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On mappings of terms determined by hypersubstitutions

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Abstract. The extensions of hypersubstitutions are mappings on the set of all terms. In the present paper we characterize all hypersubstitutions which provide bijections on the set of all terms. The set of all such hypersubstitutions forms a monoid.

On the other hand, one can modify each hypersubstitution to any mapping on the set of terms. For this we can consider mappings ρ from the set of all hypersubstitutions into the set of all mappings on the set of all terms. If for each hypersubstitution σ the application of $\rho(\sigma)$ to any identity in a given variety V is again an identity in V, so that variety is called ρ-solid. The concept of a ρ-solid variety generalizes the concept of a solid variety. In the present paper, we determine all ρ-solid varieties of semigroups for particular mappings ρ.

1. Basic Definitions and Notations

We fix a type $\tau = (n_i)_{i\in I}$, $n_i > 0$ for all $i \in I$, and a set of operation symbols $\Omega := \{f_i \mid i \in I\}$ where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet $X = \{x_1, x_2, \ldots\}$. Terms in $W_\tau(X_n)$ with $X_n = \{x_1, \ldots, x_n\}$, $n \geq 1$, are called n-ary. For natural numbers $m, n \geq 1$ we define a mapping $S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$ in the following way: For $(t_1, \ldots, t_n) \in W_\tau(X_m)^n$ we put:

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(i) $S^n_m(x_i, t_1, \dots, t_n) := t_i$ for $1 \le i \le n$;
(ii) $S^n_m(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n) := f_i(S^n_m(s_1, t_1, \dots, t_n), \dots, S^n_m(s_{n_i}, t_1, \dots, t_n))$ for $i \in I, s_1, \dots, s_{n_i} \in W_\tau(X_n)$ where $S^n_m(s_1, t_1, \dots, t_n), \dots, S^n_m(s_{n_i}, t_1, \dots, t_n)$ will be assumed to be already defined.

If it is obvious what m is, we write S^n. For $t \in W_\tau(X)$ we define the depth of t in the following inductive way:
(i) $depth(t) := 0$ for $t \in X$;
(ii) $depth(t) := \max\{depth(t_1), \dots, depth(t_{n_i})\} + 1$ for $t = f_i(t_1, \dots, t_{n_i})$with $i \in I, t_1, \dots, t_{n_i} \in W_\tau(X)$ where $depth(t_1), \dots, depth(t_{n_i})$ will be assumed to be already defined.

By $c(t)$ we denote the length of a term t (i.e. the number of the variables occurring in t), $var(t)$ denotes the set of all variables occurring in t and $cv(t)$ means the number of elements in the set $var(t)$. Instead of $x_1, x_2, x_3, \dots$ we write also $x, y, z, \dots$.

The concept of a hypersubstitution was introduced in [2].

Definition 1. *A mapping $\sigma : \Omega \to W_\tau(X)$ which assigns to every n_i-ary operation symbol f_i, $i \in I$, an n_i-ary term is called a hypersubstitution of type τ (shortly hypersubstitution). The set of all hypersubstitutions of type τ will be denoted by $Hyp(\tau)$.*

To each hypersubstitution σ there belongs a mapping from the set of all terms of the form $f_i(x_1, \dots, x_{n_i})$ to the terms $\sigma(f_i)$. It follows that every hypersubstitution of type τ then induces a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows:
(i) $\widehat{\sigma}[w] := w$ for $w \in X$;
(ii) $\widehat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^n(\sigma(f_i), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_{n_i}])$ for $i \in I, t_1, \dots, t_{n_i} \in W_\tau(X)$ where $\widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_{n_i}]$ will be assumed to be already defined.

By $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ is defined an associative operation on $Hyp(\tau)$ where $\circ$ denotes the usual composition of mappings. By ε we denote the hypersubstitution with $\varepsilon(f_i) = f_i(x_1, \dots, x_{n_i})$ for $i \in I$, where ε deals as identity element. Then $(Hyp(\tau); \circ_h, \varepsilon)$ forms a monoid, denoted by $\mathbf{Hyp}(\tau)$.

2. Bijections on $W_\tau(X)$

By $Bij(\tau)$ we denote the set of all $\sigma \in Hyp(\tau)$ such that $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ is a bijection on $W_\tau(X)$. Such hypersubstitutions have a high importance in computer science.

The product of two bijections is again a bijection. Further, for two hypersubstitutions σ_1 and σ_2 we have

$$(\sigma_1 \circ_h \sigma_2)^\frown = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$$

(see [3]). So we have the following result.

Proposition 1. *$(Bij(\tau); \circ_h, \varepsilon)$ forms a submonoid of* $\mathbf{Hyp}(\tau)$.

For the characterization of $Bij(\tau)$ we need the following notations:

(i) $\mathcal{B}$ denotes the set of all bijections on Ω preserving the arity.

(ii) Let S_n be the set of all permutations of the set $\{1, \ldots, n\}$ for $1 \leq n \in \mathbb{N}$.

(iii) $A := \bigcup\limits_{1 \leq n \in \mathbb{N}} S_n$.

(iv) $\mathcal{P} := \{p \in A^I \mid p(i) \in S_{n_i} \text{ for } i \in I\}$.

The following theorem characterizes $Bij(\tau)$ for any type τ.

Theorem 1. Let $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, be any type. For each $\sigma \in Hyp(\tau)$ the following statements are equivalent:

(i) $\sigma \in Bij(\tau)$.

