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RESEARCH ARTICLE

Topological semigroups of matrix units

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Abstract. We prove that the semigroup of matrix units is stable. Compact, countably compact and pseudocompact topologies τ on the infinite semigroup of matrix units B_λ such that (B_λ, τ) is a semitopological (inverse) semigroup are described. We prove the following properties of an infinite topological semigroup of matrix units. On the infinite semigroup of matrix units there exists no semigroup pseudocompact topology. Any continuous homomorphism from the infinite topological semigroup of matrix units into a compact topological semigroup is annihilating. The semigroup of matrix units is algebraically h-closed in the class of topological inverse semigroups. Some H-closed minimal semigroup topologies on the infinite semigroup of matrix units are considered.

In this paper all topological spaces are Hausdorff.

A semigroup is a set with a binary associative operation. The semigroup operation is called a *multiplication*. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. An element y of S is called *inverse* to x and is denoted by x^{-1}. If S is an inverse semigroup, then the map which takes $x \in S$ to the inverse element of x is called the *inversion*.

A topological space S that is algebraically semigroup with a separately continuous semigroup operation is called a *semitopological semigroup*. If the multiplication on S is jointly continuous, then S is called a *topological semigroup*.

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A *topological* (*semitopological*) *inverse semigroup* is a topological (semitopological) semigroup S that is algebraically an inverse semigroup with continuous inversion. Obviously, any topological (inverse) semigroup is a semitopological (inverse) semigroup.

If τ is a topology on a (inverse) semigroup S such that (S,τ) is a topological (inverse) semigroup, then τ is called a (*inverse*) *semigroup topology* on S.

We follow the terminology of [6, 7, 11, 16], and [22].

If S is a semigroup, then by $E(S)$ we denote the subset of idempotents of S. By ω we denote the first infinite ordinal. Further, we identify all cardinals with their corresponding initial ordinals.

Let S be a semigroup and I_λ be a set of cardinality $\lambda \geq 2$. On the set $B_\lambda(S) = I_\lambda \times S^1 \times I_\lambda \cup \{0\}$ we define the semigroup operation $'\cdot'$ as follows

$$(\alpha, a, \beta)\cdot(\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta=\gamma,\\ 0, & \text{if } \beta\neq\gamma, \end{cases}$$

and $(\alpha, a, \beta)\cdot 0 = 0\cdot(\alpha, a, \beta) = 0\cdot 0 = 0$ for $\alpha, \beta, \gamma, \delta \in I_\lambda$, $a, b \in S^1$. The semigroup $B_\lambda(S)$ is called a *Brandt-Howie semigroup of the weight* λ *over* S [13] or a *Brandt* λ*-extension of the semigroup* S [14]. Obviously $B_\lambda(S)$ is the Rees matrix semigroup $M^0(S^1; I_\lambda, I_\lambda, \mathcal{M})$, where $\mathcal{M}$ is the $I_\lambda \times I_\lambda$ identity matrix. If a semigroup S is trivial (i.e. if S contains only one element), then $B_\lambda(S)$ is *the semigroup of* $I_\lambda \times I_\lambda$*-matrix units* [7], which we shall denote by B_λ.

A semigroup $\mathcal{B}(p,q)$ generated by elements p and q which satisfy the condition $pq = 1$ is called *bicyclic*. The bicyclic semigroup plays the important role in the Algebraic Theory of Semigroups and in the Theory of Topological Semigroups. For example the well-known O. Andersen's result [1] states that a (0–) simple semigroup is completely (0–) simple if and only if it does not contain the bicyclic semigroup (see Theorem 2.54 of [7]). L. W. Anderson, R. P. Hunter and R. J. Koch in [2] proved that the bicyclic semigroup cannot be embedded into a stable semigroup. Also any Γ-compact topological semigroup (and hence compact topological semigroup) does not contain the bicyclic semigroup [15] and therefore every (0–) simple Γ-compact topological semigroup is completely (0–) simple.

In this paper we discuss semigroup topologies on the semigroup of matrix units. At the beginning we shall prove that the semigroup of matrix units is stable. Further we shall show that on any semigroup of matrix units B_λ there exists a unique compact topology τ such that (B_λ,τ) is a semitopological semigroup. Also we shall prove that on the infinite semigroup of matrix units there exists no semigroup pseudocompact topology and this implies the structure theorem for 0-simple compact topological inverse semigroups. Moreover, any continuous homomorphism from the infinite topological semigroup of matrix units into a compact topological semigroup is annihilating. Also we shall prove that if a topological inverse semigroup S contains a semigroup of matrix units B_λ, then B_λ is a closed subsemigroup of S, i.e. B_λ is algebraically h-closed in the class of topological inverse semigroups. Some H-closed minimal semigroup topologies on the infinite semigroup of matrix units will be considered.

Lemma 1. *Let $a,b\in S^1$, $\alpha,\beta\in I_\lambda$. Then the following conditions are equivalent:*

(i) $aS^1\subseteq bS^1 \quad [S^1a\subseteq S^1b]$;

(ii) $(\alpha,a,\beta)B_\lambda(S)\subseteq(\alpha,b,\beta)B_\lambda(S) \quad [B_\lambda(S)(\alpha,a,\beta)\subseteq B_\lambda(S)(\alpha,b,\beta)]$.

