mid i$, we see that $f_1 \in Z(\Lambda_2(\mathbb{K}))$, i.e. $\sum_{k\geq 1} (f_1)_X^{(k)}(X)Y^{k+1} = 0$. Assume that $f_1 \neq 0$. Then $\deg_X f_1 = n_1$, where $p \nmid n_1$. Let $\tilde{\psi}_{n_1}(Y^p) = \sum_{j=0}^{l} a_j Y^{pj}$, where $a_l \neq 0$. Then the coefficient in $X^{n_1-1}Y^{pl+2}$ in $\sum_{k\geq 1} (f_1)_X^{(k)}(X)Y^{k+1} = 0$ is equal to $n_1 a_l \neq 0$, a contradiction. Therefore, $f_1 = 0$ and $f = \sum_2 X^i \tilde{\psi}_i (Y^p)$. Now the proof follows.

(ii)It is easy to check that if $w = \sum_{i\geq 0, j\geq 0} \alpha_{ij} X^i Y^j \in \Lambda_1(\mathbb{K}, \lambda)$, then

$$wX = \sum_{i\geq 0, j\geq 0} \alpha_{ij} X^i Y^j X = \sum_{i\geq 0, j\geq 0} \alpha_{ij} \lambda^j X^{i+1} Y^j,$$
$$Yw = \sum_{i\geq 0, j\geq 0} \alpha_{ij} Y X^i Y^j = \sum_{i\geq 0, j\geq 0} \alpha_{ij} \lambda^i X^i Y^{j+1}.$$

Let $f = \sum_{i=0}^{n} X^i \psi_i(Y)$, $\psi_n(Y) \neq 0$, be a non-zero central element in $\Lambda_1(\mathbb{K}, \lambda)$. Assume that λ is not a root of unity. Since $Yf = fY$, we have $\sum_{i=0}^{n} \lambda^i X^i \psi_i(Y) Y = \sum_{i=0}^{n} X^i \psi_i(Y) Y$. In particular, $\lambda^n \psi_n(Y) = \psi_n(Y)$, i.e. $\lambda^n = 1$. Since λ is not a root of unity, we can conclude that $n = 0$, i.e. $f = \psi_0(Y)$. Let $f = \sum_{j=0}^{m} \alpha_j Y^j$, where $\alpha_m \neq 0$. Since $Xf = fX$, we have $\sum_{j=0}^{m} \alpha_j \lambda^j X Y^j = \sum_{j=0}^{m} \alpha_j X Y^j$. In particular, $\alpha_m \lambda^m = \alpha_m$, i.e. $\lambda^m = 1$. Since λ is not a root of unity, we can conclude that $m = 0$, $f = \alpha_0$. Thus, $Z(\Lambda_1(\mathbb{K}, \lambda)) = \mathbb{K}$. The case when λ is a root of unity of degree m is similar. □

Proposition 2.3. *If* $char\mathbb{K} = 0$ *and* I *is a proper two-sided ideal of the algebra* $\Lambda_2(\mathbb{K})$, *then* $I \cap \mathbb{K}[Y] = (Y^n)$ *for some* $n \in \mathbb{N}$.

Proof. We first claim that $I \cap \mathbb{K}[Y] \neq (0)$. Choose an element $f = \sum_{i=0}^{m} X^i \psi_i(Y) \in I \setminus \{0\}$ of least possible $X-$degree, m say, i.e. $\psi_m(Y) \neq 0$. Assume that $m \neq 0$. Consider the element $Yf - fY = \sum_{k\geq 1} f_X^{(k)} Y^{k+1} \in I$, $\deg_X(Yf - fY) \leq m-1$. Because of the choice of f, we have $Yf - fY = 0$. The coefficient in X^{m-1} in $Yf - fY = 0$ is equal to $m\psi_m(Y)Y^2 \neq 0$, a contradiction. So, $m = 0$ and $f = \psi_0(Y) \in I \cap \mathbb{K}[Y]$. Clearly, $I \cap \mathbb{K}[Y] \triangleleft \mathbb{K}[Y]$. Since $\mathbb{K}[Y]$ is a principal ideal ring, we have that

$I \cap \mathbb{K}[Y] = (\psi(Y))$. Let $\psi(Y) = \sum_{i=0}^{n} a_i Y^i$ and $a_n \neq 0$, $n \geq 1$. Consider the element $\psi(Y)X - X\psi(Y) = \psi'(Y)Y^2 \in I$. Then $\psi'(Y)Y^2 \in (\psi(Y))$, i.e. $\psi'(Y)Y^2 = \psi(Y)(\alpha Y + \beta)$. Let $a_0 = \ldots = a_{j-1} = 0$ and $a_j \neq 0$. Considering the coefficients in Y^j, we get $(j-1)a_{j-2} = \alpha a_{j-1} + \beta a_j$, i.e. $\beta = 0$. Therefore, $\psi'(Y)Y^2 = \alpha\psi(Y)Y$. Comparing coefficients in Y^r in both sides, we get $(r-1)a_{r-1} = \alpha a_{r-1}$. Thus $(r-1-\alpha)a_{r-1} = 0$. In particular $\alpha = n$ and therefore $a_i = 0$ for all $i < n$. □

Theorem 2.4. *If* $char\mathbb{K} = 0$ *and* I *is a proper prime ideal of* $\Lambda_2(\mathbb{K})$*, then either* $I = (Y)$*, or* $I = (Y, \psi(X))$ *for some irreducible polynomial* $\psi(X) \in \mathbb{K}[X]$*.*

Proof. It is easy to check that the ideals (Y) and $(Y, \psi(X))$, $\psi(X) \in \mathbb{K}[X]$ is irreducible, are prime. We claim that there is no other prime ideal in $\Lambda_2(\mathbb{K})$. It follows from Proposition 2.3 that if I is a proper prime ideal of $\Lambda_2(\mathbb{K})$, then $I \cap \mathbb{K}[Y] = (Y^n) \triangleleft \mathbb{K}[Y]$ for some $n \in \mathbb{N}$. If $n \geq 2$, then $Y \notin I$, $Y^{n-1} \notin I$ and $Y\Lambda_2(\mathbb{K})Y^{n-1} \subseteq \Lambda_2(\mathbb{K})Y^n \subseteq I$, i.e. the ideal I is not prime. Therefore, $n = 1$, i.e. $Y \in I$. Suppose that $I \neq (Y)$. Each element of $\Lambda_2(\mathbb{K})$ can be represented in the form $w(X,Y)Y + g(X)$ and therefore $I \cap \mathbb{K}[X] = (\psi(X)) \neq 0$ and $I = (Y, \psi(X))$. Now $\Lambda_2(\mathbb{K})/I$ is isomorphic to $\mathbb{K}[X]/I \cap \mathbb{K}[X]$ and therefore $\psi(X) \in \mathbb{K}[X]$ is irreducible. □

Similarly, one can prove

Theorem 2.5. *If* $\lambda \in \mathbb{K}^*$ *is not a root of unity and* I *is a proper prime ideal of* $\Lambda_1(\mathbb{K}, \lambda)$*, then* I *is one of ideals* (X)*,* (Y)*,* $(X, \psi(Y))$*,* $(Y, \psi(X))$ *for some irreducible polynomial* ψ *in one variable.*

3. Endomorphisms of $\Lambda_2(\mathbb{K})$

Now we shall study endomorphisms of algebra $\Lambda_2(\mathbb{K})$.

Theorem 3.1. *If* $char\mathbb{K} = 0$ *and* φ *is an automorphism of the algebra* $\Lambda_2(\mathbb{K})$*, then* $\varphi(X) = \gamma X + g(Y)$*,* $\varphi(Y) = \gamma Y$ *for some* $\gamma \in \mathbb{K}^*$ *and* $g(Y) \in \mathbb{K}[Y]$*.*

Proof. From Theorem 2.4 it follows that (Y) is a minimal nonzero prime ideal of $\Lambda_2(\mathbb{K})$. Therefore, $\varphi((Y)) = (Y)$, i.e. $\varphi(Y) = \gamma Y$ for some $\gamma \in \mathbb{K}^*$. If $\varphi^{-1}(X) = \sum_{i=0}^{n} X^i \psi_i(Y)$ then $X = \varphi\left(\varphi^{-1}(X)\right) = \sum_{i=0}^{n} (\varphi(X))^i \psi_i(\gamma Y)$. It is clear that $\deg_X \varphi(X) \geq 1$ and $n \geq 1$. Then $\deg_X \varphi(X) = 1$ and $n = 1$, i.e. $\varphi(X) = Xf(Y) + g(Y)$ for some $f \neq 0$. Then $X = \varphi\left(\varphi^{-1}(X)\right) = (Xf(Y) + g(Y))\psi_1(\gamma Y) + \psi_0(\gamma Y)$. In particular, $1 =$

$f(Y)\psi_1(\gamma Y)$. Then $f = \alpha \in \mathbb{K}^*$, i.e. $\varphi(X) = \alpha X + g(Y)$. Since $YX = XY + Y^2$, we have $\varphi(Y)\varphi(X) = \varphi(X)\varphi(Y) + (\varphi(Y))^2$. Then we have $\gamma Y(\alpha X + g(Y)) = (\alpha X + g(Y))\gamma Y + \gamma^2 Y^2$. Therefore, $\alpha Y^2 = \gamma Y^2$, i.e. $\alpha = \gamma$. □