(ii) There are $h \in \mathcal{B}$ and $p \in \mathcal{P}$ such that $\sigma(f_i) = h(f_i)(x_{p(i)(1)}, \ldots, x_{p(i)(n_i)})$ for all $i \in I$.

Proof. $(ii) \Rightarrow (i)$: We show by induction that $\widehat{\sigma}$ is injective and surjective.

Injectivity: Let $s, t \in W_\tau(X)$ with $\widehat{\sigma}[s] = \widehat{\sigma}[t]$.

Suppose that the $depth(s) = 0$. Then $depth(t) = 0$ and s, t are variables with $s = \widehat{\sigma}[s] = \widehat{\sigma}[t] = t$.

Suppose that from $\widehat{\sigma}[s'] = \widehat{\sigma}[t']$ there follows $s' = t'$ for any $s', t' \in W_\tau(X)$ with $depth(s') \leq n$.

Let $depth(s) = n + 1$. Then $depth(t) \geq 1$ and there are $i, j \in I$ with $s = f_i(s_1, \ldots, s_{n_i})$ and $t = f_j(t_1, \ldots, t_{n_j})$. Now we have

$\widehat{\sigma}[s] = S^{n_i}(h(f_i)(x_{p(i)(1)}, \ldots, x_{p(i)(n_i)}), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_i}])$

and

$\widehat{\sigma}[t] = S^{n_j}(h(f_j)(x_{p(j)(1)}, \ldots, x_{p(j)(n_j)}), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_j}])$. From $\widehat{\sigma}[s] = \widehat{\sigma}[t]$ it follows that $h(f_i) = h(f_j)$ and thus $f_i = f_j$, i.e. $i = j$, since h is a bijection. Hence $S^{n_i}(h(f_i)(x_{p(i)(1)}, \ldots, x_{p(i)(n_i)}), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_i}])$ $= S^{n_i}(h(f_i)(x_{p(i)(1)}, \ldots, x_{p(i)(n_i)}), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_j}])$ and, consequently, $\widehat{\sigma}[s_k] = \widehat{\sigma}[t_k]$ for $1 \leq k \leq n_i$. By our hypothesis we get $s_k = t_k$ for $1 \leq k \leq n_i$. Consequently, $s = f_i(s_1, \ldots, s_{n_i}) = f_j(t_1, \ldots, t_{n_j}) = t$.

Surjectivity: For $w \in X$ we have $\widehat{\sigma}[w] = w$.

Suppose that for any $s \in W_\tau(X)$ with $depth(s) \leq n$ there is an $\widetilde{s} \in W_\tau(X)$ with $\widehat{\sigma}[\widetilde{s}] = s$.

Let now $t \in W_\tau(X)$ be a term with $depth(t) = n + 1$. Then there is an $i \in I$ with $t = f_i(t_1, \ldots, t_{n_i})$ and by our hypothesis there are $\widetilde{t}_1, \ldots, \widetilde{t}_{n_i} \in W_\tau(X)$ such that $\widehat{\sigma}[\widetilde{t}_k] = t_k$ for $1 \leq k \leq n_i$. Further there is a $j \in I$ with $h(f_j) = f_i$ and $n_i = n_j$. Now we consider the term

$\widetilde{t} := f_j(\widetilde{t}_{p(j)^{-1}(1)}, \ldots, \widetilde{t}_{p(j)^{-1}(n_i)})$. There holds

$$\begin{aligned}\widehat{\sigma}[\widetilde{t}] &= S^{n_i}(h(f_j)(x_{p(j)(1)}, \ldots, x_{p(j)(n_j)}), \widehat{\sigma}[\widetilde{t}_{p(j)^{-1}(1)}], \ldots, \widehat{\sigma}[\widetilde{t}_{p(j)^{-1}(n_i)}]) = \\ &= S^{n_i}(f_i(x_{p(j)(1)}, \ldots, x_{p(j)(n_i)}), t_{p(j)^{-1}(1)}, \ldots, t_{p(j)^{-1}(n_i)})\end{aligned}$$

(by hypothesis)

$$= f_i(t_1, \ldots, t_{n_i}) = t.$$

$(i) \Rightarrow (ii)$: Since $\widehat{\sigma}$ is surjective for each $j \in I$ there is an $s_j \in W_\tau(X)$ with $\widehat{\sigma}[s_j] = f_j(x_1, \ldots, x_{n_j})$ which is minimal with respect to the depth. Obviously, the case $depth(s_j) = 0$ is impossible. Thus there are a $k \in I$ and $r_1, \ldots, r_{n_k} \in W_\tau(X)$ with $s_j = f_k(r_1, \ldots, r_{n_k})$. So $\widehat{\sigma}[s_j] = \widehat{\sigma}[f_k(r_1, \ldots, r_{n_k})] = S^{n_k}(\sigma(f_k), \widehat{\sigma}[r_1], \ldots, \widehat{\sigma}[r_{n_k}]) = f_j(x_1, \ldots, x_{n_j})$. This is only possible if $\sigma(f_k) \in X$ or $\sigma(f_k) = f_j(a_1, \ldots, a_{n_j})$ with $a_1, \ldots, a_{n_j} \in \{x_1, \ldots, x_{n_k}\}$, $\mid \{a_1, \ldots, a_{n_j}\} \mid= n_j$, and thus $n_k \geq n_j$. But the case $\sigma(f_k) \in X$ is impossible. Otherwise there is an $i \in \{1, \ldots, n_k\}$ with $\sigma(f_k) = x_i$ and $\widehat{\sigma}[s_j] = \widehat{\sigma}[r_i]$ where $depth(s_j) > depth(r_i)$, this contradicts the minimallity of s_j. This shows that for all $j \in I$ there are a $k(j) \in I$ with $n_{k(j)} \geq n_j$ and $a_1, \ldots, a_{n_j} \in X_{n_{k(j)}}$ with $\mid \{a_1, \ldots, a_{n_j}\} \mid= n_j$ such that $\sigma(f_{k(j)}) = f_j(a_1, \ldots, a_{n_j})$.