Proof. (i)⇒(ii).

$$(\alpha,a,\beta)B_\lambda(S)= \\ =\bigcup_{\gamma\in I_\lambda}(\alpha,aS^1,\gamma)\cup\{0\}\subseteq\bigcup_{\gamma\in I_\lambda}(\alpha,bS^1,\gamma)\cup\{0\}=(\alpha,b,\beta)B_\lambda(S).$$

(ii)⇒(i). Let $(\alpha,a,\beta)B_\lambda(S)\subseteq(\alpha,b,\beta)B_\lambda(S)$, then

$$\bigcup_{\gamma\in I_\lambda}(\alpha,aS^1,\gamma)\cup\{0\}\subseteq\bigcup_{\gamma\in I_\lambda}(\alpha,bS^1,\gamma)\cup\{0\}$$

and hence $\bigcup_{\gamma\in I_\lambda}(\alpha,aS^1,\gamma)\subseteq\bigcup_{\gamma\in I_\lambda}(\alpha,bS^1,\gamma)$. Therefore, $aS^1\subseteq bS^1$.

The proof of equivalency of the dual conditions is similar. □

A semigroup S is called *stable* if and only if

(i) for $a,b\in S$, $Sa\subseteq Sab$ implies $Sa=Sab$;

(ii) for $c,d\in S$, $cS\subseteq dcS$ implies $cS=dcS$.

Stable semigroups were first investigated by R. J. Koch and A. D. Wallace in [17]. A semigroup S is called *weakly stable* if S^1 is stable [20]. Every stable semigroup S is weakly stable [17]. If S is a regular semigroup, then the converse holds. L. O'Carroll [20] proved that the converse does not hold in general.

Theorem 1. *A semigroup S is weakly stable if and only if $B_\lambda(S)$ is stable for each cardinal $\lambda\geq 2$.*

Proof. ($\Leftarrow$) Suppose $aS^1 \subseteq baS^1$ $[S^1a \subseteq S^1ab]$ for $a, b \in S^1$. By Lemma 1 we get

$$(\alpha, a, \beta)B_\lambda(S) \subseteq (\alpha, b, \alpha)(\alpha, a, \beta)B_\lambda(S) = (\alpha, ba, \beta)B_\lambda(S)$$

$$[B_\lambda(S)(\alpha, a, \beta) \subseteq B_\lambda(S)(\alpha, a, \beta)(\beta, b, \beta) = B_\lambda(S)(\alpha, ab, \beta)]$$

for all $\alpha, \beta \in I_\lambda$. Since the semigroup $B_\lambda(S)$ is stable for each cardinal $\lambda \geq 2$, then

$$(\alpha, a, \beta)B_\lambda(S) = (\alpha, b, \alpha)(\alpha, a, \beta)B_\lambda(S) = (\alpha, ba, \beta)B_\lambda(S)$$

$$[B_\lambda(S)(\alpha, a, \beta) = B_\lambda(S)(\alpha, a, \beta)(\beta, b, \beta) = B_\lambda(S)(\alpha, ab, \beta)]$$

and by Lemma 1, $aS^1 = baS^1$ $[S^1a = S^1ab]$. Therefore, the semigroup S is weakly stable.

($\Rightarrow$) Let $\lambda \geq 2$. Suppose $(\alpha, a, \beta)B_\lambda(S) \subseteq (\gamma, b, \delta)(\alpha, a, \beta)B_\lambda(S)$ for $\alpha, \beta, \gamma, \delta \in I_\lambda$, $a, b \in S^1$. Obviously, $\alpha = \gamma = \delta$. Thus $(\alpha, a, \beta)B_\lambda(S) \subseteq (\alpha, ba, \beta)B_\lambda(S)$. By Lemma 1 we get $aS^1 \subseteq baS^1$. Since S is a weakly stable semigroup, then $aS^1 = baS^1$. Then by Lemma 1 we get

$$\begin{aligned}&(\alpha, a, \beta)B_\lambda(S) = \\ &= (\alpha, ba, \beta)B_\lambda(S) = (\alpha, b, \alpha)(\alpha, a, \beta)B_\lambda(S) = (\gamma, b, \delta)(\alpha, a, \beta)B_\lambda(S).\end{aligned}$$

The proof of the dual statement is similar.

Therefore, the semigroup $B_\lambda(S)$ is stable for all cardinals $\lambda \geq 2$. □

Corollary 1. *For every cardinal $\lambda \geq 2$ the semigroup B_λ is stable.*

In [10] C. Eberhart and J. Selden proved that the bicyclic semigroup $\mathcal{B}(p, q)$ admits only the discrete Hausdorff semigroup topology. M. O. Bertman and T. T. West generalized this result and showed that any Hausdorff topology τ on $\mathcal{B}(p, q)$ such that $(\mathcal{B}(p, q), \tau)$ is a semitopological semigroup is discrete [5]. Lemma 2 implies that the semigroup of matrix units has similar properties.

Further by 0 we denote the zero of the semigroup B_λ.

Lemma 2. *Let τ be a topology on B_λ such (B_λ, τ) is a semitopological semigroup. Then any nonzero element of B_λ is an isolated point of (B_λ, τ).*

Proof. Since (B_λ, τ) is a semitopological semigroup, every left internal translation $l_s \colon B_\lambda \to B_\lambda$ and every right internal translation $r_s \colon B_\lambda \to B_\lambda$ are continuous maps for any $s \in B_\lambda$. Thus for $s = (\alpha, \alpha) \in B_\lambda$ the sets $l_s^{-1}(0) = \{0\} \cup \{(\gamma, \beta) \mid \gamma \in I_\lambda \setminus \{\alpha\}, \beta \in I_\lambda\}$ and $r_s^{-1}(0) =$

$\{0\} \cup \{(\gamma, \beta) \mid \gamma \in I_\lambda, \beta \in I_\lambda \setminus \{\alpha\}\}$ are closed in (B_λ, τ), and hence the sets $B_\lambda \setminus l_s^{-1}(0) = \{(\alpha, \gamma) \mid \gamma \in I_\lambda\}$ and $B_\lambda \setminus r_s^{-1}(0) = \{(\gamma, \alpha) \mid \gamma \in I_\lambda\}$ are open in (B_λ, τ) for any $\alpha \in I_\lambda$. Therefore any nonzero element of B_λ is an isolated point of (B_λ, τ). □

In [5] M. O. Bertman and T. T. West showed that the bicyclic semigroup is embedded into a compact semitopological semigroup. The next example shows that on the infinite semigroup of matrix units B_λ there exists a topology τ_c such that (B_λ, τ_c) is a compact semitopological inverse semigroup.