Proposition 3.2. *If $char\mathbb{K} = 0$ and φ is an endomorphism of the algebra $\Lambda_2(\mathbb{K})$ with nonzero kernel, then $\varphi(Y) = 0$.*

Proof. Since the algebra $\Lambda_2(\mathbb{K})$ is a domain, we can conclude that the ideal $\ker\varphi$ is prime. From Theorem 2.4 it follows that $Y \in \ker\varphi$, i.e. $\varphi(Y) = 0$. Note that φ can take X to any element of $\Lambda_2(\mathbb{K})$. □

Notes. So, in the case $char\mathbb{K} = 0$ we have described the group of automorphisms of $\Lambda_2(\mathbb{K})$ and all endomorphisms φ of $\Lambda_2(\mathbb{K})$ when $\ker\varphi \neq 0$. It is clear from Theorem 3.1 that $Aut\Lambda_2(\mathbb{K}) \cong \mathbb{K}^* \times \mathbb{K}[Y]$ with respect to the operation $\circ$ such that $(\gamma_2, g_2(Y)) \circ (\gamma_1, g_1(Y)) = (\gamma_1\gamma_2, \gamma_1 g_2(Y) + g_1(\gamma_2 Y))$. The semigroup $\mathrm{End}\Lambda_2(\mathbb{K})$ has not been described yet. Note that there exist some endomorphisms $\varphi : \Lambda_2(\mathbb{K}) \to \Lambda_2(\mathbb{K})$ such that $\ker\varphi = 0$ and $Im\varphi \neq \Lambda_2(\mathbb{K})$. It is easy to check that maps $X \mapsto n^{-1}XY^{n-1} + g(Y)$, $Y \mapsto Y^n$ for all $g(Y) \in \mathbb{K}[Y]$, $n \in \mathbb{N}$ and $X \mapsto \alpha X^2 + \beta XY$, $Y \mapsto 2\alpha XY + 2(\alpha+\beta)Y^2$ for all $\alpha, \beta \in \mathbb{K}$ satisfy these properties. One can prove that if $\varphi \in \mathrm{End}\Lambda_2(\mathbb{K})$, then $\varphi(Y) = w(X,Y)Y$ for some $w(X,Y) \in \Lambda_2(\mathbb{K})$. Note that endomorphisms of $\Lambda_1(\mathbb{K}, \lambda)$ are classified in [3].

4. Derivations of $\Lambda_2(\mathbb{K})$

In this section we shall consider derivations of the algebra $\Lambda_2(\mathbb{K})$. All derivations of $\Lambda_1(\mathbb{K}, \lambda)$ in the case when λ is not a root of unity were classified in [2].

Notes. Let Λ be an algebra over field $\mathbb{K}$. Recall that a $\mathbb{K}-$linear map $\partial : \Lambda \to \Lambda$ is a **derivation** of Λ if for all $a, b \in \Lambda$ we have $\partial(ab) = \partial(a)b + a\partial(b)$. Given an element $w \in \Lambda$ consider the **inner derivation** $\mathrm{ad}\, w$ such that $\mathrm{ad}\, w(a) = wa - aw$, $a \in \Lambda$. The space of all derivations of Λ is a Lie algebra with respect to the operation of commutation. Denote this algebra by $Der\Lambda$. The subspace $Derint\Lambda$ of inner derivations is always an ideal in $Der\Lambda$. Let

$$L = {}^{Der\Lambda}\!/_{Derint\Lambda}$$

be an **algebra of outer derivations of** Λ.

Proposition 4.1. *Let* $w = X^a Y^b \in \Lambda_2(\mathbb{K})$. *Then*

$$\operatorname{ad} w(X) = bX^a Y^{b+1} \quad \operatorname{ad} w(Y) = -\sum_{k\geq 1} (X^a)^{(k)} Y^{b+k+1}.$$

Proof. By Proposition 2.1 we have

$$\begin{aligned} \operatorname{ad} w(X) &= wX - Xw = X^a Y^b X - X^{a+1} Y^b \\ &= X^a(XY^b + bY^{b+1}) - X^{a+1}Y^b = bX^a Y^{b+1}, \\ \operatorname{ad} w(Y) &= wY - Yw = X^a Y^{b+1} - YX^a Y^b \\ &= X^a Y^{b+1} - (\sum_{k\geq 0} (X^a)^{(k)} Y^{k+1})Y^b = -\sum_{k\geq 1} (X^a)^{(k)} Y^{b+k+1}. \end{aligned}$$

□

Proposition 4.2 (Derivations of $\Lambda_2(\mathbb{K})$).
(*I*) *If char*$\mathbb{K} = 0$, *then each derivation* ∂ *of* $\Lambda_2(\mathbb{K})$ *can be represented in the form*

$$\partial(X) = \alpha Y + \psi(X) + \operatorname{ad} w(X), \ \partial(Y) = \psi'(X) Y + \operatorname{ad} w(Y)$$

for some $\alpha \in \mathbb{K}$, $\psi \in \mathbb{K}[X]$, $w \in \Lambda_2(\mathbb{K})$.
(*II*) *If char*$\mathbb{K} = p > 2$, *then each derivation* ∂ *of* $\Lambda_2(\mathbb{K})$ *can be represented in the form*

$$\begin{aligned} \partial(X) &= \psi(X) + T(X^p, Y^p) Y + \operatorname{ad} w(X), \\ \partial(Y) &= \psi'(X) Y + S(X^p, Y^p) Y X^{p-1} Y + \operatorname{ad} w(Y) \end{aligned}$$

for some $\psi \in \mathbb{K}[X]$, $T, S \in Z(\Lambda_2(\mathbb{K}))$, $w \in \Lambda_2(\mathbb{K})$.
(*III*) *If char*$\mathbb{K} = 2$, *the each derivation* ∂ *of* $\Lambda_2(\mathbb{K})$ *can be represented in the form*

$$\begin{aligned} \partial(X) &= \psi(X) + T(X^2, Y^2) Y + \operatorname{ad} w(X), \\ \partial(Y) &= \varphi(X) + (\varphi'(X) + \psi'(X)) Y + S(X^2, Y^2) YXY + \operatorname{ad} w(Y) \end{aligned}$$

for some $\varphi, \psi \in \mathbb{K}[X]$, $T, S \in Z(\Lambda_2(\mathbb{K}))$, $w \in \Lambda_2(\mathbb{K})$.

Proof. The linear map $\partial : \Lambda_2(\mathbb{K}) \to \Lambda_2(\mathbb{K})$ is a derivation of $\Lambda_2(\mathbb{K})$ if and only if $\partial(YX) = \partial(XY) + \partial(Y^2)$, i.e. $\partial(Y)X + Y\partial(X) = \partial(X)Y + X\partial(Y) + \partial(Y)Y + Y\partial(Y)$. It can easily be checked that if the linear map $\partial : \Lambda_2(\mathbb{K}) \to \Lambda_2(\mathbb{K})$ satisfies the conditions of Proposition 4.1, then ∂ is the derivation of $\Lambda_2(\mathbb{K})$. We claim that there is no other derivations of $\Lambda_2(\mathbb{K})$. Let $\partial \in Der\Lambda_2(\mathbb{K})$. Put $U = \partial(X)$, $V = \partial(Y)$.

Then $VX + YU = UY + XV + VY + YV$. If $U = \sum_{i=0}^{m} \psi_i(X) Y^i$ and $V = \sum_{i=0}^{n} \varphi_i(X) Y^i$, then by Proposition 2.1 we get $VX = XV + V_Y' Y^2$, $YV = \sum_{k\geq 0} V_X^{(k)} Y^{k+1}$, $YU = \sum_{k\geq 0} U_X^{(k)} Y^{k+1}$. Hence,

$$V_Y' Y^2 + \sum_{k\geq 1} U_X^{(k)} Y^{k+1} = 2VY + \sum_{k\geq 1} V_X^{(k)} Y^{k+1}. \qquad (4.2.1)$$

We shall consider three cases.

(I) Let $char\mathbb{K} = 0$. If $m \geq 2$, then put $w = \sum_{k=2}^{m} (k-1)^{-1} \psi_k(X) Y^{k-1}$, $\partial_1 = \partial - \operatorname{ad} w$. From Proposition 4.1 we get $\partial_1(X) = U - \operatorname{ad} w(X) = \psi_1(X) Y + \psi_0(Y)$. Without loss of generality we can assume that $\partial = \partial_1$, i.e. $\partial(X) = \psi_1(X) Y + \psi_0(Y)$. Consider the coefficients in Y, Y^2 and Y^3 in (4.2.1). We have, respectively, $2\varphi_0 = 0$, $\varphi_1 + \psi_0' = 2\varphi_1 + \varphi_0'$, $2\varphi_2 + \psi_1' + \psi_0'' = 2\varphi_2 + \varphi_1' + \varphi_0''$. Then $\varphi_0 = 0$, $\psi_0' = \varphi_1$, $\psi_1' = 0$.