Assume that $n_{k(j)} > n_j$ for some $j \in I$. Then there is an $x \in X_{n_{k(j)}} \setminus var(\sigma(f_{k(j)}))$, i.e. x is not essential in $\sigma(f_{k(j)})$ and thus $\widehat{\sigma}$ is not a bijection on $W_\tau(X)$ (see [1], [6]), a contradiction. Thus $n_{k(j)} = n_j$ and $\sigma(f_{k(j)}) = f_j(x_{\pi_j(1)}, \ldots, x_{\pi_j(n_j)})$ for some $\pi_j \in S_{n_j}$.

Assume that there are $j, l \in I$ with $l \neq k(j)$ such that f_j is the first operation symbol in $\sigma(f_l)$. We put $t := \widehat{\sigma}[f_l(x_1, \ldots, x_{n_l})]$. Then $t = f_j(t_1, \ldots, t_{n_j})$ for some $t_1, \ldots, t_{n_j} \in W_\tau(X)$. Since $\widehat{\sigma}$ is surjective, there are $s_1, \ldots, s_{n_j} \in W_\tau(X)$ with $\widehat{\sigma}[s_i] = t_i$ for $1 \leq i \leq n_j$. Then

$$\begin{aligned}&\widehat{\sigma}[f_{k(j)}(s_{\pi_j^{-1}(1)}, \ldots, s_{\pi_j^{-1}(n_j)})] \\ &= S^{n_j}(\sigma(f_{k(j)}), \widehat{\sigma}[s_{\pi_j^{-1}(1)}], \ldots, \widehat{\sigma}[s_{\pi_j^{-1}(n_j)}]) \\ &= S^{n_j}(f_j(x_{\pi_j(1)}, \ldots, x_{\pi_j(n_j)}), t_{\pi_j^{-1}(1)}, \ldots, t_{\pi_j^{-1}(n_j)}) \\ &= f_j(t_1, \ldots, t_{n_j}).\end{aligned}$$

Since $f_{k(j)}(s_{\pi_j^{-1}(1)}, \ldots, s_{\pi_j^{-1}(n_j)}) \neq f_l(x_1, \ldots, x_{n_l})$, $\widehat{\sigma}$ is no injective, a contradiction. Altogether this shows that the mapping $h : \Omega \to \Omega$ where $h(f)$ is the first operation symbol in $\sigma(f)$ is a bijection on Ω preserving the arity. Further, let $p \in A^I$ with $p(i) := \pi_i$ for $i \in I$. Then $p \in \mathcal{P}$.

Consequently, we have $\sigma(f_i) = h(f_i)(x_{p(i)(1)}, \ldots, x_{p(i)(n_i)})$ for all $i \in I$. □

Let us give the following examples.

Example 1. *Let $2 \leq n \in \mathbb{N}$. We consider the type $\tau_n = (n)$, where f denotes the n-ary operation symbol. For $\pi \in S_n$ we define:*

$$\sigma_\pi : f \mapsto f(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

These hypersubstitutions are precisely the bijections, i.e. $Bij(\tau_n) = \{\sigma_\pi \mid \pi \in S_n\}$.

In particular, if $n = 2$ then $Bij(\tau_2) = \{\varepsilon, \sigma_d\}$ where σ_d is defined by

$$\sigma_d : f \mapsto f(x_2, x_1).$$

Example 2. *Let now $\tau = (2, 2)$ where f and g are the both binary operation symbols. Then we define the following eight hypersubstitutions $\sigma_1, \ldots, \sigma_8$ by:*

	$f \mapsto$	$g \mapsto$
$\sigma_1 :$	$f(x_1, x_2)$	$g(x_1, x_2)$
$\sigma_2 :$	$f(x_1, x_2)$	$g(x_2, x_1)$
$\sigma_3 :$	$f(x_2, x_1)$	$g(x_1, x_2)$
$\sigma_4 :$	$f(x_2, x_1)$	$g(x_2, x_1)$
$\sigma_5 :$	$g(x_1, x_2)$	$f(x_1, x_2)$
$\sigma_6 :$	$g(x_1, x_2)$	$f(x_2, x_1)$
$\sigma_7 :$	$g(x_2, x_1)$	$f(x_1, x_2)$
$\sigma_8 :$	$g(x_2, x_1)$	$f(x_2, x_1)$

These hypersubstitutions are precisely the bijections, so

$$Bij(\tau) = \{\sigma_1, \ldots, \sigma_8\}.$$

3. ρ-solid varieties

In Section 1, we mentioned that any hypersubstitution σ can be uniquely extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ ($\widehat{\sigma} \in W_\tau(X)^{W_\tau(X)}$). Thus a mapping $\rho : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$ is defined by setting $\rho(\sigma) = \widehat{\sigma}$ for all $\sigma \in Hyp(\tau)$.