Example 1. Let $\lambda \geq \omega$. A topology τ_c on B_λ is defined as follows:

a) all nonzero elements of B_λ are isolated points in B_λ;

b) $\mathcal{B}(0) = \{A \subseteq B_\lambda \mid 0 \in A \text{ and } |B_\lambda \setminus A| < \omega\}$ is the base of the topology τ_c at the point $0 \in B_\lambda$.

Lemma 3. (B_λ, τ_c) *is a compact semitopological inverse semigroup.*

Proof. Obviously, τ_c is a compact topology on B_λ.

For any $U = B_\lambda \setminus \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\} \in \mathcal{B}(0)$, where $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n \in I_\lambda$ we have

1) $U_1 \cdot \{(\alpha, \beta)\} = \{0\} \bigcup \{(\gamma, \beta) \mid \gamma \in I_\lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\} \subseteq U$, where $U_1 = U \setminus \{(\alpha_1, \alpha_1), \ldots, (\alpha_n, \alpha_n)\} \in \mathcal{B}(0)$;

2) $\{(\alpha, \beta)\} \cdot U_2 = \{0\} \bigcup \{(\gamma, \beta) \mid \gamma \in I_\lambda \setminus \{\beta_1, \ldots, \beta_n\}\} \subseteq U$, where $U_2 = U \setminus \{(\beta_1, \beta_1), \ldots, (\beta_n, \beta_n)\} \in \mathcal{B}(0)$;

3) $\{(\alpha, \beta)\} \cdot \{(\gamma, \delta)\} = \{0\} \subseteq U$ if $\beta \neq \gamma$;

4) $\{0\} \cdot U = \{0\} \subseteq U$ and $U \cdot \{0\} = \{0\} \subseteq U$;

5) $\{(\alpha, \beta)\} \cdot \{(\beta, \gamma)\} = \{(\alpha, \gamma)\}$;

6) $(U_3)^{-1} \subseteq U$, where $U_3 = B_\lambda \setminus \{(\beta_1, \alpha_1), \ldots, (\beta_n, \alpha_n)\} \in \mathcal{B}(0)$.

Therefore, (B_λ, τ_c) is a compact semitopological inverse semigroup. □

Remark 1. In [21] A. B. Paalman-de-Miranda proved that the zero of a compact completely 0-simple topological semigroup S is an isolated point in S. Example 1 implies that the zero of a completely 0-simple compact semitopological inverse semigroup (B_λ, τ) is not necessarily an isolated point in (B_λ, τ).

Lemmas 2 and 3 imply

Corollary 2. *If $\lambda \geq \omega$ then there exists no other topology τ on B_λ different from τ_c such that (B_λ, τ) is a compact semitopological semigroup.*

A topological space X is called *countably compact* if any countable open cover of X contains a finite subcover [11]. A topological space X is called *pseudocompact* [*discretely pseudocompact*] if any locally finite [discrete] collection of open subsets of X is finite. Obviously any countably compact space and any discretely pseudocompact space are pseudocompact.

Theorem 2. *Let $\lambda \geq \omega$ and let τ be a topology on the semigroup of matrix units B_λ such that (B_λ, τ) is a semitopological semigroup. Then the following statements are equivalent:*

(i) (B_λ, τ) *is a compact semitopological semigroup;*

(ii) (B_λ, τ) *is a countably compact semitopological semigroup;*

(iii) (B_λ, τ) *is a discretely pseudocompact semitopological semigroup;*

(iv) (B_λ, τ) *is a pseudocompact semitopological semigroup;*

(v) (B_λ, τ) *is topologically isomorphic to* (B_λ, τ_c).

Proof. The implications $(i) \Rightarrow (ii)$, $(i) \Rightarrow (iii)$, $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$ Suppose there exists a topology τ on the infinite semigroup of matrix units B_λ such that (B_λ, τ) is a pseudocompact noncompact semitopological semigroup. Then there exists an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ which contains no finite subcover. Let $U_{\alpha_1} \in \mathcal{U}$ such that $U_{\alpha_1} \ni 0$. Then the set $B_\lambda \setminus U_{\alpha_1}$ is infinite. We put

$$\mathcal{U}^* = \{U_{\alpha_1}\} \cup \{x \mid x \in B_\lambda \setminus U_{\alpha_1}\}.$$

Then $\mathcal{U}^*$ is an infinite locally finite family, which contradicts the pseudocompactness of the topological space (B_λ, τ).

Corollary 2 implies the equivalency $(i) \Leftrightarrow (v)$. □

Since the bicyclic semigroup $\mathcal{B}(p, q)$ admits only discrete semigroup topology [10], $\mathcal{B}(p, q)$ admits no compact (countably compact, pseudocompact) semigroup topology. The next proposition is a similar result for the infinite semigroup of matrix units and it follows from Theorem 2.

Proposition 1. *If $\lambda \geq \omega$, then there exists no compact (countably compact, pseudocompact) semigroup topology on B_λ.*

A topological semigroup S is called Γ-compact if

$$\Gamma(x) = \overline{\{x, x^2, x^3, \ldots, x^n, \ldots\}}$$

is a compact subsemigroup of S for every $x \in S$. Obviously every compact semigroup is Γ-compact. J. A. Hildebrant and R. J. Koch proved that every Γ-compact topological semigroup and hence compact topological semigroup does not contain the bicyclic semigroup [15]. Since for any element a of the semigroup of matrix units we have either $aa = a$ or $aa = 0$, the semigroup of matrix units is Γ-compact.