Lemma. *$\varphi_r = \varphi_2^{(r-2)}$ for all $r \geq 2$.*

Proof. We shall proceed by induction on r. The case $r = 2$ is clear. Assume that $\varphi_r = \varphi_2^{(r-2)}$ for all $r = 2, ..., k$ and consider the coefficients of Y^{k+2} in (4.2.1). We have $(k+1)\varphi_{k+1} + \sum_{i=0}^{k} \psi_i^{(k+1-i)} = 2\varphi_{k+1} + \sum_{i=0}^{k} \varphi_i^{(k+1-i)}$, where $\psi_2 = ... = \psi_k = 0$, $\varphi_0 = 0$ and $\psi_0^{(k+1)} = \varphi_1^{(k)}$. By induction $\varphi_i^{(k+1-i)} = \varphi_2^{(k-1)}$ for all $i = 2, ..., k$. Therefore, $(k+1)\varphi_{k+1} = 2\varphi_{k+1} + (k-1)\varphi_2^{(k-1)}$, i.e. $\varphi_{k+1} = \varphi_2^{(k-1)}$. □

So, $\partial(Y) = \psi_0'(X) Y + \sum_{k\geq 0} \varphi_2^{(k)}(X) Y^{k+2}$. From Proposition 4.1 it follows that $\sum_{k\geq 0} \varphi_2^{(k)}(X) Y^{k+2} = -(\operatorname{ad}\Phi_2(X))(Y)$, where $\Phi_2 \in \mathbb{K}[X]$ and $\Phi_2'(X) = \varphi_2(X)$. Clearly, $(\operatorname{ad}\Phi_2(X))(X) = 0$ and so

$$\begin{aligned} \partial(X) &= \psi_1(X) Y + \psi_0(Y) - (\operatorname{ad}\Phi_2(X))(X), \\ \partial(Y) &= \psi_0'(X) Y - (\operatorname{ad}\Phi_2(X))(Y). \end{aligned}$$

(II) Let $char\mathbb{K} = p > 2$. Then for any $\psi \in \mathbb{K}[X]$ we have $\psi^{(p)}(X) = 0$. From (4.2.1) we get

$$V_Y' Y^2 + \sum_{1\leq k\leq p-1} U_X^{(k)} Y^{k+1} = 2VY + \sum_{1\leq k\leq p-1} V_X^{(k)} Y^{k+1}. \qquad (4.2.2)$$

From Proposition 4.1 it follows that $\psi_k(X)Y^k = \left(\mathrm{ad}\,\frac{\psi_k(X)}{k-1}Y^{k-1}\right)(X)$ when $p \nmid (k-1)$. Put $w = \sum_1 \frac{\psi_k(X)}{k-1}Y^{k-1}$, where the sum $\sum_1$ is taken over all $k \geq 1$ such that $p \nmid (k-1)$, $\partial_1 = \partial - \mathrm{ad}\,w$. Then $\partial_1(X) = \psi_0(X) + \sum_{k=0}^{l} \psi_{kp+1}(X)Y^{kp+1}$ for some $l \in \mathbb{N}_0$. Without loss of generality we can assume that $\partial = \partial_1$. As in the case $char\mathbb{K} = 0$ it follows from (4.2.2) that $\varphi_0 = 0$ and $\psi_0' = \varphi_1$.

Lemma. *$\psi'_{np+1} = 0$ and $\varphi_{np+2+i} = \varphi^{(i)}_{np+2}$ for all $n \geq 0$ and $i = 0, \ldots, p-1$.*

Proof. We shall proceed by induction on n. Let $n = 0$. As in the case $char\mathbb{K} = 0$ by (4.2.2) we get $\psi_1' = 0$. Let us check by induction on i that $\varphi_{2+i} = \varphi_2^{(i)}$ for all $i = 0, ..., p-1$. The case $i = 0$ is trivial. Assume that $\varphi_{2+i} = \varphi_2^{(i)}$ when $0 \leq i \leq k \leq p-2$ and consider the coefficients in Y^{k+4} in (4.2.2). We have

$$(k+3)\varphi_{k+3} + \sum_{j=0}^{k+2} \psi_j^{(k+3-j)} = 2\varphi_{k+3} + \sum_{j=0}^{k+2} \varphi_j^{(k+3-j)},$$

where $\varphi_0 = 0$, $\psi_1' = 0$, $\psi_0^{(k+3)} = \varphi_1^{(k+2)}$. Since $k+2 \leq p$, we have $\psi_2 = ... = \psi_{k+2} = 0$. By inductive assumption $\varphi_j^{(k+3-j)} = \varphi_2^{(k+1)}$ when $2 \leq j \leq k+2$. Combining these results, we get $(k+1)\varphi_{k+1} = (k+1)\varphi_2^{(k-1)}$. Since $1 \leq k+1 \leq p-1$, we have $k+1 \neq 0$ in the field $\mathbb{K}$. So, $\varphi_{k+1} = \varphi_2^{(k-1)}$. Assume that the statement of lemma holds for all $n = 0, ..., m$ and consider the coefficients of $Y^{p(m+1)+3}$ in (4.2.2). We have

$$\begin{aligned}(2 + p(m+1))\varphi_{p(m+1)+2} &+ \sum_{j=pm+3}^{p(m+1)+1} \psi_j^{(p(m+1)+2-j)} \\ = 2\varphi_{p(m+1)+2} &+ \sum_{j=pm+3}^{p(m+1)+1} \varphi_j^{(p(m+1)+2-j)},\end{aligned}$$

where $\psi_{pm+3} = ... = \psi_{p(m+1)} = 0$. By inductive assumption

$$\varphi_j^{(p(m+1)+2-j)} = \varphi_{pm+2}^{(p)} = 0$$

when $pm+3 \leq j \leq p(m+1)+1$. So, $\psi'_{p(m+1)+1} = (p-1)\varphi^{(p)}_{pm+2} = 0$. Now one can check by induction on i as above that $\varphi_{(m+1)p+2+i} = \varphi^{(i)}_{(m+1)p+2}$ for all $i = 0, ..., p-1$. □

Since $\psi'_{pj+1} = 0,\ 0 \le j \le l$, we have $\psi_{pj+1}(X) = \tilde{\psi}_j(X^p)$. So,

$$U = \psi_0(X) + \sum_{j=0}^{l} \tilde{\psi}_j(X^p)Y^{pj+1} = \psi_0(X) + YT(X^p, Y^p),$$

where $\sum\limits_{j=0}^{l} \tilde{\psi}_j(X^p)Y^{pj} = T(X^p, Y^p) \in Z(\Lambda_2(\mathbb{K}))$. We have already proved that $V = \psi'_0(X)Y + \sum\limits_{k\ge 0}\sum\limits_{i=0}^{p-1} \varphi^{(i)}_{pk+2}(X)Y^{pk+2+i}$. If $\psi(X) = X^{pr+i}$, $\Psi(X) = \frac{X^{pr+i+1}}{i+1}$, $r \in \mathbb{N}_0$, $0 \le i \le p-2$, then $\Psi'(X) = \psi(X)$ and from Proposition 4.1 we get

$$\sum_{j=0}^{p-1} \psi^{(j)}(X)\, Y^{pk+2+j} = \sum_{j=1}^{p} \Psi^{(j)}(X)\, Y^{pk+1+j} = -\left(\operatorname{ad}\Psi(X)\, Y^{pk}\right)(Y),$$

where $\left(\operatorname{ad}\Psi(X)\, Y^{pk}\right)(X) = 0$ for all $k \ge 0$. Let $\varphi_{pk+2}(X) = \sum\limits_{i=0}^{s_k} \alpha^k_i X^i$, $w = \sum\limits_{k=0}^{l} \sum\limits_{0\le i\le s_k, p(i+1)} \alpha^k_i \frac{X^{i+1}}{i+1} Y^{pk}$, $\partial = \partial_2 - \operatorname{ad} w$. Then $\partial_2(X) = \partial(X)$ and

$$\begin{aligned}\partial_2(Y) &= \psi'_0(X)\, Y + \sum_{k\ge 0}\sum_{i\ge 1, pi-1\le s_k} \alpha^k_{pi-1} Y^{pk} \sum_{j=1}^{p} \tfrac{(p-1)!}{(p-j)!} X^{ip-j} Y^{j+1} \\ &= \psi'_0(X)\, Y + \Big(\sum_{k\ge 0}\sum_{i\ge 1, pi-1\le s_k} \alpha^k_{pi-1} X^{i(p-1)} Y^{pk}\Big)\Big(\sum_{j=1}^{p} \tfrac{(p-1)!}{(p-j)!} X^{p-j} Y^{j+1}\Big) \\ &= \psi'_0(X)\, Y + R(X,Y)\, S(X^p, Y^p),\end{aligned}$$

where

$$S(X^p, Y^p) = \sum_{k\ge 0}\sum_{i\ge 1, pi-1\le s_k} \alpha^k_{pi-1} X^{i(p-1)} Y^{pk} \in Z(\Lambda_2(\mathbb{K})),$$

$$R(X,Y) = \sum_{j=1}^{p} \frac{(p-1)!}{(p-j)!} X^{p-j} Y^{j+1} = YX^{p-1}Y.$$

(III) Let $char\mathbb{K} = 2$. Then for any $\psi \in \mathbb{K}[X]$ we have $\psi''(X) = 0$. From (4.2.1) we get $V'_Y Y^2 + U'_X Y^2 = V'_X Y^2$, i.e.