In [4], the concept of a solid variety was introduced. By Birkhoff, a variety V is a class of algebras of type τ satisfying a set Σ of identities, i.e. $V = Mod\Sigma$. For a variety V of type τ we denote by IdV the set of all identities in V. The variety V is said to be solid iff $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV$ for all $s \approx t \in IdV$ and all $\sigma \in Hyp(\tau)$. For a submonoid $\mathbf{M}$ of $\mathbf{Hyp}(\tau)$, the variety V is said to be M-solid iff $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV$ for all $s \approx t \in IdV$ and all $\sigma \in M$ (see [3]). If $M = Hyp(\tau)$ then we have solid varieties.

In this section we will study mappings $\rho : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$ and generalize the concept of an M-solid variety to the concept of an M-ρ-solid variety. For convenience, we put $\sigma^\rho := \rho(\sigma)$ for $\sigma \in Hyp(\tau)$.

Definition 2. *Let $\rho : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$ be a mapping and V be a variety of type τ and* $\mathbf{M}$ *be a submonoid of* $\mathbf{Hyp}(\tau)$*. V is called M-ρ-solid iff $\sigma^\rho(s) \approx \sigma^\rho(t) \in IdV$ for all $s \approx t \in IdV$ and all $\sigma \in M$.*

If $M = Hyp(\tau)$ then V is said to be ρ-solid.

Example 3. *Let $\rho : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$ be defined by $\rho(\sigma) = \widehat{\sigma}$ for all $\sigma \in Hyp(\tau)$. Then the ρ-solid varieties are exactly the solid varieties, which is clear by the appropriate definitions. L. Polák has determined all solid varieties of semigroups in [5]. Besides the trivial variety, exactly the self-dual varieties in the interval between the normalization $Z \vee RB$ of the variety of all rectangular bands and the variety defined by the identities $x^2 \approx x^4$, $x^2y^2z \approx x^2yx^2yz$, $xy^2z^2 \approx xyz^2yz^2$, and $xyzyx \approx xyxzxyx$ as well as the varieties RB of all rectangular, NB of all normal, and $RegB$ of all regular bands are solid.*

In Section 2 we have checked that $Bij(\tau)$ forms a monoid. For particular mappings $\rho : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$ the $Bij(\tau)$-ρ-solid varieties are of special interest, in particular for type $\tau = (2)$ and semigroup varieties. They realize substitutions of operations in terms which are useful in some calculational aspects of computer algebra systems. In the following we will consider such mappings $\rho : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$.

Definition 3. *Let*

$$fa : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)} \text{ and } sa : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$$

be the following mappings: For $\sigma \in Hyp(\tau)$ we put

(i) $\sigma^{fa}(x) := \sigma^{sa}(x) := x$ *for* $x \in X$;

(ii) $\sigma^{fa}(f_i(t_1, \ldots, t_{n_i})) := S^{n_i}(\sigma(f_i), \sigma^{sa}(t_1), \ldots, \sigma^{sa}(t_{n_i}))$ *and* $\sigma^{sa}(f_i(t_1, \ldots, t_{n_i})) := f_i(\sigma^{fa}(t_1), \ldots, \sigma^{fa}(t_{n_i}))$ *for $i \in I$ and $t_1, \ldots, t_{n_i} \in W_\tau(X)$ where $\sigma^{sa}(t_1), \ldots, \sigma^{sa}(t_{n_i}), \sigma^{fa}(t_1), \ldots, \sigma^{fa}(t_{n_i})$ will be assumed to be already defined.*

If we consider M-ρ-solid varieties of semigroups we have the type $\tau = (2)$ and thus $\rho : Hyp(2) \to W_{(2)}(X)^{W_{(2)}(X)}$ (where $Hyp(2) := Hyp((2))$). If one considers semigroup identities, we have the associative law and we can renounce of the operation symbol f and the brackets, i.e. we write semigroup words only as sequences of variables.

Theorem 2. The trivial variety TR and the variety Z of all zero semigroups (defined by $xy \approx zt$) are the only sa-solid varieties of semigroups.

Proof. Clearly, TR is sa-solid.

We show that for any $\sigma \in Hyp(2)$ and any $t \in W_{(2)}(X)$ there holds $\sigma^{sa}(t) \approx t \in IdZ$.

If $t \in X$ then $\sigma^{sa}(t) = t$.

If $t \notin X$ then $t = f(t_1, t_2)$ for some $t_1, t_2 \in W_{(2)}(X)$. Thus $c(t) \geq 2$ and $t \approx xy \in IdZ$. Further, there holds $\sigma^{sa}(t) = f(\sigma^{fa}(t_1), \sigma^{fa}(t_2)) \approx xy \in IdZ$. Consequently, $\sigma^{sa}(t) \approx t \in IdZ$.

This shows that $\sigma^{sa}(s) \approx s \approx t \approx \sigma^{sa}(t)$ holds in Z for all $s \approx t \in IdZ$ and all $\sigma \in Hyp(2)$, i.e. Z is sa-solid.