Question 1. *Does there exist a compact topological inverse semigroup which contains the semigroup B_ω?*

In this paper we give a negative answer to Question 1. Moreover, we show that if $\lambda \geq \omega$, then every continuous homomorphism of the topological semigroup B_λ into a compact topological semigroup is annihilating and B_λ as a topological inverse semigroup is absolutely H-closed in the class of topological inverse semigroups.

Lemma 4. *Let T be a dense subsemigroup of a topological semigroup S and 0 be the zero of T. Then 0 is the zero of S.*

Proof. Suppose that there exists $a \in S \setminus T$ such that $0 \cdot a = b \neq 0$. Then for every open neighbourhood $U(b) \not\ni 0$ in S there exists an open neighbourhood $V(a) \not\ni 0$ in S such that $0 \cdot V(a) \subseteq U(b)$. But $|V(a) \cap T| \geq \omega$, and hence $0 \in 0 \cdot V(a) \subseteq U(b)$, a contradiction with the choice of $U(b)$. Therefore $0 \cdot a = 0$ for all $a \in S \setminus T$.

The proof of the equality $a \cdot 0 = 0$ is similar. □

Lemma 5. *If A is non-singleton in B_λ, then $0 \in A \cdot A$.*

Proof. Suppose $|A| = 2$. Obviously if $0 \in A$, then $0 \in A \cdot A$.

Let $0 \notin A$ and $A = \{(\alpha, \beta), (\gamma, \delta)\}$. If (α, β), (γ, δ) are idempotents of B_λ, then $\alpha = \beta$, $\gamma = \delta$, $\beta \neq \gamma$, and hence $0 = (\alpha, \beta) \cdot (\gamma, \delta)$. If the set A contains a non-idempotent element (α, β), then $\alpha \neq \beta$ and $0 = (\alpha, \beta) \cdot (\alpha, \beta) \in A \cdot A$. □

Lemma 6. *Let $\lambda \geq \omega$ and let B_λ be a dense subsemigroup of a topological semigroup S. Then $a \cdot a = 0$ for all $a \in S \setminus B_\lambda$.*

Proof. Suppose $a \cdot a = b \neq 0$ for some $a \in S \setminus B_\lambda$. Then for any open neighbourhood $U(b) \not\ni 0$ in S there exists an open neighbourhood $V(a) \not\ni 0$ in S such that $V(a) \cdot V(a) \subseteq U(b)$. But $|V(a) \cap B_\lambda| \geq \omega$ and by Lemma 5, $0 \in V(a) \cdot V(a) \subseteq U(b)$, a contradiction. □

Theorem 3. *If $\lambda \geq \omega$, then there exists no semigroup topology τ on B_λ such that (B_λ, τ) is embedded into a compact topological semigroup.*

Proof. Suppose, on the contrary, that there exists a semigroup topology τ on B_λ such that (B_λ, τ) is a subsemigroup of some compact topological semigroup S. By Proposition 1, B_λ is not a closed subsemigroup of S. Without loss of generality we assume that B_λ is a dense subsemigroup of S. We denote $X = B_\lambda \setminus \{0\}$. By Lemma 2, X is a discrete subspace of B_λ and hence X is a locally compact subspace in B_λ. Thus, by Theorem 3.5.8 [11], $\mathcal{J} = S \setminus X$ is a closed subspace of S. Therefore, X is a discrete subspace of S.

Further, we shall show that $\mathcal{J}$ is an ideal of the semigroup S. By Lemmas 5 and 6 it is sufficient to prove that $ax, xa, ab \in \mathcal{J}$ for every $x \in B_\lambda \setminus \{0\}$, $a, b \in \mathcal{J} \setminus \{0\}$.

Assume that there exist $x \in B_\lambda \setminus \{0\}$ and $a \in \mathcal{J} \setminus \{0\}$ such that $ax = c \neq \mathcal{J}$. Then c is an isolated point in S. Thus for every open neighbourhood $U(a) \not\ni 0$ at least one of the following conditions holds

(i) $|\,(U(a) \setminus \{a\}) \cdot x| \geq \omega$,

(ii) $0 \in (U(a) \setminus \{a\}) \cdot x$.

But c is an isolated point in S, a contradiction. The proof for xa is similar.

Suppose $ab = c \notin \mathcal{J}$ for some $a, b \in \mathcal{J}$. Then c is an isolated point in S. For every open neighbourhoods $U(a) \not\ni 0$ and $U(b) \not\ni 0$ at least one of the following conditions holds

(iii) $|\,(U(a) \setminus \{a\}) \cdot (U(b) \setminus \{b\})\,| \geq \omega$,

(iv) $0 \in (U(a) \setminus \{a\}) \cdot (U(b) \setminus \{b\})$.

But c is an isolated point in S, a contradiction. Thus, $ab = c \in \mathcal{J}$.

Therefore, $\mathcal{J}$ is a compact ideal of S.

By Theorem A.2.23 [16] the Rees quotient-semigroup $S/\mathcal{J}$ is a compact topological semigroup. But the semigroup $S/\mathcal{J}$ is algebraically isomorphic to B_λ, a contradiction with Proposition 1. □

A semigroup S is called *congruence-free* (*congruence-simple, h-simple*) if it has only two congruences: identical and universal [23]. Such semigroups E. S. Lyapin [18] and L. M. Gluskin [12] called *simple*. Obviously, a semigroup S is congruence-free if and only if every homomorphism h of S into an arbitrary semigroup T is an isomorphism "into" or is an annihilating homomorphism (i. e. there exists $c \in T$ such that $h(a) = c$ for all $a \in S$).