$$V'_Y + U'_X = V'_X \tag{4.2.3}$$

From Proposition 4.1 it follows that $\psi_k(X)\, Y^k = \left(\operatorname{ad} \frac{\psi_k(X)}{k-1} Y^{k-1}\right)(X)$ when $k \in \mathbb{N}$, $2 \mid k$. Put $w = \sum_1 \frac{\psi_k(X)}{k-1} Y^{k-1}$, where the sum $\sum_1$ is taken over all $k \geq 1$ such that $2 \mid k$, $\partial_1 = \partial - \operatorname{ad} w$. Then $\partial_1(X) = \psi_0(X) + \sum_{k=0}^{l} \psi_{2k+1}(X) Y^{2k+1}$ for some $l \in \mathbb{N}_0$. As above we shall assume that $\partial = \partial_1$. From (4.2.3) it follows that φ_0 can be equal to any element of $\mathbb{K}[X]$. Considering monomials in X in (4.2.3) we have $\varphi_1 + \psi_0' = \varphi_0'$, i.e. $\varphi_1 = \psi_0' + \varphi_0'$. Then $\varphi_1' = \psi_0'' + \varphi_0'' = 0$. Consider the coefficients in Y^{2n}, $n \in \mathbb{N}$, in (4.2.3). We have $\varphi_{2n+1} + \psi_{2n}' = \varphi_{2n}'$, where $\psi_{2n} = 0$. Then $\varphi_{2n+1} = \varphi_{2n}'$ and $\varphi_{2n+1}' = \varphi_{2n}'' = 0$. Considering the coefficients of Y^{2n-1}, $n \in \mathbb{N}$, in (4.2.3) we have $\psi_{2n-1}' = \varphi_{2n-1}' = 0$. Thus,

$$\begin{aligned}\partial(X) &= \psi_0(X) + \sum_{j=0}^{l} \tilde{\psi}_j(X^2)Y^{2j+1} = \psi_0(X) + T(X^2, Y^2)Y,\\ \partial(Y) &= \varphi_0(X) + (\varphi_0'(X) + \psi_0'(X))Y\\ &\quad + \sum_{j\geq 1} (\varphi_{2j}(X)Y^{2j} + \varphi_{2j}'(X)Y^{2j+1}),\end{aligned}$$

where $T\left(X^2, Y^2\right) = \sum_{j=0}^{l} \tilde{\psi}_j\left(X^2\right) Y^{2j} \in Z\left(\Lambda_2(\mathbb{K})\right)$. As in the case $char\mathbb{K} = p > 2$ it is easily shown that

$$\partial(Y) = \varphi_0(X) + \left(\varphi_0'(X) + \psi_0'(X)\right) Y + R(X, Y)\, S\left(X^2, Y^2\right) + \operatorname{ad} w(Y),$$

where $S\left(X^2, Y^2\right) \in Z\left(\Lambda_2(\mathbb{K})\right)$, $R(X, Y) = XY^2 + Y^3 = YXY$, $w \in \Lambda_2(\mathbb{K})$ and $\operatorname{ad} w(X) = 0$. □

The next propositions are technical. We briefly indicate their proofs.

Proposition 4.3. *For any $n \in \mathbb{N}$*

$$\begin{aligned}Q_n(X, Y) &= \sum_{k=0}^{n-1} X^k Y X^{n-1-k} = \sum_{k=0}^{n-1} (n-1-k)! \binom{n}{k} X^k Y^{n-k}\\ &= \sum_{k=0}^{n-1} \frac{n!}{k!\,(n-k)} X^k Y^{n-k}.\end{aligned}$$

The proof is a direct calculation based on Proposition 2.1.

Proposition 4.4. *If $\partial \in Der\Lambda_2(\mathbb{K})$ and $\varphi \in \mathbb{K}[X]$, then $\partial(\varphi(X)) = \varphi'(X)\,\partial(X) + \operatorname{ad} w_\varphi(X)$ for some $w_\varphi \in \Lambda_2(\mathbb{K})$.*

Proof. From Proposition 4.2 it follows that $\partial(X) = zY + \psi(X)$ for some $z \in Z\left(\Lambda_2(\mathbb{K})\right)$ and $\psi \in \mathbb{K}[X]$. It is enough to check our statement for

$\varphi(X) = X^n$, $n \in \mathbb{N}_0$. The cases $n = 0$ and $n = 1$ are clear. If $n \geq 2$, then from Proposition 4.3 we obtain

$$\partial(X^n) = n\psi(X)X^{n-1} + znX^{n-1}Y + z\sum_{k=0}^{n-2}(n-1-k)!\binom{n}{k}X^kY^{n-k}.$$

From Proposition 4.1 it follows that

$$z\sum_{k=0}^{n-2}(n-1-k)!\binom{n}{k}X^kY^{n-k} = \operatorname{ad} w_\varphi(X),$$

where $w_\varphi = z\sum_{k=0}^{n-2}(n-2-k)!\binom{n}{k}X^kY^{n-k-1}$. Thus,

$$\partial(X^n) = (X^n)'\,\partial(X) + \operatorname{ad} w_\varphi(X).$$

□

Proposition 4.5. *Let* $char\mathbb{K} = 0$ *and* $\partial_1, \partial_2 \in Der\Lambda_2(\mathbb{K})$, *where*

$$\partial_i(X) = \alpha_i Y + \psi_i(X), \quad \partial_i(Y) = \psi_i'(X)Y$$

for some $\alpha_i \in \mathbb{K}$, $\psi_i \in \mathbb{K}[X]$, $i = 1, 2$. *Then there exists an element* $w \in \Lambda_2(\mathbb{K})$ *such that*

$$[\partial_1, \partial_2](X) = \psi_1(X)\psi_2'(X) - \psi_1'(X)\psi_2(X) + \operatorname{ad} w(X),$$
$$[\partial_1, \partial_2](Y) = \big(\psi_1(X)\psi_2''(X) - \psi_1''(X)\psi_2(X)\big)Y + \operatorname{ad} w(Y).$$

Proof. From Proposition 4.4 we get

$$[\partial_1, \partial_2](X) = \psi_1(X)\psi_2'(X) - \psi_1'(X)\psi_2(X) + \operatorname{ad}\tilde{w}(X)$$

for some $\tilde{w} \in \Lambda_2(\mathbb{K})$. If $\partial = [\partial_1, \partial_2] - \operatorname{ad}\tilde{w}$, then

$$\partial(X) = \psi_1(X)\psi_2'(X) - \psi_1'(X)\psi_2(X).$$

As in the proof of Proposition 4.2 it is easily shown that

$$\partial(Y) = \big(\psi_1(X)\psi_2'(X) - \psi_1'(X)\psi_2(X)\big)'Y + \operatorname{ad}\tilde{\tilde{w}}(Y)$$

for some $\tilde{\tilde{w}} \in \Lambda_2(\mathbb{K})$, where $\operatorname{ad}\tilde{\tilde{w}}(X) = 0$. Let $w = \tilde{w} + \tilde{\tilde{w}}$. Finally, we obtain

$$[\partial_1, \partial_2](X) = \psi_1(X)\psi_2'(X) - \psi_1'(X)\psi_2(X) + \operatorname{ad} w(X),$$

$$[\partial_1, \partial_2](Y) = \psi_1(X)\psi_2''(X) - \psi_1''(X)\psi_2(X) + \operatorname{ad} w(Y).$$

□

Combining Propositions 4.2 and 4.5 we obtain

Theorem 4.6. *If* $\mathrm{char}\mathbb{K} = 0$*, then each derivation* ∂ *of* $\Lambda_2(\mathbb{K})$ *can be represented in the form* $\partial(X) = \alpha Y + \psi(X) + \operatorname{ad} w(X)$*,* $\partial(Y) = \psi'(X)Y + \operatorname{ad} w(Y)$*, where* $\alpha \in \mathbb{K}$*,* $\psi \in \mathbb{K}[X]$*,* $w \in \Lambda_2(\mathbb{K})$ *and the Lie algebra of outer derivations of* $\Lambda_2(\mathbb{K})$ *is isomorphic to the algebra* $\mathbb{K} \oplus \mathbb{K}[X]$ *with respect to the operation* $[\cdot,\cdot]$ *such that*

$$[(\alpha_1, \psi_1(X)), (\alpha_2, \psi_2(X))] = \left(0, \psi_1(X)\psi_2'(X) - \psi_1'(X)\psi_2(X)\right).$$

Note that if Λ is an algebra over field $\mathbb{K}$, $\partial \in Der\Lambda$ and $z \in Z(\Lambda)$, then $\partial(z) \in Z(\Lambda)$. Indeed, for any $w \in \Lambda$ we have $\partial(w)z + w\partial(z) = \partial(wz) = \partial(zw) = \partial(z)w + z\partial(w) = \partial(z)w + \partial(w)z$, i.e. $\partial(z)w = w\partial(z)$. From Theorem 2.2 it follows that if $z \in Z(\Lambda_2(\mathbb{K}))$, then $z = \sum\limits_{i\geq 0, j\geq 0} \alpha_{ij} X^{ip} Y^{jp}$.

Put

$$z'_{X^p} = \sum_{i\geq 1, j\geq 0} i\alpha_{ij} X^{(i-1)p} Y^{jp}, \quad z'_{Y^p} = \sum_{i\geq 0, j\geq 1} j\alpha_{ij} X^{ip} Y^{(j-1)p}.$$

Proposition 4.7. *If* $\mathrm{char}\mathbb{K} = p > 0$*,* $z \in Z(\Lambda_2(\mathbb{K}))$*,* $\partial \in Der\Lambda_2(\mathbb{K})$*,* $\partial(X) = \psi + YT$*,* $\partial(Y) = \psi' Y + YX^{p-1}YS$*, where* $\psi \in \mathbb{K}[X]$*,* $T, S \in Z(\Lambda_2(\mathbb{K}))$*, then* $\partial(z) = -\left(z'_{X^p} T(X^p, Y^p) Y^p + z'_{Y^p} S(X^p, Y^p) Y^{2p}\right)$*.*

The proof is a direct calculation based on Propositions 1.2 and 4.3.