Conversely, let V be an sa-solid variety of semigroups. By σ_x (σ_y) we will denote the hypersubstitution which maps the binary operation symbol f to the term x_1 (x_2). Then $\sigma_x^{sa}(f(f(x,y),z)) \approx \sigma_x^{sa}(f(x, f(y,z))) \in IdV$. This provides $xz \approx xy \in IdV$. From $\sigma_y^{sa}(f(f(x,y),z)) \approx \sigma_y^{sa}(f(x, f(y,z))) \in IdV$ it follows $yz \approx xz \in IdV$. Both identities $xz \approx xy$ and $yz \approx xz$ provide $yz \approx xt$, i.e. $V \subseteq Z$. But TR and Z are the only subvarieties of Z. $\square$

Proposition 2. *A variety V of semigroups is $Bij(2)$-sa-solid iff*

(i) $V \subseteq Mod\{x(yz) \approx (xy)z,\ xyz \approx zxy\}$ and

(ii) $V \subseteq Mod\{x(yz) \approx (xy)z,\ xyz \approx xzy \approx zxy\}$ if there is an identity $s \approx t \in IdV$ with $cv(s) = c(s) = 3$ and $c(t) \neq 3$ or $cv(t) \neq 3$ or $var(t) \neq var(s)$.

Proof. We have already mentioned that $Bij(2) = \{\varepsilon, \sigma_d\}$.

Suppose that V is $Bij(2)$-sa-solid. Then $\sigma_d^{sa}(f(f(x,y),z)) \approx \sigma_d^{sa}(f(x, f(y,z))) \in IdV$, so $yxz \approx xzy \in IdV$. Let now $s \approx t \in IdV$ with $cv(s) = c(s) = 3$.

If $c(t) \leq 2$ then $\sigma_d^{sa}(t) = t$.

If $c(t) \geq 4$ then $\sigma_d^{sa}(t) \approx t \in IdV$ is easy to check using $yxz \approx xzy \in IdV$.

If $c(t) = 3$ and $cv(t) = 1$ then $\sigma_d^{sa}(t) \approx t \in IdV$ is obvious.

If $c(t) = 3$ and $cv(t) = 2$ then there are $w_1, w_2 \in X$ such that $t = (w_1w_2)w_2$ or $t = (w_2w_1)w_2$ or $t = (w_2w_2)w_1$ or $t = w_1(w_2w_2)$ or $t = w_2(w_1w_2)$ or $t = w_2(w_2w_1)$. Using $yxz \approx xzy \in IdV$ we get that $w_1w_2w_2 \approx w_2w_1w_2 \approx w_2w_2w_1$ in V. This shows that $\sigma_d^{sa}(t) \approx t \in IdV$.

From $cv(s) = c(s) = 3$ it follows $s = (w_1w_2)w_3$ or $s = w_1(w_2w_3)$ for some $w_1, w_2, w_3 \in X$. Without loss of generality let $s = w_1(w_2w_3)$, so $\sigma_d^{sa}(s) = w_1w_3w_2$.

If $c(t) \neq 3$ or $cv(t) \neq 3$, from $\sigma_d^{sa}(s) \approx \sigma_d^{sa}(t) \in IdV$ it follows $w_1w_3w_2 \approx t \in IdV$. Consequently, $w_1w_3w_2 \approx w_1w_2w_3 \in IdV$.

If $cv(t) = c(t) = 3$ and $var(t) \neq var(s)$ then there is a $w \in var(t) \setminus var(s)$. Substituting w by w^2 we get $s \approx r \in IdV$ from $s \approx t \in IdV$ where $c(r) = 4$. Then we get $xyz \approx zxy \in IdV$ as above.

Suppose that (i) and (ii) are satisfied. Let $s \approx t \in IdV$. Then $\varepsilon^{sa}(s) \approx \varepsilon^{sa}(t) \in IdV$. We have to show that $\sigma_d^{sa}(s) \approx \sigma_d^{sa}(t) \in IdV$ and consider the following cases:

1) If $c(s) \neq 3$ or $cv(s) \neq 3$ and $c(t) \neq 3$ or $cv(t) \neq 3$ then we have $\sigma_d^{sa}(s) \approx s \in IdV$ and $\sigma_d^{sa}(t) \approx t \in IdV$ as we have shown already. This provides $\sigma_d^{sa}(s) \approx \sigma_d^{sa}(t) \in IdV$.

2.1) If $cv(s) = c(s) = 3$ and $c(t) \neq 3$ or $cv(t) \neq 3$ or $var(t) \neq var(s)$ then $xyz \approx xzy \approx zxy$ holds in V (by (ii)) and it is easy to see that $\sigma_d^{sa}(s) \approx s \in IdV$ and $\sigma_d^{sa}(t) \approx t \in IdV$, so $\sigma_d^{sa}(s) \approx \sigma_d^{sa}(t) \in IdV$.

2.2) If $cv(s) = c(s) = 3$ and $c(t) = 3$ and $cv(t) = 3$ and $var(t) = var(s)$ then there are $w_1, w_2, w_3 \in X$ such that $s, t \in \{r_1, \dots, r_{12}\}$ where

$$\begin{array}{llll} r_1 = w_2(w_1w_3) & r_2 = (w_2w_1)w_3 & r_3 = w_3(w_2w_1) & r_4 = (w_3w_2)w_1 \\ r_5 = w_1(w_3w_2) & r_6 = (w_1w_3)w_2 & r_7 = w_2(w_3w_1) & r_8 = (w_2w_3)w_1 \\ r_9 = w_3(w_1w_2) & r_{10} = (w_3w_1)w_2 & r_{11} = w_1(w_2w_3) & r_{12} = (w_1w_2)w_3. \end{array}$$