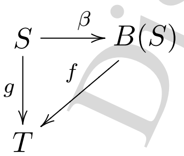
Theorem 1 [12] implies

Corollary 3. *The semigroup B_λ is congruence-free for every cardinal $\lambda \geq 2$.*

Theorem 3 and Corollary 3 imply

Proposition 2. *Let $\lambda \geq \omega$. Then every continuous homomorphism of the topological semigroup B_λ into a compact topological semigroup is annihilating.*

Recall [8] that a *Bohr compactification of a topological semigroup S* is a pair $(\beta, B(S))$ such that $B(S)$ is a compact topological semigroup, $\beta\colon S \to B(S)$ is a continuous homomorphism, and if $g\colon S \to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f\colon B(S) \to T$ such that the diagram



commutes.

Let S be a topological semigroup. Let $\{(T_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ be a family of pairs of compact topological semigroups and continuous homomorphisms $\varphi_\alpha\colon S \to T_\alpha$, respectively, such that $\varphi_\alpha(S)$ is a dense subsemigroup of T_α for any $\alpha \in \mathcal{A}$. Then $B(S)$ is a subsemigroup of $\Pi_{\alpha\in\mathcal{A}} T_\alpha$ (see the proofs of Lemma 2.43 and Theorem 2.44 [6, Vol. 1]), where $|\mathcal{A}| \leqslant 2^{2^{|S|}}$. Therefore Proposition 2 implies

Corollary 4. *If $\lambda \geq \omega$, then the Bohr compactification of the topological semigroup B_λ is a trivial semigroup.*

Theorem 4. *Let λ be a cardinal ≥ 2 and B_λ be a subsemigroup of a topological inverse semigroup S. Then B_λ is a closed subsemigroup of S.*

Proof. If $\lambda < \omega$ then B_ω is finite and hence B_λ is a closed subsemigroup of S.

Suppose $\lambda \geq \omega$.

Let $\overline{B_\lambda} = S_1$ in S. Then by Proposition II.2 [10] S_1 is a topological inverse semigroup and by Lemma 4 the zero 0 of the semigroup B_λ is the zero of the semigroup S_1.

Let b be any element of $S_1 \setminus B_\lambda$. We consider two cases: $b \in E(S_1)$ and $b \in S_1 \setminus E(S_1)$.

1) Let $b \in E(S_1)$. Then for every open neighbourhood $W(b) \not\ni 0$ there exists an open neighbourhood $U(b) \not\ni 0$ such that $U(b) \cdot U(b) \subseteq$

$W(b)$. Since $|U(b) \cap B_\lambda| \geq \omega$, there exist $\alpha, \beta, \gamma, \delta \in I_\lambda$ such that $(\alpha, \beta), (\gamma, \delta) \in U(b)$, and $\beta \neq \gamma$ or $\alpha \neq \delta$. Then $0 \in U(b) \cdot U(b) \subseteq W(b)$, a contradiction. Therefore, $E(S_1) = E(B_\lambda)$.

2) Let $b \in S_1 \setminus E(S_1)$. Then $b^{-1} \in S_1 \setminus E(S_1)$. Since 0 is the zero of the topological semigroup S_1, then $b \cdot b^{-1} \neq 0$ and $b^{-1} \cdot b \neq 0$. Otherwise, if $b \cdot b^{-1} = 0$ or $b^{-1} \cdot b = 0$, then $b = b \cdot b^{-1} \cdot b = 0 \cdot b = 0$ or $b = b \cdot b^{-1} \cdot b = b \cdot 0 = 0$, a contradiction with $b \in S_1 \setminus E(S_1)$.

Therefore, there exist $e, f \in E(S_1) = E(B_\lambda)$ such that $b \cdot b^{-1} = e$, $b^{-1} \cdot b = f$ and $e \neq f$. Let $W(e)$ and $W(f)$ be open neighbourhoods of e and f in S_1, respectively, such that $0 \notin W(e)$ and $0 \notin W(f)$. Then there exist disjoint open neighbourhoods $U(b) \not\ni 0$ and $U(b^{-1}) \not\ni 0$ such that $U(b) \cdot U(b^{-1}) \subseteq W(e)$ and $U(b^{-1}) \cdot U(b) \subseteq W(f)$. Since $|U(b) \cap B_\lambda| \geq \omega$ and $|U(b^{-1}) \cap B_\lambda| \geq \omega$, there exist $(\alpha, \beta) \in U(b)$ and $(\gamma, \delta) \in U(b^{-1})$ such that $\beta \neq \gamma$ or $\alpha \neq \delta$. Therefore, $0 \in U(b) \cdot U(b^{-1}) \subseteq W(e)$ or $0 \in U(b^{-1}) \cdot U(b) \subseteq W(f)$, a contradiction with $0 \notin W(e)$ and $0 \notin W(f)$.

If $e = f$ the proof of the statement is similar.

The obtained contradictions imply the statement of the theorem. □

Since a compact topological semigroup is stable (see Theorem 3.31 [6]) a compact 0-simple topological inverse semigroup S is completely 0-simple and by Theorem 3.9 [7] S is algebraically isomorphic to the Brandt λ-extension of a group. Theorems 3 and 4 imply

Corollary 5. *Let S be a compact 0-simple topological inverse semigroup. Then $E(S)$ is finite.*

Definition 1 ([25])**.** Let $\mathcal{S}$ be a class of topological semigroups. A semigroup $S \in \mathcal{S}$ is called *H-closed in $\mathcal{S}$*, if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as subsemigroup. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*.

We remark that in [25] H-closed semigroups are called *maximal*.