Proposition 4.8. *If* $\mathrm{char}\mathbb{K} = p > 0$*,* $\partial \in Der\Lambda_2(\mathbb{K})$*,* $\partial(X) = \psi + YT$*,* $\partial(Y) = \psi' Y + YX^{p-1}YS$*, where* $\psi \in \mathbb{K}[X]$*,* $T, S \in Z(\Lambda_2(\mathbb{K}))$*, then for some coefficients* $\alpha_{ij} \in \mathbb{K}$

$$\partial\left(YX^{p-1}Y\right) = \psi' YX^{p-1}Y + 2YX^{p-1}YX^{p-1}YS + \sum \alpha_{ij} X^i Y^j,$$

where the sum is taken over all i, j *such that* $j \geq 2$*,* $p \nmid i+1$*.*

The proof is a direct calculation based on Propositions 2.1 and 4.4.

Proposition 4.9. *If* $\mathrm{char}\mathbb{K} = p > 0$*,* $\partial_i \in Der\Lambda_2(\mathbb{K})$*,* $i = 1, 2$*,* $\partial_i(X) = \psi_i + YT_i$*,* $\partial_i(Y) = \psi_i' Y + YX^{p-1}YS_i$*, where* $\psi_i \in \mathbb{K}[X]$*,* $T_i, S_i \in Z(\Lambda_2(\mathbb{K}))$*, then for some* $w \in \Lambda_2(\mathbb{K})$

$$\begin{aligned}
[\partial_1, \partial_2](X) &= \operatorname{ad} w(X) + \psi_1\psi_2' - \psi_1'\psi_2 - Y^{p+1}(S_1T_2 - S_2T_1) \\
&\quad + Y\left(Y^p\left((T_1)'_{X^p} T_2 - (T_2)'_{X^p} T_1\right)\right. \\
&\quad \left. + Y^{2p}\left((T_1)'_{Y^p} S_2 - (T_2)'_{Y^p} S_1\right)\right), \\
[\partial_1, \partial_2](Y) &= \operatorname{ad} w(Y) + \left(\psi_1\psi_2'' - \psi_1''\psi_2\right)Y \\
&\quad + YX^{p-1}Y\left(\left((S_1)'_{X^p} T_2 - (S_2)'_{X^p} T_1\right)Y^p\right. \\
&\quad \left. + \left((S_1)'_{Y^p} S_2 - (S_2)'_{Y^p} S_1\right)Y^{2p}\right).
\end{aligned}$$

The proof is a direct calculation based on Propositions 2.1, 4.1, 4.2, 4.4, 4.7, 4.8.

Combining Propositions 4.2 and 4.9 we obtain

Theorem 4.10. *If* $char\mathbb{K} = p > 2$, *then each derivation* ∂ *of* $\Lambda_2(\mathbb{K})$ *can be written in the form*

$$\partial(X) = \psi(X) + T(X^p, Y^p)Y + \operatorname{ad} w(X),$$
$$\partial(Y) = \psi'(X)Y + S(X^p, Y^p)YX^{p-1}Y + \operatorname{ad} w(Y),$$

where $\psi \in \mathbb{K}[X]$, $T, S \in Z(\Lambda_2(\mathbb{K}))$, $w \in \Lambda_2(\mathbb{K})$ *and the Lie algebra of outer derivations of* $\Lambda_2(\mathbb{K})$ *is isomorphic to the algebra* $\mathbb{K}[X] \oplus \mathbb{K}[X^p, Y^p] \oplus \mathbb{K}[X^p, Y^p]$ *with respect to the operation* $[\cdot,\cdot]$ *such that*

$$[(\psi_1, T_1, S_1), (\psi_2, T_2, S_2)] = (\psi, T, S),$$

where $\psi = \psi_1\psi_2' - \psi_1'\psi_2$,

$$T = -Y^p(S_1T_2 - S_2T_1) + Y^p\left((T_1)'_{X^p}T_2 - (T_2)'_{X^p}T_1\right) + Y^{2p}\left((T_1)'_{Y^p}S_2 - (T_2)'_{Y^p}S_1\right),$$
$$S = Y^p\left((S_1)'_{X^p}T_2 - (S_2)'_{X^p}T_1\right) + Y^{2p}\left((S_1)'_{Y^p}S_2 - (S_2)'_{Y^p}S_1\right).$$

Now we shall consider the Lie algebra of outer derivations of $\Lambda_2(\mathbb{K})$ in the case $char\mathbb{K} = 2$. As above we shall omit technical details.

Proposition 4.11. *If* $char\mathbb{K} = 2$ *and* $\theta \in \mathbb{K}[X]$, *then* $\theta' \in Z(\Lambda_2(\mathbb{K}))$.

Proof. It is enough to consider the case $\theta = X^n$, $n \in N$. If $n = 2k$, then $\theta' = nX^{n-1} = 0$. If $n = 2k+1$, then $\theta' = nX^{n-1} = X^{2k}$. From Theorem 2.2 we get $\theta' \in Z(\Lambda_2(\mathbb{K}))$. □

Proposition 4.12. *If* $char\mathbb{K} = 2$, $z \in Z(\Lambda_2(\mathbb{K}))$, $\partial \in Der\Lambda_2(\mathbb{K})$, $\partial(X) = \psi + YT$, $\partial(Y) = \varphi + (\psi' + \varphi')Y + YXYS$, *where* $\psi, \varphi \in \mathbb{K}[X]$, $T, S \in Z(\Lambda_2(\mathbb{K}))$, *then* $\partial(z) = z'_{X^2}TY^2 + z'_{Y^2}\left(SY^2 + \varphi'\right)Y^2$.

Proof. Consider derivations $\partial_i \in Der\Lambda_2(\mathbb{K})$, $i = 1, 2$, $\partial_1(X) = \psi + YT$, $\partial_1(Y) = \psi'Y + YXYS$, $\partial_2(X) = 0$, $\partial(Y) = \varphi + \varphi'Y$. Then $\partial = \partial_1 + \partial_2$. It remains to use Propositions 2.1 and 4.7. □

Proposition 4.13. *If* $char\mathbb{K} = 2$, $\partial \in Der\Lambda_2(\mathbb{K})$, $\partial(X) = \psi + YT$, $\partial(Y) = \varphi + (\psi' + \varphi')Y + YXYS$, *where* $\psi, \varphi \in \mathbb{K}[X]$, $T, S \in Z(\Lambda_2(\mathbb{K}))$, *then for some coefficients* $\alpha_{ij} \in \mathbb{K}$ $\partial(YXY) = \psi'YXY + \sum \alpha_{ij}X^iY^j$, *where the sum is taken over all* i, j *such that* $j \geq 2$, $2 \mid i$.

Proof. Consider derivations $\partial_i \in Der\Lambda_2(\mathbb{K})$, $i = 1, 2$, $\partial_1(X) = \psi + YT$, $\partial_1(Y) = \psi' Y + YXYS$, $\partial_2(X) = 0$, $\partial(Y) = \varphi + \varphi' Y$. Then $\partial = \partial_1 + \partial_2$. It follows from Proposition 4.8 that for some coefficients $\beta_{ij} \in \mathbb{K}$ $\partial_1(YXY) = \psi' YXY + \sum_1 \beta_{ij} X^i Y^j$, where the sum $\sum_1$ is taken over all i, j such that $j \geq 2$, $2 \mid i$. From Propositions 2.1 and 4.11 we get for some coefficients $\gamma_{ij} \in \mathbb{K}$

$$\begin{aligned}\partial_2(YXY) &= \partial_2(Y)XY + YX\partial_2(Y) \\ &= \left(\varphi + \varphi' Y\right)XY + YX\left(\varphi + \varphi' Y\right) \\ &= \varphi XY + Y\varphi X = \varphi XY + \varphi XY + (\varphi X)' Y^2 \\ &= (\varphi X)' Y^2 = \textstyle\sum_1 \gamma_{ij} X^i Y^j\end{aligned}$$

Finally, we obtain for some coefficients $\alpha_{ij} \in \mathbb{K}$

$$\partial(YXY) = \partial_1(YXY) + \partial_2(YXY) = \psi' YXY + \textstyle\sum_1 \alpha_{ij} X^i Y^j.$$

□

Proposition 4.14. *If* $char\mathbb{K} = 2$, $\theta \in \mathbb{K}[X]$, $\partial \in Der\Lambda_2(\mathbb{K})$, $\partial(X) = \psi + YT$, $\partial(Y) = \varphi + (\psi' + \varphi')Y + YXYS$, *where* $\psi, \varphi \in \mathbb{K}[X]$, $T = P + QY^2$, $P = P\left(X^2\right)$, $P, Q, S \in Z(\Lambda_2(\mathbb{K}))$, *then for some coefficients* $\gamma_{ij} \in \mathbb{K}$ $\partial(\theta) = \psi\theta' + YP\theta' + YXY(\theta')'_{X^2} T + \sum \gamma_{ij} X^i Y^j$, *where the sum is taken over all* i, j *such that* $j \geq 2$, $2 \mid i$.