Then $\sigma_d^{sa}(r_1) = r_7$, $\sigma_d^{sa}(r_2) = r_{12}$, $\sigma_d^{sa}(r_3) = r_9$, $\sigma_d^{sa}(r_4) = r_8$, $\sigma_d^{sa}(r_5) = r_{11}$, $\sigma_d^{sa}(r_6) = r_{10}$, $\sigma_d^{sa}(r_7) = r_1$, $\sigma_d^{sa}(r_8) = r_4$, $\sigma_d^{sa}(r_9) = r_3$, $\sigma_d^{sa}(r_{10}) = r_6$, $\sigma_d^{sa}(r_{11}) = r_5$, and $\sigma_d^{sa}(r_{12}) = r_2$. This shows that $\sigma_d^{sa}(r_i) \approx \sigma_d^{sa}(r_j) \in IdV$ for $1 \leq i, j \leq 6$ or $7 \leq i, j \leq 12$ by $xyz \approx zxy \in IdV$. If $r_i \approx r_j \in IdV$ with $1 \leq i \leq 6$ or $7 \leq j \leq 12$ or conversely, then $xyz \approx xzy \in IdV$. Together with $xyz \approx zxy \in IdV$ it is easy to check that then $\sigma_d^{sa}(r_i) \approx r_i \in IdV$ and $\sigma_d^{sa}(r_j) \approx r_j \in IdV$, i.e. $\sigma_d^{sa}(r_i) \approx \sigma_d^{sa}(r_j) \in IdV$. Altogether this shows that $\sigma_d^{sa}(s) \approx \sigma_d^{sa}(t) \in IdV$.

3) If $cv(t) = c(t) = 3$ then we get dually $\sigma_d^{sa}(s) \approx \sigma_d^{sa}(t) \in IdV$. □

Theorem 3. TR is the only fa-solid variety of semigroups.

Proof. Clearly, TR is fa-solid. Let V be an fa-solid variety of semigroups. From $\sigma_x^{fa}(f(f(x,y),z)) \approx \sigma_x^{fa}(f(x,f(y,z))) \in IdV$ it follows $xy \approx x \in IdV$. Moreover, $\sigma_y^{fa}(f(f(x,y),z)) \approx \sigma_y^{fa}(f(x,f(y,z))) \in IdV$ provides $z \approx yz \in IdV$. Both identities $xy \approx x$ and $z \approx yz$ give $z \approx y$, i.e. $V = TR$. □

Proposition 3. *A variety V of semigroups is $Bij(2)$-fa-solid iff*

(i) $V \subseteq Mod\{x(yz) \approx (xy)z,\ xyz \approx zxy\}$ and

(ii) V is a variety of commutative semigroups if there is an identity $s \approx t \in IdV$ with $cv(s) = c(s) = 2$ and $c(t) \neq 2$ or $cv(t) \neq 2$ or $var(t) \neq var(s)$.

Proof. We have already mentioned that $Bij(2) = \{\varepsilon, \sigma_d\}$.

Suppose that V is $Bij(2)$-fa-solid. Then $\sigma_d^{fa}(f(f(x,y),z)) \approx \sigma_d^{fa}(f(x,f(y,z))) \in IdV$, so $zxy \approx yzx \in IdV$. Let now $s \approx t \in IdV$ with $cv(s) = c(s) = 2$.

If $c(t) = 1$ then $\sigma_d^{fa}(t) = t$.

If $c(t) \geq 3$ then $\sigma_d^{fa}(t) \approx t \in IdV$ is easy to check using $zxy \approx yzx \in IdV$.

If $c(t) = 2$ and $cv(t) = 1$ then $\sigma_d^{fa}(t) \approx t \in IdV$ is obvious.

From $cv(s) = c(s) = 2$ it follows $s = w_1w_2$, so $\sigma_d^{fa}(s) = w_2w_1$.

If $c(t) \neq 2$ or $cv(t) \neq 2$ from $\sigma_d^{fa}(s) \approx \sigma_d^{fa}(t) \in IdV$ it follows $w_2w_1 \approx t \in IdV$ and, consequently, $w_1w_2 \approx w_2w_1 \in IdV$.

If $cv(t) = c(t) = 2$ and $var(t) \neq var(s)$ then there is a $w \in var(t) \setminus var(s)$. Substituting w by w^2 we get $s \approx r \in IdV$ from $s \approx t \in IdV$ where $c(r) = 3$. Then we get $xy \approx yx \in IdV$ as above.

Suppose that (i) and (ii) are satisfied. Let $s \approx t \in IdV$. Then $\varepsilon^{fa}(s) \approx \varepsilon^{fa}(t) \in IdV$. We have to show that $\sigma_d^{fa}(s) \approx \sigma_d^{fa}(t) \in IdV$ and consider the following cases:

1) If $c(s) \neq 2$ or $cv(s) \neq 2$ and $c(t) \neq 2$ or $cv(t) \neq 2$ then we have $\sigma_d^{fa}(s) \approx s \in IdV$ and $\sigma_d^{fa}(t) \approx t \in IdV$ as we have shown already. This provides $\sigma_d^{fa}(s) \approx \sigma_d^{fa}(t) \in IdV$.