Theorem 4 implies

Corollary 6. *Let λ be a cardinal ≥ 2 and τ be a semigroup inverse topology on B_λ. Then (B_λ, τ) is H-closed in the class of topological inverse semigroups.*

Definition 2 ([26])**.** Let $\mathcal{S}$ be a class of topological semigroups. A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class $\mathcal{S}$* if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

Corollary 3 and Theorem 4 imply

Corollary 7. *Let λ be a cardinal ≥ 2 and τ be a semigroup inverse topology on B_λ. Then (B_λ, τ) is absolutely H-closed in the class of topological inverse semigroups.*

Let $\mathcal{S}$ be a class of topological semigroups. A semigroup S is called *algebraically h-closed in $\mathcal{S}$* if S with discrete topology d is absolutely H-closed in $\mathcal{S}$ and $(S, d) \in \mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*.

Absolutely H-closed semigroups and algebraically h-closed semigroups were introduced by J. W. Stepp in [26]. There they were called *absolutely maximal* and *algebraic maximal*, respectively.

Corollary 7 implies

Proposition 3. *For any cardinal $\lambda \geq 2$ the semigroup B_λ is algebraically h-closed in the class of topological inverse semigroups.*

The following example shows that B_ω with the discrete topology is not H-closed.

Example 2. Let $B_\omega = I_\omega \times I_\omega \cup \{0\}$ be the semigroup of matrix units and $a \notin B_\omega$. Let $S = B_\omega \cup \{a\}$. We put

$$a \cdot a = a \cdot 0 = 0 \cdot a = a \cdot (\alpha, \beta) = (\alpha, \beta) \cdot a = 0$$

for all $(\alpha, \beta) \in B_\omega \setminus \{0\}$.

Further we enumerate the elements of the set I_ω by natural numbers. Let $A_n = \{(2k-1, 2k) \mid k \geq n\}$ for each $n \in \mathbb{N}$. A topology τ on S is defined as follows:

1) all points of B_ω are isolated in S;

2) $\mathcal{B}(a) = \{U_n(a) = \{a\} \cup A_n \mid n \in \mathbb{N}\}$ is the base of the topology τ at the point $a \in S$.

Then

a) $\{(l, m)\} \cdot U_n(a) = U_n(a) \cdot \{(l, m)\} = \{0\}$ for all $(l, m) \in B_\omega \setminus \{0\}$, $n \geq \max\{l, m\}$;

b) $U_n(a) \cdot U_n(a) = U_n(a) \cdot \{0\} = \{0\} \cdot U_n(a) = \{0\}$ for any $n \in \mathbb{N}$;

c) $U_n(a)$ is a compact subset of S for each $n \in \mathbb{N}$.

Therefore (S, τ) is a locally compact topological semigroup. Obviously B_ω is not a closed subset of (S, τ).

Remark 2. Let λ_1 and λ_2 be cardinals and $\lambda_1 \leq \lambda_2$. Then B_{λ_1} is a subsemigroup of B_{λ_2}. Example 2 implies that for any infinite cardinal λ the semigroup B_λ is not H-closed.

Definition 3. A Hausdorff topological (inverse) semigroup (S,τ) is said to be *minimal* if no Hausdorff semigroup (inverse) topology on S is strictly contained in τ. If (S,τ) is minimal topological (inverse) semigroup, then τ is called *minimal semigroup (inverse) topology.*

The concept of minimal topological groups was introduced independently in the early 1970's by Doïtchinov [9] and Stephenson [24]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time (cf. [4]). More than 20 years earlier L. Nachbin [19] had studied minimality in the context of division rings, and B. Banaschewski [3] investigated minimality in the more general setting of topological algebras.

Question 2 (T. O. Banakh)**.** *Is it true that for any cardinal $\lambda \geq \omega$ the semigroup B_λ admits minimal (inverse) semigroup topology?*

For each $\alpha,\beta \in I_\lambda$ we define

$$V_\alpha = B_\lambda \setminus \{(\alpha,\gamma) \mid \gamma \in I_\lambda\} \quad \text{and} \quad H_\beta = B_\lambda \setminus \{(\gamma,\beta) \mid \gamma \in I_\lambda\}.$$

Put

$$U^{\alpha_1,\ldots,\alpha_n} = \bigcap_{i=1}^{n} V_{\alpha_i},\ U_{\beta_1,\ldots,\beta_m} = \bigcap_{i=1}^{m} H_{\beta_i} \text{ and}$$

$$U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} = U^{\alpha_1,\ldots,\alpha_n} \cap U_{\beta_1,\ldots,\beta_m},$$

where $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m \in I_\lambda$, $n,m \in \mathbb{N}$. Further we define the following families

$$\mathcal{B}_{mv} = \{U^{\alpha_1,\ldots,\alpha_n} \mid \alpha_1,\ldots,\alpha_n \in I_\lambda, n \in \mathbb{N}\} \cup \{(\alpha,\beta) \mid \alpha,\beta \in I_\lambda\},$$

$$\mathcal{B}_{mh} = \{U_{\beta_1,\ldots,\beta_m} \mid \beta_1,\ldots,\beta_m \in I_\lambda, m \in \mathbb{N}\} \cup \{(\alpha,\beta) \mid \alpha,\beta \in I_\lambda\},$$

$$\mathcal{B}_{mi} = \{U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} \mid \alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m \in I_\lambda, n,m \in \mathbb{N}\} \cup$$
$$\cup \{(\alpha,\beta) \mid \alpha,\beta \in I_\lambda\}.$$

Obviously, the conditions (BP1)—(BP3) [11] hold for families $\mathcal{B}_{mv}$, $\mathcal{B}_{mh}$ and $\mathcal{B}_{mi}$, and hence $\mathcal{B}_{mv}$, $\mathcal{B}_{mh}$ and $\mathcal{B}_{mi}$ are bases on B_λ of topologies τ_{mv}, τ_{mh} and τ_{mi}, respectively.