Proof. Put $z = (X\theta)'$. From Proposition 4.11 we get $z \in Z(\Lambda_2(\mathbb{K}))$. It is clear that $\theta = X\theta' + z$. From Proposition 4.12 we obtain for some coefficients $\alpha_{ij} \in \mathbb{K}$

$$\partial(z) = z'_{X^2} TY^2 + z'_{Y^2}\left(SY^2 + \varphi'\right)Y^2 = z'_{X^2} TY^2 = \textstyle\sum_1 \alpha_{ij} X^i Y^j,$$

where the sum $\sum_1$ is taken over all i, j such that $j \geq 2$, $2 \mid i$. Since $XY^2 = YXY + Y^3$, we can conclude that for some coefficients $\beta_{ij} \in \mathbb{K}$

$$\begin{aligned}\partial\left(X\theta'\right) &= \partial(X)\theta' + X\partial\left(\theta'\right) = \left(\psi + Y\left(P + QY^2\right)\right)\theta' + XY^2\left(\theta'\right)'_{X^2} T \\ &= \psi\theta' + YP\theta' + YXY\left(\theta'\right)'_{X^2} T + \left(Q\theta' + \left(\theta'\right)'_{X^2} T\right)Y^3 \\ &= \psi\theta' + YP\theta' + YXY\left(\theta'\right)'_{X^2} T + \textstyle\sum_1 \beta_{ij} X^i Y^j\end{aligned}$$

Finally, we obtain for some coefficients $\gamma_{ij} \in \mathbb{K}$

$$\partial(\theta) = \psi\theta' + YP\theta' + YXY\left(\theta'\right)'_{X^2} T + \textstyle\sum_1 \gamma_{ij} X^i Y^j.$$

□

Proposition 4.15. *If* $char\mathbb{K} = 2$, $\partial_i \in Der\Lambda_2(\mathbb{K})$, $i = 1, 2$, $\partial_i(X) = \psi_i + YT_i$, $\partial_i(Y) = \varphi_i + (\psi_i' + \varphi_i')Y + YXYS_i$, *where* $\psi_i, \varphi_i \in \mathbb{K}[X]$, $T_i = P_i + Q_iY^2$, $P_i = P_i(X^2)$, $P_i, Q_i, S_i \in Z(\Lambda_2(\mathbb{K}))$, *then*

$$[\partial_1, \partial_2](X) = \psi + YT + \operatorname{ad} w(X),$$
$$[\partial_1, \partial_2](Y) = \varphi + (\psi' + \varphi')Y + YXYS + \operatorname{ad} w(Y),$$

where $\psi = (\psi_1\psi_2)' + \varphi_1P_2 + \varphi_2P_1$, $\varphi = (\psi_1\varphi_2 + \psi_2\varphi_1 + \varphi_1\varphi_2)'$,

$$T = \varphi_1'T_2 + \varphi_2'T_1 + \left((T_2)'_{Y^2}\varphi_1' + (T_1)'_{Y^2}\varphi_2'\right)Y^2 + (S_1T_2 + S_2T_1)Y^2$$
$$+ \left((T_2)'_{X^2}T_1 + (T_1)'_{X^2}T_2\right)Y^2 + \left((T_2)'_{Y^2}S_1 + (T_1)'_{Y^2}S_2\right)Y^4,$$
$$S = (\varphi_2')'_{X^2}T_1 + (\varphi_1')'_{X^2}T_2 + \varphi_2'S_1 + \varphi_1'S_2$$
$$+ \left((S_2)'_{Y^2}\varphi_1' + (S_1)'_{Y^2}\varphi_2'\right)Y^2 + \left((S_2)'_{X^2}T_1 + (S_1)'_{X^2}T_2\right)Y^2$$
$$+ \left((S_2)'_{Y^2}S_1 + (S_1)'_{Y^2}S_2\right)Y^4.$$

Proof. Apply Proposition 4.1, 4.4, 4.12, 4.13, 4.14. □

Combining Propositions 4.2 and 4.15 we obtain

Theorem 4.16. *If* $char\mathbb{K} = 2$, *then each derivation* ∂ *of* $\Lambda_2(\mathbb{K})$ *can be represented in the form*

$$\partial(X) = \psi(X) + T(X^2, Y^2)Y + \operatorname{ad} w(X),$$
$$\partial(Y) = \varphi(X) + (\varphi'(X) + \psi'(X))Y + S(X^2, Y^2)YXY + \operatorname{ad} w(Y),$$

where $\varphi, \psi \in \mathbb{K}[X]$, $T = P + QY^2$, $P = P(X^2)$, $P, Q, S \in Z(\Lambda_2(\mathbb{K}))$, $w \in \Lambda_2(\mathbb{K})$, *and the Lie algebra of outer derivations of* $\Lambda_2(\mathbb{K})$ *is isomorphic to the algebra* $\mathbb{K}[X] \oplus \mathbb{K}[X] \oplus \mathbb{K}[X^2] \oplus \mathbb{K}[X^2, Y^2] \oplus \mathbb{K}[X^2, Y^2]$ *with respect to the operation* $[\cdot, \cdot]$ *such that*

$$[(\psi_1, \varphi_1, P_1, Q_1, S_1), (\psi_1, \varphi_1, P_1, Q_1, S_1)] = (\psi, \varphi, P, Q, S),$$

where $\psi = (\psi_1\psi_2)' + \varphi_1P_2 + \varphi_2P_1$,

$$\varphi = (\psi_1\varphi_2 + \psi_2\varphi_1 + \varphi_1\varphi_2)', \quad P = \varphi_1'P_2 + \varphi_2'P_1,$$
$$Q = \varphi_1'Q_2 + \varphi_2'Q_1 + (T_2)'_{Y^2}\varphi_1' + (T_1)'_{Y^2}\varphi_1' + S_1T_2 + S_2T_1$$
$$+ (T_2)'_{X^2}T_1 + (T_1)'_{X^2}T_2 + \left((T_2)'_{Y^2}S_1 + (T_1)'_{Y^2}S_2\right)Y^2,$$
$$S = (\varphi_2')'_{X^2}T_1 + (\varphi_1')'_{X^2}T_2 + \varphi_2'S_1 + \varphi_1'S_2$$
$$+ \left((S_2)'_{Y^2}\varphi_1' + (S_1)'_{Y^2}\varphi_2'\right)Y^2$$
$$+ \left((S_2)'_{X^2}T_1 + (S_1)'_{X^2}T_2\right)Y^2 + \left((S_2)'_{Y^2}S_1 + (S_1)'_{Y^2}S_2\right)Y^4,$$

where $T_i = P_i + Q_iY^2$, $i = 1, 2$.

The Lie algebra of derivations of $\Lambda_2(\mathbb{K})$ in characteristic $p > 0$ is a $p-$algebra. We shall consider this structure. Let $\frac{d}{dx} : \mathrm{k}[X] \to \mathbb{K}[X]$ be the operator of formal differentiation, i.e. $\frac{d}{dx}(\psi(X)) = \psi'(X)$ and $m_\theta : \mathbb{K}[X] \to \mathbb{K}[X]$ be the operator of multiplication by $\theta(X) \in \mathbb{K}[X]$, i.e. $m_\theta(\psi(X)) = \theta(X)\psi(X)$.

Proposition 4.17. *If $\partial \in Der\Lambda_2(\mathbb{K})$, $\partial(X) = \psi$, $\partial(Y) = \psi' Y$, where $\psi \in \mathbb{K}[X]$, then for all $n \in \mathbb{N}$ $\partial^n(X) = (m_\psi \circ \frac{d}{dx})^{n-1}(\psi)$, $\partial^n(Y) = (\frac{d}{dx} \circ m_\psi)^n(1)Y$.*

The proof is a direct calculation.

Suppose that $char\mathbb{K} = p > 0$. Let $\frac{\partial}{\partial X^p} : Z(\Lambda_2(\mathbb{K})) \to Z(\Lambda_2(\mathbb{K}))$ be the operator of formal differentiation by X^p, i.e. $\frac{\partial}{\partial X^p}(z) = z_{X^p}$, $\frac{\partial}{\partial Y^p} : Z(\Lambda_2(\mathbb{K})) \to Z(\Lambda_2(\mathbb{K}))$ be the operator of formal differentiation by Y^p, i.e. $\frac{\partial}{\partial Y^p}(z) = z_{Y^p}$ and $m_w : Z(\Lambda_2(\mathbb{K})) \to Z(\Lambda_2(\mathbb{K}))$ be the operator of multiplication by $w \in Z(\Lambda_2(\mathbb{K}))$, i.e. $m_w(z) = wz$. Put $d = m_T \circ \frac{\partial}{\partial X^p} + m_S \circ m_{Y^p} \circ \frac{\partial}{\partial Y^p} : Z(\Lambda_2(\mathbb{K})) \to Z(\Lambda_2(\mathbb{K}))$.

Proposition 4.18. *If $char\mathbb{K} = p > 2$, $\partial \in Der\Lambda_2(\mathbb{K})$, $\partial(X) = YT$, $\partial(Y) = YX^{p-1}YS$, where $T, S \in Z(\Lambda_2(\mathbb{K}))$, then for some $w \in \Lambda_2(\mathbb{K})$*

$$\partial^p(X) = Y(d \circ m_{Y^p})^{p-1}(T) + \operatorname{ad} w(X),$$
$$\partial^p(Y) = YX^{p-1}Y(m_S + d) \circ (d \circ m_{Y^p})^{p-1}(1) + \operatorname{ad} w(Y).$$

The proof is a direct calculation based on Propositions 4.2, 4.7.