2.1) If $cv(s) = c(s) = 2$ and $c(t) \neq 2$ or $cv(t) \neq 2$ or $var(t) \neq var(s)$ then V is a variety of commutative semigroups (by (ii)) and it is easy to see that $\sigma_d^{fa}(s) \approx s \in IdV$ and $\sigma_d^{fa}(t) \approx t \in IdV$, so $\sigma_d^{fa}(s) \approx \sigma_d^{fa}(t) \in IdV$.

2.2) If $cv(s) = c(s) = 2$ and $c(t) = cv(t) = 2$ and $var(t) = var(s)$ then there are $w_1, w_2 \in X$ such that $s = w_1w_2$ or $s = w_2w_1$ and $t = w_1w_2$ or $t = w_2w_1$.

If $s = t$ then $\sigma_d^{fa}(s) = \sigma_d^{fa}(t)$.

If $s \neq t$ then $s \approx t$ is the commutative law and we have $\sigma_d^{fa}(s) \approx \sigma_d^{fa}(t) \in IdV$.

3) If $cv(t) = c(t) = 2$ then we get dually $\sigma_d^{fa}(s) \approx \sigma_d^{fa}(t) \in IdV$. $\square$

Definition 4. *We define a mapping $\gamma_n : Hyp(\tau) \to W_\tau(X)^{W_\tau(X)}$ for each natural number n as follows: For $\sigma \in Hyp(\tau)$ we put*

(i) $\sigma^{\gamma_0} := \widehat{\sigma}$;

(ii) $\sigma^{\gamma_n}(x) := x$ *for* $x \in X$ *and* $1 \leq n \in \mathbb{N}$;

(iii) $\sigma^{\gamma_n}(f_i(t_1, \ldots, t_{n_i})) := f_i(\sigma^{\gamma_{n-1}}(t_1), \ldots, \sigma^{\gamma_{n-1}}(t_{n_i}))$ *for* $1 \leq n \in \mathbb{N}$, $i \in I$, *and* $t_1, \ldots, t_{n_i} \in W_\tau(X)$.

We put $Hyp^{(n)}(\tau) := \{\sigma^{\gamma_n} \mid \sigma \in Hyp(\tau)\}$ *for* $n \in \mathbb{N}$.

For the hypersubstitution $\varepsilon \in Hyp(\tau)$ (the identity element in $Hyp(\tau)$) there holds $\varepsilon^{\gamma_n} = \widehat{\varepsilon}$ for all $n \in \mathbb{N}$. This becomes clear by the following considerations: We have $\varepsilon^{\gamma_0} = \widehat{\varepsilon}$ and suppose that $\varepsilon^{\gamma_n} = \widehat{\varepsilon}$ for some natural number n then there holds $\varepsilon^{\gamma_{n+1}}(x) = x = \widehat{\varepsilon}[x]$ and

$$\begin{aligned}
&\varepsilon^{\gamma_{n+1}}(f_i(t_1, \ldots, t_{n_i})) \\
&= f_i(\varepsilon^{\gamma_n}(t_1), \ldots, \varepsilon^{\gamma_n}(t_{n_i})) \\
&= f_i(\widehat{\varepsilon}[t_1], \ldots, \widehat{\varepsilon}[t_{n_i}]) \\
&= f_i(t_1, \ldots, t_{n_i}) \\
&= \widehat{\varepsilon}[f_i(t_1, \ldots, t_{n_i})].
\end{aligned}$$

Proposition 4. *The monoids* $(Hyp^{(n)}(\tau); \circ, \widehat{\varepsilon})$ *and* $\mathbf{Hyp}(\tau)$ *are isomorphic for each natural number* n.

Proof. Let n be a natural number. We define a mapping $h : Hyp(\tau) \to Hyp^{(n)}(\tau)$ by $h(\sigma) := \sigma^{\gamma_n}$ for $\sigma \in Hyp(\tau)$. We show that h is injective. For this let $\sigma_1, \sigma_2 \in Hyp(\tau)$ with $\sigma_1^{\gamma_n} = \sigma_2^{\gamma_n}$. Assume that $\sigma_1 \neq \sigma_2$. Then there is an $i \in I$ with $\sigma_1(f_i) \neq \sigma_2(f_i)$ and we have $\widehat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})] \neq \widehat{\sigma}_2[f_i(x_1, \ldots, x_{n_i})]$. Then we define:

(i) $t_0 := f_i(x_1, \ldots, x_{n_i})$;

(ii) $t_{p+1} := f_i(t_p, x_2, \ldots, x_{n_i})$ for $p \in \mathbb{N}$.

It is easy to check that $\sigma_1^{\gamma_n}(t_n) \neq \sigma_2^{\gamma_n}(t_n)$ because of $\widehat{\sigma}_1[t_0] \neq \widehat{\sigma}_2[t_0]$, which contradicts $\sigma_1^{\gamma_n} = \sigma_2^{\gamma_n}$. This shows that h is injective.

Clearly, h is surjective. Consequently, h is a bijective mapping.

It is left to show that h satisfies the homomorphic property. We will show by induction on n that $h(\sigma_1 \circ_h \sigma_2) = h(\sigma_1) \circ h(\sigma_2)$, i.e. $(\sigma_1 \circ_h \sigma_2)^{\gamma_n} = \sigma_1^{\gamma_n} \circ \sigma_2^{\gamma_n}$.

If $n = 0$ then we have $\sigma_1^{\gamma_0} \circ \sigma_2^{\gamma_0} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2 = \widehat{(\sigma_1 \circ_h \sigma_2)} = (\sigma_1 \circ_h \sigma_2)^{\gamma_0}$ (see [3]).