Lemma 7. *Let λ be an infinite cardinal. Then*

(i) (B_λ, τ_{mv}) is a topological semigroup;

(ii) (B_λ, τ_{mh}) *is a topological semigroup;*

(iii) (B_λ, τ_{mi}) *is a topological inverse semigroup.*

Proof. (i) Since the following conditions hold

$$U^{\alpha_1,\ldots,\alpha_n}\cdot U^{\alpha_1,\ldots,\alpha_n}\subseteq U^{\alpha_1,\ldots,\alpha_n}, \quad (\alpha,\beta)\cdot U^{\alpha_1,\ldots,\alpha_n,\beta}=\{0\}\subseteq U^{\alpha_1,\ldots,\alpha_n},$$

$$U^{\alpha_1,\ldots,\alpha_n}\cdot(\alpha,\beta)=\{0\}\cup\{(\gamma,\delta)\mid \gamma\in I_\lambda\setminus\{\alpha_1,\ldots,\alpha_n\}\}\subseteq U^{\alpha_1,\ldots,\alpha_n}$$

for every open neighbourhood $U^{\alpha_1,\ldots,\alpha_n}$ of the zero of B_λ and for any $(\alpha,\beta)\in B_\lambda\setminus\{0\}$, (B_λ,τ_{mv}) is a topological semigroup.

The proof of statement (ii) is similar to the proof of item (i).

(iii) For every open neighbourhood $U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}$ of the zero of B_λ and for any $(\beta,\alpha)\in B_\lambda\setminus\{0\}$ we have:

$$U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}\cdot U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}\subseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}, \qquad (\beta,\alpha)\cdot U^{\alpha_1,\ldots,\alpha_n}_{\alpha,\beta_1,\ldots,\beta_m}=\{0\}\subseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m},$$

$$U^{\beta,\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}\cdot(\beta,\alpha)=\{0\}\subseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}, \qquad \left(U^{\beta_1,\ldots,\beta_m}_{\alpha_1,\ldots,\alpha_n}\right)^{-1}\subseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}.$$

Therefore, (B_λ,τ_{mi}) is a topological inverse semigroup. □

We remark that τ_{mv}, τ_{mh} and τ_{mi} are not locally compact topologies on B_λ for $\lambda\geq\omega$.

For $A\subseteq I_\lambda$ and $a\in I_\lambda$ we denote $A^\alpha=\{(\alpha,\beta)\in B_\lambda\mid\beta\in A\}$ and $A_\alpha=\{(\beta,\alpha)\in B_\lambda\mid\beta\in A\}$.

Lemma 8. *Let λ be an infinite cardinal, B_λ be a topological semigroup, and A^α $[A_\alpha]$ be a closed subset in B_λ for some $\alpha\in I_\lambda$. Then A^β $[A_\beta]$ is a closed subset of B_λ for any $\beta\in I_\lambda$.*

Proof. Since B_λ is a topological semigroup, then the map $\lambda_{(\alpha,\beta)}\colon B_\lambda\to B_\lambda$ $[\rho_{(\beta,\alpha)}\colon B_\lambda\to B_\lambda]$ defined by the formula $\lambda_{(\alpha,\beta)}(x)=(\alpha,\beta)\cdot x$ $[\rho_{(\beta,\alpha)}(x)=x\cdot(\beta,\alpha)]$ is continuous. Therefore $A^\beta=\left(\lambda_{(\alpha,\beta)}\right)^{-1}(A^\alpha)$ $[A_\beta=\left(\rho_{(\beta,\alpha)}\right)^{-1}(A_\alpha)]$ is a closed subset of B_λ. □

For any $A\subseteq I_\lambda$, $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m\in I_\lambda$, $n,m\in\mathbb{N}$ we denote

$$U^{\alpha_1,\ldots,\alpha_n}(A)=U^{\alpha_1,\ldots,\alpha_n}\cup\{(\alpha_i,x)\mid x\in A, i=1,\ldots,n\},$$

$$U_{\beta_1,\ldots,\beta_m}(A)=U_{\beta_1,\ldots,\beta_m}\cup\{(x,\beta_i)\mid x\in A, i=1,\ldots,m\},$$

$$U(\alpha_1,\ldots,\alpha_n;A)=U^{\alpha_1,\ldots,\alpha_n}(A)\cap U_{\alpha_1,\ldots,\alpha_n}(A).$$

The following theorem gives a positive answer to Question 2.

Theorem 5. *Let λ be an infinite cardinal. Then the following conclusions hold:*

(i) τ_{mv} *is a minimal semigroup topology on* B_λ*;*

(ii) τ_{mh} *is a minimal semigroup topology on* B_λ*;*

(iii) τ_{mi} *is the coarsest semigroup inverse topology on* B_λ*, and hence is minimal semigroup inverse.*

Proof. (i) Suppose that there exists a Hausdorff semigroup topology τ_0 on B_λ which is coarser than τ_{mv}. Let V_0 be an element of a base of the topology τ_0 at the zero of B_λ. Then by Lemma 8 there exist an infinite subset A in I_λ and $\alpha_1, \ldots, \alpha_m \in I_\lambda$ ($m \in \mathbb{N}$) such that $U^{\alpha_1,\ldots,\alpha_m}(A) \subseteq V_0$. For any $\beta \in I_\lambda$, $\gamma \in I_\lambda \setminus \{\alpha_1, \ldots, \alpha_m\}$ the following conditions hold:

a) $(\alpha_i, \beta) = (\alpha_i, \delta) \cdot (\delta, \beta)$, where $\delta \in A \setminus \{\alpha_1, \ldots, \alpha_m\}$ and, obviously, $(\alpha_i, \delta), (\delta, \beta) \in U^{\alpha_1,\ldots,\alpha_m}(A)$ $(i = 1, \ldots, m)$;

b) $(\gamma, \beta) = (\gamma, \gamma) \cdot (\gamma, \beta)$, and $(\gamma, \gamma), (\gamma, \beta) \in U^{\alpha_1,\ldots,\alpha_m}(A)$.