As in [5] it is easy to prove

Proposition 4.19. *If Λ is a $\mathbb{K}$-algebra, $char\mathrm{k} = p > 0$, $\partial_i \in Der\Lambda$, $i = 1, 2$, $[\partial_1, \partial_2] = \operatorname{ad} w$ for some $w \in \Lambda$, then $(\partial_1 + \partial_2)^p = \partial_1^p + \partial_2^p + \operatorname{ad} u$ for some $u \in \Lambda$.*

Theorem 4.20. *If $char\mathbb{K} = p > 2$, $\partial \in Der\Lambda_2(\mathbb{K})$, $\partial(X) = \psi + YT$, $\partial(Y) = \psi' Y + YX^{p-1}YS$, where $\psi \in \mathbb{K}[X]$, $T, S \in Z(\Lambda_2(\mathbb{K}))$, then*

$$\partial^p(X) = \tilde{\psi} + Y\tilde{T} + \operatorname{ad} w(X), \partial^p(Y) = \tilde{\psi}' Y + YX^{p-1}Y\tilde{S} + \operatorname{ad} w(Y),$$

where $w \in \Lambda_2(\mathbb{K})$, $\tilde{\psi} = \left(m_\psi \circ \frac{d}{dx}\right)^{p-1}(\psi)$, $\tilde{T} = (d \circ m_{Y^p})^{p-1}(T)$, $\tilde{S} = (m_S + d) \circ (d \circ m_{Y^p})^{p-1}(1)$.

Proof. Consider derivations $\partial_i \in Der\Lambda_2(\mathbb{K})$, $i = 1, 2$, where $\partial_1(X) = \psi$, $\partial_1(Y) = \psi' Y$, $\partial_2(X) = YT$, $\partial(Y) = YX^{p-1}YS$. Then $\partial = \partial_1 + \partial_2$. It follows from Proposition 4.9 that $[\partial_1, \partial_2] = \operatorname{ad} w$ for some $w \in \Lambda_2(\mathbb{K})$. Now the statement of Proposition 4.20 follows from Propositions 4.17, 4.18 and 4.19. □

Proposition 4.21. *If* $char\mathbb{K} = 2$*,* $\partial \in Der\Lambda_2(\mathbb{K})$*,* $\partial(X) = YT$*,* $\partial(Y) = YXYS$*, where* $T, S \in Z(\Lambda_2(\mathbb{K}))$*, then for some* $w \in \Lambda_2(\mathbb{K})$

$$\partial^2(X) = Y(d \circ m_{Y^2})(T) + \operatorname{ad} w(X),$$
$$\partial^2(Y) = YXY(m_S + d) \circ (d \circ m_{Y^2})(1) + \operatorname{ad} w(Y).$$

Proof. Since $\partial^2 \in Der\Lambda_2(\mathbb{K})$, it follows from Proposition 4.2 that

$$\partial^2(X) = \tilde{\psi} + Y\tilde{T} + \operatorname{ad} w(X),$$
$$\partial^2(Y) = \tilde{\varphi} + \left(\tilde{\psi}' + \tilde{\varphi}'\right) Y + YXY\tilde{S} + \operatorname{ad} w(Y),$$

here $w \in \Lambda_2(\mathbb{K})$, $\tilde{\psi}, \tilde{\varphi} \in \mathbb{K}[X]$, $\tilde{T}, \tilde{S} \in Z(\Lambda_2(\mathbb{K}))$. As in the proof of Proposition 4.18 it is easily shown that $\tilde{\psi} = 0$. Since $\partial^2(Y) = \partial((YXS)Y) = (\partial(YXS) + YXSYXS)Y$, we get $\tilde{\varphi} = 0$. The following argumentation is the same as in the proof of Proposition 4.18. □

Proposition 4.22. *If* $char\mathbb{K} = 2$*,* $\partial \in Der\Lambda_2(\mathbb{K})$*,* $\partial(X) = 0$*,* $\partial(Y) = \varphi + \varphi' Y$*, where* $\varphi \in \mathbb{K}[X]$*, then* $\partial^2(X) = 0$*,* $\partial^2(Y) = \varphi\varphi' + (\varphi')^2 Y$*.*

Proof. We have $\partial^2(X) = 0$, $\partial^2(Y) = \partial(\varphi + \varphi' Y) = \varphi'\partial(Y) = \varphi\varphi' + (\varphi')^2 Y$. □

Theorem 4.23. *If* $char\mathbb{K} = 2$*,* $\partial \in Der\Lambda_2(\mathbb{K})$*,* $\partial(X) = \psi + YT$*,* $\partial(Y) = \varphi + (\psi' + \varphi')Y + YXYS$*, where* $\varphi, \psi \in \mathbb{K}[X]$*,* $T = P + QY^2$*,* $P = P(X^2)$*,* $P, Q, S \in Z(\Lambda_2(\mathbb{K}))$*, then*

$$\partial^2(X) = \tilde{\psi} + Y\tilde{T} + \operatorname{ad} w(X),$$
$$\partial^2(Y) = \tilde{\varphi} + (\tilde{\psi}' + \tilde{\varphi}')Y + YXY\tilde{S} + \operatorname{ad} w(Y),$$

where $\tilde{T} = (d \circ m_{Y^2})(T) + \varphi'\left(T + T'_{Y^2}\right)$*,* $\tilde{S} = (m_S + d) \circ (d \circ m_{Y^2})(1) + (\varphi')'_{X^2} T + \varphi' S + \varphi' S'_{Y^2} Y^2$*,* $\tilde{\psi} = \left(m_\psi \circ \frac{d}{dx}\right)^2(X) + \varphi P$*,* $w \in \Lambda_2(\mathbb{K})$*,* $\tilde{\varphi} = \varphi\varphi'$*.*

Proof. Consider derivations $\partial_i \in Der\Lambda_2(\mathbb{K})$, $i = 1, 2, 3$, where $\partial_1(X) = \psi$, $\partial_1(Y) = \psi' Y$, $\partial_2(X) = YT$, $\partial(Y) = YX^{p-1}YS$, $\partial_3(X) = 0$, $\partial_3(Y) = \varphi + \varphi' Y$. Then $\partial = \partial_1 + \partial_2 + \partial_3$. From Proposition 4.15 we get $[\partial_1, \partial_2] = \operatorname{ad} w_{12}$ for some $w_{12} \in \Lambda_2(\mathbb{K})$, $[\partial_1, \partial_3] = \partial_{13} + \operatorname{ad} w_{13}$, where $\partial_{13}(X) = 0$, $\partial_{13}(Y) = (\psi\varphi)'$ and $w_{13} \in \Lambda_2(\mathbb{K})$, $[\partial_2, \partial_3] = \partial_{23} + \operatorname{ad} w_{23}$, where $\partial_{23}(X) = \varphi P + \varphi'\left(T + T'_{Y^2}\right)Y$, $\partial_{23}(Y) = (\varphi P)' Y + YXY\left((\varphi')'_{X^2} T + \varphi' S + \varphi' S'_{Y^2} Y^2\right)$, $w_{23} \in \Lambda_2(\mathbb{K})$. We have

$$\begin{aligned}\partial^2 &= (\partial_1 + \partial_2 + \partial_3)^2 \\ &= \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_1\partial_2 + \partial_2\partial_1 + \partial_1\partial_3 + \partial_3\partial_1 + \partial_2\partial_3 + \partial_3\partial_2 \\ &= \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_{13} + \partial_{23} + \operatorname{ad}(w_{12} + w_{13} + w_{23}).\end{aligned}$$

It remains to use Propositions 4.17, 4.21 and 4.22. □

5. Classification Theorems

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be an associative graded algebra over field $A_0 = \mathbb{K}$ generated by elements $X, Y \in A_1$. Suppose that $\dim A_2 = 3$. Then monomials X^2, Y^2, XY and YX are linear dependent over $\mathbb{K}$, so there exists a unique to proportionality set of coefficients $(\alpha, \beta, \gamma, \delta) \in \mathbb{K}^4 \backslash \{0\}$ such that

$$\alpha X^2 + \beta Y^2 + \gamma XY + \delta YX = 0. \tag{5.0.4}$$

Note that similar algebras over field of zero characteristic are considered in [8].

Proposition 5.1. *If A is an algebra without zero divisors, then $\alpha\beta - \gamma\delta \neq 0$.*

Proof. Assume that $\alpha\beta - \gamma\delta = 0$. If $\delta = 0$ then either $\alpha = 0$ or $\beta = 0$. Consider the case $\alpha = 0$. Since $(\alpha, \beta, \gamma, \delta) \in \mathbb{K}^4 \backslash \{0\}$, we see that either $\beta \neq 0$ or $\gamma \neq 0$. Since $\dim A_2 = 3$, we can conclude that X and Y are linear independent over $\mathbb{K}$. Thus, $\beta Y + \gamma X \neq 0$ and $0 = \beta Y^2 + \gamma XY = (\beta Y + \gamma X)Y$, which is impossible since A has no zero divisors. Similarly, if $\beta = 0$, then $\alpha X + \gamma Y \neq 0$ and $0 = \alpha X^2 + \gamma XY = X(\alpha X + \gamma Y)$, a contradiction. Therefore $\delta \neq 0$. Then $\alpha X + \delta Y \neq 0$, $\delta X + \beta Y \neq 0$ and

$$(\alpha X + \delta Y)(\delta X + \beta Y) = \delta(\alpha X^2 + \beta Y^2 + \gamma XY + \delta YX) = 0.$$