For $n = m$ we suppose that $\sigma_1^{\gamma_m} \circ \sigma_2^{\gamma_m} = (\sigma_1 \circ_h \sigma_2)^{\gamma_m}$.

Let now $n = m + 1$. Obviously, we have $(\sigma_1^{\gamma_{m+1}} \circ \sigma_2^{\gamma_{m+1}})(x) = x = (\sigma_1 \circ_h \sigma_2)^{\gamma_{m+1}}(x)$.

Let $i \in I$ and $t_1, \ldots, t_{n_i} \in W_\tau(X)$. Then there holds
$(\sigma_1^{\gamma_{m+1}} \circ \sigma_2^{\gamma_{m+1}})(f_i(t_1, \ldots, t_{n_i})) = \sigma_1^{\gamma_{m+1}}(f_i(\sigma_2^{\gamma_m}(t_1), \ldots, \sigma_2^{\gamma_m}(t_{n_i})))$
$= f_i((\sigma_1^{\gamma_m} \circ \sigma_2^{\gamma_m})(t_1), \ldots, (\sigma_1^{\gamma_m} \circ \sigma_2^{\gamma_m})(t_{n_i}))$
$= f_i((\sigma_1 \circ_h \sigma_2)^{\gamma_m}(t_1), \ldots, (\sigma_1 \circ_h \sigma_2)^{\gamma_m}(t_{n_i}))$ (by hypothesis)
$= (\sigma_1 \circ_h \sigma_2)^{\gamma_{m+1}}\ (f_i(t_1, \ldots, t_{n_i}))$.

Altogether, this shows that $\sigma_1^{\gamma_{m+1}} \circ \sigma_2^{\gamma_{m+1}} = (\sigma_1 \circ_h \sigma_2)^{\gamma_{m+1}}$. □

By definition, a variety V of type τ is M-γ_0-solid iff V is M-solid. The class of all solid varieties of semigroups was determined in [5]. We will now characterize the γ_n-solid varieties of semigroups for $1 \leq n \in \mathbb{N}$. Here we need some else notations. For a fixed variable $w \in X$ we put:

$F_0 := \{f(f(x, y), z) \approx f(x, f(y, z))\}$ and

$F_{m+1} := \{f(s, w) \approx f(t, w) \mid s \approx t \in F_m\} \cup \{f(w, s) \approx f(w, t) \mid s \approx t \in F_m\}$ for $m \in \mathbb{N}$.

Theorem 4. Let $1 \leq n \in \mathbb{N}$ and V be a variety of semigroups. Then V is γ_n-solid iff

$$x_1 \ldots x_{n+1} \approx y_1 \ldots y_{n+1} \in IdV.$$

Proof. Suppose that V is γ_n-solid.

Since the associative law is satisfied in V there holds $F_{n-1} \subseteq IdV$. Since V is γ_n-solid the application of $\sigma_x^{\gamma_n}$ to the identities of F_{n-1} gives again identities in V:

$$I_1 := \{w^a xzw^b \approx w^a xyw^b \mid a, b \in \mathbb{N},\ a + b = n - 1\} \subseteq IdV.$$

The application of $\sigma_y^{\gamma_n}$ to the identities of F_{n-1} provides

$$I_2 := \{w^a yzw^b \approx w^a xzw^b \mid a, b \in \mathbb{N},\ a + b = n - 1\} \subseteq IdV.$$

It is easy to check that one can derive $x_1 \dots x_{n+1} \approx y_1 \dots y_{n+1}$ from $I_1 \cup I_2$. Thus $x_1 \dots x_{n+1} \approx y_1 \dots y_{n+1} \in IdV$.

Suppose now that $x_1 \dots x_{n+1} \approx y_1 \dots y_{n+1} \in IdV$. We show that for any $\sigma \in Hyp(2)$ and any $t \in W_{(2)}(X)$ there holds $\sigma^{\gamma_n}(t) \approx t \in IdV$.

If t contains at most n operation symbols then $\sigma^{\gamma_n}(t) = t$ by definition of the mapping σ^{γ_n}.

If t contains more than n operation symbols then $c(t) \geq n + 1$ and $t \approx x_1 \dots x_{n+1} \in IdV$ because of $x_1 \dots x_{n+1} \approx y_1 \dots y_{n+1} \in IdV$. Since t contains more than n operation symbols, by definition of the mapping σ^{γ_n}, the term $\sigma^{\gamma_n}(t)$ contains at least n operation symbols and thus $c(\sigma^{\gamma_n}(t)) \geq n + 1$. Using $x_1 \dots x_{n+1} \approx y_1 \dots y_{n+1} \in IdV$ we get $\sigma^{\gamma_n}(t) \approx x_1 \dots x_{n+1} \in IdV$. Consequently, $\sigma^{\gamma_n}(t) \approx t \in IdV$.

This shows that $\sigma^{\gamma_n}(s) \approx s \approx t \approx \sigma^{\gamma_n}(t)$ holds in V for $s \approx t \in IdV$ and $\sigma \in Hyp(2)$, i.e. V is γ_n-solid. $\square$

Corollary 1. *TR and Z are the only γ_1-solid varieties of semigroups.*

Proof. By Theorem 4, a variety V of semigroups is γ_1-solid iff $x_1x_2 \approx y_1y_2 \in IdV$, i.e. $V \subseteq Z$. But TR and Z are the only subvarieties of Z. $\square$

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