Thus,

$$B_\lambda = U^{\alpha_1,\ldots,\alpha_m}(A) \cdot U^{\alpha_1,\ldots,\alpha_m}(A) \subseteq V_0 \cdot V_0$$

for any element V_0 of a base of the topology τ_0 at the zero of B_λ. This gives a contradiction with the continuity of the semigroup operation in (B_λ, τ_0). Therefore (B_λ, τ_{mv}) is a minimal topological semigroup.

The proof of item (ii) is similar to the proof of item (i).

(iii) Let τ be any Hausdorff semigroup inverse topology on B_λ. We define the maps: $\varphi\colon B_\lambda \to E(B_\lambda)$ and $\psi\colon B_\lambda \to E(B_\lambda)$ by formulae $\varphi(x) = xx^{-1}$ and $\psi(x) = x^{-1}x$. Since the topology τ is Hausdorff then the sets $\varphi^{-1}((\alpha, \alpha)) = \{(\alpha, \gamma) \mid \gamma \in I_\lambda\}$ and $\psi^{-1}((\alpha, \alpha)) = \{(\gamma, \alpha) \mid \gamma \in I_\lambda\}$ are closed for each $\alpha \in I_\lambda$, and hence $U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} \in \tau$ for all $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in I_\lambda$. Therefore $\tau_{mi} \subseteq \tau$. □

Theorem 6. *Let λ be an infinite cardinal. Then (B_λ, τ_{mv}), (B_λ, τ_{mh}), (B_λ, τ_{mi}) are H-closed topological semigroups.*

Proof. We shall show that the semigroup (B_λ, τ_{mi}) is H-closed. The proofs of H-closedness of the semigroups (B_λ, τ_{mh}) and (B_λ, τ_{mv}) are similar.

Suppose that there exists a topological semigroup S which contains (B_λ, τ_{mi}) as a non-closed subsemigroup. Then there exists $x \in \overline{B_\lambda} \setminus B_\lambda \subseteq S$. By Lemma 4, $x \cdot 0 = 0 \cdot x = 0$. Then for every open neighbourhood $W(0)$ in S there exist open neighbourhoods $U(0)$, $V(0)$, and $V(x)$ in S such that $V(0) \cap V(x) = \varnothing$, $U(0) \cap V(x) = \varnothing$, $V(0) \subseteq W(0)$, $U(0) \subseteq$

$W(0)$, $V(x)\cdot V(0)\subseteq U(0)$, and $V(0)\cdot V(x)\subseteq U(0)$. We can suppose that $U^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m}=U(0)\cap B_\lambda$ for some $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m\in I_\lambda$.

Since $|V(x)\cap B_\lambda|\geq\omega$, one of the following conditions holds:

1) the set $B_{i_0}=V(x)\cap\{(\alpha_{i_0},\gamma)\mid\gamma\in I_\lambda\}$ is infinite for some $i_0\in\{1,\dots,n\}$;

2) the set $B^{j_0}=V(x)\cap\{(\gamma,\alpha_{j_0})\mid\gamma\in I_\lambda\}$ is infinite for some $j_0\in\{1,\dots,m\}$.

In the first case we put

$$\Gamma_{i_0}=\{\gamma\in I_\lambda\mid(\alpha_{i_0},\gamma)\in V(x)\}.$$

Then the set $\{(\gamma,\gamma)\mid\gamma\in\Gamma_{i_0}\}\cap U^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_l}$ is infinite for any basic neighbourhood $U^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_l}$, $\gamma_1,\dots,\gamma_k,\delta_1,\dots,\delta_l\in I_\lambda$. Thus

$$B_{i_0}\cdot U^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_l}\nsubseteq U^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m},$$

a contradiction with $V(x)\cdot V(0)\subseteq U(0)$.

In the other case we put

$$\Gamma^{j_0}=\{\gamma\in I_\lambda\mid(\gamma,\alpha_{j_0})\in V(x)\}.$$

Then the set $\{(\gamma,\gamma)\mid\gamma\in\Gamma^{j_0}\}\cap U^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_l}$ is infinite for every basic neighbourhood $U^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_l}$, $\gamma_1,\dots,\gamma_k,\delta_1,\dots,\delta_l\in I_\lambda$. Hence

$$U^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_l}\cdot B^{j_0}\nsubseteq U^{\alpha_1,\dots,\alpha_n}_{\beta_1,\dots,\beta_m},$$

a contradiction with $V(0)\cdot V(x)\subseteq U(0)$.

Therefore the topological semigroup (B_λ,τ_{mi}) is H-closed. □

Theorem 6 and Corollary 3 imply

Corollary 8. *Let λ be an infinite cardinal. Then (B_λ,τ_{mv}) and (B_λ,τ_{mh}) are absolutely H-closed topological semigroups.*

Theorem 7. *For every cardinal $\lambda\geq\omega$ any continuous homomorphism from (B_λ,τ_{mv}) $[(B_\lambda,\tau_{mh})]$ into a locally compact topological semigroup S is annihilating.*

Proof. Let $h\colon(B_\lambda,\tau_{mv})\to S$ be a continuous homomorphism. If h is not annihilating, then by Corollary 3, h is algebraic isomorphism, and hence, since (B_λ,τ_{mv}) is a minimal topological semigroup, $h\colon B_\lambda\to S$ is a topological embedding.

By Theorem 6, $h(B_\lambda)$ is a closed subsemigroup of S and by Theorem 3.3.8 [11], $h(B_\lambda)$ is a locally compact topological semigroup. This is a contradiction with the fact that τ_{mv} is not a locally compact semigroup topology on B_λ.

The proof of the theorem for the semigroup (B_λ, τ_{mh}) is similar. □

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Add to Proof. The authors express their sincere thanks to the referee for very careful reading the manuscript and valuable remarks improving the presentation.

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