This contradiction proves the 5.1. □

Proposition 5.2. *Suppose that $\mathbb{K}$ has no quadratic extensions and $\alpha\beta - \gamma\delta \neq 0$. Then there exist generators X_1 and Y_1 such that either $Y_1 X_1 = \lambda X_1 Y_1$ for some $\lambda \in \mathbb{K}^*$ or $Y_1 X_1 = X_1 Y_1 + Y_1^2$.*

Proof. We shall consider two cases. Let first $\alpha \neq 0$, $\beta = 0$. Put $X = Y_1$, $Y = X_1$. Suppose secondly that $\alpha \neq 0$, $\beta \neq 0$. We shall find X_1 and Y_1 such that $X = X_1$, $Y = \xi X_1 + Y_1$, where $\xi \in \mathbb{K}$. We shall latter specify the value of parameter ξ. If we replace X by X_1 and Y by $\xi X_1 + Y_1$ in (5.0.4), then we get

$$\alpha X_1^2 + \beta (\xi X_1 + Y_1)^2 + \gamma X_1 (\xi X_1 + Y_1) + \delta (\xi X_1 + Y_1) X_1 =$$

$$= \tilde{\alpha} X_1^2 + \tilde{\beta} Y_1^2 + \tilde{\gamma} X_1 Y_1 + \tilde{\delta} Y_1 X_1 = 0,$$

where $\tilde{\alpha} = \alpha + \beta\xi^2 + \gamma\xi + \delta\xi$. Since $\beta \neq 0$, it follows that there exists an element $\xi \in \mathbb{K}$ such that $\tilde{\alpha} = 0$. We claim that $\tilde{\alpha}\tilde{\beta} - \tilde{\gamma}\tilde{\delta} \neq 0$. Indeed, the coefficients of quadratic form $\alpha X^2+\beta Y^2+\gamma XY+\delta YX$ are changed under the substitution $X=X_1$, $Y=\xi X_1+Y_1$ according to the rule

$$\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde\alpha & \tilde\gamma \\ \tilde\delta & \tilde\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}^T \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}.$$

Then

$$\tilde\alpha\tilde\beta-\tilde\gamma\tilde\delta=-\tilde\gamma\tilde\delta=\det\begin{pmatrix} \tilde\alpha & \tilde\gamma \\ \tilde\delta & \tilde\beta \end{pmatrix}=(\alpha\beta-\gamma\delta)\neq 0.$$

Thus, without loss of generality, we can assume that $\alpha=0$. Then $\gamma\delta\neq 0$ and

$$YX=-\gamma\delta^{-1}XY-\beta\delta^{-1}Y^2=\gamma_1 YX+\beta_1 Y^2, \qquad (5.2.5)$$

where $\gamma_1\neq 0$. If we replace X by $X_1+\zeta Y_1$ for some $\zeta\in\mathbb{K}$ and Y by Y_1 in (5.2.5), then we get $Y_1(X_1+\zeta Y_1)=\gamma_1(X_1+\zeta Y_1)Y_1+\beta_1 Y_1^2$. Therefore, $Y_1X_1=\gamma_1X_1Y_1+(\beta_1+\gamma_1\zeta-\zeta)Y_1^2$. If either $\gamma_1\neq 1$ or $\beta_1=0$, then there exists an element $\zeta\in\mathbb{K}$ such that $\beta_1+\gamma_1\zeta-\zeta=0$. Thus we have $Y_1X_1=\gamma_1X_1Y_1$. In the converse case $\gamma_1=1$ and $\beta_1\neq 0$, i.e. $YX=XY+\beta_1Y^2$. Let $X=\beta_1X_1$, $Y=Y_1$. Then $Y_1X_1=X_1Y_1+Y_1^2$. □

So, without loss of generality, we can assume that generators X and Y satisfy either the equality $YX=\lambda XY$, $\lambda\in\mathbb{K}^*$, or the equality $YX=XY+Y^2$.

The following theorems shows that if some additional conditions hold true, then there exist only two classes of these algebras, namely quantum polynomials in two variables and Jordanian plane.

Theorem 5.3. *If $\mathbb{K}$ has no quadratic extensions, A is a central algebra and $\alpha\beta-\gamma\delta\neq 0$, then either $A=\Lambda_1(\mathbb{K},\lambda)$ and $\lambda\in\mathbb{K}^*$ is not a root of unity, or $A=\Lambda_2(\mathbb{K})$ and char$\mathbb{K}=0$. In particular, A is a domain and* $\dim A_n=n+1$, $n\in\mathbb{N}$.

Proof. From Proposition 5.2 it follows that $A={}^{B}/_{I}$, where either $B=\Lambda_1(\mathbb{K},\lambda)$ or $B=\Lambda_2(\mathbb{K})$ and I is a homogeneous prime ideal of the algebra B, $I\neq B$. We are going to prove that $I=0$. Assume the converse and consider two cases.

Case 1: let $B=\Lambda_1(\mathbb{K},\lambda)$, where $\lambda\in\mathbb{K}^*$. If $\lambda^m=1$ for some $m\in\mathbb{N}$, then from Theorem 2.2 we can conclude that X^m, Y^m are central in $\Lambda_1(\mathbb{K},\lambda)$. Consider the canonical homomorphism

$$\pi:\Lambda_1(\mathbb{K},\lambda)\to {}^{\Lambda_1(\mathbb{K},\lambda)}/_{I}.$$

Since π is surjective we have $\pi(X^p),\pi(Y^p)\in Z\left({}^{\Lambda_1(\mathbb{K},\lambda)}/_{I}\right)$. But the algebra ${}^{\Lambda_1(\mathbb{K},\lambda)}/_{I}$ is central and so $\pi(X^m)=\alpha\in\mathbb{K}$ and $\pi(Y^m)=\beta\in$

$\mathbb{K}$. Therefore, $X^m - \alpha, Y^m - \beta \in I$. Since the ideal I is homogeneous, we get $\alpha = \beta = 0$, i.e. $X^m, Y^m \in I$. But the ideal I is prime, so $X, Y \in I$ and $I = (X, Y)$. But $\dim A_2 = 3$. This contradiction shows that λ is not a root of unity. Then by Theorem 2.5 it follows that I is one of ideals (X), (Y) or (X, Y). In each case $\dim A_2 < 3$. Thus, if $B = \Lambda_1(\mathbb{K}, \lambda)$, then $I = 0$.

Case 2: $B = \Lambda_2(\mathbb{K})$. If $char\mathbb{K} = p > 0$, then from Theorem 2.2 we get Y^p is central in $\Lambda_2(\mathbb{K})$. Consider the canonical homomorphism $\pi : \Lambda_2(\mathbb{K}) \to \Lambda_2(\mathbb{K})/I$. Since π is surjective we have $\pi(Y^p) \in Z\left(\Lambda_1(\mathbb{K}, \lambda)/I\right)$. Then we can apply the same arguments as in the preceding case. □

Theorem 5.4. *If $\mathbb{K}$ has no quadratic extensions,* $\dim A_n = n+1$*, $n \in \mathbb{N}$, and $\alpha\beta - \gamma\delta \neq 0$, then either $A = \Lambda_1(\mathbb{K}, \lambda)$ or $A = \Lambda_2(\mathbb{K})$. In particular, A is a domain.*

Proof. Without loss of generality, we can assume that generators X and Y satisfy either the equality $YX = \lambda XY$, $\lambda \in \mathbb{K}^*$, or the equality $YX = XY + Y^2$. Consider the case $YX = XY + Y^2$. Put

$$\Lambda_2(\mathbb{K}) = \mathbb{K}\left\langle \tilde{X}, \tilde{Y} \right\rangle \Big/ \left(\tilde{Y}\tilde{X} - \tilde{X}\tilde{Y} - \tilde{Y}^2\right) = \bigoplus_{n=0}^{\infty} \tilde{A}_n,$$

where $\tilde{A}_0 = \mathbb{K}$, $\tilde{A}_n$, $n \in \mathbb{N}$, is a linear span of monomials of degree n in $\tilde{X}$, $\tilde{Y}$. From Proposition 1.2 we get $\dim \tilde{A}_n = n + 1$. There exists a graded algebra homomorphism $\varphi : \Lambda_2(\mathbb{K}) \to A$, $\tilde{X} \mapsto X$, $\tilde{Y} \mapsto Y$. Then $\ker \varphi = 0$, i.e. $A = \Lambda_2(\mathbb{K})$. In the case $YX = \lambda XY$, $\lambda \in \mathbb{K}^*$, using the same arguments we get $A = \Lambda_1(\mathbb{K}, \lambda)$. □

Corollary 5.5. *If* $\dim A_n = n + 1$*, $n \in \mathbb{N}$, $\alpha\beta - \gamma\delta \neq 0$, then A is a domain.*

Proof. Put $\bar{A} = \bar{\mathbb{K}} \otimes_{\mathbb{K}} A$. Then $\bar{A}$ is generated over $\bar{\mathbb{K}}$ by elements $\bar{X} = 1 \otimes X$ and $\bar{Y} = 1 \otimes Y$ and $\bar{A} = \bigoplus_{n=0}^{\infty} \bar{A}_n$, where $\bar{A}_n$, $n \in \mathbb{N}$, is the linear span of all monomials of degree n in $\bar{X}$ and $\bar{Y}$. In particular, $\dim \bar{A}_n = n + 1$. It is evident that $\alpha\bar{X}^2 + \beta\bar{Y}^2 + \gamma\bar{X}\bar{Y} + \delta\bar{Y}\bar{X} = 0$. Then from Theorem 5.4 it follows that $\bar{A}$ is a domain. □

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